

# Positivity-preserving interpolation of positive data by rational cubics

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## Abstract

This work is a contribution towards the graphical display of data when it is positive. The data are required to be represented in such a way that its visual display looks smooth and pleasant, its positive shape is preserved everywhere and the computation cost is economical. A  $C^1$  piecewise rational cubic function, in its most general form, has been utilized for this objective. The method is implemented for the 1D data initially and then it is extended to an interpolating rational bicubic form for the data arranged over a rectangular grid. Simple sufficient conditions are developed on the free parameters in the description of the rational function to visualize the positive data in the form of positive curves and surfaces.

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## 1. Introduction

In recent years, a good amount of work has been published [1–19] that focuses on shape preserving curves and surfaces. Positivity is an important shape property. There are many physical situations where entities only have a meaning when their values are positive. For example, in a probability distribution, the presentation is always positive. Similarly, when dealing with the samples of populations, the data are always in positive figures. Another application area is in the observation of gas discharge when certain chemical experiments are in process. One can also find applications in three dimensions which include interpolation for process with positivity as a typical requirement. Therefore, it is important to discuss positive interpolation to provide a computationally economical and visually pleasing solution to the problems of different scientific phenomena.

Some of the work, on positivity, has already been shown in [2–5,9–11,14,18,19]. Schmidt and Hess [19] used cubic polynomials, derived the necessary and sufficient conditions to make the interpolant positive. They also have given comparatively simpler sufficient conditions that make the basis of their interpolation algorithm in which the slopes at data points are estimated. The problem of positivity, when the slopes at data points are given, is discussed in [4]. Butt and Brodlie [4] have used cubic polynomials in the development of their scheme where as Sarfraz [14] has used piecewise rational cubic functions. The algorithm of Butt and Brodlie [4] works by inserting one or two extra knots,

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wherever necessary, to preserve the shape of positive data. The rational function used in [14] has one free parameter that is used to generate the desired curves when further refinement is needed.

Piah et al. [11] have discussed the problem of positivity preserving for scattered data interpolation. Nadler [10], Chang and Sederberg [5] have also discussed the problem of nonnegative interpolation. They have considered nonnegative data arranged over a triangular mesh and have interpolated each triangular patch using a bivariate quadratic function. Brodlie et al. [3] have used a bicubic polynomial to visualize 3D positive data. The algorithm in [3] works by inserting one or two extra knots to preserve the shape of 3D positive data wherever necessary. Asim [2] has discussed the problem of positivity using the Shepard interpolation. A general-purpose curve method has been described in [6]. It is meant for computing interpolating polynomial splines with arbitrary constraints on their shape and satisfying separable or nonseparable boundary conditions. The scheme helps having periodic shape-preserving spline interpolation and the construction of visually pleasing curves. A local shape-preserving interpolation method has been described in [1]. This method uses space curves in its description. A curve and surface method has been reported in [7], it uses variable degree polynomial splines in its construction.

This paper is concerned with the preservation of positive data using a  $C^1$  piecewise rational cubic function. The rational function involves four free parameters in such a way that two of them are constrained for presentation of positive curves through positive data while the other two provide extra freedom to the user with the shape of the curve to modify as desired. The curve method proposed is the extension of the curve scheme in [17] where the user had no freedom to play with the further modification of the visual display of the positive curve description. Further more, as a major contribution in this paper, the problem has been extended to visualize 3D positive data by arranging the data over rectangular mesh. To interpolate each rectangular patch, we have used a rational bicubic function which is the extension of the rational cubic function proposed in Section 2. The method, under consideration in this paper, has the important and advantageous features. It is designed in such a way that no additional points (knots) need to be supplied. It improves its predecessor methods in [14,15] by providing the extra freedom to the user to refine the curves and surfaces.

The organization of the paper has been made in such a way that Section 2 describes about the rational cubic function to be used in the curve and surface schemes. The positivity problem is discussed in Section 3 for the generation of a  $C^1$  spline which can preserve the shape of positive data. The sufficient constraints, on the shape parameters, have also been derived in this section to preserve and control the positive interpolant. Extension to the surfaces has been made in Section 4, where the rational cubic function is extended to a rational bicubic function. In Section 5, a scheme is developed for the positivity of 3D data. The curve and surface schemes have been demonstrated in Section 6. Finally, Section 7 concludes the paper.

## 2. Rational cubic function

In this section, the  $C^1$  piecewise rational cubic function used in this paper is introduced which was initially developed by Sarfraz [13]. Let  $(x_i, f_i)$ ,  $i = 0, 1, 2, \dots, n$ , be a given set of data points where  $x_0 < x_1 < x_2 < \dots < x_n$ . Let

$$h_i = x_{i+1} - x_i, \Delta_i = \frac{f_{i+1} - f_i}{h_i}. \quad (1)$$

In each interval  $I_i = [x_i, x_{i+1}]$ , a rational cubic function  $S_i(x)$  may be defined as

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \quad (2)$$

with

$$\begin{aligned} p_i(\theta) &= r_i f_i (1 - \theta)^3 + (u_i f_i + h_i r_i \Delta_i) (1 - \theta)^2 \theta + (v_i f_{i+1} - h_i w_i \Delta_{i+1}) (1 - \theta) \theta^2 + w_i f_{i+1} \theta^3, \\ q_i(\theta) &= r_i (1 - \theta)^3 + u_i (1 - \theta)^2 \theta + v_i (1 - \theta) \theta^2 + w_i \theta^3, \end{aligned}$$

where

$$\theta = \frac{x - x_i}{h_i}. \quad (3)$$

The rational cubic function (2) has the following properties:

$$\left. \begin{aligned} S(x_i) &= f_i, S(x_{i+1}) = f_{i+1}, \\ S^{(1)}(x_i) &= d_i, S^{(1)}(x_{i+1}) = d_{i+1}. \end{aligned} \right\} \quad (4)$$

Here  $S^{(1)}$  denotes the derivative with respect to  $x$  and  $d_i$  denotes the derivative values (given or estimated by some method) at the knots  $x_i$ .  $S(x) \in C^1[x_0, x_n]$  has freedom of  $r_i, u_i, v_i, w_i$  as parameters in the interval  $[x_i, x_{i+1}]$ .

It is noted that in each interval  $I_i$ , when  $r_i = w_i = 1$  and  $u_i = v_i = 3$ , the piecewise rational cubic function reduces to the standard cubic Hermite. The case of  $r_i = w_i = 1$  and  $u_i = v_i$  provides the special case of rational cubic introduced in [4].

### 3. Positive curve interpolation

The problem of positive interpolation can be described as follows: for given data points  $(x_0, f_0), (x_1, f_1), \dots, (x_n, f_n)$  with  $x_0 < x_1 < \dots < x_n$  and  $f_0 > 0, f_1 > 0, \dots, f_n > 0$ , construct an interpolant  $S(x) \in C^1[x_0, x_n]$  which is positive on the whole interval  $[x_0, x_n]$ . i.e.

$$S(x) > 0, \quad x_0 \leq x \leq x_n.$$

The key idea is to determine conditions for  $r_i, u_i, v_i, w_i$  which guarantee positivity. It can be noted that  $r_i, u_i, v_i, w_i > 0$  guarantee strictly positive denominator  $q_i(\theta)$ . Therefore the conditions

$$r_i > 0, u_i > 0, v_i > 0, w_i > 0 \quad (5)$$

are sufficient for  $q_i(\theta) > 0$ . Thus, the positivity of  $S_i(x) = p_i(\theta)/q_i(\theta)$  depends on the positivity of cubic polynomial  $p_i(\theta)$ , and the problem reduces to the determination of appropriate values of  $r_i, u_i, v_i, w_i$ , for which the polynomial  $p_i(\theta) > 0$ .  $p_i(\theta)$  can be re-written as

$$p_i(\theta) = \alpha_i \theta^3 + \beta_i \theta^2 + \gamma_i \theta + \delta_i, \quad (6)$$

where

$$\alpha_i = (w_i - v_i)f_{i+1} - (r_i - u_i)f_i + (w_i d_{i+1} + r_i d_i)h_i,$$

$$\beta_i = v_i f_{i+1} + (3r_i - 2u_i)f_i - (w_i d_{i+1} + 2r_i d_i)h_i,$$

$$\gamma_i = r_i d_i h_i - (3r_i - u_i)f_i,$$

$$\delta_i = r_i f_i.$$

For strict inequality (for positive data) in (6) [19],  $p_i(\theta) > 0$  if

$$(p'_i(0), p'_i(1)) \in R_1 \cup R_2, \quad (7)$$

where

$$R_1 = \left\{ (a, b) : a > \frac{-3f_i}{h_i}, b < \frac{3f_{i+1}}{h_i} \right\},$$

$$R_2 = \left\{ (a, b) : 36f_i f_{i+1}(a^2 + b^2 + ab - 3A_i(a+b) + 3A_i^2) + 3(f_{i+1}a - f_i b)(2h_i ab - 3f_{i+1}a) \right. \\ \left. + 3f_i b) + 4h_i(f_{i+1}a^3 - f_i b^3) - h_i^2 a^2 b^2 > 0 \right\}$$

Since

$$p'_i(0) = \frac{-3r_i f_i}{h_i} + \frac{u_i f_i}{h_i} + r_i d_i,$$

$$p'_i(1) = \frac{3w_i f_{i+1}}{h_i} + \frac{-v_i f_{i+1}}{h_i} + w_i d_{i+1}.$$

Eq. (7) is true when

$$(p'_i(0), p'_i(1)) \in R_1 \quad (8)$$

or

$$p'_i(0) > \frac{-3f_i}{h_i}, \quad p'_i(1) < \frac{3f_{i+1}}{h_i}.$$

This leads to the following constraints:

$$u_i > \frac{-h_i r_i d_i}{f_i}, \quad v_i > \frac{h_i w_i d_{i+1}}{f_{i+1}}. \quad (9)$$

Further  $(p'_i(0), p'_i(1)) \in R_2$  if

$$\begin{aligned} &\phi(r_i, u_i, v_i, w_i) \\ &= 36f_i f_{i+1} [\phi_1^2(r_i, u_i) + \phi_2^2(v_i, w_i) + \phi_1(r_i, u_i)\phi_2(v_i, w_i) - 3\Delta_i(\phi_1(r_i, u_i) + \phi_2(v_i, w_i)) + 3\Delta_i^2] \\ &\quad + 3[f_{i+1}\phi_1(r_i, u_i) - f_i\phi_2(v_i, w_i)][2h_i\phi_1(r_i, u_i)\phi_2(v_i, w_i) - 3f_{i+1}\phi_1(r_i, u_i) + 3f_i\phi_2(v_i, w_i)] \\ &\quad + 4h_i[f_{i+1}\phi_1^3(r_i, u_i) - f_i\phi_2^3(v_i, w_i)] - h_i^2\phi_1^2(r_i, u_i)\phi_2^2(v_i, w_i) \geq 0, \end{aligned} \quad (10)$$

where

$$\phi_1(r_i, u_i) = p'_i(0), \quad \phi_2(v_i, w_i) = p'_i(1).$$

Also constraints on  $u_i$  and  $v_i$  can be determined from Eq. (10), but it requires a lot of computations. So an efficient and reasonably acceptable choice is to use conditions given in (9). This leads to the following:

**Theorem 3.1.** *The rational cubic polynomial (2) preserves positivity if*

$$u_i > \text{Max} \left\{ 0, \frac{-h_i r_i d_i}{f_i} \right\}, \quad v_i > \text{Max} \left\{ 0, \frac{h_i w_i d_{i+1}}{f_{i+1}} \right\}. \quad (11)$$

**Remark 3.1.** In most applications, the derivative parameters  $\{d_i\}$  are not given and hence must be determined either from the given data or by some other means. In this article, they are computed from the given data in such a way that the  $C^1$  smoothness of the interpolant is maintained. These methods are the approximations based on various mathematical theories. Arithmetic, geometric, and harmonic mean descriptions of such approximations can be found in the current literature [14–17]. The proposed scheme here adopted arithmetic mean choice for its manipulation.

**Remark 3.2.** The parameters  $u_i$  and  $v_i$  have been constrained to preserve the shape of positive data. However, the user can choose freely the values of parameters  $r_i$  and  $w_i$  to obtain a desired positive curve. This freedom is utilized in producing curves in Section 6. Some more description on the suitability of these parameters is also given in Section 6.

#### 4. Bivariate description

The rational cubic function (2) is extended to a rational bicubic function defined over a rectangular mesh  $D = [x_0, x_m] \times [y_0, y_n]$ . Let  $\pi : a = x_0 < x_1 < \dots < x_m = b$  be a partition of  $[a, b]$  and let  $\hat{\pi} : c = y_0 < y_1 < \dots < y_n = d$  be a partition of  $[c, d]$ . Rational bicubic function is defined over each rectangular patch  $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ ,  $i = 0, 1, \dots, m-1$ ;  $j = 0, 1, \dots, n-1$  as follows:

$$S(x, y) = S_{i,j}(x, y) = A_i(\theta)F(i, j)\hat{A}_j^T(\hat{\theta}), \quad (12)$$

where

$$\hat{\theta} = \frac{y - y_j}{\hat{h}_j},$$

$$F(i, j) = \begin{bmatrix} F_{i,j} & F_{i,j+1} & F_{i,j}^y & F_{i,j+1}^y \\ F_{i+1,j} & F_{i+1,j+1} & F_{i+1,j}^y & F_{i+1,j+1}^y \\ F_{i,j}^x & F_{i,j+1}^x & F_{i,j}^{xy} & F_{i,j+1}^{xy} \\ F_{i+1,j}^x & F_{i+1,j+1}^x & F_{i+1,j}^{xy} & F_{i+1,j+1}^{xy} \end{bmatrix}, \quad (13)$$

$$\begin{aligned} A_i(\theta) &= [a_0(\theta) \ a_1(\theta) \ a_2(\theta) \ a_3(\theta)], \\ \hat{A}_j(\hat{\theta}) &= [\hat{a}_0(\hat{\theta}) \ \hat{a}_1(\hat{\theta}) \ \hat{a}_2(\hat{\theta}) \ \hat{a}_3(\hat{\theta})], \end{aligned} \quad (14)$$

and

$$\begin{aligned} a_0 &= \frac{r_{i,j}(1-\theta)^3 + u_{i,j}\theta(1-\theta)^2}{q_i(\theta)}, \quad a_1 = \frac{\omega_{i,j}\theta^3 + v_{i,j}\theta^2(1-\theta)}{q_i(\theta)}, \\ a_2 &= \frac{h_i r_{i,j}\theta(1-\theta)^2}{q_i(\theta)}, \quad a_3 = \frac{-h_i w_{i,j}\theta^2(1-\theta)}{q_i(\theta)}, \\ q_i(\theta) &= r_{i,j}(1-\theta)^3 + u_{i,j}(1-\theta)^2\theta + v_{i,j}(1-\theta)\theta^2 + w_{i,j}\theta^3, \\ \hat{a}_0 &= \frac{\hat{r}_{i,j}(1-\hat{\theta})^3 + \hat{u}_{i,j}\hat{\theta}(1-\hat{\theta})^2}{q_j(\hat{\theta})}, \quad \hat{a}_1 = \frac{\hat{w}_{i,j}\hat{\theta}^3 + \hat{v}_{i,j}\hat{\theta}^2(1-\hat{\theta})}{q_j(\hat{\theta})}, \\ \hat{a}_2 &= \frac{\hat{h}_j \hat{r}_{i,j}\hat{\theta}(1-\hat{\theta})^2}{q_j(\hat{\theta})}, \quad \hat{a}_3 = \frac{-\hat{h}_j \hat{w}_{i,j}\hat{\theta}^2(1-\hat{\theta})}{q_j(\hat{\theta})}, \\ q_j(\hat{\theta}) &= \hat{r}_{i,j}(1-\hat{\theta})^3 + \hat{u}_{i,j}(1-\hat{\theta})^2\hat{\theta} + \hat{v}_{i,j}(1-\hat{\theta})\hat{\theta}^2 + \hat{w}_{i,j}\hat{\theta}^3. \end{aligned}$$

$F(i, j)$  is a square matrix of order 4 with its entries as the first and mixed first partial derivatives at the four corner positions of the cubic patch.

## 5. Positive surface interpolation

Let  $\pi : a = x_0 < x_1 < \dots < x_m = b$  be a partition of  $[a, b]$  and let  $\hat{\pi} : c = y_0 < y_1 < \dots < y_n = d$  be a partition of  $[c, d]$ . Let  $\{F_{i,j} : i = 0, 1, 2, \dots, m; j = 0, 1, 2, \dots, n\}$  be a set of  $(m+1) \times (n+1)$  data values such that  $F_{i,j} > 0, \forall i = 0, 1, 2, \dots, m; j = 0, 1, 2, \dots, n$ . The aim is to construct a piecewise rational bicubic function  $S(x, y)$  on  $D = [a, b] \times [c, d]$  such that

$$S(x_i, y_j) = F_{i,j}, \quad i = 0, 1, 2, \dots, m; \quad j = 0, 1, 2, \dots, n,$$

and  $S(x, y) > 0$  for  $(x, y) \in D$ . We can express (12) as

$$S(x, y) = \frac{\alpha_{i,j}(1-\theta)^3 + \beta_{i,j}(1-\theta)^2\theta + \gamma_{i,j}(1-\theta)\theta^2 + \delta_{i,j}\theta^3}{r_{i,j}(1-\theta)^3 + u_{i,j}(1-\theta)^2\theta + v_{i,j}(1-\theta)\theta^2 + w_{i,j}\theta^3}, \quad (15)$$

where

$$\begin{aligned} \alpha_{i,j} &= r_{i,j}F_{i,j}\hat{a}_0 + r_{i,j}F_{i,j+1}\hat{a}_1 + r_{i,j}F_{i,j}^y\hat{a}_2 + r_{i,j}F_{i,j+1}^y\hat{a}_3, \\ \beta_{i,j} &= (u_{i,j}F_{i,j} + h_i r_{i,j}F_{i,j}^x)\hat{a}_0 + (u_{i,j}F_{i,j+1} + h_i r_{i,j}F_{i,j+1}^x)\hat{a}_1 + (u_{i,j}F_{i,j}^y + h_i r_{i,j}F_{i,j}^{xy})\hat{a}_2 + (u_{i,j}F_{i,j+1}^y \\ &\quad + h_i r_{i,j}F_{i,j+1}^{xy})\hat{a}_3, \\ \gamma_{i,j} &= (v_{i,j}F_{i+1,j} - h_i w_{i,j}F_{i+1,j}^x)\hat{a}_0 + (v_{i,j}F_{i+1,j+1} - h_i w_{i,j}F_{i+1,j+1}^x)\hat{a}_1 + (v_{i,j}F_{i+1,j}^y - h_i w_{i,j}F_{i+1,j}^{xy})\hat{a}_2 \\ &\quad + (v_{i,j}F_{i+1,j+1}^y - h_i w_{i,j}F_{i+1,j+1}^{xy})\hat{a}_3, \\ \delta_{i,j} &= w_{i,j}F_{i+1,j}\hat{a}_0 + w_{i,j}F_{i+1,j+1}\hat{a}_1 + w_{i,j}F_{i+1,j}^y\hat{a}_2 + w_{i,j}F_{i+1,j+1}^y\hat{a}_3. \end{aligned}$$

So  $S(x, y)$  is positive if

$$\alpha_{i,j}(1-\theta)^3 + \beta_{i,j}\theta(1-\theta)^2 + \gamma_{i,j}\theta^2(1-\theta) + \delta_{i,j}\theta^3 > 0, \quad (16)$$

$$r_{i,j} > 0, u_{i,j} > 0, v_{i,j} > 0, w_{i,j} > 0, \quad (17)$$

$$\hat{r}_{i,j} > 0, \hat{u}_{i,j} > 0, \hat{v}_{i,j} > 0, \hat{w}_{i,j} > 0. \quad (18)$$

Throughout the rest of the paper, we will have the assumption that the constraints in (17) and (18) always exist. Thus, (16) is satisfied if the following constraints hold:

$$\alpha_{i,j} > 0, \beta_{i,j} > 0, \gamma_{i,j} > 0, \delta_{i,j} > 0.$$

Consider  $\alpha_{i,j}$  which, after simplification, leads to

$$\alpha_{i,j} = \frac{\left\{ r_{i,j}\hat{r}_{i,j}F_{i,j}(1-\hat{\theta})^3 + (r_{i,j}\hat{u}_{i,j}F_{i,j} + \hat{h}_j r_{i,j}\hat{r}_{i,j}F_{i,j}^y)(1-\hat{\theta})^2\hat{\theta} \right.}{\left. + (r_{i,j}\hat{v}_{i,j}F_{i,j+1} - \hat{h}_j r_{i,j}\hat{w}_{i,j}F_{i,j+1}^y)(1-\hat{\theta})\hat{\theta}^2 + r_{i,j}\hat{w}_{i,j}F_{i,j+1}\hat{\theta}^3 \right\}}{\hat{r}_{i,j}(1-\hat{\theta})^3 + \hat{u}_{i,j}(1-\hat{\theta})^2\hat{\theta} + \hat{v}_{i,j}(1-\hat{\theta})\hat{\theta}^2 + \hat{w}_{i,j}\hat{\theta}^3}. \quad (19)$$

Note that  $\alpha_{i,j} > 0$  if

$$\hat{u}_{i,j} > \frac{-\hat{h}_j\hat{r}_{i,j}F_{i,j}^y}{F_{i,j}}, \quad \hat{v}_{i,j} > \frac{\hat{h}_j\hat{w}_{i,j}F_{i,j+1}^y}{F_{i,j+1}}. \quad (20)$$

Similarly it can be shown that  $\beta_{i,j} > 0$  if

$$\hat{u}_{i,j} > \frac{-\hat{h}_j\hat{r}_{i,j}F_{i,j}^y}{F_{i,j}}, \quad \hat{v}_{i,j} > \frac{\hat{h}_j\hat{w}_{i,j}F_{i,j+1}^y}{F_{i,j+1}}, \quad (21)$$

and

$$u_{i,j} > \text{Max} \left\{ \frac{-h_i r_{i,j} F_{i,j}^x}{F_{i,j}}, \frac{-h_i r_{i,j} F_{i,j+1}^x}{F_{i,j+1}}, \frac{-h_i r_{i,j} (\hat{u}_{i,j} F_{i,j}^x + \hat{h}_j \hat{r}_{i,j} F_{i,j}^{xy})}{\hat{u}_{i,j} F_{i,j} + \hat{h}_j \hat{r}_{i,j} F_{i,j}^y}, \frac{-h_i r_{i,j} (\hat{v}_{i,j} F_{i,j+1}^x - \hat{h}_j \hat{w}_{i,j} F_{i,j+1}^{xy})}{\hat{v}_{i,j} F_{i,j+1} - \hat{h}_j \hat{w}_{i,j} F_{i,j+1}^y} \right\}. \quad (22)$$

Following the same reasons, it can be shown that  $\gamma_{i,j} > 0$  if

$$\hat{u}_{i,j} > \frac{-\hat{h}_j\hat{r}_{i,j}F_{i+1,j}^y}{F_{i+1,j}}, \quad \hat{v}_{i,j} > \frac{\hat{h}_j\hat{w}_{i,j}F_{i+1,j+1}^y}{F_{i+1,j+1}}, \quad (23)$$

and

$$v_{i,j} > \text{Max} \left\{ \frac{h_i w_{i,j} F_{i+1,j}^x}{F_{i+1,j}}, \frac{h_i w_{i,j} F_{i+1,j+1}^x}{F_{i+1,j+1}}, \frac{h_i w_{i,j} (\hat{u}_{i,j} F_{i+1,j}^x + \hat{h}_j \hat{r}_{i,j} F_{i+1,j}^{xy})}{\hat{u}_{i,j} F_{i+1,j} + \hat{h}_j \hat{r}_{i,j} F_{i+1,j}^y}, \frac{h_i w_{i,j} (\hat{v}_{i,j} F_{i+1,j+1}^x - \hat{h}_j \hat{w}_{i,j} F_{i+1,j+1}^{xy})}{\hat{v}_{i,j} F_{i+1,j+1} - \hat{h}_j \hat{w}_{i,j} F_{i+1,j+1}^y} \right\}, \quad (24)$$

and finally  $\delta_{i,j} > 0$  if

$$\hat{u}_{i,j} > \frac{-\hat{h}_j \hat{r}_{i,j} F_{i+1,j}^y}{F_{i+1,j}}, \quad \hat{v}_{i,j} > \frac{\hat{h}_j \hat{w}_{i,j} F_{i+1,j+1}^y}{F_{i+1,j+1}}. \quad (25)$$

The above can be summarized as:

**Theorem 5.1.** *The piecewise rational bicubic interpolant  $S(x, y)$  defined over the rectangular mesh  $D = [x_0, x_m] \times [y_0, y_n]$ , in (12), is positive if the following sufficient conditions are satisfied:*

$$S(x_i, y_j) = F_{i,j} > 0, \quad i = 0, 1, 2, \dots, m; \quad j = 0, 1, 2, \dots, n, \quad (26)$$

$$r_{i,j} > 0, \hat{r}_{i,j} > 0, w_{i,j} > 0, \hat{w}_{i,j} > 0, \quad (27)$$

$$\hat{u}_{i,j} > \text{Max} \left\{ 0, \frac{-\hat{h}_j \hat{r}_{i,j} F_{i,j}^y}{F_{i,j}}, \frac{-\hat{h}_j \hat{r}_{i,j} F_{i+1,j}^y}{F_{i+1,j}} \right\}, \quad (28)$$

$$\hat{v}_{i,j} > \text{Max} \left\{ 0, \frac{\hat{h}_j \hat{w}_{i,j} F_{i,j+1}^y}{F_{i,j+1}}, \frac{\hat{h}_j \hat{w}_{i,j} F_{i+1,j+1}^y}{F_{i+1,j+1}} \right\}, \quad (29)$$

$$u_{i,j} > \text{Max} \left\{ 0, \frac{-h_i r_{i,j} F_{i,j}^x}{F_{i,j}}, \frac{-h_i r_{i,j} F_{i,j+1}^x}{F_{i,j+1}}, \frac{-h_i r_{i,j} (\hat{u}_{i,j} F_{i,j}^{xy} + \hat{h}_j \hat{r}_{i,j} F_{i,j}^{xy})}{\hat{u}_{i,j} F_{i,j} + \hat{h}_j \hat{r}_{i,j} F_{i,j}^y}, \frac{-h_i r_{i,j} (\hat{v}_{i,j} F_{i,j+1}^x - \hat{h}_j \hat{w}_{i,j} F_{i,j+1}^{xy})}{\hat{v}_{i,j} F_{i,j+1} - \hat{h}_j \hat{w}_{i,j} F_{i,j+1}^y} \right\}, \quad (30)$$

$$v_{i,j} > \text{Max} \left\{ 0, \frac{h_i w_{i,j} F_{i+1,j}^x}{F_{i+1,j}}, \frac{h_i w_{i,j} F_{i+1,j+1}^x}{F_{i+1,j+1}}, \frac{h_i w_{i,j} (\hat{u}_{i,j} F_{i+1,j}^x + \hat{h}_j \hat{w}_{i,j} F_{i+1,j}^{xy})}{\hat{u}_{i,j} F_{i+1,j} + \hat{h}_j \hat{w}_{i,j} F_{i+1,j}^y}, \frac{h_i w_{i,j} (\hat{v}_{i,j} F_{i+1,j+1}^x - \hat{h}_j \hat{w}_{i,j} F_{i+1,j+1}^{xy})}{\hat{v}_{i,j} F_{i+1,j+1} - \hat{h}_j \hat{w}_{i,j} F_{i+1,j+1}^y} \right\}. \quad (31)$$

**Proof.** Eqs. (28) and (29) can be obtained by the combination of Eqs. (18), (20), (21), (23), (25). Eq. (30) can be derived from Eqs. (17) and (22). Finally, we obtain Eq. (31) from Eqs. (17) and (24).  $\square$

## 6. Demonstration

This section is meant for the practical demonstration of the proposed curve and surface schemes. First of all, let us take the example of a positive data in Table 1. The default values  $r_i = w_i = 1$  and  $u_i = v_i = 3$ , as mentioned in Section 2, demonstrate the curve in Fig. 1. It should be noted that it is a standard Hermite cubic spline. It is also a special case of rational cubic of [14]. Its visual display very obviously reflects the loss of positivity in the designed curve description. After implementing the proposed curve scheme, Fig. 2 demonstrates the positive curve through positive data using the positive rational function of Section 3 with the settings of the free parameters as  $r_i = w_i = 2$ . It is important to mention that  $u_i$  and  $v_i$  are constrained as in Section 3. Fig. 3 also shows the positive curve through positive data in Table 1 using the positive rational function of Section 3, here the selection of parameters has been made as  $r_i = w_i = 1.6$ . Further

Table 1  
A positive data

$x$	1	2	3	8	10	11	12	14
$f$	14	8	2	0.8	0.5	0.25	0.40	0.37

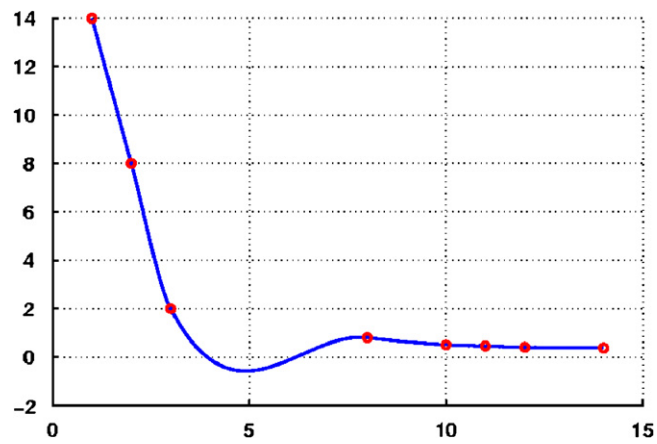


Fig. 1. Cubic Hermite function.

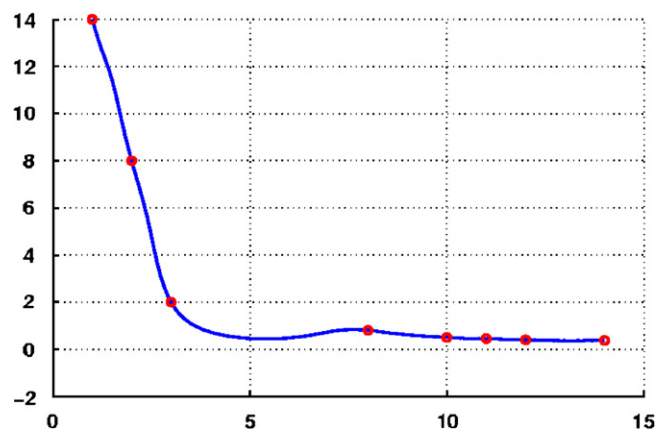
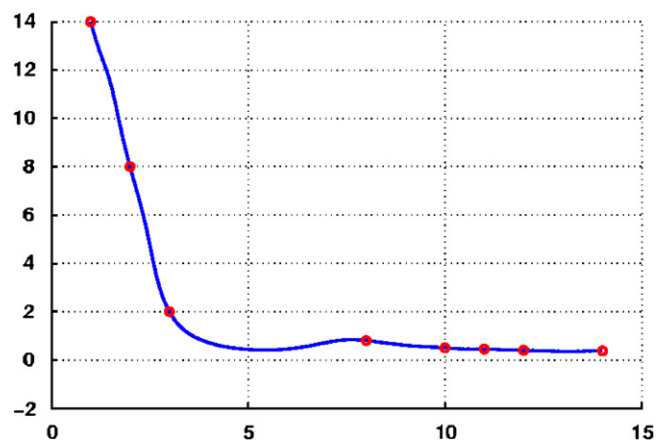
Fig. 2. Positive rational cubic spline with  $r_i = w_i = 2$ .Fig. 3. Positive rational cubic spline with  $r_i = w_i = 1.6$ .



Table 2

A positive data of known volume of NaOH

$x$	2	3	7	8	9	13	14
$f$	10	2	3	7	2	3	10

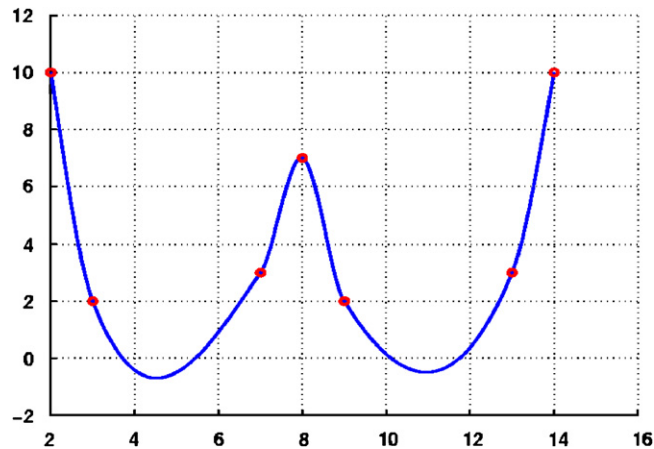


Fig. 4. Cubic Hermite function.

modifications in Figs. 2 and 3 are also possible. This can be achieved by assigning the appropriate values to  $r_i > 0$  and  $w_i > 0$ .

**Remark 6.1.** What are the appropriate values  $r_i > 0$  and  $w_i > 0$ ? It is an interesting question. In this document, we have assumed them as default values  $r_i = w_i = 1$ . The default proposed positive curve is enough for the smooth data presentation. However, the existence of  $r_i$  and  $w_i$  provide extra degrees of freedom for the user to make modifications wherever one desires. Making such modifications is useful feature in the proposed scheme which can help to visualize the curve presentation in (may be) more desired form when ever it is needed. This is true that providing these values manually is hectic and not convenient. One can think to automate these values according to the problem requirement. Some deterministic or non-deterministic optimization techniques may be used for this purpose, this is again a research problem and is left for future work.

Second example is a demonstration of known volume of NaOH, which was taken in a beaker and its conductivity was determined. HCl solution was added from the burette in steps drop by drop. After each addition, volume of HCl ( $x$ ) was stirred by gentle shaking and conductance ( $f$ ) was determined as shown in Table 2. The default cubic Hermite spline method produces the curve in Fig. 4. This curve shows the negative value of conductance which is ridiculous. This flaw is recovered nicely in Fig. 5 using the positivity preserving rational cubic scheme of this paper, the free parameters have been assigned the values as  $r_i = w_i = 0.1$ . Fig. 6 is generated using the positivity preserving constraints with the choice of free parameters as  $r_i = 0.5$  and  $w_i = 0.5$ .

For the implementation of surface scheme, let us consider 3D data sets. Two samples of 3D positive data, in Tables 3 and 4, have been chosen for the demonstration purpose. The data in Table 3 is generated from the following smooth function:

$$F(x, y) = e^{-x-y} + 0.01, \quad 0.01 \leq x, y \leq 300. \quad (32)$$

These data is generated by taking the values truncated to four decimal places. Fig. 7 is produced from this data set using default rational spline surface case. It is (like the curve case) actually a bicubic Hermite, which loses the positivity in its display. Fig. 8 is a different view of Fig. 7 obtained after making a rotation, it confirms quite clearly that the surface is not preserving positivity feature. Fig. 9 is produced from the same data set by the surface scheme developed in Section 5.

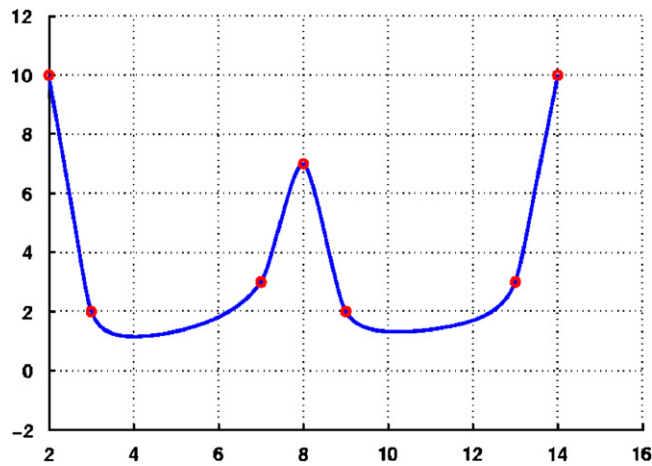
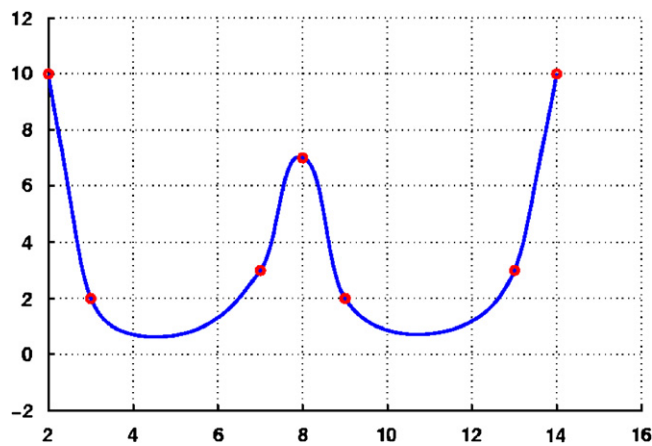
Fig. 5. Positive rational cubic spline with  $r_i = w_i = 0.1$ .Fig. 6. Positive rational function with  $r_i = w_i = 0.5$ .

Table 3

A 3D data set taken from a positive function in Eq. (32)

$y/x$	0.01	100	200	300
0.01	0.9902	0.0100	0.0100	0.0100
100	0.0100	0.0100	0.0100	0.0100
200	0.0100	0.0100	0.0100	0.0100
300	0.0100	0.0100	0.0100	0.0100

As there is a freedom for the choices of  $r$ 's and  $w$ 's, we have computed the surface with  $r_{i,j} = \hat{r}_{i,j} = w_{i,j} = \hat{w}_{i,j} = 0.001$ . One can visualize that the shape of the data is preserved in a positive manner.

Another surface example is for the data in Table 4. This data has been generated approximately from the following smooth function:

$$F(x, y) = \frac{4}{((x^2 + y^2)^2 - 1)}, \quad -3 \leq x, y \leq 3, x, y \neq 0. \quad (33)$$

This data is reported by taking the values truncated to four decimal places. Fig. 10 is produced from the data set in Table 4 using bicubic Hermite, which loses the positivity. Fig. 11 is produced from the same data set by the proposed

Table 4  
A 3D data set taken from a positive function in Eq. (33)

$y/x$	−3	−2	−1	1	2	3
−3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124
−2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
−1	0.0404	0.1667	1.3333	1.3333	0.1667	0.0404
1	0.0404	0.1667	1.3333	1.3333	0.1667	0.0404
2	0.0238	0.0635	0.1667	0.1667	0.0635	0.0238
3	0.0124	0.0238	0.0404	0.0404	0.0238	0.0124

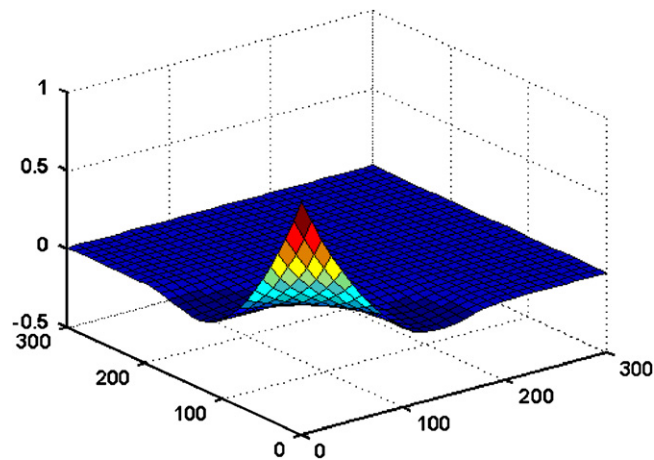


Fig. 7. Bicubic Hermite surface to the 3D data set taken from the positive function in Eq. (32).

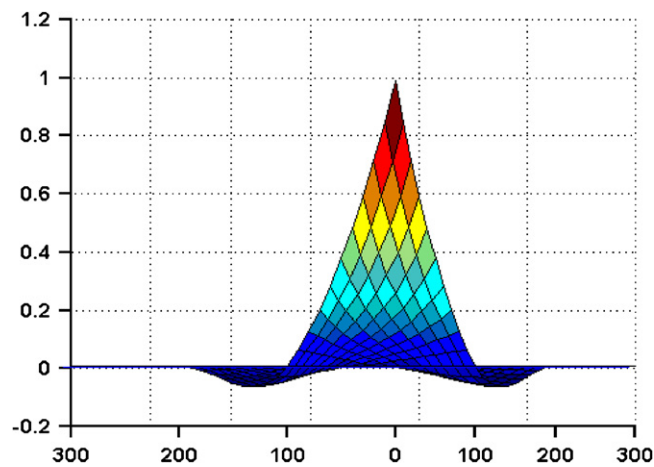


Fig. 8. A different view of the bicubic Hermite surface, of Fig. 7, after rotation.

positive surface scheme developed in Section 5 with the choice of free parameters as  $r_{i,j} = \hat{r}_{i,j} = w_{i,j} = \hat{w}_{i,j} = 0.005$ . It is obvious to see that the shape of the data has been preserved by the surface representation.

**Remark 6.2.** Derivative computation in the description of the curve and surface methods have been made through arithmetic mean choice [16]. However, any other numerical derivative schemes, like geometric or harmonic mean choices can also be used. The fact that the user has so much freedom to specify derivatives may actually be a disadvantage too. Estimating derivatives by methods such as arithmetic, geometric or harmonic mean, which have known serious

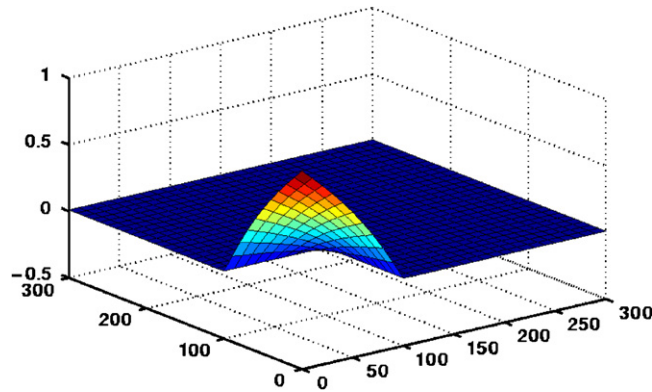


Fig. 9. Positive rational bicubic surface with  $r_{i,j} = \hat{r}_{i,j} = w_{i,j} = \hat{w}_{i,j} = 0.001$ .

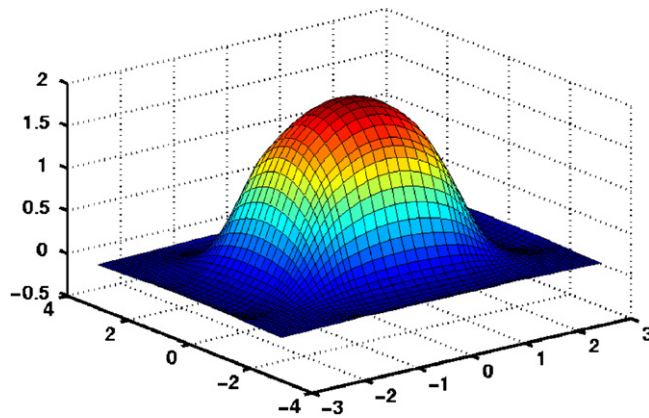


Fig. 10. Bicubic Hermite surface to the 3D data set taken from the positive function in Eq. (33).

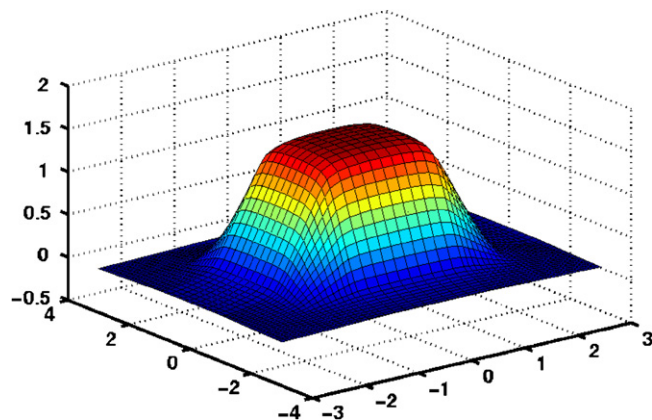


Fig. 11. Positive rational bicubic surface with  $r_{i,j} = \hat{r}_{i,j} = w_{i,j} = \hat{w}_{i,j} = 0.005$ .

problems, may not be a good strategy. The fundamental issue of how the derivatives of this method be estimated automatically for large data sets is open. This problem is under discussion of the authors and might be reported in some future findings.

## 7. Conclusion

A  $C^1$  piecewise rational cubic Hermite interpolant, in its most generalized form with the description of four shape parameters, has been utilized to visualize positive data arose from some scientific experiments. Data dependent shape constraints are derived on two shape parameters to assure the positive shape preservation of the data while the other two are left at the user's freedom. The choice of arithmetic mean has been adopted for derivative computation. But, in general, choice of the derivative parameters is left at the wish of the user as well. Any numerical derivatives, like arithmetic, geometric or harmonic mean choices can be used. The method has been implemented for scalar valued curves, whereas the case of parametric curves is left for future work. We have also extended the proposed idea of positive  $C^1$  rational cubic spline to a positive rational bicubic surface. It assumes that the data are arranged over a rectangular grid. The scheme has been derived by imposing some constraints upon the middle parameters in the description of rational bicubic spline, whereas the remaining parameters have been left free at the disposal of the users. However, a default choice of the free parameters is set as unity in both curve as well as surface cases which can also be changed when needed. The proposed curve and surface schemes have been demonstrated over the practical data sets and they proved visually pleasant results.

How pleasant the proposed positive curves or surfaces should be for the positive data? Although, it is partially dependent upon the constrained and free parameters as was observed in Section 6, but one can reach to a satisfactory choice based upon a particular application. This is an interesting problem and is currently under discussion of the authors and is left for future work. Similarly, having most optimal derivative computation is also a research issue which needs some consideration. This work is also left as future task.

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