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Metric approximation of set-valued functions of bounded variation

Elena E. Berdysheva ^{*}, Nira Dyn [†], Elza Farkhi ^{† ‡}, Alona Mokhov [§]

Abstract. In this paper we approximate univariate set-valued functions (SVFs) of bounded variation with range consisting of general (not necessarily convex) compact sets. The approximation operators adapted to SVFs are local operators such as the symmetric Schoenberg spline operator, the Bernstein polynomial operator and the Steklov function. All operators are adapted by using metric linear combinations. Error bounds, obtained in the averaged Hausdorff metric, provide rates of approximation similar to those for real-valued functions of bounded variation.

Key words: compact sets, set-valued functions, metric selections, metric linear combinations, metric integral, positive linear operators.

Mathematics Subject Classification 2010: 26E25, 28B20, 26A99, 54C60, 54C65

1. Introduction

The theoretical interest in the approximation of set-valued functions (SVFs, multifunctions) is relevant to various applications in fields where SVFs are used, such as economic optimization, dynamical systems, control theory, game theory, differential inclusions, geometric modeling. In particular, SVFs are relevant to the problem of the reconstruction of 3D objects from their parallel 2D cross-sections (see e.g. [9]). The 3D object determines a set-valued function whose samples are the given cross-sections, and whose graph is the 3D object. SVFs corresponding to 3D objects are not necessarily continuous and in many cases can be of bounded variation only.

Approximation of continuous set-valued functions with general compact images by adaptation of classical approximation operators are studied first in [15, 1] and recently in [5, 6]. Extensive research on approximation of SVFs with range consisting of convex compact sets can be found in [15, 6, 11, 3]. The adaptation of the Bernstein operator to set-valued functions using Minkowski linear combinations is considered by Vitale in [15]. The approximating functions in this case converge to the convex hull of the function to be approximated and hence cannot approximate SVFs with range consisting of general compact sets. This effect is called convexification in [4].

In [5, 6] approximation operators for continuous SVFs with general compact images based on metric linear combinations and other adaptations are investigated, and corresponding error bounds in the Hausdorff metric are obtained.

In this paper we study approximation of SVFs of bounded variation which are not necessarily continuous, and derive error bounds measured in the averaged Hausdorff metric. Our work extends the research on the approximation of single-valued functions of bounded variation (see e.g. [8], [13]).

First we investigate the adaptations of two types of sample-based approximation operators, namely, local operators like the Schoenberg spline operator (Section 3) and the Bernstein operator (Section 4). Both types are adapted to SVFs by replacing linear combinations of reals by metric linear combinations of sets, similar to what was done in [5, 6] for continuous SVFs. The third type of approximation we consider is a set-valued analogue of the Steklov function, adapted by replacing the Riemann integral by the metric integral of SVFs introduced in [7]. The metric integral is free of the convexification drawback, which is typical to the commonly used Aumann integral of SVFs. The metric Steklov functions are investigated in Section 5.

2. Preliminaries

In this section we introduce some notation and basic notions related to sets and set-valued functions. We also present notions of regularity of functions in metric spaces, and discuss the new notion of the metric integral of set-valued functions.

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2.1. On sets

We denote by $K(\mathbb{R}^d)$ the collection of all compact non-empty subsets of $\mathbb{R}^d$. All sets considered from now on are from $K(\mathbb{R}^d)$. By $\text{Co}(\mathbb{R}^d)$ we denote the collection of all convex sets in $K(\mathbb{R}^d)$. The convex hull of a set A is denoted by $\text{co}(A)$, the closure of A by $\text{cl}(A)$, the interior of A by $\text{int}(A)$ and the boundary of A by ∂A . The metric in $\mathbb{R}^d$ is of the form $\rho(u, v) = |u - v|$, where $|\cdot|$ is a norm on $\mathbb{R}^d$. An important fact is that all norms in $\mathbb{R}^d$ are equivalent, namely for two norms $|\cdot|_\alpha, |\cdot|_\beta$ there exist constants $C_1^{\alpha, \beta} \leq C_2^{\alpha, \beta}$ such that for all $u \in \mathbb{R}^d$

$$C_1^{\alpha, \beta} |u|_\beta \leq |u|_\alpha \leq C_2^{\alpha, \beta} |u|_\beta. \quad (1)$$

To measure the distance between two sets $A, B \in K(\mathbb{R}^d)$ we use the Hausdorff metric based on ρ

$$\text{haus}(A, B)_\rho = \max \left\{ \max_{a \in A} \text{dist}(a, B)_\rho, \max_{b \in B} \text{dist}(b, A)_\rho \right\}, \quad (2)$$

where $\text{dist}(a, B)_\rho = \min_{b \in B} \rho(a, b)$.

It is not difficult to see that the equivalence (1) induces a similar equivalence between the corresponding Hausdorff distances

$$C_1^{\alpha, \beta} \text{haus}(A, B)_\beta \leq \text{haus}(A, B)_\alpha \leq C_2^{\alpha, \beta} \text{haus}(A, B)_\beta, \quad (3)$$

where $\text{haus}(A, B)_\alpha$ denotes the Hausdorff distance in (2) corresponding to the metric ρ based on the norm $|\cdot|_\alpha$.

Remark 2.1. In the following we introduce notions that were defined before relative to the Euclidean distance. Here we extend these definitions to any norm-based metric on $\mathbb{R}^d$. The extension is straightforward. From now on, we omit indices indicating the underlying metric on $\mathbb{R}^d$ where the specific metric is not essential.

We denote by $|A| = \text{haus}(A, \{0\})$ the “norm” of the set. It is well known that $K(\mathbb{R}^d)$ and $\text{Co}(\mathbb{R}^d)$ are complete metric spaces with respect to the Hausdorff metric [12, 13].

The set of projections of $a \in \mathbb{R}^d$ on a set $B \in K(\mathbb{R}^d)$ is

$$\Pi_B(a) = \{b \in B : |a - b| = \text{dist}(a, B)\},$$

and the set of metric pairs of two sets $A, B \in K(\mathbb{R}^d)$ is

$$\Pi(A, B) = \{(a, b) \in A \times B : a \in \Pi_A(b) \text{ or } b \in \Pi_B(a)\}.$$

Using the metric pairs, we can rewrite

$$\text{haus}(A, B) = \max \{|a - b| : (a, b) \in \Pi(A, B)\}.$$

We recall here the notion of a metric chain and of a metric linear combination [7].

Definition 2.2. Given a finite sequence of sets $A_0, \dots, A_n \in K(\mathbb{R}^d)$, $n \geq 1$, a metric chain of $A_0, \dots, A_n$ is an $(n+1)$ -tuple $(a_0, \dots, a_n)$ such that $(a_i, a_{i+1}) \in \Pi(A_i, A_{i+1})$, $i = 0, 1, \dots, n-1$. We denote the collection of all metric chains of $A_0, \dots, A_n$ by

$$\text{CH}(A_0, \dots, A_n) = \{(a_0, \dots, a_n) : (a_i, a_{i+1}) \in \Pi(A_i, A_{i+1}), i = 0, 1, \dots, n-1\}. \quad (4)$$

The metric linear combination of the sets $A_0, \dots, A_n \in K(\mathbb{R}^d)$, $n \geq 1$, is

$$\bigoplus_{i=0}^n \lambda_i A_i = \left\{ \sum_{i=0}^n \lambda_i a_i : (a_0, \dots, a_n) \in \text{CH}(A_0, \dots, A_n) \right\}.$$

Remark 2.3. In [5] and [6 Ch. 1] the definition of a metric chain is slightly different from the one in (4). A vector $(a_0, \dots, a_n)$ satisfying $a_{i-1} \in \Pi_{A_{i-1}}(a_i)$, $1 \leq i \leq j$, $a_{i+1} \in \Pi_{A_{i+1}}(a_i)$, $j \leq i \leq n-1$, for some $j \in \{0, \dots, n\}$, is termed in [5, 6] a metric chain of $\{A_0, \dots, A_n\}$. Note that in [5, 6] only the Euclidean distance is used.

The collection (4) includes all the metric chains in the sense of [5, 6], in case (4) is defined with respect to the Euclidean distance.

Note that the metric linear combination depends on the order of the sets, in contrast to the Minkowski linear combination of sets which is defined by

$$\sum_{i=0}^n \lambda_i A_i = \left\{ \sum_{i=0}^n \lambda_i a_i : a_i \in A_i \right\}, \quad n \geq 1.$$

For a sequence of sets $\{A_n\}_{n=1}^\infty$ the upper Kuratowski limit is the set of all limit points of converging subsequences $\{a_{n_k}\}_{k=1}^\infty$, where $a_{n_k} \in A_{n_k}$, $k \in \mathbb{N}$, namely

$$\limsup_{n \rightarrow \infty} A_n = \left\{ a : \exists \{n_k\}_{k=1}^\infty \text{ with } \lim_{k \rightarrow \infty} n_k = \infty, \exists a_{n_k} \in A_{n_k} \text{ such that } \lim_{k \rightarrow \infty} a_{n_k} = a \right\}.$$

2.2. Regularity measures of functions with values in a metric space

In this work we consider functions defined on a fixed interval $[a, b] \subset \mathbb{R}$ with values in a complete metric space (X, ρ) , where X is either $\mathbb{R}^d$ or $K(\mathbb{R}^d)$. In $K(\mathbb{R}^d)$ the metric is the Hausdorff metric.

We denote by $\text{Lip}(\mathcal{L})$ the collection of all Lipschitz continuous functions satisfying

$$\rho(f(x), f(y)) \leq \mathcal{L}|x - y|, \quad \forall x, y \in [a, b].$$

Next we recall the notion of the variation of a function $f : [a, b] \rightarrow X$. Let $\chi = \{x_0, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$, be a partition of the interval $[a, b]$ with the norm

$$|\chi| = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i).$$

The variation of f on the partition χ is defined as

$$V(f, \chi) = \sum_{i=1}^n \rho(f(x_i), f(x_{i-1})).$$

The total variation of f on $[a, b]$ is

$$V_a^b(f) = \sup_{\chi} V(f, \chi),$$

where the supremum is taken over all partitions of $[a, b]$.

A function f is of bounded variation if $V_a^b(f) < \infty$. We call functions of bounded variation BV functions and write $f \in \text{BV}[a, b]$.

The equivalence of the norms in $\mathbb{R}^d$ (see (1)) and the induced equivalence of the Hausdorff distances in $K(\mathbb{R}^d)$ (see (3)) implies the equivalence of the variations of functions with values in $\mathbb{R}^d$ or $K(\mathbb{R}^d)$

$$C_1^{\alpha, \beta} V_a^b(f)_\beta \leq V_a^b(f)_\alpha \leq C_2^{\alpha, \beta} V_a^b(f)_\beta. \quad (5)$$

Any function $f : [a, b] \rightarrow X$ is extended in this paper to a function defined on $\mathbb{R}$ in a way preserving its variation, namely,

$$f(x) = \begin{cases} f(a), & x < a, \\ f(x), & a \leq x \leq b, \\ f(b), & x > b. \end{cases} \quad (6)$$

The L_1 -modulus of continuity of f is defined by

$$\omega(f, \delta)_{L_1} = \frac{1}{b-a} \sup_{0 \leq h \leq \delta} \int_a^b \rho(f(x+h), f(x)) dx.$$

In [13] it has been shown that for a BV function $f : \mathbb{R} \rightarrow \mathbb{R}$ the estimate $\int_a^b V_{x-\delta}^{x+\delta}(f) dx \leq 2\delta V_a^b(f)$ holds. We extend this important result to metric-space valued functions, repeating the proof from [13] almost verbatim.

Lemma 2.4. *For a BV function $f : \mathbb{R} \rightarrow X$ the following property holds*

$$\int_a^b V_{x-\delta}^{x+\delta}(f) dx \leq 2\delta V_a^b(f).$$

Proof. We have

$$\begin{aligned} \int_a^b V_{x-\delta}^{x+\delta}(f) dx &= \int_a^b V_{a-\delta}^{x+\delta}(f) dx - \int_a^b V_{a-\delta}^{x-\delta}(f) dx = \int_{a+\delta}^{b+\delta} V_{a-\delta}^t(f) dt - \int_{a-\delta}^{b-\delta} V_{a-\delta}^t(f) dt \\ &= \int_{b-\delta}^{b+\delta} V_{a-\delta}^t(f) dt - \int_{a-\delta}^{a+\delta} V_{a-\delta}^t(f) dt \leq \int_{b-\delta}^{b+\delta} V_a^b(f) dt = 2\delta V_a^b(f). \end{aligned}$$

□

2.3. SVFs, metric selections and the metric integral

Now we present some notions related to set-valued functions mapping $[a, b]$ to $K(\mathbb{R}^d)$.

The graph of a multifunction F is

$$\text{Graph}(F) = \{(x, y) : y \in F(x), x \in [a, b]\}.$$

Note that the graph is a subset of $\mathbb{R}^{d+1}$. It is easy to see that if $F \in \text{BV}[a, b]$ then $\text{Graph}(F)$ is a bounded set. F has a bounded range if $\left| \bigcup_{x \in [a, b]} F(x) \right| < \infty$. We denote the class of SVFs with compact graphs and bounded variation by $\mathcal{F}[a, b]$.

For a set-valued function $F : [a, b] \rightarrow K(\mathbb{R}^d)$, a single-valued function $s : [a, b] \rightarrow \mathbb{R}^d$ such that $s(x) \in F(x)$ for all $x \in [a, b]$ is called a selection of F .

The Aumann integral of a multifunction F is the set

$$\int_a^b F(x)dx = \left\{ \int_a^b s(x)dx : s \text{ is any integrable selection of } F \right\}.$$

It is well-known [2] that the Aumann integral is convex and compact for $F \in \mathcal{F}[a, b]$, even if the values of F are not convex. This is the reason why the Aumann integral is not a candidate for adapting integral approximation operators to SVFs with general (i.e. not necessarily convex) compact sets as images.

A new set-valued integral, which is not necessarily convex, has been introduced in [7]. This integral is used successfully for the approximation of SVFs from $\mathcal{F}[a, b]$ in Section 3.

Here we present some definitions and results of [7] which are also valid for any fixed norm-based metric in $\mathbb{R}^d$. In particular, we recall the definition of metric selections and the metric integral. Note that these notions depend on the metric in $\mathbb{R}^d$. In [7] these notions are defined with respect to the Euclidean norm only.

Given a multifunction $F : [a, b] \rightarrow K(\mathbb{R}^d)$, a partition $\chi = \{x_0, \dots, x_n\} \subset [a, b]$, $a = x_0 < \dots < x_n = b$, and a corresponding metric chain $\phi = (y_0, \dots, y_n) \in \text{CH}(F(x_0), \dots, F(x_n))$, the chain function based on χ and ϕ is

$$c_{\chi, \phi}(x) = \begin{cases} y_i, & x \in [x_i, x_{i+1}), \quad i = 0, \dots, n-1, \\ y_n, & x = x_n. \end{cases} \quad (7)$$

Definition 2.5. A selection s is called a metric selection if it is the pointwise limit function of a sequence $\{c_{\chi_k, \phi_k}\}_{k \in \mathbb{N}}$ of chain functions of F , with $\lim_{k \rightarrow \infty} |\chi_k| = 0$.

We denote the set of all metric selections of F by $\mathcal{S}$.

Note that (7) and Definition 2.5 imply that a metric selection s of a multifunction F is constant in any open interval where the graph of s stays in $\text{int}(\text{Graph}(F))$.

Result 2.6. For $F \in \mathcal{F}[a, b]$, through any point in $\text{Graph}(F)$ there exists a metric selection. Moreover, any $F \in \mathcal{F}[a, b]$ has a representation by metric selections, namely

$$F(x) = \{s_\alpha(x) : \alpha \in \text{Graph}(F)\},$$

where s_α denotes a metric selection that passes through α .

Result 2.7. Let s be a metric selection of $F \in \mathcal{F}[a, b]$. Then

$$V_a^b(s) \leq V_a^b(F).$$

Moreover, if $F \in \text{Lip}(\mathcal{L})$ then $s \in \text{Lip}(\mathcal{L})$.

Next we recall the definition of the metric integral.

Definition 2.8. For a multifunction $F : [a, b] \rightarrow K(\mathbb{R}^d)$ and for a partition $\chi = \{x_0, \dots, x_n\}$, the metric Riemann sum of F is defined by

$$(\mathcal{M})S_\chi F = \bigoplus_{i=0}^{n-1} (x_{i+1} - x_i) F(x_i).$$

The metric integral of F is defined as the Kuratowski upper limit of metric Riemann sums corresponding to partitions with norm tending to zero. We denote this integral by $(\mathcal{M}) \int_a^b F(x)dx$.

By its definition, the set $(\mathcal{M}) \int_a^b F(x)dx$ is non-empty if F has a bounded range.

Remark 2.9. It is easy to see from the definition of the metric integral that if F is bounded so is its metric integral, namely if $|F(x)| \leq \widetilde{M}$ for all $x \in [a, b]$, and $F \in \mathcal{F}[a, b]$, then

$$\left| (\mathcal{M}) \int_a^b F(x) dx \right| \leq (b-a) \widetilde{M}.$$

The following result relates the metric integral of F to its metric selections.

Result 2.10. Let $F \in \mathcal{F}[a, b]$. Then

$$(\mathcal{M}) \int_a^b F(x) dx = \left\{ \int_a^b s(x) dx : s \in \mathcal{S} \right\}.$$

3. Local metric operators

In this section we adapt to set-valued functions certain local approximation operators for real-valued functions. Let $n \in \mathbb{N}$, and let $\chi = \{x_0, \dots, x_n\} \subset [a, b]$, $a = x_0 < x_1 < \dots < x_n = b$, be a partition of the interval $[a, b]$. For $m \ll n$, let $c_i : [a, b] \rightarrow \mathbb{R}$, $i = 0, \dots, n$ have supports $\text{supp}(c_i) \subseteq [x_{i-m}, x_{i+m+1}]$, with $x_i = x_0$ for $i = -m, \dots, -1$, $x_i = x_n$ for $i = n+1, \dots, n+m+1$ and let $\sum_{i=0}^n c_i(x) = 1$ for $x \in [x_m, x_{n-m+1}]$. We consider local approximation operators of the form $\mathcal{A}_m f(x) = \sum_{i=0}^n c_i(x) f(x_i)$, where m the locality parameter.

For a set-valued function F we define the adapted metric operators (see [6])

$$\mathcal{A}_m^M F(x) = \bigoplus_{i=0}^n c_i(x) F(x_i) \quad x \in [x_m, x_{n-m+1}]. \quad (8)$$

Below we obtain a bound on the integral of the approximation error

Theorem 3.1. For $F \in \mathcal{F}[a, b]$ we have

$$\int_{x_m}^{x_{n-m+1}} \text{haus} \left(F(x), \mathcal{A}_m^M F(x) \right) dx \leq \widetilde{\gamma}(2m+1) |\chi| V_a^b(F),$$

where

$$\widetilde{\gamma} = \sup_{x \in [x_m, x_{n-m+1}]} \sum_{i=0}^n |c_i(x)|. \quad (9)$$

Proof. For a given $j \in \{m, m+1, \dots, n-m\}$, the formula (8) can be written as

$$\mathcal{A}_m^M F(x) = \bigoplus_{i=j-m}^{j+m} c_i(x) F(x_i), \quad x \in [x_j, x_{j+1}].$$

Thus

$$\text{haus} \left(F(x), \mathcal{A}_m^M F(x) \right) = \text{haus} \left(F(x), \bigoplus_{i=j-m}^{j+m} c_i(x) F(x_i) \right), \quad x \in [x_j, x_{j+1}].$$

Fix $x \in [x_j, x_{j+1}]$. First we estimate $\text{dist}(y, \mathcal{A}_m^M F(x))$ for an arbitrary $y \in F(x)$. We have

$$\text{dist} \left(y, \bigoplus_{i=j-m}^{j+m} c_i(x) F(x_i) \right) \leq \left| y - \sum_{i=j-m}^{j+m} c_i(x) \phi_i \right|$$

for any $\phi = (\phi_{j-m}, \dots, \phi_{j+m}) \in \mathcal{CH}(F(x_{j-m}), \dots, F(x_{j+m}))$. Denote by ϕ^0 a chain satisfying $\phi_j^0 \in \Pi_{F(x_j)}(y)$, i.e. $\text{dist}(y, F(x_j)) = |y - \phi_j^0|$. Then

$$\begin{aligned} \text{dist} \left(y, \bigoplus_{i=j-m}^{j+m} c_i(x) F(x_i) \right) &\leq \left| \sum_{i=j-m}^{j+m} c_i(x) y - \sum_{i=j-m}^{j+m} c_i(x) \phi_i^0 \right| \leq \sum_{i=j-m}^{j+m} |c_i(x)| |y - \phi_i^0| \\ &\leq \sum_{i=j-m}^{j+m} |c_i(x)| (|y - \phi_j^0| + |\phi_j^0 - \phi_i^0|) \\ &\leq \sum_{i=j-m}^{j+m} |c_i(x)| \text{haus}(F(x), F(x_j)) + \max_{j-m \leq i \leq j+m} |\phi_j^0 - \phi_i^0| \sum_{i=j-m}^{j+m} |c_i(x)| \\ &\leq (\text{haus}(F(x), F(x_j)) + V_{x_{j-m}}^{x_{j+m}}(F)) \sum_{i=j-m}^{j+m} |c_i(x)|. \end{aligned}$$

Next we estimate $\text{dist}(z, F(x))$ for $z \in \mathcal{A}_m^M F(x)$ and for $x \in [x_j, x_{j+1}]$. Each element $z \in \bigoplus_{i=j-m}^{j+m} c_i(x)F(x_i)$ is built upon a chain $\phi = (\phi_0, \dots, \phi_n) \in \text{CH}(F(x_0), \dots, F(x_n))$, i.e. $z = \sum_{i=j-m}^{j+m} c_i(x)\phi_i$. Choosing $f^0 \in F(x)$ such that $f^0 \in \Pi_{F(x)}(\phi_j)$, we get

$$\begin{aligned} \text{dist}(z, F(x)) &\leq |z - f^0| \leq \left| \sum_{i=j-m}^{j+m} c_i(x)\phi_i - \sum_{i=j-m}^{j+m} c_i(x)f^0 \right| \leq \sum_{i=j-m}^{j+m} |c_i(x)| (|\phi_i - \phi_j| + |\phi_j - f^0|) \\ &\leq (\text{haus}(F(x), F(x_j)) + V_{x_{j-m}}^{x_{j+m}}(F)) \sum_{i=j-m}^{j+m} |c_i(x)|. \end{aligned}$$

Thus,

$$\begin{aligned} \text{haus}(F(x), \mathcal{A}_m^M F(x)) &\leq \tilde{C} (\text{haus}(F(x), F(x_j)) + V_{x_{j-m}}^{x_{j+m}}(F)) \\ &\leq \tilde{C} (V_{x_j}^{x_{j+1}}(F) + V_{x_{j-m}}^{x_{j+m}}(F)), \quad x \in [x_j, x_{j+1}]. \end{aligned}$$

Now

$$\begin{aligned} \int_{x_m}^{x_{n-m+1}} \text{haus}(F(x), \mathcal{A}_m^M F(x)) dx &\leq \tilde{C} \sum_{j=m}^{n-m} \int_{x_j}^{x_{j+1}} (V_{x_j}^{x_{j+1}}(F) + V_{x_{j-m}}^{x_{j+m}}(F)) dx \\ &\leq \tilde{C} |\chi| \sum_{j=m}^{n-m} (V_{x_j}^{x_{j+1}}(F) + V_{x_{j-m}}^{x_{j+m}}(F)) \\ &= \tilde{C} |\chi| \sum_{j=m}^{n-m} \left(V_{x_{j-m}}^{x_{j+1}}(F) + \sum_{i=j-m}^{j+m-1} V_{x_i}^{x_{i+1}}(F) \right) \\ &\leq \tilde{C} |\chi| (2(n-m+1)) \sum_{j=0}^{n-1} V_{x_j}^{x_{j+1}}(F) \\ &= \tilde{C} |\chi| (2(n-m+1)) V_a^b(F). \end{aligned}$$

Thus we finally obtain the claim of the theorem $\square$

A direct conclusion from Theorem 3.1 is the following statement.

Corollary 3.2. *Let $\{\mathcal{A}_{n,m}^M\}_{n \geq 0}$ be a sequence of operators of the form (8), where $\mathcal{A}_{n,m}^M$ corresponds to a partition $\chi^n = \{x_0^n, \dots, x_n^n\}$ and to functions c_i^n , $i = 0, \dots, n$. Let, for each n , $\tilde{C}^n$ be the constant as defined in (9). If $\lim_{n \rightarrow \infty} \tilde{C}^n |\chi^n| = 0$, then*

$$\lim_{n \rightarrow \infty} \int_{x_m}^{x_{n-m+1}} \text{haus}(F(x), \mathcal{A}_{n,m}^M F(x)) dx = 0.$$

Remark 3.3. *One can define local operators with $\text{supp}(c_i) \subseteq [x_{i-m}, x_{i+m}]$, $i = 0, \dots, n$, when $x_i = x_0$ for $i = -m, \dots, -1$, $x_i = x_n$ for $i = n+1, \dots, n+m$ and $\sum_{i=0}^n c_i(x) = 1$ for $x \in [x_{m-1}, x_{n-m+1}]$, as in the next example. The proof of Theorem 3.1 can be easily modified for this case and we get the estimate*

$$\int_{x_{m-1}}^{x_{n-m+1}} \text{haus}(F(x), \mathcal{A}_m^M F(x)) dx \leq \tilde{C} 2m |\chi| V_a^b(F),$$

with $\tilde{C} = \sup_{x \in [x_{m-1}, x_{n-m+1}]} \sum_{i=0}^n |c_i(x)|$. A corollary similar to Corollary 3.2 holds also in this case.

Example 3.4. *The symmetric Schoenberg spline operators for $f : [0, 1] \rightarrow \mathbb{R}$ are defined as*

$$S_{2m} f(x) = \sum_{i=0}^n B_{2m-1}(nx - i) f\left(\frac{i}{n}\right), \quad x \in \left[\frac{m-1}{n}, \frac{n-m+1}{n}\right],$$

where B_{2m-1} is a B-spline of degree $2m-1$ with integer knots and support $[-m, m]$. Since $B_{2m-1} \geq 0$, $x \in \mathbb{R}$, these operators are positive. Clearly, they are local with the locality parameter m . Moreover, $\tilde{C} = 1$, since $\sum_{i=-\infty}^{\infty} B_{2m-1}(x-i) = 1$, $x \in \mathbb{R}$.

The adapted metric operators are

$$S_{n,m}^M F(x) = \bigoplus_{i=0}^n B_{2m-1}(nx-i) F\left(\frac{i}{n}\right), \quad x \in \left[\frac{m-1}{n}, \frac{n-m+1}{n}\right],$$

and Theorem 3.1 yields for $F \in \mathcal{F}[0, 1]$

$$\int_{\frac{m-1}{n}}^{\frac{n-m+1}{n}} \text{haus}(F(x), S_{n,m}^M F(x)) dx \leq \frac{2m+1}{n} V_0^1(F).$$

In this example we can extend F by (6) and the operators by

$$\tilde{S}_{n,m}^M F(x) = \bigoplus_{i=-m}^{n+m} B_{2m-1}(nx-i) F\left(\frac{i}{n}\right)$$

to get the estimate

$$\int_0^1 \text{haus}(F(x), \tilde{S}_{n,m}^M F(x)) dx \leq \frac{2m+1}{n} V_0^1(F).$$

An example corresponding to the operators (8) of Theorem 3.1 are the Schoenberg-type operators of the form

$$\tilde{S}_{n,m} f(x) = \sum_{i=0}^n B_{2m}(nx-i) f\left(\frac{i}{n}\right), \quad x \in \left[\frac{m}{n}, \frac{n-m+1}{n}\right],$$

where B_{2m} is a B-spline of degree $2m$ with integer knots and support $[-m, m+1]$.

4. Metric Bernstein operators

The classical Bernstein operator is defined for a function $f : [0, 1] \rightarrow \mathbb{R}$ by the formula

$$B_n f(x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1],$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Recall that $p_{n,k} \geq 0$ for $x \in [0, 1]$ and $\sum_{k=0}^n p_{n,k}(x) = 1$ for any $x \in \mathbb{R}$. If f is continuous, the sequence of polynomials $B_n f$ converges to f uniformly on $[0, 1]$. The next important result, taken from [10, Ch.1], says that although the Bernstein operator is not local in the sense of the previous section, it possesses a certain localization property.

Result 4.1. For each $s = 0, 1, 2, \dots$, there exists a constant A_s such that for all $x \in [0, 1]$ and $\delta > 0$ the following estimate holds

$$\sum_{k: |\frac{k}{n} - x| \geq \delta} p_{n,k}(x) \leq A_s \delta^{-2s} n^{-s}.$$

For $F : [0, 1] \rightarrow K(\mathbb{R}^d)$, the metric Bernstein operator of degree n (similarly to [6, Ch. 2]) is given by

$$B_n^M F(x) = \bigoplus_{k=0}^n p_{n,k}(x) F\left(\frac{k}{n}\right) = \left\{ \sum_{k=0}^n p_{n,k}(x) f_k : (f_0, \dots, f_n) \in \text{CH} \left(F(0), F\left(\frac{1}{n}\right), \dots, F(1) \right) \right\},$$

where $\text{CH} \left(F(0), F\left(\frac{1}{n}\right), \dots, F(1) \right)$ is the collection of metric chains defined in (4).

This definition of the metric Bernstein operator is slightly different from the one in [6, Ch.2] and in [5] due to the change in the definition of metric chains (see Remark 2.3).

Theorem 4.2. Let $F \in \mathcal{F}[0, 1]$ and let $\widetilde{M} \in \mathbb{R}$ be such that $|F(x)| \leq \widetilde{M}$ for all $x \in [0, 1]$, then for any $\frac{1}{n} \leq h < 1$ and each $s \in \mathbb{N}$ we have

$$\int_0^1 \text{haus} \left(F(x), B_n^M F(x) \right) dx \leq V_0^1(F) \left(\frac{1}{n} + 2h \right) + 2\widetilde{M}A_s h^{-2s} n^{-s}. \quad (10)$$

In particular, for $h = n^{-\frac{s}{1+2s}}$ we have

$$\int_0^1 \text{haus} \left(F(x), B_n^M F(x) \right) dx \leq C_{s,F} n^{-\frac{s}{1+2s}}, \quad (11)$$

where $C_{s,F}$ is a constant that depends only on s and F .

Proof. Fix $j \in \{0, \dots, n-1\}$ and $x \in [\frac{j}{n}, \frac{j+1}{n}]$. We estimate $\text{haus} \left(F(x), B_n^M F(x) \right)$.

Let, first, $f \in F(x)$. We denote by $\phi^* = (\phi_0^*, \dots, \phi_n^*)$ a chain satisfying $\phi_j^* \in \Pi_{F(\frac{j}{n})}(f)$, i.e., $\text{dist} \left(f, F \left(\frac{j}{n} \right) \right) = |f - \phi_j^*|$. Then

$$\begin{aligned} \text{dist} \left(f, B_n^M F(x) \right) &\leq \left| f - \sum_{k=0}^n p_{n,k}(x) \phi_k^* \right| \leq |f - \phi_j^*| + \sum_{k=0}^n p_{n,k}(x) |\phi_j^* - \phi_k^*| \\ &= \text{dist} \left(f, F \left(\frac{j}{n} \right) \right) + \sum_{k=0}^n p_{n,k}(x) |\phi_j^* - \phi_k^*| \leq \text{haus} \left(F(x), F \left(\frac{j}{n} \right) \right) + \sum_{k=0}^n p_{n,k}(x) |\phi_j^* - \phi_k^*|. \end{aligned}$$

We extend the function F to a larger interval $[-h, 1+h]$ by putting $F(x) = F(0)$ for $x < 0$ and $F(x) = F(1)$ for $x > 1$. Note that for k with $|x - \frac{k}{n}| < h$ and $h \geq \frac{1}{n}$ both points $\frac{k}{n}$ and $\frac{j}{n}$ lie in the interval $[x-h, x+h]$, so that $|\phi_j^* - \phi_k^*| \leq V_{x-h}^{x+h}(F)$ in this case. Thus, for the second term we obtain

$$\begin{aligned} \sum_{k=0}^n p_{n,k}(x) |\phi_j^* - \phi_k^*| &\leq \sum_{k: |x - \frac{k}{n}| < h} p_{n,k}(x) |\phi_j^* - \phi_k^*| + \sum_{k: |x - \frac{k}{n}| \geq h} p_{n,k}(x) |\phi_j^* - \phi_k^*| \\ &\leq V_{x-h}^{x+h}(F) \sum_{k: |x - \frac{k}{n}| < h} p_{n,k}(x) + 2\widetilde{M} \sum_{k: |x - \frac{k}{n}| \geq h} p_{n,k}(x) \\ &\leq V_{x-h}^{x+h}(F) + 2\widetilde{M}A_s h^{-2s} n^{-s}, \end{aligned}$$

where we have used the facts that $\sum_{k: |x - \frac{k}{n}| < h} p_{n,k}(x) \leq 1$, $|\phi_j^* - \phi_k^*| \leq 2\widetilde{M}$, and Result 4.1. Thus,

$$\text{dist} \left(f, B_n^M F(x) \right) \leq \text{haus} \left(F(x), F \left(\frac{j}{n} \right) \right) + V_{x-h}^{x+h}(F) + 2\widetilde{M}A_s h^{-2s} n^{-s}. \quad (12)$$

Now, for $g \in B_n^M F(x)$, we estimate $\text{dist}(g, F(x))$. By the definition of the metric Bernstein operator we have

$$g = \sum_{k=0}^n p_{n,k}(x) f_k$$

with some metric chain $(f_0, f_1, \dots, f_n) \in \text{CH} \left(F(0), F \left(\frac{1}{n} \right), \dots, F(1) \right)$. Take $f^* \in \Pi_{F(x)}(f_j)$. We have

$$\text{dist}(g, F(x)) \leq |g - f^*| = \left| \sum_{k=0}^n p_{n,k}(x) (f_k - f^*) \right| \leq \sum_{k: |x - \frac{k}{n}| < h} p_{n,k}(x) |f_k - f^*| + \sum_{k: |x - \frac{k}{n}| \geq h} p_{n,k}(x) |f_k - f^*|.$$

For k satisfying $|x - \frac{k}{n}| < h$ and $h \geq \frac{1}{n}$, we have

$$|f_k - f^*| \leq |f_k - f_j| + |f_j - f^*| \leq V_{x-h}^{x+h}(F) + \text{haus} \left(F \left(\frac{j}{n} \right), F(x) \right).$$

To estimate the second term, we use Result 4.1 again:

$$\sum_{k: |x - \frac{k}{n}| \geq h} p_{n,k}(x) |f_k - f^*| \leq 2\widetilde{M}A_s h^{-2s} n^{-s}.$$

Thus, we obtain

$$\text{dist}(g, F(x)) \leq \text{haus} \left(F(x), F \left(\frac{j}{n} \right) \right) + V_{x-h}^{x+h}(F) + 2\widetilde{M}A_s h^{-2s} n^{-s}. \quad (13)$$

It follows from (12) and (13) that

$$\text{haus} \left(F(x), B_n^M F(x) \right) \leq \text{haus} \left(F(x), F \left(\frac{j}{n} \right) \right) + V_{x-h}^{x+h}(F) + 2\widetilde{M}A_s h^{-2s} n^{-s}.$$

For the first term, using $\text{haus} \left(F(x), F \left(\frac{j}{n} \right) \right) \leq V_{j/n}^{(j+1)/n}(F)$, $x \in [\frac{j}{n}, \frac{j+1}{n}]$, we obtain

$$\begin{aligned} \sum_{j=0}^{n-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} \text{haus} \left(F(x), F \left(\frac{j}{n} \right) \right) dx &\leq \sum_{j=0}^{n-1} \int_{j/n}^{(j+1)/n} V_{j/n}^{(j+1)/n}(F) dx \\ &= \frac{1}{n} \sum_{j=0}^{n-1} V_{j/n}^{(j+1)/n}(F) = \frac{1}{n} V_0^1(F). \end{aligned}$$

To estimate the second term, we use Lemma 2.4 to get

$$\int_0^1 V_{x-h}^{x+h}(F) dx \leq 2hV_0^1(F).$$

Summarizing, we obtain the estimate

$$\int_0^1 \text{haus} \left(F(x), B_n^M F(x) \right) dx \leq V_0^1(F) \left(\frac{1}{n} + 2h \right) + 2\widetilde{M}A_s h^{-2s} n^{-s}.$$

In a special case when we take $h = n^{-\alpha}$, $\alpha > 0$, we obtain $h^{-2s} n^{-s} = n^{s(2\alpha-1)}$, and thus we require $\alpha \in (0, 1/2)$ in order to guarantee that the second term in the estimate (13) tends to zero. In particular, $h \geq \frac{1}{n}$ in this case, and the first term has the order $n^{-\alpha}$. It is clear that the best rate of convergence will be achieved when $n^{-\alpha} = n^{s(2\alpha-1)}$, that is when $\alpha = \frac{s}{1+2s}$. $\square$

It is well known (e.g., [10]) that the optimal rate of convergence of the Bernstein operator for Lipschitz continuous real-valued function on $[0, 1]$ is $n^{-\frac{1}{2}}$. In our Theorem 4.2, we obtain the rate of convergence $n^{-\frac{s}{1+2s}}$ with an arbitrary $s \in \mathbb{N}$. Since $\frac{s}{1+2s}$ tends to $\frac{1}{2}$ as s tends to infinity, we can achieve the rate $n^{-(\frac{1}{2}-\varepsilon)}$ with an arbitrary $\varepsilon > 0$.

5. Steklov functions

In this section we introduce and study the metric set-valued Steklov functions. First we prove two basic properties of Steklov functions with values in $\mathbb{R}^d$.

In the following any function $f : [a, b] \rightarrow X$ is extended by (6) to $\mathbb{R}$.

5.1. Properties of $\mathbb{R}^d$ -valued Steklov functions

For a function $f : \mathbb{R} \rightarrow \mathbb{R}^d$, the derivative and the integral are applied to each component of f .

For an integrable single-valued function $f : \mathbb{R} \rightarrow \mathbb{R}^d$ the function defined by

$$f_\delta(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t) dt, \quad \delta > 0, \quad x \in \mathbb{R},$$

is called the Steklov function of f . It is easy to see that the Steklov function f_δ satisfies

$$f'_\delta(x) = \frac{1}{2\delta} (f(x+\delta) - f(x-\delta)) \quad (14)$$

almost everywhere.

The following result holds for $V_a^b(\cdot)_1$, the total variation based on the ℓ_1 -norm in $\mathbb{R}^d$. We recall the definition of the ℓ_1 -norm, $\|u\|_1 = \sum_{i=1}^d |u_i|$ for $u = (u_1, \dots, u_d) \in \mathbb{R}^d$.

Theorem 5.1. For $f : [a, b] \rightarrow \mathbb{R}^d$, $f \in \text{BV}[a, b]$ and $\rho(u, v) = |u - v|_1$, $u, v \in \mathbb{R}^d$, it holds

$$V_a^b(f_\delta)_1 \leq V_a^b(f)_1.$$

Proof. First we prove that $V_a^b(g)_1 \leq \int_a^b |g'(x)|_1 dx$ for any function $g : [a, b] \rightarrow \mathbb{R}^d$ such that $|g'(x)|_1$ is integrable. Let $g(x) = (g_1(x), \dots, g_d(x))$, then

$$V_a^b(g)_1 = \sup_{\chi} \sum_{j=1}^n |g(x_j) - g(x_{j-1})|_1 = \sup_{\chi} \sum_{j=1}^n \sum_{i=1}^d |g_i(x_j) - g_i(x_{j-1})|,$$

where the supremum is taken over all partitions $\chi = \{x_0, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$, of $[a, b]$. Thus

$$V_a^b(g)_1 = \sup_{\chi} \sum_{i=1}^d \sum_{j=1}^n |g_i(x_j) - g_i(x_{j-1})| \leq \sum_{i=1}^d \sup_{\chi} \sum_{j=1}^n |g_i(x_j) - g_i(x_{j-1})| = \sum_{i=1}^d V_a^b(g_i).$$

Since $V_a^b(g_i) = \int_a^b |g'_i(x)| dx$, $i = 1, \dots, d$, we finally get

$$V_a^b(g)_1 \leq \sum_{i=1}^d \int_a^b |g'_i(x)| dx = \int_a^b |g'(x)|_1 dx. \quad (15)$$

To continue with the proof of the theorem we use (15), (14), and Lemma 2.4

$$V_a^b(f_\delta)_1 \leq \int_a^b |f'_\delta(x)|_1 dx = \frac{1}{2\delta} \int_a^b |f(x+\delta) - f(x-\delta)|_1 dx \leq \frac{1}{2\delta} \int_a^b V_{x-\delta}^{x+\delta}(f)_1 dx \leq V_a^b(f)_1.$$

□

For the variation $V_a^b(\cdot)_\alpha$ based on any norm $|\cdot|_\alpha$ in $\mathbb{R}^d$ we conclude

Corollary 5.2. For $\rho(u, v) = |u - v|_\alpha$, $u, v \in \mathbb{R}^d$, there exists a constant C^α such that for any function $f : [a, b] \rightarrow \mathbb{R}^d$, $f \in \text{BV}[a, b]$, it holds

$$V_a^b(f_\delta)_\alpha \leq C^\alpha V_a^b(f)_\alpha.$$

Indeed by Theorem 5.1 and (5) we get the claim of the corollary with $C^\alpha = C_2^{\alpha,1} C_2^{1,\alpha}$.

The next results show that for a bounded vector-valued function its Steklov function is Lipschitz continuous. First we consider the ℓ_1 -norm. The inequality

$$\left| \int_c^d f(t) dt \right|_1 \leq \int_c^d |f(t)|_1 dt, \quad (16)$$

which is easily verified, explains our choice of the norm. In what follows, $\text{Lip}(\mathcal{L})_\alpha$ denotes the Lipschitz class based on the norm $|\cdot|_\alpha$.

Lemma 5.3. Let $f : \mathbb{R} \rightarrow \mathbb{R}^d$. If $|f(x)|_1 \leq M$ for all $x \in \mathbb{R}$, then

$$f_\delta \in \text{Lip}(\mathcal{L})_1, \quad \mathcal{L} = \frac{M}{\delta}.$$

Proof. Consider

$$f_\delta(x+h) - f_\delta(x) = \frac{1}{2\delta} \int_{x+h-\delta}^{x+h+\delta} f(t) dt - \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t) dt, \quad h > 0.$$

For $h \geq 2\delta$, by (16) we get

$$|f_\delta(x+h) - f_\delta(x)|_1 \leq 2M \leq 2M \frac{h}{2\delta} = \frac{M}{\delta} h.$$

If $h < 2\delta$, the intersection of the intervals $(x-\delta, x+\delta)$ and $(x+h-\delta, x+h+\delta)$ is non-empty, and therefore

$$f_\delta(x+h) - f_\delta(x) = \frac{1}{2\delta} \left(\int_{x+\delta}^{x+\delta+h} f(t) dt - \int_{x-\delta}^{x-\delta+h} f(t) dt \right).$$

Using the substitution $t = \tau + 2\delta$ in the first integral and (16), we obtain

$$|f_\delta(x+h) - f_\delta(x)|_1 = \left| \frac{1}{2\delta} \int_{x-\delta}^{x-\delta+h} (f(\tau+2\delta) - f(\tau)) d\tau \right|_1 = \frac{1}{2\delta} \int_{x-\delta}^{x-\delta+h} |f(\tau+2\delta) - f(\tau)|_1 d\tau \leq \frac{M}{\delta} h.$$

Thus in both cases

$$|f_\delta(x+h) - f_\delta(x)|_1 \leq \frac{M}{\delta} h.$$

□

The last lemma and (1) imply

Corollary 5.4. *Let $\rho(u, v) = |u - v|_\alpha$, $u, v \in \mathbb{R}^d$, and let $f : \mathbb{R} \rightarrow \mathbb{R}^d$ such that $|f(x)|_\alpha \leq M$ for all $x \in \mathbb{R}$. Then*

$$f_\delta \in \text{Lip}(\mathcal{L})_\alpha, \quad \mathcal{L} = C^\alpha \frac{M}{\delta},$$

with C^α the constant from Corollary 5.2.

5.2. The metric Steklov set-valued function

Here we extend the definition of the Steklov function to SFVs with a bounded range using the metric integral.

In this subsection all the notions such as the metric integral, as well as the obtained results, are based on the metric $\rho(u, v)_1 = |u - v|_1 = \sum_{i=1}^d |u_i - v_i|$, without indicating it explicitly in the discussion.

Definition 5.5. *Let $F : \mathbb{R} \rightarrow K(\mathbb{R}^d)$ have a bounded range. We define its metric Steklov set-valued function by*

$$F_\delta(x) = \frac{1}{2\delta}(\mathcal{M}) \int_{x-\delta}^{x+\delta} F(t) dt, \quad x \in \mathbb{R}.$$

For $F \in \mathcal{F}[a, b]$, it follows from Result 2.10 that

$$F_\delta(x) = \left\{ s_\delta(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} s(t) dt : s \in \mathcal{S} \right\}, \quad (17)$$

where $\mathcal{S}$ is the collection of all metric selections of F .

By Remark 2.9 and the definition of F_δ , we have

Corollary 5.6. *Let $F \in \mathcal{F}[a, b]$. If $|F| \leq \widetilde{M}$ for all $x \in [a, b]$ then $|F_\delta(x)| \leq \widetilde{M}$ for all $x \in [a, b]$.*

The next result shows that the variation of the metric Steklov function is bounded by that of F .

Proposition 5.7. *Let $F \in \mathcal{F}[a, b]$, then*

$$V_a^b(F_\delta) \leq V_a^b(F).$$

Proof. First we show that

$$\text{haus}(F_\delta(x), F_\delta(z)) \leq V_x^z(F), \quad x < z. \quad (18)$$

Consider

$$\text{haus}(F_\delta(x), F_\delta(z)) = \text{haus} \left(\frac{1}{2\delta}(\mathcal{M}) \int_{x-\delta}^{x+\delta} F(t) dt, \frac{1}{2\delta}(\mathcal{M}) \int_{z-\delta}^{z+\delta} F(t) dt \right).$$

Let $y_0 \in F_\delta(x)$ be arbitrary, then by (17) there exists a metric selection of F , $\tilde{s} \in \mathcal{S}$, such that

$$y_0 = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \tilde{s}(t) dt.$$

Then

$$\begin{aligned} \text{dist} \left(y_0, \frac{1}{2\delta}(\mathcal{M}) \int_{z-\delta}^{z+\delta} F(t) dt \right) &= \text{dist} \left(\frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \tilde{s}(t) dt, \left\{ \frac{1}{2\delta} \int_{z-\delta}^{z+\delta} s(t) dt : s \in \mathcal{S} \right\} \right) \\ &\leq \left| \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \tilde{s}(t) dt - \frac{1}{2\delta} \int_{z-\delta}^{z+\delta} \tilde{s}(t) dt \right| = |\tilde{s}_\delta(x) - \tilde{s}_\delta(z)| \leq V_x^z(\tilde{s}) \leq V_x^z(F), \end{aligned}$$

where the last two inequalities are obtained in view of Theorem 5.1 and Result 2.7.

Analogously, one can see that for any $y_1 \in F_\delta(z)$

$$\text{dist} \left(y_1, \frac{1}{2\delta}(\mathcal{M}) \int_{x-\delta}^{x+\delta} F(t) dt \right) \leq V_x^z(F),$$

which establishes (18).

Let now $\chi = \{x_0, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$, be an arbitrary partition of $[a, b]$. Using (18) we obtain

$$\sum_{i=0}^{n-1} \text{haus}(F_\delta(x_{i+1}), F_\delta(x_i)) \leq \sum_{i=0}^{n-1} V_{x_i}^{x_{i+1}}(F) = V_{x_0}^{x_n}(F) \leq V_a^b(F)$$

and thus

$$V_a^b(F_\delta) \leq V_a^b(F).$$

□

Similarly to the case of vector-valued functions we prove that F_δ is Lipschitz continuous.

Proposition 5.8. *Let $F \in \mathcal{F}[a, b]$, and let $|F(x)| \leq \widetilde{M}$ for all $x \in [a, b]$. Then*

$$F_\delta \in \text{Lip}(\mathcal{L}), \quad \mathcal{L} = \frac{\widetilde{M}}{\delta}.$$

Proof. As in the proof of Proposition 5.7 we have

$$\text{haus}(F_\delta(x), F_\delta(z)) \leq \sup_{s \in \mathcal{S}} |s_\delta(x) - s_\delta(z)|.$$

Then the claim follows from Lemma 5.3. $\square$

Another interesting relation between the regularity of F_δ and F , known for single-valued functions, is extended to SVFs below.

Corollary 5.9. *Let $F \in \mathcal{F}[a, b]$, then*

$$\omega(F_\delta, h)_{L_1} \leq \frac{h}{b-a} V_a^b(F).$$

Proof. By (18) we have

$$\omega(F_\delta, h)_{L_1} = \frac{1}{b-a} \sup_{0 \leq t \leq h} \int_a^b \text{haus}(F_\delta(x+t), F_\delta(x)) dx \leq \frac{1}{b-a} \sup_{0 \leq t \leq h} \int_a^b V_x^{x+t}(F_\delta) dx.$$

In view of Proposition 5.7 and Lemma 2.4 we get

$$\omega(F_\delta, h)_{L_1} \leq \frac{1}{b-a} \sup_{0 \leq t \leq h} \int_a^b V_x^{x+t}(F) dx \leq \frac{1}{b-a} \sup_{0 \leq t \leq h} t \cdot V_a^b(F) = \frac{h}{b-a} V_a^b(F).$$

$\square$

The next result provides a bound on the deviation of the metric Steklov multifunction F_δ from $F \in \mathcal{F}[a, b]$.

Theorem 5.10. *For $F \in \mathcal{F}[a, b]$ we have*

$$\int_a^b \text{haus}(F(x), F_\delta(x)) dx \leq 2\delta V_a^b(F).$$

Proof. First we show that for $F \in \mathcal{F}[a, b]$ and $x_0 \in [a, b]$

$$\text{haus}(F(x_0), F_\delta(x_0)) \leq V_{x_0-\delta}^{x_0+\delta}(F). \quad (19)$$

First let $y_0 \in F(x_0)$ and let $\mathcal{S}_{(x_0, y_0)} = \{s \in \mathcal{S} : s(x_0) = y_0\}$. Using Definition 5.5, (17), (16) and Result 2.7 we obtain

$$\begin{aligned} \text{dist}(y_0, F_\delta(x_0)) &= \text{dist}\left(y_0, \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} F(t) dt\right) = \text{dist}\left(y_0, \left\{\frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} s(t) dt : s \in \mathcal{S}\right\}\right) \\ &\leq \text{dist}\left(y_0, \left\{\frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} s(t) dt : s \in \mathcal{S}_{(x_0, y_0)}\right\}\right) \\ &\leq \sup \left\{ \frac{1}{2\delta} \left| \int_{x_0-\delta}^{x_0+\delta} s(x_0) dt - \int_{x_0-\delta}^{x_0+\delta} s(t) dt \right| : s \in \mathcal{S}_{(x_0, y_0)} \right\} \\ &\leq \sup \left\{ \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} |s(x_0) - s(t)| dt : s \in \mathcal{S}_{(x_0, y_0)} \right\} \\ &\leq \sup \left\{ \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} V_{x_0-\delta}^{x_0+\delta}(s) dt : s \in \mathcal{S}_{(x_0, y_0)} \right\} \\ &\leq \sup \left\{ V_{x_0-\delta}^{x_0+\delta}(s) : s \in \mathcal{S}_{(x_0, y_0)} \right\} \leq V_{x_0-\delta}^{x_0+\delta}(F). \end{aligned} \quad (20)$$

Similarly for any $y_\delta \in F_\delta(x_0)$ one can guarantee in view of (17) the existence of a metric selection $s \in \mathcal{S}$ such that $y_\delta = \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} s(t) dt$. Then

$$\text{dist}(y_\delta, F(x_0)) \leq |y_\delta - s(x_0)| \leq \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} |s(x_0) - s(t)| dt \leq V_{x_0-\delta}^{x_0+\delta}(F). \quad (21)$$

Inequalities (20) and (21) imply (19). Now by (19) and Lemma 2.4 we get

$$\int_a^b \text{haus}(F(x), F_\delta(x)) dx \leq \int_a^b V_{x-\delta}^{x+\delta}(F) dx \leq 2\delta V_a^b(F).$$

□

Remark 5.11. All the results of this subsection hold also in the case when F_δ is defined with respect to any norm-based metric in $\mathbb{R}^d$, possibly with other constants which can be derived, e.g., from (1).

5.3. Metric Bernstein operators applied on Steklov SVFs

Since the Steklov operator is a smoothing operator, a natural idea how to approximate discontinuous SVF is to first smooth it by the metric Steklov operator and then to apply an approximation method designed for continuous functions to the Steklov function. In this section we consider the combination of the metric Steklov operator with the metric Bernstein operator and obtain an error estimate for the approximation of functions from the class $\mathcal{F}[0, 1]$. The error estimate we derive is not better than the one in Theorem 4.2. Indeed, the estimate in Theorem 4.2 is arbitrarily close to the optimal one, so that it cannot be essentially improved for functions of bounded variation. However, we find this combination interesting. It might be useful for approximation of functions that do not have bounded variation — the situation where the classical Bernstein operator is not appropriate anymore. This is a subject of future research.

As in Subsection 5.2, we assume here that the underlying norm in $\mathbb{R}^d$ is the ℓ_1 -norm. Nevertheless, the obtained estimates, up to a constant, remain the same for any other norm in $\mathbb{R}^d$.

The Steklov-based Bernstein operator for $F \in \mathcal{F}[0, 1]$ is

$$B_{n,\delta}^M F(x) = B_n^M F_\delta(x) = \bigoplus_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \frac{1}{2\delta} \int_{\frac{k}{n}-\delta}^{\frac{k}{n}+\delta} F(x) dx, \quad x \in [0, 1].$$

Using our previous results, we get

Corollary 5.12. Let $F \in \mathcal{F}[0, 1]$, then for each $s \in \mathbb{N}$ we have

$$\int_0^1 \text{haus}(B_{n,\delta}^M F(x), F(x)) dx \leq C_{s,F} n^{-\frac{s}{1+2s}} + 2\delta V_0^1(F).$$

In particular, taking $\delta = O(n^{-\frac{s}{1+2s}})$ and letting s go to infinity, we obtain the rate $n^{-(\frac{1}{2}-\varepsilon)}$ like in Theorem 4.2.

Proof. The estimate (11) applied to the Bernstein multifunction F_δ and Theorem 5.10 yield

$$\begin{aligned} \int_0^1 \text{haus}(B_{n,\delta}^M F(x), F(x)) dx &\leq \int_0^1 \text{haus}(B_{n,\delta}^M F(x), F_\delta(x)) dx + \int_0^1 \text{haus}(F_\delta(x), F(x)) dx \\ &\leq C_{s,F} n^{-\frac{s}{1+2s}} + 2\delta V_0^1(F). \end{aligned}$$

Note that the constant $C_{s,F}$ here depends only on s and F . Indeed, the estimate (11) in Theorem 4.2 is a consequence of (10) that includes only the total variation $V_0^1(F)$ and the bound $\sup_{x \in [a,b]} |F(x)|$. Here we apply (11) to the function F_δ . By Corollary 5.6 and Proposition 5.7 both values $V_0^1(F_\delta)$ and $\sup_{x \in [a,b]} |F_\delta(x)|$ are bounded by those of F , and thus the constant in the estimate above can be taken the same for all values of δ . Thus we obtain the claim of the Corollary. □

Recalling that F_δ is Lipschitz continuous, we can apply known error estimates for Lipschitz continuous SVFs. In [5], [6] it is shown that if $F \in \text{Lip}(\mathcal{L})$ on $[0, 1]$, then for any $x \in [0, 1]$

$$\text{haus}(B_n^M F(x), F(x)) \leq 2\frac{\mathcal{L}}{n} + C\mathcal{L}\sqrt{\frac{x(1-x)}{n}} \quad (22)$$

when the underlying norm is the Euclidean norm. Here $\mathcal{C}$ is a certain constant.

By Proposition 5.8 we have $F_\delta \in \text{Lip}(\mathcal{L})$ with $\mathcal{L} = \frac{\widetilde{M}}{\delta}$ where $\widetilde{M}$ is such that $|F(x)| \leq \widetilde{M}$ for all $x \in [0, 1]$. These estimates with respect to the ℓ_1 -norm. Thus (22) applied to F_δ and (1) give

$$\text{haus}(B_n^M F_\delta(x), F_\delta(x)) \leq \frac{C_1}{\delta n} + \frac{C_2}{\delta} \sqrt{\frac{x(1-x)}{n}}$$

with certain constants C_1, C_2 .

Using the above result and Theorem 5.10 we conclude that

$$\int_0^1 \text{haus} \left(B_{n,\delta}^M F(x), F(x) \right) dx \leq \frac{C_1}{\delta n} + \frac{C_3}{\delta \sqrt{n}} + 2\delta V_0^1(F).$$

Obviously, $\int_0^1 \text{haus}(B_{n,\delta}^M F(x), F(x)) dx$ tends to zero when $\delta \rightarrow 0$ and $\sqrt{n}\delta \rightarrow \infty$. To get the optimal rate of convergence in the above estimate, we choose $\delta = \frac{1}{n^{1/4}}$, since in this case $\frac{1}{\delta \sqrt{n}} = \frac{1}{n^{3/4}}$. We see that the rate we obtain using this approach is worse than the one in Corollary 5.12.

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