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Impulsive anti-periodic boundary value problem of first-order integro-differential equations

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ABSTRACT

This paper is concerned with the anti-periodic boundary value problem of first-order nonlinear impulsive integro-differential equations. We first establish a new comparison principle, and then obtain the existence of extremal solutions by upper–lower solution and monotone iterative techniques. Some examples are presented to illustrate the main results.

1. Introduction

In recent years, there has been a great deal of research into the equations of existence and uniqueness of solutions to boundary value problems for differential equations [1]. Meanwhile, impulsive differential equations have also attracted more and more attention [2–6] because it is an important tool to study practical problems of biology, engineering and physics. The periodic boundary value problems involving impulsive differential equations have been studied by many authors; see [7–15] and the references therein. The first-order integro-differential equation with periodic boundary value condition has also been considered by many authors; see [12,16–18]. However, in real problems, some problems come down to anti-periodic boundary value problems. As far as we know, the papers concerned with anti-periodic boundary value problems are few; see [19–24].

In this paper, we consider the following nonlinear problem for first-order integro-differential equation with impulse at fixed points

$$y'(t) = f(t, y(t), (Ty)(t), (Sy)(t)), \quad t \in J_0, \tag{1.1}$$

$$\Delta y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \dots, p, \tag{1.2}$$

$$y(0) = -y(T), \tag{1.3}$$

where $J = [0, T]$, $J_0 = J \setminus \{t_1, t_2, \dots, t_p\}$, $0 < t_1 < t_2 < \cdots < t_p < T$, $f \in C(J \times R \times R \times R, R)$, $I_k \in C(R, R)$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$. And $y(t_k^+)$, $y(t_k^-)$ denote the right and left limits of $y(t)$ at t_k, $k = 1, 2, \dots, p$,

$$(Ty)(t) = \int_0^t k(t, s)y(s)\mathrm{d}s, \qquad (Sy)(t) = \int_0^T h(t, s)y(s)\mathrm{d}s,$$

$k \in C(D, R^+)$, $D = \{(t, s) \in J \times J : t \geq s\}$, $h \in C(J \times J, R^+)$.

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Problems (1.1) and (1.2) were considered in several papers with different types of boundary conditions. In [12,25], Chen and Sun studied problems (1.1) and (1.2) and a similar problem to (1.1) and (1.2), respectively, under the boundary condition $g(y(0), y(T)) = 0$. There, the authors established the existence of extremal solutions by the upper and lower solutions and the monotone iterative technique when g satisfied some monotonicity conditions. Those results are applicable in some important case such as the initial or the periodic case. However, they are not valid for anti-periodic boundary conditions, i.e. $y(0) = -y(T)$.

Just recently, Luo and Nieto [17] proved some new comparison principles for periodic boundary value problem and extended the earlier results. Encouraged by paper [17], in this paper we first establish a new comparison principle for anti-periodic boundary value problem and then obtain the existence of extremal solutions for Eqs. (1.1)–(1.3) by the upper–lower solution and monotone iterative techniques.

2. Preliminaries and lemmas

In this section, we first introduce some definitions in order to define the concept of solution for Eqs. (1.1)–(1.3). Let

$$\begin{aligned} PC(J) &= \{y : J \to R : y \text{ is continuous at } t \in J_0; \\ &\qquad y(0^+), y(T^-),\ y(t_k^+) \text{ and } y(t_k^-) \text{ exist and } y(t_k^-) = y(t_k), k = 1, \dots, p\}, \\ PC^1(J) &= \{y \in PC(J) : y \text{ is continuously differentiable for any } t \in J_0; \\ &\qquad y'(0^+), y'(T^-) \text{ and } y'(t_k^+),\ y'(t_k^-) \text{ exist}, k = 1, \dots, p\}. \end{aligned}$$

$PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|y\|_{PC(J)} = \sup\{|y(t)| : t \in J\}, \qquad \|y\|_{PC^1(J)} = \|y\|_{PC(J)} + \|y'\|_{PC(J)}.$$

We say that a function y is a solution for Eqs. (1.1)–(1.3) if $y \in PC^1(J)$ and satisfies Eqs. (1.1)–(1.3).

In order to obtain the existence of solution for Eqs. (1.1)–(1.3), we need the following key lemma.

Lemma 2.1. *Assume that $y \in PC^1(J)$ satisfies*

$$\begin{cases} y'(t) + My(t) + N(Ty)(t) + N_1(Sy)(t) \le 0, & t \in J_0, \\ \Delta y(t_k) \le -L_k y(t_k), & k = 1, 2, \dots, p, \\ y(0) \le 0, \end{cases} \tag{2.1}$$

where $M > 0$, $N, N_1 \ge 0$, $L_k < 1$, $k = 1, 2, \dots, p$, and

$$\int_0^T q(s)\mathrm{d}s \le \prod_{j=1}^{p}(1 - \bar{L}_j) \tag{2.2}$$

with $\bar{L}_k = \max\{L_k, 0\}$, $k = 1, 2, \dots, p$,

$$q(t) = N \int_0^t k(t, s)\mathrm{e}^{M(t-s)} \prod_{s<t_k<T}(1 - L_k)\mathrm{d}s + N_1 \int_0^T h(t, s)\mathrm{e}^{M(t-s)} \prod_{s<t_k<T}(1 - L_k)\mathrm{d}s,$$

then $y \le 0$.

Proof. Let $c_k = 1 - L_k$ and $x(t) = (\prod_{t<t_k<T} c_k^{-1})y(t)\mathrm{e}^{Mt}$, then we have

$$\begin{cases} x'(t) \le -\left(\prod_{t<t_k<T} c_k^{-1}\right)\left[N\int_0^t k(t, s)x(s)\mathrm{e}^{M(t-s)}\left(\prod_{s<t_k<T} c_k\right)\mathrm{d}s + N_1\int_0^T h(t, s)x(s)\mathrm{e}^{M(t-s)}\left(\prod_{s<t_k<T} c_k\right)\mathrm{d}s\right], \\ \quad t \in J_0, \\ x(t_k^+) \le c_k x(t_k), \quad k = 1, 2, \dots, p, \\ x(0) \le 0. \end{cases} \tag{2.3}$$

Obviously, the function y and x have the same sign.

Suppose the contrary, then there exists a $t^* \in J$ such that $x(t^*) > 0$. Let $x(t_*) = \min_{t\in[0,t^*]} x(t) = b$, then $b < 0$. Otherwise, it follows the first equation of (2.3) that $x'(t) \le 0$ on $[0, t^*] \bigcap J_0$ so x is non-increasing. Thus $x(t^*) \le x(0) \le 0$, which is a contradiction. Therefore, Eq. (2.3) becomes

$$\begin{cases} x'(t) \le -b\left(\prod_{t<t_k<T} c_k^{-1}\right)q(t), & t \in J_0, \\ x(t_k^+) \le c_k x(t_k), & k = 1, 2, \dots, p, \\ x(0) \le 0. \end{cases} \tag{2.4}$$

Suppose $t^* \in (t_j, t_{j+1})$ and $t_* \in (t_m, t_{m+1})$, then integrate the first equation of (2.4) on $(t_j, t^*), (t_{j-1}, t_j), \ldots, (t_m, t_*)$ and we have

$$\begin{aligned} x(t^*) &\le x(t_j^+) - b\int_{t_j}^{t^*}\left(\prod_{s<t_k<T} c_k^{-1}\right) q(s)\mathrm{d}s,\\ &\le c_j x(t_j) - b\int_{t_j}^{t^*}\left(\prod_{s<t_k<T} c_k^{-1}\right) q(s)\mathrm{d}s, \end{aligned}$$

$$x(t_j) \le c_{j-1}x(t_{j-1}^+) - b\int_{t_{j-1}}^{t_j}\left(\prod_{s<t_k<T} c_k^{-1}\right) q(s)\mathrm{d}s,$$

$$\vdots$$

$$x(t_{m+1}) \le x(t_*) - b\int_{t_*}^{t_{m+1}}\left(\prod_{s<t_k<T} c_k^{-1}\right) q(s)\mathrm{d}s,$$

which yield

$$\left(\prod_{t^*<t_k<T} c_k^{-1}\right)\int_{t_*}^{t^*} q(s)\mathrm{d}s > \prod_{t_*<t_k<t^*} c_k.$$

Therefore

$$\int_0^T q(s)\mathrm{d}s \ge \int_{t_*}^{t^*} q(s)\mathrm{d}s > \left(\prod_{t^*<t_k<T} c_k\right) \ge \prod_{j=1}^{p}\bar{c}_j = \prod_{j=1}^{p}(1-\bar{L}_j)$$

which is a contradiction. This completes the proof. □

Remark 2.1. As [17], in Lemma 2.1, we do not require $L_k \ge 0$, $k = 1, 2, \ldots, p$. But usually in references in the literature, for example, Lemma 2.4 in [21] and Lemma 1 in [25], they all assume $0 \le L_k \le 1$. So our result improves all known corresponding conclusions.

Similar to the proof of Lemma 2.1, we can prove the following more general result.

Corollary 2.1. *Assume that $y \in PC^1(J)$ satisfies*

$$\begin{cases} y'(t) + My(t) + N(t)(Ty)(t) + N_1(t)(Sy)(t) \le 0, \quad t \in J_0,\\ \Delta y(t_k) \le -L_k y(t_k), \quad k = 1, 2, \ldots, p,\\ y(0) \le 0, \end{cases}$$

where $M > 0$, $N(t)$, $N_1(t)$ are non-negative bounded integrable functions and satisfy

$$\int_0^T q(s)\mathrm{d}s \le \prod_{j=1}^{p}(1-\bar{L}_j)$$

with $\bar{L}_k = \max\{L_k, 0\}$, $k = 1, 2, \ldots, p$,

$$q(t) = N(t)\int_0^t k(t,s)\mathrm{e}^{\int_s^t M(\tau)\mathrm{d}\tau}\prod_{s<t_k<T}(1-L_k)\mathrm{d}s + N_1(t)\int_0^T h(t,s)\mathrm{e}^{\int_s^t M(\tau)\mathrm{d}\tau}\prod_{s<t_k<T}(1-L_k)\mathrm{d}s.$$

Then $y \le 0$.

Consider the following linear equation

$$\begin{cases} y'(t) + My(t) = \sigma(t) - N(Ty)(t) - N_1(Sy)(t), \quad t \in J_0,\\ \Delta y(t_k) = -L_k y(t_k) + I_k(u(t_k)) + L_k u(t_k), \quad k = 1, 2, \ldots, p,\\ y(0) = -y(T), \end{cases} \tag{2.5}$$

where $M > 0$, $N, N_1 \ge 0$, $L_k < 1$ are constants and $\sigma(t) \in PC(J, R)$.

Lemma 2.2. *$y \in PC^1(J)$ is a solution of (2.5) if and only if $y \in PC(J)$ is a solution of the impulsive integral equation*

$$y(t) = \int_0^T G(t,s)[\sigma(s) - N(Ty)(s) - N_1(Sy)(s)]\mathrm{d}s + \sum_{k=1}^{p} G(t,t_k)[-L_k y(t_k) + I_k(u(t_k)) + L_k u(t_k)],$$

where

$$G(t,s)=\begin{cases}\dfrac{e^{M(T-t+s)}}{e^{MT}+1}, & 0\le s<t\le T,\\[2mm] \dfrac{-e^{M(s-t)}}{e^{MT}+1}, & 0\le t\le s\le T.\end{cases}$$

The proof is similar to Lemma 2.1 [21], so we omit it.

Remark 2.2. If the constants N, N_1 would be replaced by $N(t), N_1(t)$, we will get a similar result to Lemma 2.2. This result and Corollary 2.1 improve the known conclusions in the literature.

Lemma 2.3. *Assume that $M>0$, $N, N_1\ge 0$, $L_k<1$, $k=1,2,\ldots,p$ and the following inequality holds*

$$\sup_{t\in J}\int_0^T G(t,s)\left[N\int_0^s k(s,\tau)d\tau+N_1\int_0^T h(s,\tau)d\tau\right]ds+\frac{e^{MT}}{e^{MT}+1}\sum_{j=1}^{p}|L_k|<1, \tag{2.6}$$

where $G(t,s)$ is defined as in Lemma 2.2. Then (2.5) has a unique solution.

The proof is easy and we omit it.

Remark 2.3. As Remark 2.2, if the constants N, N_1 would be replaced by $N(t), N_1(t)$, then the Lemma 2.3 also holds provided that the inequality (2.6) is replaced by

$$\sup_{t\in J}\int_0^T G(t,s)\left[N(s)\int_0^s k(s,\tau)d\tau+N_1(s)\int_0^T h(s,\tau)d\tau\right]ds+\frac{e^{MT}}{e^{MT}+1}\sum_{j=1}^{p}|L_k|<1, \tag{2.7}$$

where $G(t,s)$ is defined as in Lemma 2.2.

Now we recall the concepts of the upper and lower solutions of anti-periodic boundary value problem.

Definition 2.1. Functions $\alpha_0,\ \beta_0\in PC^1(J)$ are called a coupled lower and upper solution of Eqs. (1.1)–(1.3) if $\alpha_0\le\beta_0$ and

$$\begin{cases}\alpha_0'(t)\le f(t,\alpha_0(t),(T\alpha_0)(t),(S\alpha_0)(t)), & t\in J_0,\\ \Delta\alpha_0(t_k)\le I_k(\alpha_0(t_k)), & k=1,2,\ldots,p,\\ \alpha_0(0)\le-\beta_0(T).\end{cases}$$

$$\begin{cases}\beta_0'(t)\ge f(t,\beta_0(t),(T\beta_0)(t),(S\beta_0)(t)), & t\in J_0,\\ \Delta\beta_0(t_k)\ge I_k(\beta_0(t_k)), & k=1,2,\ldots,p,\\ \beta_0(0)\ge-\alpha_0(T).\end{cases}$$

If there exists a coupled lower and upper solution of Eqs. (1.1)–(1.3) such that $\alpha_0\le\beta_0$ and the nonlinearity f and the impulsive I_k satisfy one-side Lipschitz condition, then we will prove that Eqs. (1.1)–(1.3) have an extremal solution between the lower and upper solutions in the next section.

3. Main result

In this section, we shall prove that Eqs. (1.1)–(1.3) have at least one solution by using the upper and lower solutions and monotone iterative technique.

Theorem 3.1. *Assume that*

(A_1) *Functions α_0, β_0 are a coupled lower and upper solution of Eqs. (1.1)–(1.3) such that $\alpha_0\le\beta_0$ for $t\in J_0$;*

(A_2) *the function $f\in C(J\times R\times R\times R, R)$ satisfies*

$$f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\ge-M(x_1-x_2)-N(y_1-y_2)-N_1(z_1-z_2),$$
$$\alpha_0\le x_2\le x_1\le\beta_0,\qquad T\alpha_0\le y_2\le y_1\le T\beta_0,\qquad S\alpha_0\le z_2\le z_1\le S\beta_0,\quad t\in J,$$

where $M>0$, $N, N_1\ge 0$;

(A_3) *the functions $I_k\in C(R,R)$, $k=1,2,\ldots,p$, satisfy*

$$I_k(x)-I_k(y)\ge-L_k(x-y),$$

where $\alpha_0\le y\le x\le\beta_0$ and $L_k<1$;

(A_4) *The inequalities (2.2) and (2.6) hold.*

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ which converge uniformly on J to the extremal solutions of Eqs. (1.1)–(1.3) in $[\alpha_0,\beta_0]$.

Proof. Firstly, we construct two sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ which satisfy the following problems

$$\begin{cases}\alpha_i'(t)+M\alpha_i(t)+N(T\alpha_i)(t)+N_1(S\alpha_i)(t)\\ \quad =f(t,\alpha_{i-1}(t),(T\alpha_{i-1})(t),(S\alpha_{i-1})(t))+M\alpha_{i-1}(t)+N(T\alpha_{i-1})(t)+N_1(S\alpha_{i-1})(t), \quad t\in J_0,\\ \Delta\alpha_i(t_k)=-L_k\alpha_i(t_k)+I_k(\alpha_{i-1}(t_k))+L_k\alpha_{i-1}(t_k), \quad k=1,2,\ldots,p,\\ \alpha_i(0)=-\beta_{i-1}(T),\end{cases} \tag{3.1}$$

$$\begin{cases}\beta_i'(t)+M\beta_i(t)+N(T\beta_i)(t)+N_1(S\beta_i)(t)\\ \quad =f(t,\beta_{i-1}(t),(T\beta_{i-1})(t),(S\beta_{i-1})(t))+M\beta_{i-1}(t)+N(T\beta_{i-1})(t)+N_1(S\beta_{i-1})(t), \quad t\in J_0,\\ \Delta\beta_i(t_k)=-L_k\beta_i(t_k)+I_k(\beta_{i-1}(t_k))+L_k\beta_{i-1}(t_k), \quad k=1,2,\ldots,p,\\ \beta_i(0)=-\alpha_{i-1}(T).\end{cases} \tag{3.2}$$

It follows from Lemma 2.2 that problems (3.1) and (3.2) have a solution, respectively. So the above definitions are adequate.

Secondly, we prove that $\alpha_i\le\alpha_{i+1}$ and $\beta_i\le\beta_{i-1}$. For that we consider the following problem

$$\begin{cases}\alpha_1'(t)+M\alpha_1(t)+N(T\alpha_1)(t)+N_1(S\alpha_1)(t)=f(t,\alpha_0(t),(T\alpha_0)(t),(S\alpha_0)(t))\\ \quad +M\alpha_0(t)+N(T\alpha_0)(t)+N_1(S\alpha_0)(t), \quad t\in J_0,\\ \Delta\alpha_1(t_k)=-L_k\alpha_1(t_k)+I_k(\alpha_0(t_k))+L_k\alpha_0(t_k), \quad k=1,2,\ldots,p,\\ \alpha_1(0)=-\beta_0(T).\end{cases}$$

Letting $p(t)=\alpha_0(t)-\alpha_1(t)$, it follows that

$$\begin{aligned}p'(t)+Mp(t)+N(Tp)(t)+N_1(Sp)(t) &= \alpha_0'(t)+M\alpha_0(t)+N(T\alpha_0)(t)+N_1(S\alpha_0)(t)\alpha_1'(t)\\ &\quad -M\alpha_1(t)-N(T\alpha_1)(t)+N_1(S\alpha_1)(t)\\ &\le 0,\end{aligned}$$

$$\Delta p(t_k)\le -L_kp(t_k), \quad p(0)\le 0.$$

Then by Lemma 2.1, we get $p(t)\le 0$, that is, $\alpha_0(t)\le\alpha_1(t)$, for all $t\in J_0$. In a similar way, it can be proved that $\beta_0(t)\le\beta_1(t)$, for all $t\in J_0$. Now we prove that $\alpha_1(t)\le\beta_1(t)$, for all $t\in J_0$. Setting $p(t)=\alpha_1(t)-\beta_1(t)$ and using (A_1)–(A_2), we have

$$\begin{aligned}&p'(t)+Mp(t)+N(Tp)(t)+N_1(Sp)(t)\\ &\quad =\alpha_1'(t)-\beta_1'(t)+M(\alpha_1(t)-\beta_1(t))+N(T\alpha_1(t)-T\beta_1(t))+N_1(S\alpha_1(t)-S\beta_1(t))\\ &\quad =f(t,\alpha_0(t),(T\alpha_0)(t),(S\alpha_0)(t))+M\alpha_0(t)+N(T\alpha_0)(t)+N_1(S\alpha_0)(t)-f(t,\beta_0(t),(T\beta_0)(t),(S\beta_0)(t))\\ &\qquad -M\beta_0(t)-N(T\beta_0)(t)-N_1(S\beta_0)(t)\\ &\quad \le 0,\end{aligned}$$

$$\begin{aligned}\Delta p(t_k) &= -L_kp(t_k)+I_k(\alpha_0(t_k))-I_k(\beta_0(t_k))+(L_k\alpha_0(t_k)-L_k\beta_0(t_k))\\ &\le -L_kp(t_k),\end{aligned}$$

$$p(0)=\alpha_1(0)-\beta_1(0)=-\beta_0(T)+\alpha_0(T)\le 0.$$

Again by Lemma 2.1, we get that $p(t)\le 0$, that is, $\alpha_1(t)\le\beta_1(t)$ for all $t\in J_0$. Thus we have $\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t)$ for all $t\in J_0$. Continuing this process, by induction, one can obtain monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ such that

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le\cdots\le\beta_n(t)\le\cdots\le\beta_1(t)\le\beta_0(t), \quad t\in J_0,$$

where each $\alpha_i(t),\ \beta_i(t)\in PC^1(J)$ and satisfies (3.1) and (3.2). As the sequences $\{\alpha_n\}$ $\{\beta_n\}$ are uniformly bounded and equi-continuous, one can employ the standard arguments Ascoli–Arzela criterion [2] to conclude that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly on J_0 with

$$\lim_{n\to\infty}\alpha_n(t)=y_*(t), \qquad \lim_{n\to\infty}\beta_n(t)=y^*(t).$$

Obviously, $y_*(t),\ y^*(t)$ are the solutions of Eqs. (1.1)–(1.3). Now we prove that $y_*(t),\ y^*(t)$ are in fact the extremal solutions of Eqs. (1.1)–(1.3) in $[\alpha_0,\beta_0]$. Let $y(t)$ be any solution of Eqs. (1.1)–(1.3) such that $y(t)\in[\alpha_0,\beta_0]$, $t\in J_0$. We will prove that if $\alpha_n(t)\le y(t)\le\beta_n(t)$ for $n=0,1,\ldots$, then $\alpha_{n+1}(t)\le y(t)\le\beta_{n+1}(t)$.

Letting $p(t)=\alpha_{n+1}(t)-y(t)$, for $t\in J_0$

$$\begin{aligned}&p'(t)+Mp(t)+N(Tp)(t)+N_1(Sp)(t)=f(t,\alpha_n(t),(T\alpha_n)(t),(S\alpha_n)(t))+M\alpha_n(t)\\ &\qquad +N(T\alpha_n)(t)+N_1(S\alpha_n)(t)-f(t,y(t),(Ty)(t),(Sy)(t))-My(t)-N(Ty)(t)-N_1(Sy)(t)\\ &\quad \le 0,\end{aligned}$$

$$\begin{aligned}\Delta p(t_k) &= -L_kp(t_k)+I_k(\alpha_n(t_k))-I_k(y(t_k))+(L_k\alpha_n(t_k)-L_ky(t_k))\\ &\le -L_kp(t_k),\end{aligned}$$

$$p(0)=\alpha_n(0)-y(0)\le 0.$$

By Lemma 2.1, we have $p(t)\le 0$ for all $t\in J_0$, that is, $\alpha_{n+1}(t)\le y(t)$. Similarly, we can prove $y(t)\le\beta_{n+1}(t)$ for all $t\in J_0$. Thus $\alpha_{n+1}(t)\le y(t)\le\beta_{n+1}(t)$ for all $t\in J_0$, which implies $y_*(t)\le y(t)\le y^*(t)$. This completes the proof. $\square$

Corollary 3.1. *Assume that all assumptions in Theorem 3.1 hold with* (A_2) *and* (A_4) *replaced by* (A_2^*) *and* (A_4^*),

(A_2^*) *the function* $f \in C(J \times R \times R \times R, R)$ *satisfies*

$$f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2) \geq -M(x_1-x_2) - N(t)(y_1-y_2) - N_1(t)(z_1-z_2),$$
$$\alpha_0 \leq x_2 \leq x_1 \leq \beta_0, \qquad T\alpha_0 \leq y_2 \leq y_1 \leq T\beta_0, \qquad S\alpha_0 \leq z_2 \leq z_1 \leq S\beta_0, \quad t \in J,$$

where $M > 0$, $N(t), N_1(t)$ *are non-negative bounded integrable functions;*

(A_4^*) *The inequalities (2.2) and (2.7) hold.*

Then the result of Theorem 3.1 also holds.

The proof is almost similar to that of Theorem 3.1 and we omit it.

4. Examples

Example 1. Consider the following problem

$$\begin{cases} y'(t) = -2y(t) + \dfrac{1}{12}\displaystyle\int_0^t e^{-2(t-s)}y(s)ds - \dfrac{2}{15}\int_0^1 y(s)ds, \quad t \in \left[0, \dfrac{1}{2}\right) \bigcup \left(\dfrac{1}{2}, 1\right] \\ \Delta y\left(\dfrac{1}{2}\right) = -L_1 y\left(\dfrac{1}{2}\right), \\ y(0) = -y(1). \end{cases} \tag{4.1}$$

Let $f(t,x,y,z) = -2x + \frac{1}{12}y - \frac{2}{15}z$, $L_1 = -\frac{1}{8}$, $M = 2$, $N = \frac{1}{10}$, $N_1 = \frac{2}{15}$, $J = [0,1]$, $k(t,s) = e^{-2(t-s)}$, $h(t,s) = 1$, then for $t \in J, x_i, y_i, z_i \in R, i = 1, 2, x_1 \geq x_2, y_1 \geq y_2, z_1 \geq z_2$,

$$\begin{aligned} f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2) &= -2(x_1-x_2) + \frac{1}{12}(y_1-y_2) - \frac{2}{15}(z_1-z_2) \\ &\geq -2(x_1-x_2) - \frac{1}{10}(y_1-y_2) - \frac{2}{15}(z_1-z_2). \end{aligned}$$

Thus the condition (A_2) holds. Direct computation shows that

$$\begin{aligned} \int_0^1 q(s)ds &\leq \int_0^1 \left(N\int_0^t e^{-2(t-s)}e^{2(t-s)}(1-L_1)ds + N_1\int_0^1 e^{2(t-s)}(1-L_1)ds\right)dt \\ &= \int_0^1 \left(\frac{7}{80}t + \frac{7}{120}e^{2t}(1-e^{-2})ds\right)dt \\ &\simeq 0.20496 < 1 = 1 - \bar{L}_1, \end{aligned}$$

$$\begin{aligned} &\sup_{t\in J}\int_0^1 G(t,s)\left[N\int_0^s k(s,\tau)d\tau + N_1\int_0^1 h(s,\tau)d\tau\right]ds + \frac{e^M}{e^M+1}|L_1| \leq \sup_{t\in J}\int_0^1 |G(t,s)|\frac{1}{10}(1-e^{-2s})dt + \frac{2}{15} + \frac{1}{8} \\ &\leq \frac{43}{120} < 1. \end{aligned}$$

Therefore, the condition (A_4) holds. It is easy to verify that (4.1) admits lower solution $\alpha_0(t)$ and upper solution $\beta_0(t)$ given by

$$\alpha_0(t) = \begin{cases} -\dfrac{8}{9}, & t \in \left[0, \dfrac{1}{2}\right], \\ -1, & t \in \left(\dfrac{1}{2}, 1\right], \end{cases} \qquad \beta_0(t) = \begin{cases} -t+1, & t \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{t}{2} + \dfrac{1}{3}, & t \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{4.2}$$

Obviously, $\alpha_0(t) \leq \beta_0(t)$. And thus the conclusion of Theorem 3.1 holds for problem (4.1).

Remark 4.1. In Example 1, we remark that $L_1 = -\frac{1}{8} < 0$ is different from the earlier results, where L_1 was required to be non-negative, so we improve the earlier results.

Example 2. Consider the following problem

$$\begin{cases} y'(t) = -2y(t) + \dfrac{t^2}{360}\displaystyle\int_0^t e^{-2(t-s)}y(s)ds - \dfrac{1}{45}\left[t - \int_0^1 ty(s)ds\right]^3, \quad t \in \left[0, \dfrac{1}{2}\right) \bigcup \left(\dfrac{1}{2}, 1\right] \\ \Delta y\left(\dfrac{1}{2}\right) = -L_1 y\left(\dfrac{1}{2}\right), \\ y(0) = -y(1). \end{cases} \tag{4.3}$$

Let $f(t,x,y,z) = -2x + \frac{t^2}{360}y - \frac{1}{45}[t-z]^3, L_1 = -\frac{1}{8},\ M = 2,\ N = \frac{t^2}{10},\ N_1 = \frac{2t^2}{15}, J = [0,1], k(t,s) = \mathrm{e}^{-2(t-s)}, h(t,s) = t$. It is easy to show that (4.3) admits lower solution $\alpha_0(t)$ and upper solution $\beta_0(t)$ given by (4.2). It follows $S\alpha_0(t) < 1$ and $S\beta_0(t) < 1$ that, for $t \in J, x_i, y_i, z_i \in R, i = 1, 2, \beta_0 \geq x_1 \geq x_2 \geq \alpha_0, T\beta_0 \geq y_1 \geq y_2 \geq T\alpha_0, S\beta_0 \geq z_1 \geq z_2 \geq S\alpha_0$,

$$\begin{aligned} f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2) &= -2(x_1-x_2) + \frac{t^2}{12}(y_1-y_2) - \frac{t}{45}[(t-z_1)^3 - (t-z_2)^3] \\ &\geq -2(x_1-x_2) - \frac{t^2}{10}(y_1-y_2) - \frac{2t^2}{15}(z_1-z_2). \end{aligned}$$

Thus the condition (A_2^*) holds. Now we verify the condition (A_4^*). Note that $t \in [0,1]$, we have

$$\begin{aligned} \int_0^1 q(s)\mathrm{d}s &= \int_0^1 \left(N(t)\int_0^t \mathrm{e}^{-2(t-s)}\mathrm{e}^{2(t-s)}(1-L_1)\mathrm{d}s + N_1(t)\int_0^1 \mathrm{e}^{2(t-s)}(1-L_1)\mathrm{d}s \right)\mathrm{d}t \\ &\leq \int_0^1 \left(N\int_0^t \mathrm{e}^{-2(t-s)}\mathrm{e}^{2(t-s)}(1-L_1)\mathrm{d}s + N_1\int_0^1 \mathrm{e}^{2(t-s)}(1-L_1)\mathrm{d}s \right)\mathrm{d}t \\ &< 1 = 1 - \bar{L}_1, \end{aligned}$$

$$\begin{aligned} &\sup_{t\in J}\int_0^1 G(t,s)\left[N(s)\int_0^s k(s,\tau)\mathrm{d}\tau + N_1(s)\int_0^1 h(s,\tau)\mathrm{d}\tau\right]\mathrm{d}s + \frac{\mathrm{e}^M}{\mathrm{e}^M+1}|L_1| \\ &\quad \leq \sup_{t\in J}\int_0^1 |G(t,s)|\left[N\int_0^s k(s,\tau)\mathrm{d}\tau + N_1\int_0^1 h(s,\tau)\mathrm{d}\tau\right]\mathrm{d}s + \frac{\mathrm{e}^M}{\mathrm{e}^M+1}|L_1| < 1. \end{aligned}$$

Therefore, the result of Corollary 3.1 holds.

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