

## Accepted Manuscript

An improved PC scheme for nonlinear fractional differential equations: Error and stability analysis

Mohammad Shahbazi Asl, Mohammad Javidi

PII: S0377-0427(17)30196-6

DOI: <http://dx.doi.org/10.1016/j.cam.2017.04.026>

Reference: CAM 11105

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 15 April 2016

Revised date: 22 February 2017

Please cite this article as: M.S. Asl, M. Javidi, An improved PC scheme for nonlinear fractional differential equations: Error and stability analysis, *Journal of Computational and Applied Mathematics* (2017), <http://dx.doi.org/10.1016/j.cam.2017.04.026>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



## An improved PC scheme for nonlinear fractional differential equations: error and stability analysis

Mohammad Shahbazi Asl<sup>1</sup>, Mohammad Javidi<sup>1,\*</sup>

<sup>a</sup> Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.

---

### Abstract

A novel computationally effective fractional predictor–corrector (PC) scheme is proposed to solve fractional differential equations involving Caputo derivative. The properties of the Caputo derivative are used to reduce the fractional differential equation into a Volterra integral equation. To design high order numerical solution of FDEs, the Simpson’s 3/8 rule is applied to the Volterra type integral equation. The scheme is capable of handling both linear and nonlinear fractional differential equations. A detailed error analysis and stability analysis of the numerical scheme are rigorously established. The proposed scheme is compared with the PC schemes of literature for illustrating the effectiveness of the algorithm.

**Keywords:** Predictor–corrector scheme, Fractional differential equation, Caputo derivative, Volterra integral equation

**2010 MSC:** 26A33, 65D05, 65D25, 65D30

---

### 1. Introduction

The concept of differentiation to an arbitrary order was started in the 17-th century by the ideas of Leibniz and Hospital. At the end of the 19-th century, Liouville and Riemann introduced the first definition of the fractional derivative [1]. However, the idea of fractional calculus did not attract much attention for a long time, this idea started to be interesting for engineers only in the late 1960s, especially when it was observed that the fractional derivatives allow us to describe and model many systems in the real

---

\*Corresponding author

Email address: mo\_javidi@tabrizu.ac.ir ( Mohammad Javidi )

world more adequate than the previously used integer order model [2, 3, 1]. A possible explanation for this delay could be the fact that the fractional derivatives have no a fully acceptable geometrical or physical interpretation, or the apparent self-sufficiency of the integer order calculus, its complexity and the fact that there are multiple definitions for fractional derivatives [4, 5, 6, 7]. However, in recent decades, fractional calculus and fractional differential equations has gained a great development in both theory and application because of its powerful potential applications. The fractional calculus has been used to describe many phenomenon in almost all applied sciences [8, 9].

The integer order differential operator is a local operator but the fractional order differential operator is a non-local operator. This non-local property means that the next state of a system not only depends on its current state but also on its historical states starting from the initial time. This property is closer to reality and it is the main advantage of fractional derivatives in comparison with classical integer-order models [3, 10]. Moreover, in many natural and technical phenomena, the underground dynamics of a system depends either on its history and the environment [11]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [12, 13]. This is one reason why fractional calculus has become more and more popular in scientific and technological fields [10, 14]. Also, integer-order calculus sometimes contradicts the experimental results and therefore derivatives of fractional order may be more suitable [15, 16].

Modeling of systems using fractional derivatives, naturally leads a set of differential equations of fractional order and to the necessity to solve such equations. The analytical solutions to most of the fractional differential equations are usually difficult to obtain [17]. The exact analytical solutions are only known for a few simple cases and refer to some special functions such as the Mittag-Leffler function, the Fox H-function and the hyperbolic geometry function [18]. The complexity of computing these special functions and the difficulties of finding exact solutions for most problems limit the applications fractional differential equations in applied science, so investigating numerical solutions of these equations becomes more and more important [19]. It is worth note that, theoretical studies of the numerical methods and the error estimate of fractional order differential equation are quite limited. This is probably due to the

fact that the theoretical analysis of fractional order numerical methods has been found  
 40 to be very difficult [20, 21].

There are more than six kinds of definitions of fractional derivatives. Among these definitions, the Riemann-Liouville derivative and the Caputo derivative are most frequently used [13, 19]. When used in mathematical models the Riemann-Liouville approach needs initial conditions to be expressed in terms of fractional integrals and their  
 45 derivatives, which there is no known physical interpolation for such types of initial conditions. However, in cases with the time fractional Caputo derivative, the initial conditions take the same form as that for integer-order differential equations, namely, the initial conditions are expressed in terms of values of the unknown function and it's integer order derivatives which have clear physical meaning. Therefore Riemann-  
 50 Liouville approach is not always the most convenient definition for real applications [5, 22, 23]. Properties of the Caputo derivative allow one to reduce the fractional differential equations into a Volterra type integral equation, which is the main idea for solving fractional differential equations in this paper [10].

Much effort has been devoted during the recent years to the numerical investigations of fractional calculus. Diethelm et al. have extended Adams-Bashforth method  
 55 to numerically solve nonlinear fractional differential equations [24]. The detailed error analysis of this method was given in [25], which it was proved that the convergent order of Diethelm's PC approach is  $\min(2, 1 + \alpha)$ . After those, some efforts to improve this method was presented by researchers [26, 18, 27] and it has been used to solve some  
 60 applied fractional problems [28, 29, 30, 31]. Deng combined the short memory principle and the PC approach to achieve a new numerical scheme [18]. Daftardar-Gejji et al modified fractional Adams method by applying iterative method [26]. Li et al. designed a new high order PC approach for fractional differential equations [32]. Yan et al. used piecewise quadratic interpolation to introduce a higher order PC scheme [33].  
 65 The convergent order of this PC scheme was proved to be  $O(h^{1+2\alpha})$  for  $0 < \alpha \leq 1$  and  $O(h^3)$  for  $\alpha > 1$ .

The paper is organized as follows. A new PC scheme is presented in section (2) followed by the truncation error analysis in Section (3). The stability of the presented high order numerical method is proven in section (4). Numerous examples are pre-

70 sented in Section (5) to illustrate the performance of our numerical schemes. Solutions obtained by the presented PC scheme and literature are compared. It is observed that the proposed PC method is accurate. Conclusions are given in the last section.

## 2. The numerical method

Consider the following initial value problem for  $\alpha > 0$  :

$$\begin{cases} {}^C_0D_t^\alpha y(t) = f(t, y(t)), & 0 \leq t \leq T, \\ y^{(k)}(t_0) = y_0^{(k)}, & k = 0, 1, \dots, [\alpha] - 1, \end{cases} \quad (1)$$

where  $[\alpha]$  is the first integer not less than  $\alpha$ ,  ${}^C_0D_t^\alpha$  denotes Caputo derivative and  $y^{(n)}(\tau)$  is the classical  $n$ th-order derivative of  $y(\tau)$ . Let  $f(t, y)$  be a continuous function which satisfies the Lipschitz condition with respect to second argument, i.e.,

$$|f(t, y) - f(t, x)| \leq L|y - x|, \quad (2)$$

where  $L$  is a positive constant, then the initial value problem (1) has a unique solution on some interval  $[0, T]$  [26]. It is well known that the initial value problem (1) is equivalent to the following Volterra integral equation [26, 18, 30, 21]:

$$y(t) = \sum_{i=0}^{[\alpha]-1} \frac{t^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (3)$$

The main problem is to solve the integral on the right-hand side of (3) by numerical method. To construct the high order scheme, the Simpson's 3/8 rule is used, taken with respect to the weight function  $(t_{k+1} - \cdot)^{\alpha+1}$  to replace the integral naturally. We need nodes  $t_j$ , ( $j = 0, 1, \dots, k+1$ ),  $t_{j+\frac{1}{3}}$  and  $t_{j+\frac{2}{3}}$ , ( $j = 0, 1, \dots, k$ ). For solving Eq. (3) the interval  $[0, T]$  is divided into  $N+1$  equi-spaced nodes  $t_j$ , given by  $t_j = jh$ ,  $j = 0, 1, 2, \dots, N$ , in which  $h$  denotes the time step size. Throughout this paper, let  $F_j = f(t_j, y_j)$  and  $F(t_j) = f(t_j, y(t_j))$ , where  $y_j$  is the numerical approximation to  $y(t_j)$ . Suppose that the numerical values of  $y_j$  for  $j = 0, 1, 2, \dots, k$  has been determined (this requires that we have got  $y_{j+1/3}$  and  $y_{j+2/3}$  for  $j = 0, 1, 2, \dots, k-1$ ), now we need to calculate  $y_{k+1}$ . To do this, Firstly we need to compute the values of  $y_{k+1/3}$  and  $y_{k+2/3}$

respectively. The Simpson's 3/8 rule and the Simpson's 1/3 rule will be used to design  
 85 a corrector formula for computing  $y_{k+\frac{1}{3}}$  and  $y_{k+\frac{2}{3}}$ .

The discretized form of (3) to calculate  $y_{k+\frac{1}{3}}$  can be written as

$$y\left(t_{k+\frac{1}{3}}\right)=\sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t_{k+\frac{1}{3}}^i}{i!} y^{(i)}(0)+\frac{1}{\Gamma(\alpha)} \int_0^{t_{k+\frac{1}{3}}} \left(t_{k+\frac{1}{3}}-\tau\right)^{\alpha-1} F(\tau) d \tau . \quad (4)$$

The integral in the right hand side of Eq. (4) is compute numerically in such a way:

$$\begin{aligned} I_{k+\frac{1}{3}} &= \int_0^{t_{k+\frac{1}{3}}} \left(t_{k+\frac{1}{3}}-\tau\right)^{\alpha-1} F(\tau) d \tau = \left[ \int_0^{t_k} \hat{F}_k(\tau) + \int_{t_k}^{t_{k+\frac{1}{3}}} \hat{F}_{k+1}(\tau) \right] \left(t_{k+\frac{1}{3}}-\tau\right)^{\alpha-1} d \tau \\ &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left(t_{k+\frac{1}{3}}-\tau\right)^{\alpha-1} \hat{F}_{j+1}(\tau) d \tau + \int_{t_k}^{t_{k+\frac{1}{3}}} \left(t_{k+\frac{1}{3}}-\tau\right)^{\alpha-1} \hat{F}_{k+1}(\tau) d \tau, \end{aligned} \quad (5)$$

in which  $\hat{F}_{j+1}$  is the cubic interpolation of  $F$  at the nodes  $t_j, t_{j+\frac{1}{3}}, t_{j+\frac{2}{3}}, t_{j+1}$ , ( $j = 0, 1, 2, \dots, k-1$ ) and  $\hat{F}_{k+1}$  is the piecewise quadratic interpolation of  $F$  at the nodes  $t_{k-\frac{1}{3}}, t_k$  and  $t_{k+\frac{1}{3}}$ . After some explicit calculations, the right hand side of (5) gives

$$I_{k+\frac{1}{3}} = \sum_{j=0}^k I_{j,k+\frac{1}{3}} F(t_j) + \sum_{j=0}^k L_{j,k+\frac{1}{3}} F(t_{j+\frac{1}{3}}) + \sum_{j=0}^{k-1} M_{j,k+\frac{1}{3}} F(t_{j+\frac{2}{3}}), \quad (6)$$

where

$$I'_{j,k+\frac{1}{3}} = \begin{cases} \left[ k - \frac{2}{3} \right]^{\alpha+1} \left[ \alpha^2 - \alpha + 27k^2 + 9k(\alpha - 1) \right] - \frac{1}{6} \left[ k + \frac{1}{3} \right]^{\alpha} \left[ -6\alpha^3 - 23\alpha \right. \\ \quad \left. + \alpha^2(33k - 25) + 3\alpha k(31 - 36k) + 18k(9(k-1)k + 2) \right], & j = 0, \\ \left[ -(j - k + \frac{2}{3}) \right]^{\alpha+1} \left[ +27j^2 + 27k^2 + (\alpha - 1)\alpha - 9j(\alpha + 6k - 1) \right. \\ \quad \left. + 9k(\alpha - 1) \right] - \left[ k - j + \frac{1}{3} \right]^{\alpha+1} \left[ 54j^2 + 72 + 11\alpha(\alpha + 5) - 36j(3k + 1) \right. \\ \quad \left. + 18k(3k + 2) \right] + \left[ k - j + \frac{4}{3} \right]^{\alpha+1} \left[ \alpha^2 + \alpha(9j - 9k - 7) \right. \\ \quad \left. + 9(j - k - 1)(3j - 3k - 2) \right], & 1 \leq j \leq k-1, \\ \frac{3-\alpha-1}{2} \left[ -11\alpha^2 - 79\alpha - 144 + 2^{2\alpha+3}(18 + (\alpha - 7)\alpha) \right], & j = k, \end{cases} \quad (7)$$

$$L'_{j,k+\frac{1}{3}} = \begin{cases} \frac{3}{2} \left[ k - j - \frac{2}{3} \right]^{\alpha+1} \left[ \alpha + 6 - 3(\alpha^2 + 8\alpha(k - j) + 18(j - k)^2) \right] \\ \quad + 3 \left[ k - j + \frac{1}{3} \right]^{\alpha+1} \left[ 3\alpha^2 + 5\alpha(3j - 3k + 2) + 3(3j - 3k + 1) \right. \\ \quad \left. (3j - 3k + 2) \right], & 0 \leq j \leq k-1, \\ \frac{3-\alpha}{2} (\alpha + 3)(\alpha + 4), & j = k, \end{cases} \quad (8)$$

$$M'_{j,k+\frac{1}{3}} = \begin{cases} 3 \left[ k - j - \frac{2}{3} \right]^{\alpha+1} \left[ 3\alpha^2 + 5\alpha(-3j+3k+1) + 9(j-k)(3j-3k-1) \right] \\ - \frac{3}{2} \left[ k - j + \frac{1}{3} \right]^{\alpha+1} \left[ 3\alpha^2 + 18(j-k)(3j-3k+2) \right. \\ \left. + \alpha(24j-24k+7) \right] & 0 \leq j \leq k-2, \\ \frac{3-\alpha}{2} \left[ 5\alpha^2 + 37\alpha + 72 + 4^{\alpha+1}((17-3\alpha)\alpha - 18) \right], & j = k-1, \end{cases} \quad (9)$$

in which  $I_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} I'_{j,k+\frac{1}{3}}$ ,  $L_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} L'_{j,k+\frac{1}{3}}$  and  $M_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} M'_{j,k+\frac{1}{3}}$ . Therefore the corrector formula to determine  $y_{k+\frac{1}{3}}$  is given by

$$y_{k+\frac{1}{3}} = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{1}{3}}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \left[ \sum_{j=0}^k I_{j,k+\frac{1}{3}} F_j + \sum_{j=0}^{k-1} L_{j,k+\frac{1}{3}} F_{j+\frac{1}{3}} + \sum_{j=0}^{k-1} M_{j,k+\frac{1}{3}} F_{j+\frac{2}{3}} \right. \\ \left. + L_{k,k+\frac{1}{3}} f(t_{k+\frac{1}{3}}, y_{k+\frac{1}{3}}^P) \right]. \quad (10)$$

At the beginning ( $k = 0$ ) the trapezoidal quadrature formula is applied to calculate  $y_{\frac{1}{3}}$ .

$$I_{0,\frac{1}{3}} = 3^{-\alpha} \alpha(\alpha+2)(\alpha+3), \quad L_{0,\frac{1}{3}} = 3^{-\alpha}(\alpha+2)(\alpha+3), \quad (11)$$

From Eq. (10) it is clear that we need to predict value of  $y_{k+\frac{1}{3}}$ . To this purpose, the trapezoidal quadrature formula is used for approximate the second integral in Eq. (5) instead of the Simpson's 1/3 formula. Indeed, as before, to approximate  $I_{k+\frac{1}{3}}, \hat{F}_{j+1}$  ( $j = 0, 1, \dots, k-1$ ) is chosen same as equation (5). On the last interval  $[t_k, t_{k+\frac{1}{3}}]$ ,  $\hat{F}_{k+1}$  is taken to be the piecewise linear interpolation of  $F$  at the nodes  $t_{k-\frac{1}{3}}$  and  $t_k$ . Hence,  $y_{k+\frac{1}{3}}^P$  is calculated by

$$y_{k+\frac{1}{3}}^P = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{1}{3}}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \left[ \sum_{j=0}^k \tilde{I}_{j,k+\frac{1}{3}} F_j + \sum_{j=0}^{k-1} L_{j,k+\frac{1}{3}} F_{j+\frac{1}{3}} + \sum_{j=0}^{k-1} \tilde{M}_{j,k+\frac{1}{3}} F_{j+\frac{2}{3}} \right]. \quad (12)$$

The product rectangular rule is used at the beginning. Therefore the weights of Eq.

(12) are as below:

$$\tilde{I}'_{j,k+\frac{1}{3}} = \begin{cases} \tilde{I}'_{0,\frac{1}{3}} = (\frac{1}{3})^\alpha (\alpha+1)(\alpha+2)(\alpha+3), \\ \begin{cases} I'_{j,k+\frac{1}{3}} & 0 \leq j \leq k-1, \\ \frac{3-\alpha-1}{2} \left[ -5\alpha^2 - 37\alpha - 72 + 2^{2\alpha+3}(18 + (\alpha-7)\alpha) \right], & j = k, \end{cases} \end{cases} \quad (13)$$

$$\tilde{M}'_{j,k+\frac{1}{3}} = \begin{cases} M'_{j,k+\frac{1}{3}} & 0 \leq j \leq k-2, \\ 3^{-\alpha} \left[ 2\alpha^2 + 15\alpha + 30 + 2^{2\alpha+1}(18 - (17-3\alpha)\alpha) \right], & j = k-1, \end{cases} \quad (14)$$

where  $\tilde{I}_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \tilde{I}'_{j,k+\frac{1}{3}}$ ,  $\tilde{M}_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \tilde{M}'_{j,k+\frac{1}{3}}$  and  $L_{j,k+\frac{1}{3}}$  are the same as equation (8).

The discretized form of (3) to calculate  $y_{k+\frac{2}{3}}$  is described below

$$y\left(t_{k+\frac{2}{3}}\right) = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{2}{3}}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} F(\tau) d\tau. \quad (15)$$

To achieve an implicit formula for calculate  $y_{k+\frac{2}{3}}$ , the integral in the right hand side of Eq. (15) is approximated by the following approach

$$\begin{aligned} I_{k+\frac{2}{3}} &= \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} F(\tau) d\tau = \left[ \int_0^{t_k} \hat{F}_k(\tau) + \int_{t_k}^{t_{k+\frac{1}{3}}} \hat{F}_{k+1}(\tau) \right] (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} d\tau \\ &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} \hat{F}_{j+1}(\tau) d\tau + \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} \hat{F}_{k+1}(\tau) d\tau, \end{aligned} \quad (16)$$

where  $\hat{F}_{j+1}$  is the cubic interpolation of  $F$  at the nodes  $t_j, t_{j+\frac{1}{3}}, t_{j+\frac{2}{3}}, t_{j+1}$ , ( $j = 0, 1, 2, \dots, k-1$ ) and  $\hat{F}_{k+1}$  is the piecewise quadratic interpolation of  $F$  at the nodes  $t_k, t_{k+\frac{1}{3}}$  and  $t_{k+\frac{2}{3}}$ . From this, we have

$$I_{k+\frac{2}{3}} = \sum_{j=0}^k P_{j,k+\frac{2}{3}} F(t_j) + \sum_{j=0}^k q_{j,k+\frac{2}{3}} F(t_{j+\frac{1}{3}}) + \sum_{j=0}^k R_{j,k+\frac{2}{3}} F(t_{j+\frac{2}{3}}), \quad (17)$$

$$P'_{j,k+\frac{2}{3}} = \begin{cases} \frac{1}{6} \left[ k + \frac{2}{3} \right]^\alpha \left[ 108\alpha k^2 - 162k^3 + 2\alpha(3\alpha^2 + 7\alpha + 2) - 3k(11\alpha^2 + 7\alpha - 6) \right] \\ \left[ k - \frac{1}{3} \right]^{\alpha+1} \left[ 27k^2 + \alpha(\alpha + 2) + 9k(\alpha + 1) \right], & j = 0, \\ \left[ k - j + \frac{5}{3} \right]^{\alpha+1} \left[ \alpha^2 - 10\alpha + 27j^2 + 36 - 9j(-\alpha + 6k + 7) + 27k^2 \right. \\ \left. - 9k(\alpha - 7) \right] + \left[ - (j - k + \frac{1}{3}) \right]^{\alpha+1} \left[ 27j^2 + 27k^2 + \alpha(\alpha + 2) \right. \\ \left. - 9j(\alpha + 6k + 1) + 9k(\alpha + 1) \right] - \left[ k - j + \frac{2}{3} \right]^{\alpha+1} \\ \left[ 54j^2 + 11\alpha(\alpha + 5) - 36j(3k + 2) + 18k(3k + 4) + 90 \right], & 1 \leq j \leq k-1, \\ \left[ \frac{5}{3} \right]^{\alpha+1} \left[ 36 + \alpha(\alpha - 10) \right] - \frac{1}{3} \left[ \frac{2}{3} \right]^{\alpha+2} \left[ 45 + \alpha(5\alpha + 28) \right], & j = k, \end{cases} \quad (18)$$

$$q'_{j,k+\frac{2}{3}} = \begin{cases} 3 \left[ k - j + \frac{2}{3} \right]^{\alpha+1} \left[ 27j^2 + 27k^2 + \alpha(3\alpha + 5) + j(15\alpha - 54k + 9) \right. \\ \left. - 3k(5\alpha + 3) \right] - \frac{3}{2} \left[ k - j - \frac{1}{3} \right]^{\alpha+1} \left[ 54j^2 + 54k^2 + \alpha(3\alpha + 7) \right. \\ \left. - 12j(2\alpha + 9k + 3) + 12k(2\alpha + 3) \right], & 1 \leq j \leq k-1, \\ 4 \left( \frac{2}{3} \right)^\alpha \alpha(\alpha + 3), & j = k, \end{cases} \quad (19)$$

$$R'_{j,k+\frac{2}{3}} = \begin{cases} 3 \left[ k - j - \frac{1}{3} \right]^{\alpha+1} \left[ 3\alpha^2 + 10\alpha + 27j^2 - 3j(5\alpha + 18k + 9) \right. \\ \left. + 27k^2 + 3k(5\alpha + 9) + 6 \right] - \frac{3}{2} \left[ k - j + \frac{2}{3} \right]^{\alpha+1} \left[ 3\alpha^2 - \alpha \right. \\ \left. + 54j^2 + 24\alpha j - 108jk + 54k^2 - 24\alpha k - 6 \right], & 1 \leq j \leq k-1, \\ \left( \frac{2}{3} \right)^\alpha (2 - \alpha)(\alpha + 3), & j = k. \end{cases} \quad (20)$$

where  $P_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} P'_{j,k+\frac{1}{3}}$ ,  $q_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} q'_{j,k+\frac{1}{3}}$  and  $R_{j,k+\frac{1}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} R'_{j,k+\frac{1}{3}}$ . Therefore  $y_{k+\frac{2}{3}}$  can be calculated by

$$y_{k+\frac{2}{3}} = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{2}{3}}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \left[ \sum_{j=0}^k P_{j,k+\frac{2}{3}} F_j + \sum_{j=0}^k q_{j,k+\frac{2}{3}} F_{j+\frac{1}{3}} + \sum_{j=0}^{k-1} R_{j,k+\frac{2}{3}} F_{j+\frac{2}{3}} \right. \\ \left. + R_{k,k+\frac{2}{3}} f(t_{k+\frac{2}{3}}, y_{k+\frac{2}{3}}^P) \right]. \quad (21)$$

At the beginning ( $k = 0$ ) the Simpson's 1/3 rule at the nodes  $t_0$ ,  $t_{\frac{1}{3}}$  and  $t_{\frac{2}{3}}$  is applied to calculate  $y_{\frac{2}{3}}$ . Hence if  $k = 0$  we have

$$P'_{0,\frac{2}{3}} = \alpha^2(\alpha + 3) \left( \frac{2}{3} \right)^\alpha, \quad q'_{0,\frac{2}{3}} = q'_{k,\frac{2}{3}}, \quad R'_{0,\frac{2}{3}} = R'_{k,\frac{2}{3}}. \quad (22)$$

An explicit formula for compute  $y_{k+\frac{2}{3}}$  is obtained by a very similar way to that used above for calculate  $y_{k+\frac{1}{3}}^P$ . To approximate  $I_{k+\frac{2}{3}}$ ,  $\hat{F}_{j+1}$  ( $j = 0, 1, \dots, k-1$ ) is chosen same as equation (16), but on the last interval  $[t_k, t_{k+\frac{2}{3}}]$  the trapezoidal quadrature formula is utilized instead of the Simpson's 1/3 formula. That means  $\hat{F}_{k+1}$  in (16) is taken to be the piecewise linear interpolation of  $F$  at the nodes  $t_k$  and  $t_{k+\frac{1}{3}}$ . After some elementary calculations, we get

$$y_{k+\frac{2}{3}}^P = \sum_{i=0}^{[\alpha]-1} \frac{t_{k+\frac{2}{3}}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \left[ \sum_{j=0}^k \tilde{P}_{j,k+\frac{2}{3}} F_j + \sum_{j=0}^k \tilde{q}_{j,k+\frac{2}{3}} F_{j+\frac{1}{3}} + \sum_{j=0}^{k-1} R_{j,k+\frac{2}{3}} F_{j+\frac{2}{3}} \right]. \quad (23)$$

with

$$\tilde{P}'_{j,k+\frac{2}{3}} = \begin{cases} P'_{j,k+\frac{2}{3}}, & 0 \leq j \leq k-1, \\ 3^{-\alpha-1} [5\alpha+1(\alpha^2 - 10\alpha + 36) - 2^\alpha(17\alpha^2 + 109\alpha + 198)], & j = k, \end{cases} \quad (24)$$

$$\tilde{q}'_{j,k+\frac{2}{3}} = \begin{cases} q'_{j,k+\frac{2}{3}}, & 0 \leq j \leq k-1, \\ 2(\frac{2}{3})^\alpha (\alpha+2)(\alpha+3), & j = k, \end{cases} \quad (25)$$

where  $\tilde{P}_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \tilde{P}'_{j,k+\frac{2}{3}}$ ,  $\tilde{q}_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \tilde{q}'_{j,k+\frac{2}{3}}$  and  $R_{j,k+\frac{1}{3}}$  are defined in (20). At the beginning the trapezoidal quadrature formula at the nodes  $t_0$  and  $t_{\frac{1}{3}}$  is utilized to approximate  $y_{\frac{2}{3}}^P$ .

$$\tilde{P}_{0,\frac{2}{3}} = \left(\frac{2}{3}\right)^\alpha (\alpha-1)(\alpha+2)(\alpha+3), \quad \tilde{q}_{0,\frac{2}{3}} = \tilde{q}_{k,\frac{2}{3}}. \quad (26)$$

The remaining problem is the determination of the PC formula required to the node  $t_{k+1}$ . To derive a corrector formula for calculating  $y_{k+1}$  the integral in the right hand side of Eq. (3) is replaced by the Simpson's 3/8 formula as follows:

$$\begin{aligned} I_{k+1} &= \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau = \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_k(\tau) d\tau \\ I_{k+1} &= \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_{j+1}(\tau) d\tau \\ &= \sum_{j=0}^{k+1} a_{j,k+1} F(t_j) + \sum_{j=0}^k b_{j,k+1} F(t_{j+\frac{1}{3}}) + \sum_{j=0}^k c_{j,k+1} F(t_{j+\frac{2}{3}}), \end{aligned} \quad (27)$$

where  $\hat{F}_{j+1}$  is the cubic interpolation for  $F$  at the nodes  $t_j, t_{j+1/3}, t_{j+2/3}$  and  $t_{j+1}$ .

$$a'_{j,k+1} = \begin{cases} k^{\alpha+1} \left[ \alpha^2 + 5\alpha + 9\alpha k + 6 + 27k(k+1) \right] + \frac{1}{2} [k+1]^\alpha \left[ 2\alpha^3 \right. \\ \quad \left. + \alpha^2(1-11k) + \alpha(3+k(36k+17)) - 6k(2+9k(k+1)) \right], & j=0, \\ \left[ k-1 \right]^{\alpha+1} \left[ \alpha^2 + 5\alpha + 27j^2 + 6 - 9j(\alpha+6k+3) + 27k^2 + 9k(\alpha+3) \right] \\ \quad + \left[ k-j+2 \right]^{\alpha+1} \left[ \alpha^2 - 13\alpha + 27j^2 + 27k^2 + 60 - 9j(-\alpha+6k+9) \right. \\ \quad \left. - 9k(\alpha-9) \right] - \left[ k-j+1 \right]^{\alpha+1} \left[ 54j^2 + 120 + 11\alpha(\alpha+5) \right. \\ \quad \left. - 108j(k+1) + 54k(k+2) \right], & 1 \leq j \leq k, \\ \alpha^2 - 4\alpha + 6, & j = k+1, \end{cases} \quad (28)$$

$$b'_{j,k+1} = \frac{9}{2} [k-j]^{\alpha+1} \left[ 18j^2 + 18k^2(\alpha+2)(\alpha+3) - 4j(2\alpha+9k+6) + 8k(\alpha+3) \right] \\ - 9 [k-j+1]^{\alpha+1} \left[ \alpha^2 + 9j^2 + 9k^2 + j(5\alpha-18k-3) + k(3-5\alpha) \right], \quad (29)$$

$$c'_{j,k+1} = 9 [k-j]^{\alpha+1} \left[ 9j^2 + 9k^2 + (\alpha+2)(\alpha+3) - j(5(\alpha+3) + 18k) + 5k(\alpha+3) \right] \\ - \frac{9}{2} [k-j+1]^{\alpha+1} \left[ \alpha^2 + \alpha(8j-8k-3) + 6(j-k)(3j-3k-2) \right], \quad (30)$$

in which  $a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} a'_{j,k+1}$ ,  $b_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} b'_{j,k+1}$  and  $c_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} c'_{j,k+1}$ . In this way, the corrector formula to determine  $y_{k+1}$  is given by

$$y_{k+1} = \sum_{i=0}^{[\alpha]-1} \frac{t_{k+1}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \left[ \sum_{j=0}^k a_{j,k+1} F_j + \sum_{j=0}^k b_{j,k+1} F_{j+\frac{1}{3}} + \sum_{j=0}^k c_{j,k+1} F_{j+\frac{2}{3}} \right. \\ \left. + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^P) \right]. \quad (31)$$

The Simpson's 3/8 rule and the Simpson's 1/3 rule are used to achieve an approximation formula for the  $y_{k+1}^P$ . To reach this purpose,  $I_{k+1}$  in (27) is modified as follows:

$$I_{k+1} = \int_0^{t_k} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_k(\tau) d\tau + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_{k+1}(\tau) d\tau, \\ = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_{j+1}(\tau) d\tau + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_{k+1}(\tau) d\tau, \quad (32)$$

in which  $\hat{F}_{j+1}$  is the cubic interpolation for  $F$  in the nodes  $t_j, t_{j+\frac{1}{3}}, t_{j+\frac{2}{3}}, t_{j+1}$ ,  $j = 0, 1, \dots, k-1$ , and  $\hat{F}_{k+1}$  is the piecewise quadratic interpolation of  $F$  at the nodes  $t_k, t_{k+\frac{1}{3}}$  and  $t_{k+\frac{2}{3}}$ . Consequently, the algorithm for the predict of  $y_{k+1}$  can be expressed by

$$y_{k+1}^P = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+1}^i}{i!} y^{(i)}(0) + \frac{1}{\Gamma(\alpha)} \left[ \sum_{j=0}^k \tilde{a}_{j,k+1} F_j + \sum_{j=0}^k \tilde{b}_{j,k+1} F_{j+\frac{1}{3}} + \sum_{j=0}^k \tilde{c}_{j,k+1} F_{j+\frac{2}{3}} \right], \quad (33)$$

in which

$$\tilde{a}'_{j,k+1} = \begin{cases} a'_{j,k+1}, & 0 \leq j \leq k-1, \\ -114 - 10\alpha^2 - 59\alpha + 2^{\alpha+1}(60 + (\alpha - 13)\alpha), & j = k, \end{cases} \quad (34)$$

$$\tilde{b}'_{j,k+1} = \begin{cases} b'_{j,k+1}, & 0 \leq j \leq k-1, \\ 6(\alpha - 1)(\alpha + 3), & j = k, \end{cases} \quad (35)$$

$$\tilde{c}'_{j,k+1} = \begin{cases} c'_{j,k+1}, & 0 \leq j \leq k-1, \\ -\frac{3}{2}(\alpha - 4)(\alpha + 3), & j = k, \end{cases} \quad (36)$$

where  $\tilde{a}_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \tilde{a}'_{j,k+\frac{2}{3}}$ ,  $\tilde{b}_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \tilde{b}'_{j,k+\frac{2}{3}}$  and  $\tilde{c}_{j,k+\frac{2}{3}} = \frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \tilde{c}'_{j,k+\frac{2}{3}}$ . The following special case should be considered:

$$\tilde{a}'_{0,1} = \frac{1}{2}(2\alpha^2 - 3\alpha + 4)(\alpha + 3), \quad \tilde{b}'_{0,1} = \tilde{b}'_{k,k+1}, \quad \tilde{c}'_{0,1} = \tilde{c}'_{k,k+1}. \quad (37)$$

Finally the new PC algorithm for solving Eq (1), is completely described by (31), (33), (10), (12), (21) and (23).

### 90 3. Truncation error analysis

In this section, we make the local truncation error analysis for presented PC algorithm. Firstly, we propose several lemmas for giving the error analysis of the compound Simpson's 3/8 formula and a corresponding result for the compound Simpson's 1/3 formula and compound trapezoidal formula, that we have used for the presented PC

95 formulae. In the following error analysis, let  $E_l = y(t_l) - y_l$  and  $E_l^P = y(t_l) - y_l^P$  denote the errors and  $\tilde{E}_l = y_l - \tilde{y}_l$ ,  $\tilde{E}_l^P = y_l - \tilde{y}_l^P$ , we always use the same  $C$  to denote some fixed constants which may have dissimilar values at different formulae.

**Lemma 1.** *For the weights of the novel PC scheme we have*

$$\begin{aligned} \sum_{j=0}^k |I_{j,k+\frac{1}{3}}| &\leq \frac{C_L}{\alpha} T^\alpha, & \sum_{j=0}^k |L_{j,k+\frac{1}{3}}| &\leq \frac{C_L}{\alpha} T^\alpha, & \sum_{j=0}^{k-1} |M_{j,k+\frac{1}{3}}| &\leq \frac{C_M}{\alpha} T^\alpha, \\ \sum_{j=0}^k |\tilde{I}_{j,k+\frac{1}{3}}| &\leq \frac{C_L^P}{\alpha} T^\alpha, & \sum_{j=0}^{k-1} |\tilde{L}_{j,k+\frac{1}{3}}| &\leq \frac{C_L^P}{\alpha} T^\alpha, & \sum_{j=0}^{k-1} |\tilde{M}_{j,k+\frac{1}{3}}| &\leq \frac{C_M^P}{\alpha} T^\alpha, \\ \sum_{j=0}^k |P_{j,k+\frac{2}{3}}| &\leq \frac{C_P}{\alpha} T^\alpha, & \sum_{j=0}^k |q_{j,k+\frac{2}{3}}| &\leq \frac{C_q}{\alpha} T^\alpha, & \sum_{j=0}^k |R_{j,k+\frac{2}{3}}| &\leq \frac{C_R}{\alpha} T^\alpha, \\ \sum_{j=0}^k |\tilde{P}_{j,k+\frac{2}{3}}| &\leq \frac{C_P^P}{\alpha} T^\alpha, & \sum_{j=0}^k |\tilde{q}_{j,k+\frac{2}{3}}| &\leq \frac{C_q^P}{\alpha} T^\alpha, & \sum_{j=0}^{k-1} |\tilde{R}_{j,k+\frac{2}{3}}| &\leq \frac{C_R^P}{\alpha} T^\alpha, \\ \sum_{j=0}^{k+1} |a_{j,k+1}| &\leq \frac{C_a}{\alpha} T^\alpha, & \sum_{j=0}^k |b_{j,k+1}| &\leq \frac{C_b}{\alpha} T^\alpha, & \sum_{j=0}^k |c_{j,k+1}| &\leq \frac{C_c}{\alpha} T^\alpha, \\ \sum_{j=0}^k |\tilde{a}_{j,k+1}| &\leq \frac{C_a^P}{\alpha} T^\alpha, & \sum_{j=0}^k |\tilde{b}_{j,k+1}| &\leq \frac{C_b^P}{\alpha} T^\alpha, & \sum_{j=0}^k |\tilde{c}_{j,k+1}| &\leq \frac{C_c^P}{\alpha} T^\alpha, \end{aligned}$$

where the constants  $c_* > 0$  and  $c_*^P > 0$  are independent of all discretization parameters.

*Proof.* We just prove that  $\sum_{j=0}^k |I_{j,k+\frac{1}{3}}| \leq \frac{C_L}{\alpha} T^\alpha$ . All of the reminder inequalities, can be proved in an entirely similar manner. The proof will be divided into three steps.

**Step 1:**  $j = 0$ :

$$\begin{aligned} |I_{0,k+\frac{1}{3}}| &= \left| \int_{t_0}^{t_1} (t_{k+\frac{1}{3}} - \tau)^{(\alpha-1)} \frac{\tau - t_{1/3}}{t_0 - t_{1/3}} \frac{\tau - t_{2/3}}{t_0 - t_{2/3}} \frac{\tau - t_1}{t_0 - t_1} d\tau \right| \leq \left| \frac{\tilde{\tau} - t_{1/3}}{t_0 - t_{1/3}} \frac{\tilde{\tau} - t_{2/3}}{t_0 - t_{2/3}} \frac{\tilde{\tau} - t_1}{t_0 - t_1} \right| \\ &\quad \int_{t_0}^{t_1} (t_{k+\frac{1}{3}} - \tau)^{(\alpha-1)} d\tau \leq \left| \frac{h^3}{-\frac{2}{9}h^3} \right| \frac{1}{\alpha} \left[ (t_{k+\frac{1}{3}})^\alpha - (t_{k-\frac{2}{3}})^\alpha \right] \leq \frac{9}{2\alpha} (t_{k+1})^\alpha = \frac{9}{2\alpha} T^\alpha. \end{aligned}$$

Here  $\tilde{\tau} \in [t_0, t_1]$  and the second integral mean value theorem is used.

**Step 2:**  $1 \leq j \leq k-1$ :

$$\begin{aligned} \sum_{j=1}^{k-1} |I_{j,k+\frac{1}{3}}| &= \sum_{j=1}^{k-1} \left[ \left| \int_{t_{j-1}}^{t_j} (t_{k+\frac{1}{3}} - \tau)^{(\alpha-1)} \frac{\tau - t_{j-1}}{t_j - t_{j-1}} \frac{\tau - t_{j-2/3}}{t_j - t_{j-2/3}} \frac{\tau - t_{j-1/3}}{t_j - t_{j-1/3}} d\tau \right| \right. \\ &\quad \left. + \left| \int_{t_j}^{t_{j+1}} (t_{k+\frac{1}{3}} - \tau)^{(\alpha-1)} \frac{\tau - t_{j+1/3}}{t_j - t_{j+1/3}} \frac{\tau - t_{j+2/3}}{t_j - t_{j+2/3}} \frac{\tau - t_{j+1}}{t_j - t_{j+1}} d\tau \right| \right] \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{k-1} |I_{j,k+\frac{1}{3}}| &\leq \left| \frac{h^3}{\frac{2}{9}h^3} \right| \frac{1}{\alpha} \sum_{j=1}^{k-1} \left[ (t_{k+\frac{1}{3}} - t_{j-1})^\alpha - (t_{k+\frac{1}{3}} - t_j)^\alpha \right] \\ &\quad + \left| \frac{h^3}{-\frac{2}{9}h^3} \right| \frac{1}{\alpha} \sum_{j=1}^{k-1} \left[ (t_{k+\frac{1}{3}} - t_j)^\alpha - (t_{k+\frac{1}{3}} - t_{j+1})^\alpha \right], \end{aligned}$$

$$\sum_{j=1}^{k-1} |I_{j,k+\frac{1}{3}}| = \frac{9}{2\alpha} \left[ (t_{k+\frac{1}{3}})^\alpha - (t_{k+\frac{1}{3}} - t_{k-1})^\alpha \right] + \left[ (t_{k+\frac{1}{3}} - t_1)^\alpha - (t_{k+\frac{1}{3}} - t_k)^\alpha \right] \leq \frac{9}{\alpha} T^\alpha$$

**Step 3:**  $j = k$ :

$$\begin{aligned} |I_{k,k+\frac{1}{3}}| &= \left| \int_{t_{k-1}}^{t_k} (t_{k+\frac{1}{3}} - \tau)^{(\alpha-1)} \frac{\tau - t_{k-1}}{t_k - t_{k-1}} \frac{\tau - t_{k-2/3}}{t_k - t_{k-2/3}} \frac{\tau - t_{k-1/3}}{t_k - t_{k-1/3}} d\tau \right. \\ &\quad \left. + \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{(\alpha-1)} \frac{\tau - t_{k-1/3}}{t_k - t_{k-1/3}} \frac{\tau - t_{k+1/3}}{t_k - t_{k+1/3}} d\tau \right| \\ &\leq \left| \frac{h^3}{\frac{2}{9}h^3} \right| \frac{1}{\alpha} \left[ (t_{k+\frac{1}{3}} - t_{k-1})^\alpha - (t_{k+\frac{1}{3}} - t_k)^\alpha \right] + \left| \frac{h^2}{\frac{1}{9}h^2} \right| \frac{1}{\alpha} (t_{k+\frac{1}{3}} - t_k)^\alpha \\ &\leq \frac{9}{2} \frac{1}{\alpha} (t_{k+1})^\alpha + \frac{9}{\alpha} (t_{k+1})^\alpha = \frac{27}{2\alpha} T^\alpha. \end{aligned}$$

Combining the results of these three steps, yields:  $\sum_{j=0}^k |I_{j,k+\frac{1}{3}}| \leq \frac{27}{\alpha} T^\alpha$ .  $\square$

100 The errors of the compound Simpson's 3/8 formula in equations (5), (16), (27) and (32) are presented in Lemmas (2)–(4) respectively.

**Lemma 2.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_0^{t_k} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_k(\tau)) d\tau \right| \leq Ch^4. \quad (38)$$

*Proof.*

$$I = \sum_{j=0}^{k-1} \left| \int_{t_j}^{t_{j+1}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_j(\tau)) d\tau \right|$$

By using Taylor theorem, it can be checked that for all  $\tau \in [t_j, t_{j+1}]$  there exist  $\xi_j(\tau) \in$

$[t_j, t_{j+1}]$  such that

$$\begin{aligned}
 I &\leq \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left| (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \frac{F^{(4)}(\xi_j(\tau))}{4!} (\tau - t_j)(\tau - t_{j+\frac{1}{3}})(\tau - t_{j+\frac{2}{3}})(\tau - t_{j+1}) \right| d\tau \\
 &\leq \frac{M_4}{4!} \sum_{j=0}^{k-1} \left| (\tilde{t}_j - t_j)(\tilde{t}_j - t_{j+1/3})(\tilde{t}_j - t_{j+2/3})(\tilde{t}_j - t_{j+1}) \right| \int_{t_j}^{t_{j+1}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} d\tau \\
 &\leq h^4 \frac{M_4}{4!} \frac{1}{\alpha} \sum_{j=0}^{k-1} \left[ (t_{k+\frac{1}{3}} - t_j)^\alpha - (t_{k+\frac{1}{3}} - t_{j+1})^\alpha \right] = h^4 \frac{M_4}{\alpha 4!} \left[ (t_{k+\frac{1}{3}})^\alpha - (t_1)^\alpha \right] \\
 &\leq h^4 \frac{M_4}{\alpha 4!} (t_{k+1})^\alpha = \left( \frac{M_4 T^\alpha}{\alpha 4!} \right) h^4, \quad \tilde{t}_j \in [t_j, t_{j+1}], \quad M_4 = \sup_{t \in [0, T]} |F^{(4)}(t)|.
 \end{aligned}$$

□

**Lemma 3.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_0^{t_k} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_k(\tau)) d\tau \right| \leq Ch^4. \quad (39)$$

**Lemma 4.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_0^{t_k} (t_{k+1} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_{k+1}(\tau)) d\tau \right| \leq Ch^4. \quad (40)$$

The proof of Lemmas (3) and (4) are similar to the proof of Lemma (2).

**Lemma 5.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\begin{aligned}
 &\left| \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^{k+1} a_{j,k+1} F(t_j) - \sum_{j=0}^k b_{j,k+1} F(t_{j+\frac{1}{3}}) \right. \\
 &\quad \left. - \sum_{j=0}^k c_{j,k+1} F(t_{j+\frac{2}{3}}) \right| \leq Ch^4. \quad (41)
 \end{aligned}$$

*Proof.* Equation (41) can be expressed in an integral form by applying Eq (27):

$$\begin{aligned}
 I &= \left| \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k \int_{t_j}^{t_{j+1}} (t_{k+1} - \tau)^{\alpha-1} \hat{F}_{j+1}(\tau) d\tau \right| \\
 &\leq \sum_{j=0}^k \left| \int_{t_j}^{t_{j+1}} (t_{k+1} - \tau)^{\alpha-1} \frac{F^{(4)}(\xi_j(\tau))}{4!} (\tau - t_j)(\tau - t_{j+\frac{1}{3}})(\tau - t_{j+\frac{2}{3}})(\tau - t_{j+1}) d\tau \right|.
 \end{aligned}$$

The reminder of proof is similar to Lemma (2).

$$I \leq h^4 \frac{M_4}{4!} \frac{1}{\alpha} \sum_{j=0}^k \left[ (t_{k+1} - t_j)^\alpha - (t_{k+1} - t_{j+1})^\alpha \right] = h^4 \frac{M_4}{\alpha 4!} (t_{k+1})^\alpha = \left( \frac{M_4 T^\alpha}{\alpha 4!} \right) h^4.$$

□

105 The errors of the compound Simpson's 1/3 formula in Eqs. (5), (16) and (32) are as below:

**Lemma 6.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_{k+1}(\tau)) d\tau \right| \leq Ch^{3+\alpha}. \quad (42)$$

*Proof.*

$$\begin{aligned} I &= \int_{t_k}^{t_{k+\frac{1}{3}}} \left| (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \frac{F^{(3)}(\xi_k(\tau))}{3!} (\tau - t_{k-\frac{1}{3}})(\tau - t_k)(\tau - t_{k+\frac{1}{3}}) d\tau \right| \\ &\leq \frac{M_3}{3!} \left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} (\tau - t_{k-\frac{1}{3}})(\tau - t_k)(\tau - t_{k+\frac{1}{3}}) d\tau \right| \\ &= \left| \frac{M_3}{3!} \frac{3^{-\alpha-3}(\alpha+5)}{\alpha^3+6\alpha^2+11\alpha+6} \right| h^{3+\alpha} \leq Ch^{3+\alpha}, \quad \xi_k(\tau) \in [t_{k-\frac{1}{3}}, t_{k+\frac{1}{3}}]. \end{aligned}$$

□

**Lemma 7.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_{k+1}(\tau)) d\tau \right| \leq Ch^{3+\alpha}. \quad (43)$$

**Lemma 8.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_{k+1}(\tau)) d\tau \right| \leq Ch^{3+\alpha}. \quad (44)$$

The errors of the trapezoidal quadrature formula, which used to construct formulae for  $y_{k+\frac{1}{3}}^P$  and  $y_{k+\frac{2}{3}}^P$ , are given by:

**Lemma 9.** Let  $F(\tau) \in C^2[0, T]$ , then

$$\left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_{k+1}(\tau)) d\tau \right| \leq Ch^{2+\alpha}. \quad (45)$$

*Proof.*

$$\begin{aligned} I &= \left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} \frac{F^{(2)}(\xi_k(\tau))}{2} (\tau - t_{k-\frac{1}{3}})(\tau - t_k) d\tau \right| \\ &\leq \frac{M_2}{2} \left| \int_{t_k}^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} (\tau - t_{k-\frac{1}{3}})(\tau - t_k) d\tau \right| = \left| \frac{M_2}{2} \frac{3^{-\alpha-2}(\alpha+4)}{\alpha(\alpha^2+3\alpha+2)} \right| h^{2+\alpha}, \end{aligned}$$

110 where  $\xi_k(\tau) \in [t_{k-\frac{1}{3}}, t_k]$  and  $M_2 = \sup_{t \in [0, T]} |F^{(2)}(t)|$ . □

**Lemma 10.** Let  $F(\tau) \in C^2[0, T]$ , then

$$\left| \int_{t_k}^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} (F(\tau) - \hat{F}_{k+1}(\tau)) d\tau \right| \leq Ch^{2+\alpha}. \quad (46)$$

From Lemmas (2), (6) and (9) the following lemma can be formulated:

**Lemma 11.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k I_{j,k+\frac{1}{3}} F(t_j) - \sum_{j=0}^k L_{j,k+\frac{1}{3}} F(t_{j+\frac{1}{3}}) - \sum_{j=0}^{k-1} M_{j,k+\frac{1}{3}} F(t_{j+\frac{2}{3}}) \right| \leq C_1 h^{\min\{4,3+\alpha\}}, \quad (47)$$

$$\left| \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k \tilde{I}_{j,k+\frac{1}{3}} F(t_j) - \sum_{j=0}^{k-1} L_{j,k+\frac{1}{3}} F(t_{j+\frac{1}{3}}) - \sum_{j=0}^{k-1} \tilde{M}_{j,k+\frac{1}{3}} F(t_{j+\frac{2}{3}}) \right| \leq C_2 h^{\min\{4,2+\alpha\}}, \quad (48)$$

From Lemmas (3), (7) and (10) the following Lemma can be derived.

**Lemma 12.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k P_{j,k+\frac{2}{3}} F(t_j) - \sum_{j=0}^k q_{j,k+\frac{2}{3}} F(t_{j+\frac{1}{3}}) - \sum_{j=0}^k R_{j,k+\frac{2}{3}} F(t_{j+\frac{2}{3}}) \right| \leq C_1 h^{\min\{4,3+\alpha\}}, \quad (49)$$

$$\left| \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k \tilde{P}_{j,k+\frac{2}{3}} F(t_j) - \sum_{j=0}^k \tilde{q}_{j,k+\frac{2}{3}} F(t_{j+\frac{1}{3}}) - \sum_{j=0}^{k-1} R_{j,k+\frac{2}{3}} F(t_{j+\frac{2}{3}}) \right| \leq C_2 h^{\min\{4,2+\alpha\}}, \quad (50)$$

From Lemmas (4) and (8) the following lemma can be proved.

**Lemma 13.** Let  $F(\tau) \in C^3[0, T]$ , then

$$\left| \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k \tilde{a}_{j,k+1} F(t_j) - \sum_{j=0}^k \tilde{b}_{j,k+1} F(t_{j+\frac{1}{3}}) - \sum_{j=0}^k \tilde{c}_{j,k+1} F(t_{j+\frac{2}{3}}) \right| \leq C h^{\min\{4,3+\alpha\}}, \quad (51)$$

In the next based on the explained lemmas, we give the error analysis for the equations (10), (12), (21) and (23) by using some ideas similar to that employed in [33].

**Lemma 14.** Assume that  $F(\tau) \in C^3[0, T]$  for some suitable chosen  $T$ , and

$$\max_{0 \leq j \leq k} |E_j| \leq C_0 h^q \quad (52)$$

then we have

$$\max_{0 \leq j \leq k} |E_{j+\frac{1}{3}}| \leq C_1 h^p + C_0 h^q, \quad (53)$$

$$\max_{0 \leq j \leq k} |E_{j+\frac{2}{3}}| \leq C_1 h^p + C_0 h^q, \quad (54)$$

for all  $j = 0, 1, 2, \dots, k$ , in which  $p = \min\{\delta_1, \delta_2 + \alpha\}$ , and  $\delta_1 = \min\{4, 3 + \alpha\}$ ,  $\delta_2 = \min\{4, 2 + \alpha\}$ .

*Proof.* The proof is based on mathematical induction. In view of the given initial conditions, the induction basis is evident. Assume that (53) and (54) are hold for  $j = 0, 1, 2, \dots, k-1$ ; we will prove it for  $j = k$ . From (12) and (4) it can be found that:

$$\begin{aligned} |E_{k+\frac{1}{3}}^P| &\leq \frac{1}{\Gamma(\alpha)} \left[ \left| \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k \tilde{I}_{j,k+\frac{1}{3}} F(t_j) - \sum_{j=0}^{k-1} L_{j,k+\frac{1}{3}} F(t_{j+\frac{1}{3}}) \right. \right. \\ &\quad \left. \left. - \sum_{j=0}^{k-1} \tilde{M}_{j,k+\frac{1}{3}} F(t_{j+\frac{2}{3}}) \right| + \left| \sum_{j=0}^k \tilde{I}_{j,k+\frac{1}{3}} (F(t_j) - F_j) \right| \right. \\ &\quad \left. + \left| \sum_{j=0}^{k-1} L_{j,k+\frac{1}{3}} (F(t_{j+\frac{1}{3}}) - F_{j+\frac{1}{3}}) \right| + \left| \sum_{j=0}^{k-1} \tilde{M}_{j,k+\frac{1}{3}} (F(t_{j+\frac{2}{3}}) - f_{j+\frac{2}{3}}) \right| \right], \end{aligned}$$

applying (48) from lemma 11 to the first statement of the above equation one can get

$$\begin{aligned} |E_{k+\frac{1}{3}}^P| &\leq \frac{1}{\Gamma(\alpha)} \left[ C_2 h^{\delta_2} + \sum_{j=0}^k |\tilde{I}_{j,k+\frac{1}{3}}| |F(t_j) - F_j| + \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| |F(t_{j+\frac{1}{3}}) - F_{j+\frac{1}{3}}| \right. \\ &\quad \left. + \sum_{j=0}^{k-1} |\tilde{M}_{j,k+\frac{1}{3}}| |F(t_{j+\frac{2}{3}}) - f_{j+\frac{2}{3}}| \right], \end{aligned}$$

in which from lemma 11,  $\delta_2 = \min\{4, 2 + \alpha\}$ . We have, by the Lipschitz condition (2),

$$\begin{aligned} |E_{k+\frac{1}{3}}^P| &\leq \frac{1}{\Gamma(\alpha)} \left[ C_2 h^{\delta_2} + L_1 \sum_{j=0}^k |\tilde{I}_{j,k+\frac{1}{3}}| |E_j| + L_2 \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| |E_{j+\frac{1}{3}}| \right. \\ &\quad \left. + L_3 \sum_{j=0}^{k-1} |\tilde{M}_{j,k+\frac{1}{3}}| |E_{j+\frac{2}{3}}| \right] \end{aligned}$$

$$E_{k+\frac{1}{3}}^P \leq \frac{1}{\Gamma(\alpha)} \left[ C_2 h^{\delta_2} + L_1 \sum_{j=0}^k \left| \tilde{I}_{j,k+\frac{1}{3}} \right| \max_{0 \leq j \leq k} |E_j| + L_2 \sum_{j=0}^{k-1} \left| L_{j,k+\frac{1}{3}} \right| \max_{0 \leq j \leq k-1} |E_{j+\frac{1}{3}}| \right. \\ \left. + L_3 \sum_{j=0}^{k-1} \left| \tilde{M}_{j,k+\frac{1}{3}} \right| \max_{0 \leq j \leq k-1} |E_{j+\frac{2}{3}}| \right].$$

By applying Lemma 1, Eq. (52) and the induction hypothesis it can be obtained

$$\left| E_{k+\frac{1}{3}}^P \right| \leq \frac{1}{\Gamma(\alpha)} \left[ C_2 h^{\delta_2} + L_1 \frac{C_L^P}{\alpha} T^\alpha (C_0 h^q) + L_2 \frac{C_L^P}{\alpha} T^\alpha (C_{1\frac{1}{3}} h^p + C_0 h^q) \right. \\ \left. + L_3 \frac{C_M^P}{\alpha} T^\alpha (C_{1\frac{2}{3}} h^p + C_0 h^q) \right] \leq C_{1P\frac{1}{3}} h^{\delta_2} + C_{2P\frac{1}{3}} h^p + C_{3P\frac{1}{3}} h^q, \quad (55)$$

$$C_{1P\frac{1}{3}} = \frac{C_2}{\Gamma(\alpha)}, \quad C_{2P\frac{1}{3}} = \frac{L_2 C_L^P C_{1\frac{1}{3}} T^\alpha}{\Gamma(\alpha+1)} + \frac{L_3 C_M^P C_{1\frac{2}{3}} T^\alpha}{\Gamma(\alpha+1)}, \\ C_{3P\frac{1}{3}} = \frac{L_1 C_L^P C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 C_L^P C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_3 C_M^P C_0 T^\alpha}{\Gamma(\alpha+1)},$$

as a same procedure in [33, 25],  $T$  is chosen to be sufficiently small such that  $\frac{L_1 C_L^P T^\alpha}{\Gamma(\alpha+1)} \leq C_0/2$ . After fixing this value for  $T$ , we can then make the sum of the remaining expressions in  $C_{3P\frac{1}{3}}$  smaller than  $C_0/2$  too (for sufficiently small  $h$ ) by choosing  $C_0$  sufficiently large. For this reason, one can get

$$\left| E_{k+\frac{1}{3}}^P \right| \leq C_{1P\frac{1}{3}} h^{\delta_2} + C_{2P\frac{1}{3}} h^p + C_0 h^q, \quad (56)$$

In a quite similar way, for  $y_{k+\frac{2}{3}}^P$  it can be proved that

$$\left| E_{k+\frac{2}{3}}^P \right| \leq C_{1P\frac{2}{3}} h^{\delta_2} + C_{2P\frac{2}{3}} h^p + C_0 h^q, \quad (57)$$

here from lemma 12,  $\delta_2 = \min\{4, 2 + \alpha\}$ . For the error of  $y_{k+\frac{1}{3}}$  from (4) and (10) one can obtain,

$$\left| E_{k+\frac{1}{3}} \right| \leq \frac{1}{\Gamma(\alpha)} \left[ \left| \int_0^{t_{k+\frac{1}{3}}} (t_{k+\frac{1}{3}} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k I_{j,k+\frac{1}{3}} F(t_j) - \sum_{j=0}^k L_{j,k+\frac{1}{3}} F(t_{j+\frac{1}{3}}) \right. \right. \\ \left. \left. - \sum_{j=0}^{k-1} M_{j,k+\frac{1}{3}} F(t_{j+\frac{2}{3}}) \right| + \left| \sum_{j=0}^k I_{j,k+\frac{1}{3}} (F(t_j) - F_j) \right| + \left| \sum_{j=0}^{k-1} L_{j,k+\frac{1}{3}} (F(t_{j+\frac{1}{3}}) - F_{j+\frac{1}{3}}) \right| \right. \\ \left. + \left| L_{k,k+\frac{1}{3}} (F(t_{k+\frac{1}{3}}) - f(t_{k+\frac{1}{3}}, y_{k+\frac{1}{3}}^P)) \right| + \left| \sum_{j=0}^{k-1} M_{j,k+\frac{1}{3}} (F(t_{j+\frac{2}{3}}) - F_{j+\frac{2}{3}}) \right| \right],$$

applying (47) from lemma 11 to the first statement

$$\begin{aligned} |E_{k+\frac{1}{3}}| \leq & \frac{1}{\Gamma(\alpha)} \left[ C_1 h^{\delta_1} + L_1 \sum_{j=0}^k |I_{j,k+\frac{1}{3}}| |E_j| + L_2 \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| |E_{j+\frac{1}{3}}| \right. \\ & \left. + L_2 |L_{k,k+\frac{1}{3}}| |E_{k+\frac{1}{3}}^P| + L_3 \sum_{j=0}^{k-1} |M_{j,k+\frac{1}{3}}| |E_{j+\frac{2}{3}}| \right], \end{aligned}$$

where from lemma 11,  $\delta_1 = \min\{4, 3 + \alpha\}$ . We have, by the value of  $L_{k,k+\frac{1}{3}}$  in Eq. (8),

$$\begin{aligned} |E_{k+\frac{1}{3}}| \leq & \frac{1}{\Gamma(\alpha)} \left[ C_1 h^{\delta_1} + L_1 \sum_{j=0}^k |I_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k} |E_j| + L_2 \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k-1} |E_{j+\frac{1}{3}}| \right. \\ & \left. + L_2 \left| \frac{3^{-\alpha}(\alpha+4)}{2\alpha(\alpha+1)(\alpha+2)} h^\alpha \right| |E_{k+\frac{1}{3}}^P| + L_3 \sum_{j=0}^{k-1} |M_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k-1} |E_{j+\frac{2}{3}}| \right]. \end{aligned}$$

By applying Lemma 1, the induction hypothesis and Eq. (56) it can be achieved

$$\begin{aligned} |E_{k+\frac{1}{3}}| \leq & \frac{1}{\Gamma(\alpha)} \left[ C_1 h^{\delta_1} + L_1 \frac{C_I}{\alpha} T^\alpha (C_0 h^q) + L_2 \frac{C_L^P}{\alpha} T^\alpha (C_{1\frac{1}{3}} h^p + C_0 h^q) \right. \\ & + \left| \frac{L_2 3^{-\alpha}(\alpha+4)}{2\alpha(\alpha+1)(\alpha+2)} h^\alpha \right| (C_{1P\frac{1}{3}} h^{\delta_2} + C_{2P\frac{1}{3}} h^p + C_0 h^q) \\ & \left. + L_3 \frac{C_M}{\alpha} T^\alpha (C_{1\frac{1}{3}} h^p + C_0 h^q) \right] \leq C_2 h^{\delta_1} + C_3 h^{\alpha+\delta_2} + C_4 h^p + C_5 h^q, \\ C_2 = & \frac{C_1}{\Gamma(\alpha)}, \quad C_3 = \left| \frac{L_2 C_{1P\frac{1}{3}} 3^{-\alpha}(\alpha+4)}{2\Gamma(\alpha+3)} \right|, \\ C_4 = & \frac{L_2 C_L^P C_{1\frac{1}{3}} T^\alpha}{\Gamma(\alpha+1)} + \left| \frac{L_2 C_{2P\frac{1}{3}} 3^{-\alpha}(\alpha+4) h^\alpha}{2\Gamma(\alpha+3)} \right| + \frac{L_3 C_M C_{1\frac{1}{3}} T^\alpha}{\Gamma(\alpha+1)}, \\ C_5 = & \frac{L_1 C_I C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 C_L^P C_0 T^\alpha}{\Gamma(\alpha+1)} + \left| \frac{L_2 C_0 3^{-\alpha}(\alpha+4) h^\alpha}{2\Gamma(\alpha+3)} \right| + \frac{L_3 C_M C_0 T^\alpha}{\Gamma(\alpha+1)}, \end{aligned}$$

by choosing suitable  $T$  and sufficiently large  $C_0$  in a similar way to equation (55) we make  $C_5 \leq C_0$ . In view of relations  $p \leq \delta_1$  and  $p \leq \alpha + \delta_2$  and choose sufficiently large

<sup>120</sup>  $C_{1\frac{1}{3}}$  one can complete proof of (53). Eqs. (15) and (21) are used to proof Eq. (54)

$$\begin{aligned} |E_{k+\frac{2}{3}}| \leq & \frac{1}{\Gamma(\alpha)} \left[ \int_0^{t_{k+\frac{2}{3}}} (t_{k+\frac{2}{3}} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k P_{j,k+\frac{2}{3}} F(t_j) - \sum_{j=0}^k q_{j,k+\frac{2}{3}} F(t_{j+\frac{1}{3}}) \right. \\ & - \sum_{j=0}^k R_{j,k+\frac{2}{3}} F(t_{j+\frac{2}{3}}) \left. + L_1 \sum_{j=0}^k |P_{j,k+\frac{2}{3}}| \max_{0 \leq j \leq k} |E_j| + L_2 \sum_{j=0}^k |q_{j,k+\frac{2}{3}}| \max_{0 \leq j \leq k} |E_{j+\frac{1}{3}}| \right. \\ & \left. + L_3 \sum_{j=0}^{k-1} |R_{j,k+\frac{2}{3}}| \max_{0 \leq j \leq k-1} |E_{j+\frac{2}{3}}| + L_4 |R_{k,k+\frac{2}{3}}| |E_{k+\frac{1}{3}}^P| \right], \end{aligned}$$

$$\begin{aligned} |E_{k+\frac{2}{3}}| &\leq \frac{1}{\Gamma(\alpha)} \left[ C_3 h^{\delta_1} + L_1 \frac{C_P}{\alpha} T^\alpha (C_0 h^q) + L_2 \frac{C_q}{\alpha} T^\alpha (C_{1\frac{1}{3}} h^p + C_0 h^q) \right. \\ &\quad \left. + L_3 \frac{C_R^P}{\alpha} T^\alpha (C_{1\frac{2}{3}} h^p + C_0 h^q) + \left| \frac{L_3 2^\alpha (2-\alpha)}{3^\alpha \alpha (\alpha+1)(\alpha+2)} h^\alpha \right| \right. \\ &\quad \left. (C_{1P\frac{2}{3}} h^{\delta_2} + C_{2P\frac{2}{3}} h^p + C_0 h^q) \right] \leq C_1 h^{\delta_1} + C_2 h^{\alpha+\delta_2} + C_4 h^p + C_5 h^q, \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{C_3}{\Gamma(\alpha)}, \quad C_2 = \left| \frac{L_3 C_{1P\frac{2}{3}} 2^\alpha (2-\alpha)}{3^\alpha \Gamma(\alpha+3)} \right|, \\ C_4 &= \frac{L_2 C_q C_{1\frac{1}{3}} T^\alpha}{\Gamma(\alpha+1)} + \frac{L_3 C_R^P C_{1\frac{2}{3}} T^\alpha}{\Gamma(\alpha+1)} + \left| \frac{L_3 C_{2P\frac{2}{3}} 2^\alpha (2-\alpha) h^\alpha}{3^\alpha \Gamma(\alpha+3)} \right|, \\ C_5 &= \frac{L_1 C_P C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 C_q C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_3 C_R^P C_0 T^\alpha}{\Gamma(\alpha+1)} + \left| \frac{L_3 C_0 2^\alpha (2-\alpha) h^\alpha}{3^\alpha \Gamma(\alpha+3)} \right|, \end{aligned}$$

here from lemma 12,  $\delta_1 = \min\{4, 3 + \alpha\}$ . The value of  $R_{k,k+\frac{2}{3}}$  in Eq. (20), Lemma 1, induction hypothesis, Eqs. (53) and (57) are used. Like as Eq. (55) we make  $C_5 \leq C_0$  and from inequalities  $p \leq \delta_1$ ,  $p \leq \alpha + \delta_2$  and choose sufficiently large  $C_{1\frac{2}{3}}$  one can complete proof of (54). The details of this proof are similar to proof of Eq. (53).  $\square$

125 Based on the error estimate of the preceding section, we now present a general Theorem for the local truncation error the convergence result of the presented PC scheme.

**Theorem 1.** Let  $F(\tau) \in C^3[0, T]$  for some suitable chosen  $T$ , and the assumptions of Lemma (14) hold. Then we have

$$\max_{0 \leq j \leq N} |E_j| = O(h^q),$$

where  $q = \min\{4, P\}$  and  $N = \lceil T/h \rceil$ .

*Proof.* This proof will be used based on mathematical induction. it is assumed that

$$\max_{0 \leq j \leq k} |E_j| \leq C_0 h^q \quad (58)$$

is true for  $j = 0, 1, \dots, k$  for some  $k \leq N - 1$ . We must prove that it is also holds for  $j = k + 1$ . In view of the given initial conditions, the induction basis  $j = 0$  is trivial.

For  $y_{k+1}^P$ , we find that

$$\begin{aligned} |E_{k+1}^P| \leq & \frac{1}{\Gamma(\alpha)} \left[ \left| \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^k \tilde{a}_{j,k+1} F(t_j) - \sum_{j=0}^k \tilde{b}_{j,k+1} F(t_{j+\frac{1}{3}}) \right. \right. \\ & \left. \left. - \sum_{j=0}^k \tilde{c}_{j,k+1} F(t_{j+\frac{2}{3}}) \right| + L_1 \sum_{j=0}^k |\tilde{a}_{j,k+1}| |E_j| + L_2 \sum_{j=0}^k |\tilde{b}_{j,k+1}| |E_{j+\frac{1}{3}}| \right. \\ & \left. + L_3 \sum_{j=0}^k |\tilde{c}_{j,k+1}| |E_{j+\frac{2}{3}}| \right], \end{aligned}$$

applying Lemmas 1 and 13 to the first statement of the above equation one can get

$$\begin{aligned} |E_{k+1}^P| \leq & \frac{1}{\Gamma(\alpha)} \left[ C_1 h^{\delta_1} + L_1 \frac{C_a^P}{\alpha} T^\alpha \max_{0 \leq j \leq k} |E_j| + L_2 \frac{C_b^P}{\alpha} T^\alpha \max_{0 \leq j \leq k} |E_{j+\frac{1}{3}}| \right. \\ & \left. + L_3 \frac{C_c^P}{\alpha} T^\alpha \max_{0 \leq j \leq k} |E_{j+\frac{2}{3}}| \right], \end{aligned}$$

where  $\delta_1 = \min\{4, 3 + \alpha\}$ . By mathematical induction (58) and Lemma (14), we have

$$\begin{aligned} |E_{k+1}^P| \leq & \frac{1}{\Gamma(\alpha)} \left[ C_1 h^{\delta_1} + L_1 \frac{C_a^P}{\alpha} T^\alpha (C_0 h^q) + L_2 \frac{C_b^P}{\alpha} T^\alpha (C_{1\frac{1}{3}} h^p + C_0 h^q) \right. \\ & \left. + L_3 \frac{C_c^P}{\alpha} T^\alpha (C_{2\frac{2}{3}} h^p + C_0 h^q) \right] \leq C_2 h^{\delta_1} + C_3 h^p + C_4 h^q, \end{aligned}$$

$$\begin{aligned} C_2 = & \frac{C_1}{\Gamma(\alpha)}, \quad C_3 = \frac{L_2 C_b^P C_{1\frac{1}{3}} T^\alpha}{\Gamma(\alpha+1)} + \frac{L_3 C_c^P C_{1\frac{2}{3}} T^\alpha}{\Gamma(\alpha+1)}, \\ C_4 = & \frac{L_1 C_a^P C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 C_b^P C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_3 C_c^P C_0 T^\alpha}{\Gamma(\alpha+1)}, \end{aligned}$$

by choosing suitable  $T$  and sufficiently large  $C_0$  in a similar way to equation (55) we make  $C_4 \leq C_0$ . From the assumptions of Lemma 14 we have  $p \leq \delta_1$ , in this way, by choosing sufficiently large  $C_P$ , we find that  $|y(t_{k+1}) - y_{k+1}^P| \leq C_P h^p + C_0 h^q$ . Finally in the same manner the order of convergence of  $y_{k+1}$  is calculated as follows:

$$\begin{aligned} |E_{k+1}| \leq & \frac{1}{\Gamma(\alpha)} \left[ \left| \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} F(\tau) d\tau - \sum_{j=0}^{k+1} a_{j,k+1} F(t_j) - \sum_{j=0}^k b_{j,k+1} F(t_{j+\frac{1}{3}}) \right. \right. \\ & \left. \left. - \sum_{j=0}^k c_{j,k+1} F(t_{j+\frac{2}{3}}) \right| + L \sum_{j=0}^k |a_{j,k+1}| |E_j| + L |a_{k+1,k+1}| |E_{k+1}^P| \right. \\ & \left. + L_1 \sum_{j=0}^k |b_{j,k+1}| |E_{j+\frac{1}{3}}| + L_2 \sum_{j=0}^k |c_{j,k+1}| |E_{j+\frac{2}{3}}| \right], \end{aligned}$$

applying Lemmas 1, 5 and (14), mathematical induction (58) and the value of  $a_{k+1,k+1}$  in (28) leads to

$$|E_{k+1}| \leq \frac{1}{\Gamma(\alpha)} \left[ C_1 h^4 + L \frac{C_a}{\alpha} T^\alpha (C_0 h^q) + L \left| \frac{(\alpha^2 - 4\alpha + 6)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \right| h^\alpha (C_P h^p + C_0 h^q) \right. \\ \left. + L_1 \frac{C_b}{\alpha} T^\alpha (C_{1\frac{1}{3}} h^p + C_0 h^q) + L_2 \frac{C_c}{\alpha} T^\alpha (C_{1\frac{2}{3}} h^p + C_0 h^q) \right] \leq C_2 h^4 + C_3 h^p + C_4 h^q,$$

$$C_2 = \frac{C_1}{\Gamma(\alpha)}, \quad C_3 = \left| \frac{L C_P (\alpha^2 - 4\alpha + 6) h^\alpha}{\Gamma(\alpha+4)} \right| + \frac{L_1 C_b C_{1\frac{1}{3}} T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 C_c C_{1\frac{2}{3}} T^\alpha}{\Gamma(\alpha+1)}, \\ C_4 = \frac{L C_a C_0 T^\alpha}{\Gamma(\alpha+1)} + \left| \frac{L C_0 (\alpha^2 - 4\alpha + 6) h^\alpha}{\Gamma(\alpha+4)} \right| + \frac{L_1 C_b C_0 T^\alpha}{\Gamma(\alpha+1)} + \frac{L_2 C_c C_0 T^\alpha}{\Gamma(\alpha+1)}.$$

In view of the relations  $q \leq 4$  and  $q \leq p$  and by choosing  $C_0$  sufficiently large it can be written as  $|E_{k+1}| \leq C_0 h^q$ .  $\square$

#### 130 4. Stability analysis

A numerical initial value problem solver is stable if small perturbations in the initial conditions do not cause the numerical approximation to diverge away from the true solution provided the true solution of the initial value problem is bounded [34, 20, 32, 35]. In this section, stability analysis of the presented PC method is discussed.

**Lemma 15.** Suppose  $y_{k+\frac{1}{3}}$  and  $\tilde{y}_{k+\frac{1}{3}}$  are numerical solutions in (10), also  $y_{k+\frac{2}{3}}$  and  $\tilde{y}_{k+\frac{2}{3}}$  are numerical solutions in (21), the initial conditions are given by  $y_0^{(i)}$  and  $\tilde{y}_0^{(i)}$ , respectively. Then

$$\max_{0 \leq j \leq k} |\tilde{E}_{j+\frac{1}{3}}| \leq K_{1\frac{1}{3}} \|\tilde{E}_0\|_\infty + K_{2\frac{1}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \quad (59)$$

$$\max_{0 \leq j \leq k} |\tilde{E}_{j+\frac{2}{3}}| \leq K_{1\frac{2}{3}} \|\tilde{E}_0\|_\infty + K_{2\frac{2}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \quad (60)$$

135 for all  $j = 0, 1, 2, \dots, k$ .

*Proof.* The principle of mathematical induction is used to prove this Lemma. In view of the given initial condition, the induction basis is presupposed. Suppose that equations (59) and (60) are true for  $j = 0, 1, 2, \dots, k-1$ . For  $y_{k+\frac{1}{3}}^p$  from equations (12) by applying the Lipschitz property of  $f$  we have

$$\begin{aligned} |\tilde{E}_{k+\frac{1}{3}}^P| &\leq \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{1}{3}}^i}{i!} \|\tilde{E}_0\|_\infty + \frac{1}{\Gamma(\alpha)} \left[ L_1 \sum_{j=0}^k |\tilde{I}_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k} |\tilde{E}_j| + L_2 \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| \right. \\ &\quad \left. \max_{0 \leq j \leq k-1} |\tilde{E}_{j+\frac{1}{3}}| + L_3 \sum_{j=0}^{k-1} |\tilde{M}_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k-1} |\tilde{E}_{j+\frac{2}{3}}| \right], \end{aligned}$$

Substitution the induction hypothesis leads to

$$\begin{aligned} |\tilde{E}_{k+\frac{1}{3}}^P| &\leq \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{1}{3}}^i}{i!} \|\tilde{E}_0\|_\infty + \frac{1}{\Gamma(\alpha)} \left[ L_1 \sum_{j=0}^k |\tilde{I}_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k} |\tilde{E}_j| + L_2 \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| \right. \\ &\quad \left( K_{1\frac{1}{3}} \|\tilde{E}_0\|_\infty + K_{2\frac{1}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \right) + L_3 \sum_{j=0}^{k-1} |\tilde{M}_{j,k+\frac{1}{3}}| \left( K_{1\frac{2}{3}} \|\tilde{E}_0\|_\infty \right. \\ &\quad \left. \left. + K_{2\frac{2}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \right) \right] \leq K_{1P\frac{1}{3}} \|\tilde{E}_0\|_\infty + K_{2P\frac{1}{3}} \max_{0 \leq j \leq k} |y_j - \tilde{y}_j|. \end{aligned} \quad (61)$$

By a similar way for  $y_{k+\frac{2}{3}}^P$  one gets,

$$|\tilde{E}_{k+\frac{2}{3}}^P| \leq K_{1P\frac{2}{3}} \|\tilde{E}_0\|_\infty + K_{2P\frac{2}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j|. \quad (62)$$

Construction  $y_{k+\frac{1}{3}}$  and  $\tilde{y}_{k+\frac{1}{3}}$  from Eq. (10) and employ the Lipschitz property of  $f$ , leads to

$$\begin{aligned} |\tilde{E}_{k+\frac{1}{3}}| &\leq \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{1}{3}}^i}{i!} \|\tilde{E}_0\|_\infty + \frac{1}{\Gamma(\alpha)} \left[ L_1 \sum_{j=0}^k |I_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k} |\tilde{E}_j| + L_2 \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| \right. \\ &\quad \left. \max_{0 \leq j \leq k-1} |\tilde{E}_{j+\frac{1}{3}}| + L_3 |L_{k,k+\frac{1}{3}}| |\tilde{E}_{k+\frac{1}{3}}^P| + L_4 \sum_{j=0}^{k-1} |M_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k-1} |\tilde{E}_{j+\frac{2}{3}}| \right]. \end{aligned}$$

By the induction hypothesis and Eq. (61), above inequality becomes

$$\begin{aligned} |\tilde{E}_{k+\frac{1}{3}}| &\leq \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+\frac{1}{3}}^i}{i!} \|\tilde{E}_0\|_\infty + \frac{1}{\Gamma(\alpha)} \left[ L_1 \sum_{j=0}^k |I_{j,k+\frac{1}{3}}| \max_{0 \leq j \leq k} |\tilde{E}_j| + L_2 \sum_{j=0}^{k-1} |L_{j,k+\frac{1}{3}}| \right. \\ &\quad \left( K_{1\frac{1}{3}} \|\tilde{E}_0\|_\infty + K_{2\frac{1}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \right) + L_3 |L_{k,k+\frac{1}{3}}| \left( K_{1P\frac{1}{3}} \|\tilde{E}_0\|_\infty + K_{2P\frac{1}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \right) \\ &\quad \left. + L_4 \sum_{j=0}^{k-1} |M_{j,k+\frac{1}{3}}| \left( K_{1\frac{2}{3}} \|\tilde{E}_0\|_\infty + K_{2\frac{2}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \right) \right]. \end{aligned}$$

140 Therefore Eq. (59) is holds. In a quit same manner, Eq. (60) can be proved by using the induction hypothesis and equations (59) and (62).  $\square$

**Theorem 2.** Let  $y_{k+1}$  and  $\tilde{y}_{k+1}$  be numerical solutions in (31), which the initial conditions are given by  $y_0^{(i)}$  and  $\tilde{y}_0^{(i)}$ , respectively. Then

$$|\tilde{E}_{k+1}| \leq K \|\tilde{E}_0\|_\infty, \quad (63)$$

for any  $k$ , i.e. the new PC scheme is numerically stable.

*Proof.* This proof will be used based on mathematical induction. In view of the given initial condition, the induction basis is presupposed. Assuming (63) to holds for  $j = 0, 1, 2, \dots, k$ , we will prove it for  $j = k + 1$ . Consider the following formula for  $y_{k+1}^p$ .

$$\begin{aligned} |\tilde{E}_{k+1}^p| &\leq \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+1}^i}{i!} \|\tilde{E}_0\|_\infty + \frac{1}{\Gamma(\alpha)} \left[ L_1 \sum_{j=0}^k |\tilde{a}_{j,k+1}| \max_{0 \leq j \leq k} |\tilde{E}_j| + L_2 \sum_{j=0}^k |\tilde{b}_{j,k+1}| \right. \\ &\quad \left. \max_{0 \leq j \leq k} |\tilde{E}_{j+\frac{1}{3}}| + L_3 \sum_{j=0}^k |\tilde{c}_{j,k+1}| \max_{0 \leq j \leq k} |\tilde{E}_{j+\frac{2}{3}}| \right]. \end{aligned}$$

By Lemma (15) one can obtain,

$$\begin{aligned} |\tilde{E}_{k+1}^p| &\leq \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{k+1}^i}{i!} \|\tilde{E}_0\|_\infty + \frac{1}{\Gamma(\alpha)} \left[ L_1 \sum_{j=0}^k |\tilde{a}_{j,k+1}| \max_{0 \leq j \leq k} |\tilde{E}_j| + L_2 \sum_{j=0}^k |\tilde{b}_{j,k+1}| \left( K_{1\frac{1}{3}} \right. \right. \\ &\quad \left. \left. \|\tilde{E}_0\|_\infty + K_{2\frac{1}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \right) + L_3 \sum_{j=0}^k |\tilde{c}_{j,k+1}| \left( K_{1\frac{2}{3}} \|\tilde{E}_0\|_\infty + K_{2\frac{2}{3}} \max_{0 \leq j \leq k} |\tilde{E}_j| \right) \right], \\ &\leq K_{1P} \|y_0 - \tilde{y}_0\|_\infty + K_{2P} \max_{0 \leq j \leq k} |y_j - \tilde{y}_j|. \end{aligned} \quad (64)$$

The proof of Eq. (63) is quit similar by using Lemma (15) and Eq. (64).  $\square$

## 5. Numerical results and discussion

**Example 1.** Consider the following fractional order differential equation

$${}_0^C D_t^\alpha y(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - y(t) + t^2 - t; \quad y(0) = 0. \quad (65)$$

145 The exact solution to this initial value problem is  $y(t) = t^2 - t$ . The absolute error of PC schemes given in Deng and Li [3], Diethelm et al. [25], Li et al. [32] and presented scheme are shown in Tables (1)–(2) and they are compared for different values of  $h$  and  $\alpha$  for  $t = 1$ . From Tables (1)–(2), it can be seen that the errors of the presented PC scheme are improved significantly compared with the literature.

**Example 2.** Consider the following non linear FDE

$${}_0^C D_t^\alpha y(t) = \frac{40320t^{8-\alpha}}{\Gamma(9-\alpha)} - \frac{3\Gamma(5+\alpha/2)t^{4-\alpha/2}}{\Gamma(5-\alpha/2)} + \frac{9\Gamma(\alpha+1)}{4} + \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3 - y(t)^{3/2},$$

$$y(0) = 0, \quad y'(0) = 0. \quad (66)$$

150 The exact solution is  $y(t) = t^8 - 3t^{4+\alpha/2} + \frac{9}{4}t^\alpha$ . The absolute errors of the presented scheme and Diethelm's PC scheme [25] and the scheme of [22] at  $t = 1$  is shown in Table 3. Observe that once again this scheme gives lower errors in almost all cases.

**Example 3.** Consider the following fractional differential equation

$${}_0^C D_t^\alpha y(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^\alpha - \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + (t^{2\alpha} - t^2)^4 - y^4(t); \quad y(0) = 0, \quad (67)$$

whose exact solution is given by  $y(t) = t^{2\alpha} - t^2$ . Tables (4) and (5) show the absolute errors of PC scheme of Daftardar-Gejji et al. [26], the fractional Adams method [25] and presented scheme for different values of  $\alpha$  and  $h$  at the time  $t = 1$ . It is noteworthy that error of presented scheme is always smaller than error of literature in all given cases. So the new PC method is more accurate for given parameters.

## 6. conclusion

A novel high order scheme for the approximate numerical solution of FDEs based on predictor–corrector (PC) scheme is proposed. The Simpsons 3/8 rule is used to improve the accuracy of the PC scheme. The detailed error analysis is discussed to illustrate the high order accuracy of the method, also the stability analysis is proved. The numerical examples are provided and compared with the corresponding results of literature to prove the validity of the scheme. Even though the scheme is designed for a scalar FDE, it can be easily extended to the fractional systems. According to the numerical results, the computing errors are in general acceptable in engineering.

## References

## References

- [1] D. Sierociuk, A. Dzieliński, Fractional kalman filter algorithm for the states, parameters and order of fractional system estimation, *International Journal of Applied Mathematics and Computer Science* 16 (1) (2006) 129.
- [2] L.-x. Yang, W.-s. He, X.-j. Liu, Synchronization between a fractional-order system and an integer order system, *Computers & Mathematics with Applications* 62 (12) (2011) 4708–4716.
- [3] W. Deng, C. Li, Numerical schemes for fractional ordinary differential equations, in: P. Miidla (Ed.), *Numerical Modelling*, InTech, Rijeka, 2012, Ch. 16, pp. 355–374.
- [4] M.-F. Danca, Numerical approximation of a class of discontinuous systems of fractional order, *Nonlinear Dynamics* 66 (1-2) (2011) 133–139.
- [5] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Vol. 198, Academic press, 1998.
- [6] R. E. Gutiérrez, J. M. Rosário, J. Tenreiro Machado, *Fractional order calculus: basic concepts and engineering applications*, *Mathematical Problems in Engineering* 2010.
- [7] S. David, J. Linares, E. Pallone, Fractional order calculus: historical apologia, basic concepts and some applications, *Revista Brasileira de Ensino de Física* 33 (4) (2011) 4302–4302.
- [8] M. Roohi, M. P. Aghababa, A. R. Haghighi, Switching adaptive controllers to control fractional-order complex systems with unknown structure and input nonlinearities, *Complexity* 21 (2) (2015) 211–223.

- [9] W. Liu, K. Chen, Chaotic behavior in a new fractional-order love triangle system with competition, *Journal of Applied Analysis and Computation* 5 (1) (2015) 103–113.
- 195 [10] S. Gupta, D. Kumar, J. Singh, Numerical study for systems of fractional differential equations via laplace transform, *Journal of the Egyptian Mathematical Society* 23 (2) (2015) 256–262.
- [11] L. Vázquez, H. Jafari, Fractional calculus: theory and numerical methods, *Open Physics* 11 (10) (2013) 1163–1163.
- 200 [12] B. Ahmad, S. K. Ntouyas, A. Alsaedi, On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions, *Chaos, Solitons & Fractals* 83 (2016) 234–241.
- [13] Z. M. Odibat, Computational algorithms for computing the fractional derivatives of functions, *Mathematics and Computers in Simulation* 79 (7) (2009) 2013–2020.
- 205 [14] V. G. Ivancevic, T. T. Ivancevic, High-dimensional chaotic and attractor systems: a comprehensive introduction, Vol. 32, Springer Science & Business Media, 2007.
- [15] D. Tavares, R. Almeida, D. F. Torres, Caputo derivatives of fractional variable order: numerical approximations, *Communications in Nonlinear Science and Numerical Simulation* 35 (2016) 69–87.
- 210 [16] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, 2000.
- [17] H. Ding, C. Li, Numerical algorithms for the fractional diffusion-wave equation with reaction term, in: *Abstract and Applied Analysis*, Vol. 2013, 2013, p. 15 pages. doi:10.1155/2013/493406.
- 215 [18] W. Deng, Short memory principle and a predictor–corrector approach for fractional differential equations, *Journal of Computational and Applied Mathematics* 206 (1) (2007) 174–188.

- [19] G.-h. Gao, Z.-z. Sun, H.-w. Zhang, A new fractional numerical differentiation formula to approximate the caputo fractional derivative and its applications, *Journal of Computational Physics* 259 (2014) 33–50.
- [20] R. Lin, F. Liu, Fractional high order methods for the nonlinear fractional ordinary differential equation, *Nonlinear Analysis: Theory, Methods & Applications* 66 (4) (2007) 856–869.
- [21] J. Cao, C. Xu, A high order schema for the numerical solution of the fractional ordinary differential equations, *Journal of Computational Physics* 238 (2013) 154–168.
- [22] P. Kumar, O. P. Agrawal, An approximate method for numerical solution of fractional differential equations, *Signal Processing* 86 (10) (2006) 2602–2610.
- [23] E. C. De Oliveira, J. A. T. Machado, A review of definitions for fractional derivatives and integral, *Mathematical Problems in Engineering* 2014.
- [24] K. Diethelm, N. J. Ford, A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, *Nonlinear Dynamics* 29 (1–4) (2002) 3–22.
- [25] K. Diethelm, N. J. Ford, A. D. Freed, Detailed error analysis for a fractional adams method, *Numerical algorithms* 36 (1) (2004) 31–52.
- [26] V. Daftardar-Gejji, Y. Sukale, S. Bhalekar, A new predictor–corrector method for fractional differential equations, *Applied Mathematics and Computation* 244 (2014) 158–182.
- [27] W. Deng, Numerical algorithm for the time fractional fokker–planck equation, *Journal of Computational Physics* 227 (2) (2007) 1510–1522.
- [28] K. Deng, W. Deng, Finite difference/predictor–corrector approximations for the space and time fractional fokker–planck equation, *Applied Mathematics Letters* 25 (11) (2012) 1815–1821.

- [29] K. Diethelm, Efficient solution of multi-term fractional differential equations using p(ec)<sup>m</sup>e methods, *Computing* 71 (4) (2003) 305–319.
- [30] C. Yang, F. Liu, A computationally effective predictor-corrector method for simulating fractional order dynamical control system, *ANZIAM Journal* 47 (2006) 168–184.
- [31] T. Kozlinskaya, V. Kovenya, The predictor–corrector method for solving of magnetohydrodynamic problems, in: *Hyperbolic Problems: Theory, Numerics, Applications*, Springer, 2008, pp. 625–633.
- [32] C. Li, A. Chen, J. Ye, Numerical approaches to fractional calculus and fractional ordinary differential equation, *Journal of Computational Physics* 230 (9) (2011) 3352–3368.
- [33] Y. Yan, K. Pal, N. J. Ford, Higher order numerical methods for solving fractional differential equations, *BIT Numerical Mathematics* 54 (2) (2014) 555–584.
- [34] D. Zill, W. Wright, *Differential equations with boundary-value problems*, Cengage Learning, 2012.
- [35] D. Zill, W. S. Wright, M. R. Cullen, *Advanced engineering mathematics*, Jones & Bartlett Learning, 2011.

Table 1: The absolute errors of the present scheme (E) and numerical methods of [25], [3] and [32] for (65).

h	$\alpha = 0.1$			$\alpha = 0.3$		
	E	[3]	[25]	E	[32]	[25]
1/10	6.6685e-04	0.104	0.103	2.8975e-05	0.0225	0.0314
1/20	1.1413e-04	0.0466	0.0495	2.7002e-06	0.0124	0.0110
1/40	1.8217e-05	0.0187	0.0209	1.9410e-07	0.0061	0.00391
1/80	2.7464e-06	0.00739	0.00865	2.1956e-07	0.0029	0.00142

Table 2: The absolute errors of the present scheme and numerical methods of [25], [3] and [32] with  $\alpha = 0.5$  for (65).

h	Presented scheme	Deng and Li [3]	Li et al. [32]	Diethelm et el. [25]
1/10	9.4477e-05	0.00927	0.0041	0.0144
1/20	3.3462e-05	0.00229	0.0031	0.00452
1/40	1.1724e-05	5.87e-04	0.0018	0.00146
1/80	4.1148e-06	1.56e-04	0.0010	4.81e-04

Table 3: The absolute errors of the present scheme (E) and numerical methods of [25] and [22] for (66).

h	$\alpha = 0.25$			$\alpha = 1.25$		
	E	[25]	[22]	E	[25]	[22]
1/10	4.5283e-04	2.50e-01	2.51e-03	2.0390e-05	5.53e-03	1.14e-04
1/20	2.6821e-04	1.81e-02	3.47e-04	1.2473e-06	1.59e-03	3.14e-05
1/40	1.1428e-04	3.61e-03	4.26e-05	7.7539e-08	4.33e-04	3.12e-06

Table 4: The absolute errors of the present scheme ( $E_1$ ), and the methods of [26] ( $E_2$ ) and [25] ( $E_3$ ) with  $h = 0.1$ ,  $t = 1$  for (67).

	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
$E_1$	6.3508e-04	6.2571e-04	5.1209e-04	3.7711e-04
$E_2$	0.00416254	0.00571831	0.00591036	0.00556043
$E_3$	0.00464954	0.0061384	0.00590812	0.00556109
	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
$E_1$	2.5548e-04	1.6037e-04	9.1920e-05	4.2823e-05
$E_2$	0.00487407	0.0039647	0.00287399	0.00157926
$E_3$	0.00487443	0.00396479	0.00287403	0.00157926

Table 5: The absolute errors of the present scheme ( $E_1$ ), and the methods of [26] ( $E_2$ ) and [25] ( $E_3$ ) with  $h = 0.01$ ,  $t = 1$  for (67).

	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
$E_1$	4.2158e-05	3.1242e-05	2.0148e-05	1.1722e-05
$E_2$	3.8120e-05	2.2142e-04	1.8891e-04	1.4379e-04
$E_3$	2.2944e-04	2.2826e-04	1.8812e-04	1.4377e-04
	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
$E_1$	6.2289e-06	3.0286e-06	1.3242e-06	4.7924e-07
$E_2$	1.0385e-04	7.1542e-05	4.5617e-05	2.3155e-05
$E_3$	1.0385e-04	7.1542e-05	4.5617e-05	2.3155e-05