



## Letter to the Editor

Note on the complex zeros of  $H'_\nu(x) + i\zeta H_\nu(x) = 0$ 

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Received 13 September 2005; received in revised form 17 January 2006

**Abstract**

Approximations for the complex zeros in the  $\nu$ -plane of the Hankel functions  $H_\nu(x)$  and  $H'_\nu(x)$  are available in terms of the zeros of Airy functions. The corresponding zeros of a linear combination of Hankel functions are of interest when scattering from cylinders with a surface impedance is studied. A simple method to compute these zeros efficiently is presented.

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MSC: 33C10; 34E05; 78A45

Keywords: Zeros of special functions; Asymptotic expansions; Iterative solution

**1. Introduction**

The zeros of the Hankel functions in the complex  $\nu$ -plane relate to scattering from circular cylinders [2,3] and also to the Regge poles of quantum mechanics. One important function of these zeros is to determine the propagation constants of creeping waves [9,10]. Expressions for the zeros for the two basic scattering problems (two polarizations) can be found in Ref. [3]. A more difficult problem arises when an impedance boundary condition is used. One then looks for the zeros, in the complex variable corresponding to the index, of a linear combination of the Hankel function and its derivative. This problem was solved by Bouche et al. [2] by approximating the Hankel functions in terms of Airy functions and then applying a numerical procedure to the resulting approximate problem. Other combinations of numerical procedures, suitable for small values of  $x$ , have also been devised [4]. This note presents an approximate solution to the original problem.

**2. Discussion**

The zeros of  $H_\nu(x)$  and  $H'_\nu(x)$  are given in terms of the parameter  $x$  that corresponds to frequency, the zeros of the Airy function and its derivative  $a_s = -\alpha_s$ ,  $a'_s = -\beta_s$  [1,3] and the Fock parameter  $m = (x/2)^{1/3}$ . Asymptotic expansions

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will be used and the value of  $x$ ; and hence also of  $m$ , is therefore assumed to be large. Approximate zeros are given by

$$v_s = x + e^{i\pi/3} \alpha_s m - e^{-i\pi/3} \frac{\alpha_s^2}{60} m^{-1} - \frac{1}{140} \left( 1 - \frac{\alpha_s^3}{10} \right) m^{-3}, \quad (1)$$

$$\bar{v}_s = x + e^{i\pi/3} \beta_s m - \frac{e^{-i\pi/3}}{10} \left( \beta_s^{-1} + \frac{1}{6} \beta_s^2 \right) m^{-1} + \frac{1}{200} \left( \beta_s^{-3} + 4 + \frac{\beta_s^3}{7} \right) m^{-3}. \quad (2)$$

The combination of Hankel functions that is studied in this note,

$$H'_v(x) + i\zeta H_v(x) = 0, \quad (3)$$

involves the complex parameter  $\zeta$  that corresponds to either a surface impedance or a surface admittance depending on the polarization of the field [2]. We refer to the function  $H_v^{(1)}(x)$  [1]. Also in this case, the zeros can be found by extending the approach devised by Franz and Galle [6].

Asymptotic expansions for the Hankel function and its derivative are obtained from Watson's integral representation by means of the saddle-point method. From these expansions, in terms of Airy functions, the zeros in Eqs. (1) and (2) were extracted by means of Taylor expansions of the Airy functions, in the neighbourhood of their zeros [6]. By simply combining the expansions one obtains approximate zeros, corresponding to the impedance  $\zeta$ , in terms of the zeros of either the Airy function or its derivative.

$$v_{s\zeta} = x + e^{i\pi/3} \beta_s m + i\zeta e^{-i\pi/3} m^2 \left( \frac{1}{\beta_s} - \frac{e^{i\pi/3}}{15m^2} \right) - \frac{e^{-i\pi/3}}{10m} \left( \frac{1}{\beta_s} + \frac{\beta_s^2}{6} \right), \quad m|\zeta| < 1, \quad (4)$$

$$v_{s\zeta} = x + e^{i\pi/3} \alpha_s m + \frac{1}{i\zeta} \left( 1 + \frac{e^{i\pi/3} \alpha_s}{15m^2} \right) - \frac{e^{-i\pi/3} \alpha_s^2}{60m}, \quad m|\zeta| > 1. \quad (5)$$

These expressions are fairly accurate for the first zeros, i.e., small values of the index  $s$ . The error with respect to the exact zero is larger when  $m|\zeta|$  is close to 1 since neither of the Taylor expansions is accurate when the linear combination does not clearly emphasize one of the functions, cf. Eq. (3). Similar results were derived by Keller et al. [7] from a linear combination of Airy functions with constant coefficients, under the assumption that  $m|\zeta|$  is either large or small. Cochran [5] uses uniform expansions systematically to derive higher-order terms, but with no significant improvement in accuracy. Streifer [11] also derives higher order terms for large and small  $m|\zeta|$  but the method has the same deficiency for  $m|\zeta| \approx 1$ . Fig. 1 shows the difference between the approximate zero and the exact zero as a

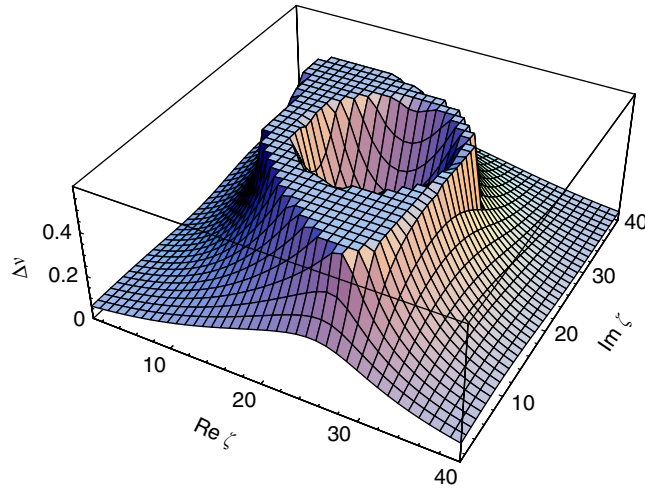


Fig. 1. The error  $\Delta v = |v_{s\zeta} - v_{se}|$ , as a function of the complex surface impedance  $\zeta$ , computed by means of asymptotic expansions of the Hankel function and Taylor expansions of the Airy function:  $x = 10$ ,  $s = 1$ . A grid that corresponds to  $-1.3 < \text{Re } \zeta < 1.3$ ,  $-1.3 < \text{Im } \zeta < 1.3$  is used. The surface is truncated at about half the maximum value for the sake of clarity. The circle  $m|\zeta| = 1$  can be discerned in the graph.

function of the impedance  $\zeta$ . The accuracy is poor in a fairly large domain  $m|\zeta| \approx 1$ . This region also presents some difficulties as regards uniformity when the boundary layer method is used [8]. In order to improve accuracy, asymptotic expansions for the Hankel function, in terms of only one type of Airy function, could be used [6]. Eq. (3) can then be written in the approximate form [2]

$$\text{Ai}'(-z) - q \text{Ai}(-z) = 0, \quad (6)$$

$$\zeta = m\zeta e^{-i\pi/6}, \quad m = \left(\frac{x}{2}\right)^{1/3}, \quad z = \frac{v-x}{m} e^{-i\pi/3},$$

$$q(z, \zeta) = \frac{\zeta \left(1 - \frac{e^{i\pi/3}z}{15m^2} - \frac{e^{-i\pi/3}z^2}{20m^4} \left(\frac{13}{63} - \frac{z^3}{360}\right)\right) + \frac{e^{i\pi/3}}{10m^2} \left(1 + \frac{z^3}{6}\right) + \frac{e^{-i\pi/3}z}{60m^4} \left(1 + \frac{z^3}{56}\right)}{1 + \frac{e^{i\pi/3}z}{15m^2} + \frac{e^{-i\pi/3}z^2}{40m^4} \left(\frac{19}{63} + \frac{z^3}{180}\right) - \zeta \left(\frac{e^{i\pi/3}z^2}{60m^2} - \frac{e^{-i\pi/3}}{140m^4} \left(1 - \frac{z^3}{3}\right)\right)}.$$

The Airy function and its derivative have asymptotic expansions [1] for large arguments and notably for large arguments close to the negative real axis where the zeros of the Airy functions are located. Rearranging Eq. (6) yields,

$$\frac{\tan(\zeta + \pi/4) \sum_0^\infty (-1)^k c_{2k} \zeta^{-2k} - \sum_0^\infty (-1)^k c_{2k+1} \zeta^{-2k-1}}{\sum_0^\infty (-1)^k d_{2k} \zeta^{-2k} + \tan(\zeta + \pi/4) \sum_0^\infty (-1)^k d_{2k+1} \zeta^{-2k-1}} = -\frac{z^{1/2}}{q}, \quad (7)$$

$$\zeta = \frac{2}{3} z^{3/2},$$

$$c_0 = 1, \quad c_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})},$$

$$d_0 = 1, \quad d_k = -\frac{6k+1}{6k-1} c_k.$$

Three or four terms are useful in the sums in Eq. (7) and with some shorthand for these sums one obtains,

$$\tan\left(\zeta + \frac{\pi}{4}\right) = \frac{S_{c1} - (z^{1/2}/q) S_{d0}}{S_{c0} + (z^{1/2}/q) S_{d1}} = -Q. \quad (8)$$

By means of the inverse tangent function it is possible to formulate a stable iteration for the zeros, cf. [1, p. 18]. The periodicity of the tangent function introduces an index  $s > 0$  that enumerates the zeros. There are two possible formulations since one could relate either to the zeros of the Airy function or to the zeros of its derivative

$$z_{n+1} = \left( (4s-1) \frac{3\pi}{8} - \frac{3}{2} \arctan Q_n \right)^{2/3}, \quad (9)$$

$$z_{n+1} = \left( (4s-3) \frac{3\pi}{8} + \frac{3}{2} \arctan \frac{1}{Q_n} \right)^{2/3}. \quad (10)$$

The iteration achieves  $|z_{n+1} - z_n| < \varepsilon \approx 10^{-5}$  after typically 5–10 steps. One notes that the first term in the expansion for the zeros of the Airy function and its derivative [1] can be retrieved by letting  $Q$  tend to extreme values in Eqs. (9) and (10). Some caution is appropriate if the value of  $\zeta$  should bring the iteration procedure close to the branch points of the arctan function. The iteration is not sensitive to the initial value and the accuracy improves with the number of terms included from the Hankel expansion, i.e., the complexity of  $q$ .

The actual zeros are obtained from,

$$v_s \zeta = x + e^{i\pi/3} z_s \zeta m. \quad (11)$$

One notes that the error, shown in Fig. 2, increases close to the branch points associated with the arctan function [1]. The branch cut, as it appears in the  $\zeta$ -plane, lies between the branch points and marks the transition to other values of the index  $s$ , cf. Eq. (9). The branch points could possibly have interpretations in the context of waves on the surface

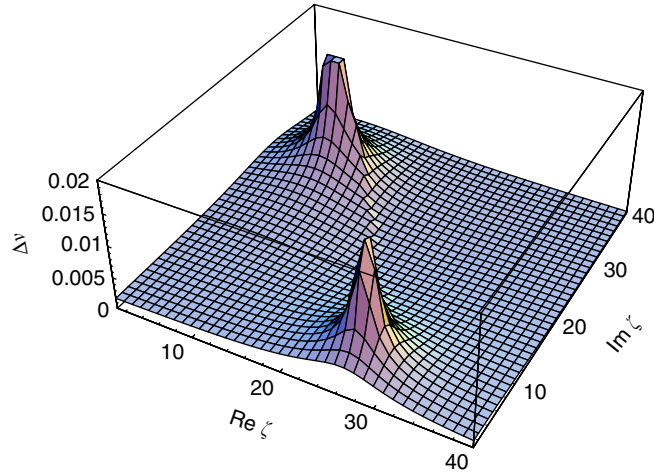


Fig. 2. The error  $\Delta v = |v_{s\zeta} - v_{se}|$ , as a function of the complex surface impedance  $\zeta$ , computed by means of asymptotic expansions of the Hankel function and the Airy function followed by the iteration in Eq. (12):  $x = 10$ ,  $s = 1$ . A grid that corresponds to  $-1.3 < \text{Re } \zeta < 1.3$ ,  $-1.3 < \text{Im } \zeta < 1.3$  is used.

[9]. Higher zeros  $s > 1$  are of interest for coated structures [10] and can be computed without difficulty even though the accuracy decreases gradually with the index  $s$ .

Yet another possibility is to use the Airy functions directly, starting from a point  $z_0$  close to the zero value. A Taylor expansion yields the following iteration:

$$z_{n+1} = z_n - \frac{\text{Ai}'(-z_n) - q_n \text{Ai}(-z_n)}{z_n \text{Ai}(-z_n) + q_n \text{Ai}'(-z_n)}, \quad (12)$$

in terms of the Airy function and its derivative and the function  $q$  introduced in Eq. (6). There is no index  $s$ , as in Eqs. (4) and (5) and (9) and (10) that specifies which zero that is sought. A fairly good initial estimate of the zero is required but such an estimate is readily available from Eqs. (9) and (10). These iterations are easily combined and the final result is shown in Fig. 2. The accuracy is much better than that of Eqs. (9) and (10) alone, particularly for  $s = 1$ , due to the asymptotic approximation of the Airy function. Accuracy increases with the parameter  $x$  up to a limit ( $x = 370$ ) where instability sets in. The accuracy is fair down to  $x = 1$ .

### 3. Conclusion

An approximate method has been devised to compute the zeros of a linear combination of Hankel functions efficiently. The accuracy of the zeros is sufficient to provide good initial values for root finding algorithms. Should the accuracy requirements be moderate, these approximate zeros may suffice as is for most, or all, values of the surface impedance.

### Acknowledgements

The support from the KK foundation, AerotechTelub AB and Chelton Applied Composites AB is gratefully acknowledged.

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