



Periodic solutions for Liénard type p -Laplacian equation with two deviating arguments[☆]

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ARTICLE INFO

Article history:

Received 25 July 2007

Received in revised form 10 June 2008

MSC:

34C25

34D40

Keywords:

p -Laplacian

Liénard equation

Deviating argument

Periodic solution

Coincidence degree

ABSTRACT

In this paper, the Liénard type p -Laplacian equation with two deviating arguments

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t)$$

is studied. By applying the coincidence degree theory, we obtain some new results on the existence of periodic solutions to this equation. Our results improve and extend some existing ones in the literature.

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1. Introduction

Consider the Liénard type p -Laplacian equation with two deviating arguments of the form

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t), \quad (1.1)$$

where $p > 1$ and $\varphi_p : R \rightarrow R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, τ_1, τ_2 and e are T -periodic, g_1 and g_2 are T -periodic in the first argument with $T > 0$.

As a model coming from physics, mechanics and engineering (for example, see [1–4]), Eq. (1.1) has been the object of intensive analysis by numerous authors. In particular, there have been extensive results on periodic solutions of (1.1) in the literature. We refer the reader to [1–3,7] and the references cited therein. Moreover, in the above-mentioned literature, we observe that the following assumption

(H₁) $g_1(t, x) + g_2(t, x) = g(x)$, $g(x) \in C(R, R)$ and there exists a constant $L > 0$ such that

$$|g(t, u) - g(t, v)| \leq L|u - v|, \quad \text{for all } t, u, v \in R.$$

has been considered as fundamental for the considered existence of periodic solutions of Eq. (1.1). However, for the existence of periodic solutions to Eq. (1.1) without (H₁), the results are scarce. Thus, it is worthwhile to study Eq. (1.1) in this case.

[☆] This work was supported by grant 06JJ2063, 07JJ46001 from the Scientific Research Fund of Hunan Provincial Natural Science Foundation of China, and the Scientific Research Fund of Hunan Provincial Education Department (06C576, 06C586).

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The main purpose of this paper is to establish sufficient conditions ensuring the existence of T -periodic solutions of Eq. (1.1) without (H_1) . By using an illustrative example, we show that our results improve and extend the main results obtained in [1–7].

For convenience, let us denote

$$C_T^1 := \{x \in C^1(R) : x \text{ is } T\text{-periodic}\},$$

$$|x|_k = \left(\int_0^T |x(t)|^k dt \right)^{1/k}, \quad |x|_\infty = \max_{t \in [0, T]} |x(t)|.$$

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = \tilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T). \quad (1.2)$$

where $\tilde{f} \in C(R^3, R)$ is T -periodic in the first variable, we have the following lemma:

Lemma 1.1 ([4]). Let Ω be an open bounded set in C_T^1 , if the following conditions hold

(i) For each $\lambda \in (0, 1)$ the problem

$$(\varphi_p(x'(t)))' = \lambda \tilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T)$$

has no solution on $\partial\Omega$.

(ii) The equation

$$F(a) := \frac{1}{T} \int_0^T \tilde{f}(t, a, 0) dt = 0,$$

has no solution on $\partial\Omega \cap R$.

(iii) The Brouwer degree of F

$$\deg(F, \Omega \cap R, 0) \neq 0.$$

Then the periodic boundary value problem (1.2) has at least one T -periodic solution on $\overline{\Omega}$.

We can easily obtain the homotopic equation of Eq. (1.1) as following:

$$(\varphi_p(x'(t)))' + \lambda f(x(t))x'(t) + \lambda g(t, x(t - \tau(t))) = \lambda e(t), \quad \lambda \in (0, 1). \quad (1.3)$$

The following lemma will be useful to prove our main results in Section 2.

Lemma 1.2. Assume that the following conditions are satisfied.

(A₁) one of the following conditions holds:

(1) $(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) > 0$, for $i = 1, 2$, $u_i \in R$, $\forall t \in R$ and $u_1 \neq u_2$,

(2) $(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) < 0$, for $i = 1, 2$, $u_i \in R$, $\forall t \in R$ and $u_1 \neq u_2$;

(A₂) there exists a constant $d > 0$ such that one of the following conditions holds:

(1) $x(g_1(t, x) + g_2(t, x) - e(t)) > 0$, for all $t \in R$, $|x| \geq d$;

(2) $x(g_1(t, x) + g_2(t, x) - e(t)) < 0$, for all $t \in R$, $|x| \geq d$.

If $x(t)$ is a T -periodic solution of (1.3), then

$$|x|_\infty \leq d + \frac{1}{2} \int_0^T |x'(s)| ds. \quad (1.4)$$

Proof. Let $x(t)$ be a T -periodic solution of Eq. (1.3). Integrating (1.3) from 0 to T , we have

$$\int_0^T [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - e(t)] dt = 0, \quad (1.5)$$

which implies that there exists a constant $t_1 \in R$ such that

$$g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - e(t_1) = 0. \quad (1.6)$$

Then, we show that the following claim holds true.

Claim. If $x(t)$ is a T -periodic solution of (1.3), then there exists a constant $t_2 \in R$ such that

$$|x(t_2)| \leq d. \quad (1.7)$$

Assume, by way of contradiction, that (1.7) does not hold. Then

$$|x(t)| > d, \quad \text{all } t \in R, \quad (1.8)$$

this, together with (A₂) and (1.5), implies that one of the following relations holds:

$$x(t_1 - \tau_1(t_1)) > x(t_1 - \tau_2(t_1)) > d; \quad (1.9)$$

$$x(t_1 - \tau_2(t_1)) > x(t_1 - \tau_1(t_1)) > d; \quad (1.10)$$

$$x(t_1 - \tau_1(t_1)) < x(t_1 - \tau_2(t_1)) < -d; \quad (1.11)$$

$$x(t_1 - \tau_2(t_1)) < x(t_1 - \tau_1(t_1)) < -d. \quad (1.12)$$

Suppose that (1.9) holds, in view of (A₁)(1), (A₁)(2), (A₂)(1) and (A₂)(2), we shall consider four cases as follows.

Case (i). If (A₂)(1) and (A₁)(1) hold, according to (1.9), we obtain

$$\begin{aligned} 0 &< g_1(t_1, x(t_1 - \tau_2(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - e(t_1) \\ &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - e(t_1), \end{aligned}$$

which contradicts that (1.6). Hence, (1.7) is true.

Case (ii). If (A₂)(1) and (A₁)(2) hold, according to (1.9), we obtain

$$\begin{aligned} 0 &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - e(t_1) \\ &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - e(t_1), \end{aligned}$$

which contradicts that (1.6). Thus, (1.7) is true.

Case (iii). If (A₂)(2) and (A₁)(1) hold, according to (1.9), we obtain

$$\begin{aligned} 0 &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - e(t_1) \\ &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - e(t_1), \end{aligned}$$

which contradicts that (1.6). Hence, (1.7) is true.

Case (iv). If (A₂)(2) and (A₁)(2) hold, according to (1.9), we obtain

$$\begin{aligned} 0 &> g_1(t_1, x(t_1 - \tau_2(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - e(t_1) \\ &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - e(t_1), \end{aligned}$$

which contradicts that (1.6). Thus, (1.7) is true.

Suppose that (1.10) (or (1.11), or (1.12)) holds, using the methods similar to those used in Case (i)–Case (iv), we can show that (1.7) holds true. Thus, the Claim is true.

Let $t_2 = mT + \bar{\xi}$, where $\bar{\xi} \in [0, T]$, and m be an integer. Then, we have

$$|x(t)| = \left| x(\bar{\xi}) + \int_{\bar{\xi}}^t x'(s) ds \right| \leq d + \int_{\bar{\xi}}^t |x'(s)| ds, \quad t \in [\bar{\xi}, \bar{\xi} + T],$$

and

$$|x(t)| = |x(t - T)| = \left| x(\bar{\xi}) - \int_{t-T}^{\bar{\xi}} x'(s) ds \right| \leq d + \int_{t-T}^{\bar{\xi}} |x'(s)| ds, \quad t \in [\bar{\xi}, \bar{\xi} + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x|_{\infty} &= \max_{t \in [0, T]} |x(t)| = \max_{t \in [\bar{\xi}, \bar{\xi} + T]} |x(t)| \\ &\leq \max_{t \in [\bar{\xi}, \bar{\xi} + T]} \left\{ d + \frac{1}{2} \left(\int_{\bar{\xi}}^t |x'(s)| ds + \int_{t-T}^{\bar{\xi}} |x'(s)| ds \right) \right\} \\ &\leq d + \frac{1}{2} \int_0^T |x'(s)| ds. \end{aligned} \quad (1.13)$$

This completes the proof of Lemma 1.2. $\square$

Remark 1.1. The methods in Refs. [1–7] are not available to show that (1.4) holds, since (H₁) does not hold in Theorem 2.1. This implies that the mathematical analysis technique in Theorem 2.1 of this paper is essentially new.

2. Main results

Theorem 2.1. Let (A_1) (1) and (A_2) (1) hold. Suppose that the following condition is satisfied.

(A_3) there exist nonnegative constants m_1, m_2, m_3 and m_4 such that

$$2^{1-p}T^p(m_1 + 2m_3) < 1,$$

and one of the following conditions holds:

(1) $|g_2(t, x)| \leq m_3|x|^{p-1} + m_4$ for all $t \in R, x \in R$, and $g_1(t, x) + g_2(t, x) - e(t) \leq m_1|x|^{p-1} + m_2$, for all $t \in R, x \geq d$;

(2) $|g_2(t, x)| \leq m_3|x|^{p-1} + m_4$ for all $t \in R, x \in R$, and $g_1(t, x) + g_2(t, x) - e(t) \geq -m_1|x|^{p-1} - m_2$, for all $t \in R, x \leq -d$.

Then Eq. (1.1) has at least one solution with period T .

Proof. We shall seek to apply Lemma 1.1. To do this, it suffices to prove that the set of all possible T -periodic solutions of Eq. (1.3) are bounded.

Let $x(t)$ be a T -periodic solution of Eq. (1.3). Integrating (1.3) from 0 to T , we have

$$\int_0^T [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - e(t)]dt = 0. \quad (2.1)$$

Set

$$[x(t - \tau_1(t)) < -d] = \{t | t \in [0, T], x(t - \tau_1(t)) < -d\},$$

$$[x(t - \tau_1(t)) \geq -d] = \{t | t \in [0, T], x(t - \tau_1(t)) \geq -d\},$$

$$[x(t - \tau_1(t)) > d] = \{t | t \in [0, T], x(t - \tau_1(t)) > d\},$$

$$[x(t - \tau_1(t)) \leq d] = \{t | t \in [0, T], x(t - \tau_1(t)) \leq d\}.$$

Then, (2.1) implies that

$$\begin{aligned} & \int_{[x(t-\tau_1(t)) < -d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\ &= - \int_{[x(t-\tau_1(t)) < -d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt \\ &= \int_{[x(t-\tau_1(t)) \geq -d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt \\ &\quad - \int_0^T [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt \\ &= \int_{[x(t-\tau_1(t)) \geq -d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt - \int_0^T [g_1(t, x(t - \tau_1(t))) \\ &\quad + g_2(t, x(t - \tau_2(t))) - e(t)]dt - \int_0^T g_2(t, x(t - \tau_1(t)))dt + \int_0^T g_2(t, x(t - \tau_2(t)))dt \\ &= \int_{[x(t-\tau_1(t)) \geq -d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt - \int_0^T g_2(t, x(t - \tau_1(t)))dt \\ &\quad + \int_0^T g_2(t, x(t - \tau_2(t)))dt, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} & \int_{[x(t-\tau_1(t)) > d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\ &= \int_{[x(t-\tau_1(t)) > d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt \\ &= - \int_{[x(t-\tau_1(t)) \leq d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt \\ &\quad + \int_0^T [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt \\ &= - \int_{[x(t-\tau_1(t)) \leq d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt + \int_0^T [g_1(t, x(t - \tau_1(t))) \end{aligned}$$

$$\begin{aligned}
& + g_2(t, x(t - \tau_2(t))) - p(t)]dt + \int_0^T g_2(t, x(t - \tau_1(t)))dt - \int_0^T g_2(t, x(t - \tau_2(t)))dt \\
& = \int_{[x(t-\tau_1(t)) \leq d]} [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)]dt + \int_0^T g_2(t, x(t - \tau_1(t)))dt \\
& \quad - \int_0^T g_2(t, x(t - \tau_2(t)))dt.
\end{aligned} \tag{2.3}$$

Now suppose that $(A_3)(1)$ (or $(A_3)(2)$) holds, we shall consider two cases as follows.

Case (1) If $(A_3)(1)$ holds, it follows from (2.2) that

$$\begin{aligned}
& \int_{[x(t-\tau_1(t)) < -d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \leq \int_{[x(t-\tau_1(t)) \geq -d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \quad + \int_0^T |g_2(t, x(t - \tau_1(t)))|dt + \int_0^T |g_2(t, x(t - \tau_2(t)))|dt \\
& \leq \int_{\{t|t \in [0, T], |x(t - \tau_1(t))| \leq d\}} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \quad + \int_{[x(t-\tau_1(t)) > d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \quad + \int_0^T (m_3|x(t - \tau_1(t))|^{p-1} + m_4)dt + \int_0^T (m_3|x(t - \tau_2(t))|^{p-1} + m_4)dt \\
& \leq T(\max\{|g_1(t, x) + g_2(t, x) - e(t)| : t \in R, |x| \leq d\}) + \int_0^T (m_1|x(t - \tau_1(t))|^{p-1} + m_2)dt \\
& \quad + 2T(m_3|x|_\infty^{p-1} + m_4) \\
& \leq T(\max\{|g_1(t, x) + g_2(t, x) - e(t)| : t \in R, |x| \leq d\}) + m_2 + 2m_4 + T(m_1 + 2m_3)|x|_\infty^{p-1} \\
& \leq T(\theta_1 + m_2 + 2m_4) + T(m_1 + 2m_3)|x|_\infty^{p-1}
\end{aligned} \tag{2.4}$$

where $\theta_1 = \max\{|g_1(t, x) + g_2(t, x) - e(t)| : t \in R, |x| \leq d\}$. Then, (2.4) implies that

$$\begin{aligned}
& \int_0^T |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - p(t)|dt \\
& = \int_{[x(t-\tau_1(t)) < -d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \quad + \int_{[x(t-\tau_1(t)) \geq -d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \leq 2T(\theta_1 + m_2 + m_4) + 2T(m_1 + m_3)|x|_\infty^{p-1}.
\end{aligned} \tag{2.5}$$

Case (2) If $(A_3)(2)$ holds, it follows from (2.3) that

$$\begin{aligned}
& \int_{[x(t-\tau_1(t)) > d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \leq \int_{[x(t-\tau_1(t)) \leq d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \quad + \int_0^T |g_2(t, x(t - \tau_1(t)))|dt + \int_0^T |g_2(t, x(t - \tau_2(t)))|dt \\
& \leq \int_{\{t|t \in [0, T], |x(t - \tau_1(t))| \leq d\}} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \quad + \int_{[x(t-\tau_1(t)) < -d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
& \quad + \int_0^T (m_3|x(t - \tau_1(t))|^{p-1} + m_4)dt + \int_0^T (m_3|x(t - \tau_2(t))|^{p-1} + m_4)dt
\end{aligned}$$

$$\begin{aligned}
&\leq T(\max\{|g_1(t, x) + g_2(t, x) - e(t)| : t \in R, |x| \leq d\}) + \int_0^T (m_1|x(t - \tau_1(t))|^{p-1} + m_2)dt \\
&\quad + 2T(m_3|x|_\infty^{p-1} + m_4) \\
&\leq T(\max\{|g_1(t, x) + g_2(t, x) - e(t)| : t \in R, |x| \leq d\} + m_2 + 2m_4) + T(m_1 + 2m_3)|x|_\infty \\
&\leq T(\theta_1 + m_2 + 2m_4) + T(m_1 + 2m_3)|x|_\infty^{p-1},
\end{aligned} \tag{2.6}$$

which implies that

$$\begin{aligned}
&\int_0^T |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
&= \int_{[x(t - \tau_1(t)) > d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
&\quad + \int_{[x(t - \tau_1(t)) \leq d]} |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \\
&\leq 2T(\theta_1 + m_2 + m_4) + 2T(m_1 + m_3)|x|_\infty^{p-1}.
\end{aligned} \tag{2.7}$$

As $x(0) = x(T)$, there exists $t_0 \in [0, T]$ such that $x'(t_0) = 0$, while $\varphi_p(0) = 0$ we see

$$|\varphi_p(x'(t))| = \left| \int_{t_0}^t (\varphi_p(x'(s)))' ds \right| \leq \lambda \int_0^T |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - e(t)|dt, \tag{2.8}$$

where $t \in [t_0, t_0 + T]$.

Since $x(t)$ is T -periodic, multiplying (1.3) by $x(t)$ and integrating it from 0 to T , in view of (2.5) and (2.7), we get

$$\begin{aligned}
\int_0^T |x'(t)|^p dt &= - \int_0^T (\varphi_p(x'(t)))' x(t) dt \\
&= \lambda \int_0^T f(x(t))x(t)x'(t) dt + \lambda \int_0^T (g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - e(t))x(t) dt \\
&= \lambda \int_0^T f(x(t))x(t)x'(t) dt + \lambda \int_0^T [g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) \\
&\quad - e(t) + g_2(t, x(t - \tau_2(t))) - g_2(t, x(t - \tau_1(t)))]x(t) dt \\
&\leq |x|_\infty \left[\int_0^T |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) - e(t)|dt \right. \\
&\quad \left. + \int_0^T |g_2(t, x(t - \tau_2(t)))|dt + \int_0^T |g_2(t, x(t - \tau_1(t)))|dt \right] \\
&\leq [2T(\theta_1 + m_2 + m_4) + 2T(m_1 + m_3)|x|_\infty^{p-1} + 2T(m_3|x|_\infty^{p-1} + m_4)]|x|_\infty \\
&\leq [2T(\theta_1 + m_2 + 2m_4) + 2T(m_1 + 2m_3)|x|_\infty^{p-1}]|x|_\infty.
\end{aligned} \tag{2.9}$$

For $x(t) \in C(R, R)$ with $x(t + T) = x(t)$, and $0 < r \leq s$, by using Hölder inequality, we obtain

$$\left(\frac{1}{T} \int_0^T |x(t)|^r dt \right)^{1/r} \leq \left(\frac{1}{T} \left(\int_0^T (|x(t)|^r)^{\frac{s}{r}} dt \right)^{\frac{r}{s}} \left(\int_0^T 1 dt \right)^{\frac{s-r}{s}} \right)^{1/r} = \left(\frac{1}{T} \int_0^T |x(t)|^s dt \right)^{1/s},$$

this yields that

$$|x|_r \leq T^{\frac{s-r}{rs}} |x|_s, \quad \text{for } 0 < r \leq s. \tag{2.10}$$

Then, in view of (1.4), (2.9) and (2.10), we can get

$$\begin{aligned}
\left(\int_0^T |x'(t)| dt \right)^p &\leq T^{p-1} |x'(t)|_p^p = T^{p-1} \int_0^T |x'(t)|^p dt \\
&\leq T^{p-1} [2T(\theta_1 + m_2 + 2m_4) + 2T(m_1 + 2m_3)|x|_\infty^{p-1}]|x|_\infty \\
&\leq T^{p-1} \left[2T(\theta_1 + m_2 + 2m_4) + 2T(m_1 + 2m_3) \left(d + \frac{1}{2} \int_0^T |x'(s)| ds \right)^{p-1} \right]
\end{aligned}$$

$$\times \left(d + \frac{1}{2} \int_0^T |x'(s)| ds \right). \quad (2.11)$$

Since $2^{1-p}T^p(m_1 + 2m_3) < 1$ and $p > 1$, (2.11) yields that we can choose some positive constant M_1 such that

$$\int_0^T |x'(s)| ds \leq M_1, \quad |x|_\infty \leq d + \frac{1}{2} \int_0^T |x'(s)| ds \leq M_1.$$

In view of (2.5), (2.7) and (2.8), we have

$$\begin{aligned} |x'|_\infty^{p-1} &= \max_{t \in [t_0, t_0+T]} \{|\varphi_p(x'(t))|\} = \max_{t \in [t_0, t_0+T]} \left\{ \left| \int_{t_0}^t (\varphi_p(x'(s)))' ds \right| \right\} \\ &\leq \int_0^T |g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - e(t)| dt \\ &\leq 2T(\theta_1 + m_2 + 2m_4) + 2T(m_1 + 2m_3)|x|_\infty^{p-1} \\ &\leq 2T(\theta_1 + m_2 + 2m_4) + 2T(m_1 + 2m_3)M_1^{p-1}, \end{aligned}$$

which implies that there exists a positive constant $M_2 > M_1 + 1$ such that for all $t \in \mathbb{R}$,

$$|x'(t)| \leq M_2.$$

Set

$$\Omega = \{x | x \in X, |x|_\infty < M, |x'|_\infty < M\}.$$

Then we know that Eq. (1.3) has no solution on $\partial\Omega$. As $\lambda \in (0, 1)$ and when $x(t) \in \partial\Omega \cap R$, $x(t) = M_2 + 1$ or $x(t) = -M_2 - 1$, from $(A_1)(1)$, we can see that

$$\begin{aligned} \frac{1}{T} \int_0^T \{-g_1(t, M_2 + 1) - g_2(t, M_2 + 1) + e(t)\} dt &< 0, \\ \frac{1}{T} \int_0^T \{-g_1(t, -M_2 - 1) - g_2(t, -M_2 - 1) + e(t)\} dt &> 0, \end{aligned}$$

so condition (ii) is also satisfied. Set

$$H(x, \mu) = -(1 - \mu)x - \mu \frac{1}{T} \int_0^T (g_1(t, x) + g_2(t, x) - e(t)) dt,$$

and when $x \in \partial\Omega \cap R$, $\mu \in [0, 1]$ we have

$$xH(x, \mu) = -(1 - \mu)x^2 - \mu \frac{1}{T} \int_0^T (g_1(t, x) + g_2(t, x) - e(t)) dt > 0.$$

Thus $H(x, \mu)$ is a homotopic transformation and

$$\begin{aligned} \deg \{F, \Omega \cap R, 0\} &= \deg \left\{ -\frac{1}{T} \int_0^T (g_1(t, x) + g_2(t, x) - e(t)) dt, \Omega \cap R, 0 \right\} \\ &= \deg \{-x, \Omega \cap R, 0\} \neq 0. \end{aligned}$$

so condition (iii) is satisfied. In view of the previous Lemma 1.1, there exists at least one solution with period T . This completes the proof. $\square$

A similar argument leads to

Theorem 2.2. Let (A_1) (2) and (A_2) (2) hold. Suppose that the following condition is satisfied.

(A_4) there exist nonnegative constants m_1, m_2, m_3 and m_4 such that

$$2^{1-p}T^p(m_1 + 2m_3) < 1,$$

and one of the following conditions holds:

- (1) $|g_2(t, x)| \leq m_3|x|^{p-1} + m_4$ for all $t \in \mathbb{R}, x \in \mathbb{R}$, and $g_1(t, x) + g_2(t, x) - e(t) \geq -m_1|x|^{p-1} - m_2$, for all $t \in \mathbb{R}, x \geq d$;
- (2) $|g_2(t, x)| \leq m_3|x|^{p-1} + m_4$ for all $t \in \mathbb{R}, x \in \mathbb{R}$, and $g_1(t, x) + g_2(t, x) - e(t) \leq m_1|x|^{p-1} + m_2$, for all $t \in \mathbb{R}, x \leq -d$.

Then Eq. (1.1) has at least one T -periodic solution.

Remark 2.1. If $g_1(t, x) + g_2(t, x) = g(t, x)$, we can find that the main results of [7] are special ones of Theorems 2.1 and 2.2. Moreover, since few authors have considered the existence on periodic solutions of Liénard type p -Laplacian equation with only one deviating argument. Then, we can easily show that the results of this paper are essentially new.

3. Examples and remarks

Example 3.1. Let $p = 4$, $g_1(t, x) = \frac{1}{2000+2\cos^2 t}x^3$ for all $t \in R, x > 0$, $g_1(t, x) = x^{33}$ for all $t \in R, x \leq 0$, and $g_2(t, x) = \frac{1}{1000} \arctan^3 x$. Then, the following Liénard type p -Laplacian equation with two deviating arguments

$$(\varphi_p(x'(t)))' + x^4(t)x'(t) + g_1(t, x(t - |\cos(t)|)) + g_2(t, x(t - |\sin(t)|)) = \cos t \quad (3.1)$$

has at least one 2π -periodic solution.

Proof. From (3.1), it is straight forward to check that all the conditions needed in Theorem 2.1 are satisfied. Therefore, by Theorem 2.1, Eq. (3.1) has at least one 2π -periodic solution. $\square$

Remark 3.1. In view of $p = 4$, $g_1(t, x) = \frac{1}{2000+2\cos^2 t}x^3$ for all $t \in R, x > 0$, $g_1(t, x) = x^{33}$ for all $t \in R, x \leq 0$, $g_2(t, x) = \frac{1}{1000} \arctan^3 x$, $\tau_1(t) = |\cos(t)|$ and $\tau_2(t) = |\sin(t)|$, it is clear to see that (H_1) do not hold for Eq. (3.1), and so the results obtained in [1–7] and the references cited therein can not be applicable to Eq. (3.1). This implies that the results of this paper are essentially new.

Remark 3.2. By using the methods similar to those used for Eq. (1.1), we can deal with the Liénard Type p -Laplacian equation with multiple deviating arguments of the following type:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + \sum_{i=1}^n g_i(t, x(t - \tau_i(t))) = e(t), \quad (3.2)$$

where $f, \tau_i (i = 1, 2, \dots, n)$, $e : R \rightarrow R$ and $g_i (i = 1, 2, \dots, n) : R \times R \rightarrow R$ are continuous functions, $\tau_i (i = 1, 2, \dots, n)$ and e are T -periodic, $g_i, i = 1, 2, \dots, n$, are T -periodic in the first argument, and $T > 0$. One may also establish the results similar to those in Theorems 2.1 and 2.2 under some minor additional assumptions on $g_i(t, x) (i = 1, 2, \dots, n)$.

Remark 3.3. In recent years, many results on periodic solutions of second order differential equations with multiple deviating arguments have been established by numerous authors (see [6,8–10]). The significant and potential applications of these results and our results can be found in some literature [6,8–10] and the references cited therein. Thus, our results have important practical applications and academic significance to some extent.

Acknowledgement

The authors would like to express their sincere appreciation to the reviewer for his/her helpful comments in improving the presentation and quality of the paper.

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