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High dimensional finite elements for two-scale Maxwell wave equations

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ABSTRACT

We develop an essentially optimal numerical method for solving two-scale Maxwell wave equations in a domain $D \subset \mathbb{R}^d$. The problems depend on two scales: one macroscopic scale and one microscopic scale. Solving the macroscopic two-scale homogenized problem, we obtain the desired macroscopic and microscopic information. This problem depends on two variables in $\mathbb{R}^d$, one for each scale that the original two-scale equation depends on, and is thus posed in a high dimensional tensorized domain. The straightforward full tensor product finite element (FE) method is exceedingly expensive. We develop the sparse tensor product FEs that solve this two-scale homogenized problem with essentially optimal number of degrees of freedom, i.e. the number of degrees of freedom differs by only a logarithmic multiplying factor from that required for solving a macroscopic problem in a domain in $\mathbb{R}^d$ only, for obtaining a required level of accuracy. Numerical correctors are constructed from the FE solution. We derive a rate of convergence for the numerical corrector in terms of the microscopic scale and the FE mesh width. Numerical examples confirm our analysis.

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1. Introduction

We study the high dimensional finite element (FE) method for solving two-scale Maxwell wave equations in a domain $D \subset \mathbb{R}^d$ which is a Lipschitz polyhedron when $d = 3$ and a Lipschitz polygon when $d = 2$. The equation depends on the macroscopic scale and a microscopic scale, and is locally periodic. We study the problem via two-scale convergence. In the limit where the microscopic scale converges to zero, we obtain the two-scale homogenized equation. This equation contains the solution to the homogenized equation which approximates the solution of the original two-scale equation macroscopically, and the scale interacting corrector terms which provides the microscopic behaviour of the solution. Solving the equation, we obtain all the necessary information. However, the two-scale homogenized equation is posed in a high dimensional tensorized domain. It depends on two variables in $\mathbb{R}^d$, one for each scale. The direct full tensor product FE method is highly expensive. We develop the sparse tensor product FEs to solve this problem which requires only an essentially equal number of degrees of freedom as for solving an equation posed in $\mathbb{R}^d$, i.e. the number of degrees of freedom differs by only a logarithmic multiplying factor, for obtaining a required level of accuracy. The complexity

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is thus essentially optimal. We note that the numerical method in this paper can be equally developed for the case of multiple scales but we restrict our consideration to the case of two scales to make the presentation concise.

As for any other two-scale problems, a direct numerical method using fine mesh to capture the microscopic scale is prohibitively expensive. There have been attempts to develop numerical methods for solving multiscale wave equations, and multiscale Maxwell equations with reduced complexity, though comparing to other types of multiscale equations, multiscale wave and multiscale Maxwell equations have been paid far less attention.

For multiscale wave equations, in [1] Owhadi and Zhang build a set of basis functions that contain microscopic information from the solutions of d multiscale equations. These equations are solved using fine mesh to capture the microscopic scales. In [2], Jiang et al. employ the Multiscale FE method (MsFEM) [3,4] to solve wave equations that depend on a continuum spectrum of scales, using limited global information. The Heterogeneous Multiscale Method (HMM) [5,6] is employed by Engquist et al. [7] using finite difference to solve two-scale wave equations that show the dispersive behaviour at large time. Abdulle and Grote [8] employ the HMM to solve multiscale equation using FEs. The approaches in these papers are general, but the complexity at each time step grows superlinearly with respect to the optimal complexity level. In [9], Xia and Hoang develop the essentially optimal sparse tensor product FE method for locally periodic multiscale wave equations; the complexity of the method only grows log-linearly at each time step. The method is employed successfully for multiscale elastic wave equations in [10]. The sparse tensor product FE approach for two-scale problems is initiated by Hoang and Schwab in [11] for elliptic equations, and is applied for other types of equations in [9,10,12,13].

There has not been much research on efficient numerical methods for multiscale Maxwell equations. The traditional method that constructs the homogenized equation by solving cell problems is considered in [14] (see also the related references therein) where a set of cell problems are solved at each macroscopic point. The complexity is thus very high. The HMM is applied for multiscale Maxwell equations in frequency domain in Ciarlet et al. [15]. Ohlberger et al. considered a locally periodic two-sale harmonic Maxwell equation in [16] though the problem is assumed uniformly coercive with respect to the microscopic scale. The HMM is analysed for the two scale homogenized problem using the approach in [17]. The complexity of the method is equivalent to that of the full tensor product FE method for solving the two scale homogenized equation. In [18], Chu and Hoang develop the sparse tensor product edge FE method for locally periodic stationary multiscale Maxwell equations. It uses edge FEs [19] which discretize functions in the $H(\mathrm{curl})$ spaces, for the solution u_0 of the homogenized equation, and for the fast variable in the scale interaction terms. To obtain a FE approximation with error $O(2^{-L})$ in the energy norm for the solution of the multiscale homogenized equation, the method requires only $O(L2^{dL})$ degrees of freedom that is essentially equivalent to that needed for solving a macroscopic scale Maxwell equation in a domain in $\mathbb{R}^d$, and is therefore optimal. Chu and Hoang [18] construct numerical correctors from the FE solutions. For two scale problems, an explicit error in terms of the FE error and the homogenization error is deduced for the numerical corrector.

We develop the sparse tensor product FE approach for two-scale Maxwell wave equations in this paper using edge FEs. We show that the complexity of the method is essentially optimal.

In the next section, we set up the two-scale Maxwell wave equation and derive the two-scale homogenized equation. In Section 3, we study FE approximation for the two-scale homogenized Maxwell wave equation using general FE spaces. In Section 3.1, we study the spatially semidiscrete problem where only the spatial variable is discretized. We follow the framework of Dupont [20] for wave equations. The approach has been applied for the multiscale homogenized equations of scalar multiscale wave equations in Xia and Hoang [9]. However, the application of the framework to two-scale Maxwell wave equations requires substantial modification for the analysis of the convergence due to the corrector terms u_1 in (2.4). In Section 3.2, we consider the fully discrete problem where both the temporal and spatial variables are discretized. The convergence of the general discretization schemes in Section 3, and the full and sparse tensor product FE approximations in Section 5 require regularity for the solution of the two-scale homogenized Maxwell wave equation. In Section 4, we prove that the required regularity holds under mild conditions. In Section 5, we apply the discretization schemes in Section 3 for the full tensor product and the sparse tensor product edge FEs. We prove that the sparse tensor product FE method obtains an approximation with essentially the same level of accuracy as the full tensor product FEs but requires only essentially the same number of degrees of freedom as for solving a macroscopic Maxwell equation in $\mathbb{R}^d$, and is thus essentially optimal. In Section 6, we construct numerical correctors from the FE solutions. We derive an explicit homogenization error in terms of the microscopic scale. From this, we derive a numerical corrector with an explicit error in terms of the homogenization error, and the FE error. Section 7 presents some numerical examples in two dimensions that confirm our analysis.

Throughout the paper, by curl and ∇ without explicitly indicating the variable, we mean the curl and the gradient with respect to x of a function of x, or of x and t; and by curl_x and ∇_x we denote the partial curl and partial gradient of a function that depends on x and other variables. Repeated indices indicate summation. The notation # denotes spaces of periodic functions with the period being the unit cube in $\mathbb{R}^d$. The notation c denotes various constants whose values can differ between appearances. For functions depending on the time variable t and other variables, when we only wish to emphasize the dependence on t, we only indicate the t dependence.

2. Two-scale Maxwell wave problems

2.1. Problem setting

Let D be a Lipschitz polygon in $\mathbb{R}^2$ or a Lipschitz polyhedron in $\mathbb{R}^3$. Let Y be the unit cube in $\mathbb{R}^d$ for $d = 2, 3$. Let a and b be continuous functions from $D \times Y$ to $\mathbb{R}^{d\times d}_{sym}$ which are periodic with respect to y with the period being Y. We assume that a and b satisfy the boundedness and coerciveness conditions: for all $x \in D$ and $y \in Y$, and all $\xi, \zeta \in \mathbb{R}^d$,

$$\begin{aligned} &\alpha|\xi|^2 \le a_{ij}(x, y)\xi_i\xi_j, \quad a_{ij}\xi_i\zeta_j \le \beta|\xi||\zeta| \\ &\alpha|\xi|^2 \le b_{ij}(x, y)\xi_i\xi_j, \quad b_{ij}\xi_i\zeta_j \le \beta|\xi||\zeta| \end{aligned} \tag{2.1}$$

where α and β are positive numbers. Let ε be a small positive value. We define the two-scale coefficients of the Maxwell equation a^ε and b^ε which are functions from D to $\mathbb{R}^{d\times d}_{sym}$ as

$$a^\varepsilon(x) = a(x, \frac{x}{\varepsilon}), \quad b^\varepsilon(x) = b(x, \frac{x}{\varepsilon}).$$

When $d = 3$ we define the space

$$W = H_0(\mathrm{curl}, D) = \{u \in L^2(D)^3, \quad \mathrm{curl}\, u \in L^2(D)^3, \quad u \times \nu = 0\}, \quad H = L^2(D)^3$$

and when $d = 2$

$$W = H_0(\mathrm{curl}, D) = \{u \in L^2(D)^2, \quad \mathrm{curl}\, u \in L^2(D), \quad u \times \nu = 0\}, \quad H = L^2(D)^2$$

where ν denotes the outward normal vector on the boundary ∂D. These spaces form the Gelfand triple $W \subset H \subset W'$. We denote by $\langle\cdot,\cdot\rangle$ the inner product in H extended by density to the duality pairing between W and W'. We note that when $d = 3$, $\mathrm{curl}\, u^\varepsilon$ is a vector function in $L^2(D)^3$ and when $d = 2$, $\mathrm{curl}\, u^\varepsilon$ is a scalar function in $L^2(D)$. Let $f \in L^2((0, T), H)$, $g_0 \in W$ and $g_1 \in H$. We consider the problem: Find $u^\varepsilon(t, x) \in L^2((0, T), W)$ so that

$$\begin{cases} b^\varepsilon(x)\dfrac{\partial^2 u^\varepsilon(t, x)}{\partial t^2} + \mathrm{curl}(a^\varepsilon(x)\mathrm{curl} u^\varepsilon(t, x)) = f(t, x), \quad (t, x) \in (0, T) \times D \\ u^\varepsilon(0, x) = g_0(x), \quad u^\varepsilon_t(0, x) = g_1(x) \end{cases}$$

with the boundary condition $u^\varepsilon \times \nu = 0$ on $(0, T) \times \partial D$. We will mostly present the analysis for the case $d = 3$ and only discuss the case $d = 2$ when there is significant difference. For notational conciseness, we denote by $H_1 = L^2(D \times Y)^3$. In variational form, this problem becomes: Find $u^\varepsilon \in L^2((0, T), W) \cap H^1((0, T), H)$ so that

$$\left\langle b^\varepsilon(x)\frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi(x)\right\rangle + \int_D a^\varepsilon(x)\mathrm{curl} u^\varepsilon(t, x) \cdot \mathrm{curl}\, \phi(x)dx = \int_D f(t, x) \cdot \phi(x)dx \tag{2.2}$$

for all $\phi \in W$ for almost all $t \in (0, T)$. The equation is for the case $d = 3$; when $d = 2$ we need to replace the vector product for curl by the scalar multiplication. Problem (2.2) has a unique solution $u^\varepsilon \in L^2((0, T), W) \bigcap H^1((0, T), H) \bigcap H^2((0, T), W')$ that satisfies

$$\|u^\varepsilon\|_{L^2((0,T),W)} + \|u^\varepsilon\|_{H^1((0,T),H)} + \|u^\varepsilon\|_{H^2((0,T),W')} \le c(\|f\|_{L^2((0,T),H)} + \|g_0\|_W + \|g_1\|_H) \tag{2.3}$$

where the constant c only depends on the constants α and β in (2.1) and T (see Wloka [21]).

We will study this problem via two-scale convergence.

2.2. Two-scale convergence

We study homogenization of problem (2.2) via two-scale convergence. We first recall its definition (see Nguetseng [22], Allaire [23] and Allaire and Briane [24]).

Definition 2.1. A sequence of functions $\{w^\varepsilon\}_\varepsilon \subset L^2((0, T), H)$ two-scale converges to a function $w_0 \in L^2((0, T), D \times Y)$ if for all smooth functions $\phi(t, x, y)$ which are Y periodic w.r.t y:

$$\lim_{\varepsilon\to 0}\int_0^T\int_D w^\varepsilon(t, x)\phi(t, x, \frac{x}{\varepsilon})dxdt = \int_0^T\int_D\int_Y w_0(t, x, y)\phi(t, x, y)dydxdt.$$

We have the following result.

Proposition 2.2. *From a bounded sequence in $L^2((0, T), H)$ we can extract a two-scale convergent subsequence.*

The definition above for functions which depend also on t is slightly different from that in [22] and [23] as we take also the integral with respect to t. However, the proof of Proposition 2.2 is similar.

For a bounded sequence in $L^2((0, T), W)$, we have the following results which are similar to those in [18] and [25] for functions which do not depend on t. The proofs for these results are similar to those in [18] so we do not present them here. As in [18], we denote by $\tilde{H}_\#(\mathrm{curl}, Y)$ the space of equivalent classes of functions in $H_\#(\mathrm{curl}, Y)$ of equal curl.

Proposition 2.3. *Let $\{w^\varepsilon\}_\varepsilon$ be a bounded sequence in $L^2((0,T),W)$. There is a subsequence (not renumbered), a function $w_0 \in L^2((0,T),W)$, a function $\mathfrak{w}_1 \in L^2((0,T)\times D, H^1_\#(Y)/\mathbb{R})$ such that*

$$w^\varepsilon \overset{\text{two-scale}}{\longrightarrow} w_0 + \nabla_y \mathfrak{w}_1.$$

Further, there is a function $w_1 \in L^2((0,T)\times D, \tilde{H}_\#(\mathrm{curl}, Y))$ such that

$$\mathrm{curl}\, w^\varepsilon \overset{\text{two-scale}}{\longrightarrow} \mathrm{curl}\, w_0 + \mathrm{curl}_y w_1.$$

From (2.3) and Proposition 2.3, we can extract a subsequence (not renumbered), a function $u_0 \in L^2((0,T),W)$, a function $\mathfrak{u}_1 \in L^2((0,T), L^2(D, H^1_\#(Y)/\mathbb{R}))$ and a function $u_1 \in L^2((0,T), L^2(D, \tilde{H}_\#(\mathrm{curl}, Y)))$ such that

$$u^\varepsilon \overset{\text{two-scale}}{\longrightarrow} u_0 + \nabla_y \mathfrak{u}_1, \tag{2.4}$$

and

$$\mathrm{curl}\, u^\varepsilon \overset{\text{two-scale}}{\longrightarrow} \mathrm{curl}\, u_0 + \mathrm{curl}_y u_1. \tag{2.5}$$

Let $W_1 = L^2(D, \tilde{H}_\#(\mathrm{curl}, Y))$ and $V_1 = L^2(D, H^1_\#(Y)/\mathbb{R})$. We define the space $\mathbf{W}$ as

$$\mathbf{W} = W \times W_1 \times V_1. \tag{2.6}$$

For $\boldsymbol{v} = (v_0, \{v_1\}, \{\mathfrak{v}_1\}) \in \mathbf{W}$, we define the norm

$$|||\boldsymbol{v}||| = \|v_0\|_{H(\mathrm{curl},D)} + \|v_1\|_{L^2(D,\tilde{H}_\#(\mathrm{curl},Y))} + \|\mathfrak{v}_1\|_{L^2(D,H^1_\#(Y))}.$$

Let $\boldsymbol{u} = (u_0, \{u_1\}, \{\mathfrak{u}_1\}) \in \mathbf{W}$. We then have the following result.

Proposition 2.4. *The function $\boldsymbol{u} = (u_0, \{u_1\}, \{\mathfrak{u}_1\})$ satisfies*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int_D\int_Y b(x,y)\big(u_0(t,x) + \nabla_y \mathfrak{u}_1(t,x,y)\big)\cdot\big(v_0(x,y) + \nabla_y \mathfrak{v}_1(x,y)\big)\,dydx\,+ \\ \int_D\int_Y a(x,y)\big(\mathrm{curl}\, u_0 + \mathrm{curl}_y u_1\big)\cdot\big(\mathrm{curl}\, v_0 + \mathrm{curl}_y v_1\big)\,dydx = \int_D f(t,x)\cdot v_0(x)dx\end{aligned} \tag{2.7}$$

for all $\boldsymbol{v} = (v_0, \{v_1\}, \{\mathfrak{v}_1\}) \in \mathbf{W}$, with the initial conditions

$$u_0(0,\cdot) = g_0, \quad \nabla_y \mathfrak{u}_1(0,\cdot,\cdot) = 0, \tag{2.8}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\int_D\int_Y b(x,y)(u_0(t,x) + \nabla_y \mathfrak{u}_1(t,x,y))\cdot(v_0(x) + \mathfrak{v}_1(x,y))dydx\Big|_{t=0} \\ = \int_D\int_Y b(x,y)g_1\cdot(v_0(x) + \nabla_y \mathfrak{v}_1(x,y))dxdy\end{aligned} \tag{2.9}$$

for all $v_0 \in W$ and $\mathfrak{v}_1 \in V_1$; the derivative $\frac{\partial}{\partial t}$ is understood in the distribution sense.

We then have

Proposition 2.5. *With the initial conditions (2.8) and (2.9), problem (2.7) has a unique solution. The whole sequences $\{u^\varepsilon\}_\varepsilon$ and $\{\mathrm{curl}\, u^\varepsilon\}_\varepsilon$ two-scale converge to the solution of problem (2.7).*

We prove Propositions 2.4 and 2.5 in Appendix A.

3. General spatially semidiscrete and fully discrete problems

We study FE approximation for problem (2.7) in this section. We first consider the semidiscrete problem where we discretize the spatial variables. We then consider the fully discrete problem where both the temporal and spatial variables are discretized.

3.1. Spatially semidiscrete problem

We consider in this section the spatial semidiscretization of the homogenized problem (2.7). For approximating u_0, we suppose that there is a hierarchy of finite dimensional subspaces

$$W^1 \subset W^2 \subset \ldots \subset W^L \ldots \subset W;$$

to approximate u_1, we assume a hierarchy of finite dimensional subspaces

$$W_1^1 \subset W_1^2 \subset \ldots \subset W_1^L \ldots \subset W_1;$$

and to approximate $\mathfrak{u}_1$, we assume a hierarchy of finite dimensional subspaces

$$V_1^1 \subset V_1^2 \subset \ldots \subset V_1^L \ldots \subset V_1.$$

Let

$$\mathbf{W}^L = W^L \times W_1^L \times V_1^L$$

which is a finite dimensional subspace of $\mathbf{W}$ defined in (2.6). We consider the spatially semidiscrete approximating problems: Find $\boldsymbol{u}^L(t) = (u_0^L, u_1^L, \mathfrak{u}_1^L) \in \mathbf{W}^L$ so that

$$\begin{aligned}\int_D\int_Y \Big[& b(x,y)\left(\frac{\partial^2}{\partial t^2}u_0^L(t,x) + \nabla_y \frac{\partial^2}{\partial t^2}\mathfrak{u}_1^L(t,x,y_1)\right)\cdot\left(v_0^L + \nabla_y \mathfrak{v}_1^L\right) \\ & + a(x,y)\left(\operatorname{curl} u_0^L + \operatorname{curl}_y u_1^L\right)\cdot\left(\operatorname{curl} v_0^L + \operatorname{curl}_y v_1^L\right)\Big] dydx = \int_D f(t,x)\cdot v_0^L(x)dx \end{aligned} \tag{3.1}$$

for all $\boldsymbol{v}^L = (v_0^L, v_1^L, \mathfrak{v}_1^L) \in \mathbf{W}^L$. Let $g_0^L \in W^L$, $g_1^L \in W^L$ which are approximations of g_0 and g_1 in W and in H respectively. The initial conditions (2.8) are approximated by:

$$u_0^L(0,\cdot) = g_0^L, \quad \nabla_y \mathfrak{u}_1^L(0,\cdot,\cdot) = 0. \tag{3.2}$$

We approximate the initial conditions (2.9) by

$$\int_D\int_Y b(x,y)\left(\frac{\partial u_0^L}{\partial t}(0,x,y) + \frac{\partial}{\partial t}\nabla_y \mathfrak{u}_1^L(0,x,y)\right)\cdot\left(v_0^L + \nabla_y\mathfrak{v}_1^L\right)dydx = \int_D\int_Y b(x,y)g_1^L(x)\cdot\left(v_0^L + \nabla_y\mathfrak{v}_1^L\right)dydx$$

for all $v_0^L \in W^L$ and $\mathfrak{v}_1^L \in V_1^L$, i.e.,

$$\int_D\int_Y b(x,y)\left(\frac{\partial u_0^L}{\partial t}(0,x) - g_1^L + \frac{\partial}{\partial t}\nabla_y \mathfrak{u}_1^L(0,x,y)\right)\cdot\left(v_0^L + \nabla_y\mathfrak{v}_1^L\right)dydx = 0.$$

Using the coercivity of the matrix $b(x,y)$, we get

$$\frac{\partial u_0^L}{\partial t}(0) = g_1^L, \quad \frac{\partial}{\partial t}\nabla_y \mathfrak{u}_1^L(0) = 0. \tag{3.3}$$

For $\boldsymbol{v} = (v_0, v_1, \mathfrak{v}_1)$ and $\boldsymbol{w} = (w_0, w_1, \mathfrak{w}_1)$ in $\mathbf{W}$, we define the bilinear forms

$$A(\boldsymbol{v},\boldsymbol{w}) = \int_D\int_Y a(x,y)\left(\operatorname{curl} v_0 + \operatorname{curl}_y v_1\right)\cdot\left(\operatorname{curl} w_0 + \operatorname{curl}_y w_1\right)dydx,$$

and

$$B(\boldsymbol{v},\boldsymbol{w}) = \int_D\int_Y b(x,y)\left(v_0 + \nabla_y\mathfrak{v}_1\right)\cdot\left(w_0 + \nabla_y\mathfrak{w}_1\right)dydx.$$

Proposition 3.1. *Problem (3.1) together with the initial conditions (3.2) and (3.3) has a unique solution.*

The proof of this proposition is standard.

For each $t \in (0,T)$, let $\boldsymbol{w}^L(t) = (w_0^L, w_1^L, \mathfrak{w}_1^L) \in \mathbf{W}^L$ be the solution of the problem

$$B(\boldsymbol{w}^L(t) - \boldsymbol{u}(t), \boldsymbol{v}^L) + A(\boldsymbol{w}^L(t) - \boldsymbol{u}(t), \boldsymbol{v}^L) = 0 \tag{3.4}$$

for all $\boldsymbol{v}^L \in \mathbf{W}^L$. As the coefficients a and b in (2.1) are both uniformly bounded and coercive for all $x \in D$ and $y \in Y$, problem (3.4) has a unique solution. Let $\boldsymbol{q}^L = \boldsymbol{w}^L - \boldsymbol{u}$. We then have the following estimate.

Lemma 3.2. *For $\boldsymbol{q}^L = \boldsymbol{w}^L - \boldsymbol{u}$ where $\boldsymbol{w}^L$ is the solution of problem (3.4) we have*

$$\left\|\boldsymbol{q}^L(t)\right\|_{\mathbf{W}} \le c \inf_{\boldsymbol{v}^L\in\mathbf{W}^L}\left\|\boldsymbol{u}(t) - \boldsymbol{v}^L\right\|_{\mathbf{W}}.$$

Proof. From (3.4), we have

$$B(\boldsymbol{w}^L - \boldsymbol{u}, \boldsymbol{w}^L - \boldsymbol{u}) + A(\boldsymbol{w}^L - \boldsymbol{u}, \boldsymbol{w}^L - \boldsymbol{u}) = B(\boldsymbol{w}^L - \boldsymbol{u}, \boldsymbol{v}^L - \boldsymbol{u}) + A(\boldsymbol{w}^L - \boldsymbol{u}, \boldsymbol{v}^L - \boldsymbol{u})$$

for all $\boldsymbol{v}^L \in \mathbf{W}^L$. From the coerciveness and boundedness of the matrices a and b we get the conclusion. □

When $\boldsymbol{u}$ is sufficiently regular with respect to t, we have the following estimates.

Lemma 3.3. *If* $\frac{\partial \boldsymbol{u}}{\partial t} \in C([0,T],\mathbf{W})$, *then*

$$\left\|\frac{\partial \boldsymbol{q}^L}{\partial t}\right\|_{L^\infty((0,T),\mathbf{W})} \le c \sup_{t\in[0,T]} \inf_{\boldsymbol{v}^L\in\mathbf{W}^L} \left\|\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{v}^L\right\|_{\mathbf{W}}.$$

If $\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^2((0,T),\mathbf{W})$, *then*

$$\left\|\frac{\partial^2 \boldsymbol{q}^L}{\partial t^2}\right\|_{L^2((0,T),\mathbf{W})} \le c \inf_{\boldsymbol{v}^L\in L^2((0,T),\mathbf{W}^L)} \left\|\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \boldsymbol{v}^L\right\|_{L^2((0,T),\mathbf{W})}.$$

Proof. If $\frac{\partial \boldsymbol{u}}{\partial t} \in C([0,T],\mathbf{W})$ from (3.4) we have

$$B\left(\frac{\partial}{\partial t}\boldsymbol{w}^L(t) - \frac{\partial}{\partial t}\boldsymbol{u}(t), \boldsymbol{v}^L\right) + A\left(\frac{\partial}{\partial t}\boldsymbol{w}^L(t) - \frac{\partial}{\partial t}\boldsymbol{u}(t), \boldsymbol{v}^L\right) = 0$$

for all $\boldsymbol{v}^L \in \mathbf{W}^L$. We then proceed as in the proof of Lemma 3.2 to show the first inequality. The proof for the second inequality is similar. □

Let $\boldsymbol{p}^L = \boldsymbol{u}^L - \boldsymbol{w}^L$, i.e., $p_1^L = u_1^L - w_1^L$, $\mathfrak{p}_1^L = \mathfrak{u}_1^L - \mathfrak{w}_1^L$ and $p_0^L = u_0^L - w_0^L$.

Proposition 3.4. *Assume that* $\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^2((0,T),\mathbf{W})$. *Then there is a constant c depending on T such that for all* $t \in (0,T)$

$$\begin{aligned}
&\left\|\frac{\partial p_0^L}{\partial t}(t) + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}(t)\right\|_{H_1} + \left\|\operatorname{curl} p_0^L(t) + \operatorname{curl}_y p_1^L(t)\right\|_{H_1} \\
&\quad \le c\left[\left\|\frac{\partial^2 q_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2} - q_0^L - \nabla_y \mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)} + \left\|\frac{\partial p_0^L}{\partial t}(0) + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}(0)\right\|_{H_1} + \left\|\operatorname{curl} p_0^L(0)\right\|_H\right].
\end{aligned}$$

Proof. Since $\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^2((0,T),\mathbf{W})$, from (2.7) and (3.1) we have for all $\boldsymbol{v}^L = (v_0^L, v_1^L, \mathfrak{v}_1^L) \in \mathbf{W}^L$

$$\begin{aligned}
&\int_D\int_Y \left[b(x,y)\left(\frac{\partial^2 p_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{p}_1^L}{\partial t^2}\right)\cdot\left(v_0^L + \nabla_y \mathfrak{v}_1^L\right) + a(x,y)\left(\operatorname{curl} p_0^L + \operatorname{curl}_y p_1^L\right)\cdot\left(\operatorname{curl} v_0^L + \operatorname{curl}_y v_1^L\right)\right] dydx \\
&\quad = -\int_D\int_Y b(x,y)\left(\frac{\partial^2 q_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2}\right)\cdot\left(v_0^L + \nabla_y \mathfrak{v}_1^L\right) dydx - A(\boldsymbol{q}^L, \boldsymbol{v}^L).
\end{aligned}$$

From (3.4) we have $A(\boldsymbol{q}^L, \boldsymbol{v}^L) = -B(\boldsymbol{q}^L, \boldsymbol{v}^L)$. Thus

$$\begin{aligned}
&\int_D\int_Y \left[b(x,y)\left(\frac{\partial^2 p_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{p}_1^L}{\partial t^2}\right)\cdot\left(v_0^L + \nabla_y \mathfrak{v}_1^L\right) + a(x,y)\left(\operatorname{curl} p_0^L + \operatorname{curl}_y p_1^L\right)\cdot\left(\operatorname{curl} v_0^L + \operatorname{curl}_y v_1^L\right)\right] dydx \\
&\quad = -\int_D\int_Y b(x,y)\left(\frac{\partial^2 q_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2} - q_0^L - \nabla_y \mathfrak{q}_1^L\right)\cdot\left(v_0^L + \nabla_y \mathfrak{v}_1^L\right) dydx. \qquad (3.5)
\end{aligned}$$

Let $\boldsymbol{v}^L = \frac{\partial \boldsymbol{p}^L}{\partial t}$. We then have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_D\int_Y \left[b(x,y)\left(\frac{\partial p_0^L}{\partial t} + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}\right)\cdot\left(\frac{\partial p_0^L}{\partial t} + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}\right) + a(x,y)\left(\operatorname{curl} p_0^L + \operatorname{curl}_y p_1^L\right)\cdot\left(\operatorname{curl} p_0^L + \operatorname{curl}_y p_1^L\right)\right] dydx \\
&\quad \le c\left\|\frac{\partial^2 q_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2} - q_0^L - \nabla_y \mathfrak{q}_1^L\right\|_{H_1}\left\|\frac{\partial p_0^L}{\partial t} + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}\right\|_{H_1} \\
&\quad \le \frac{c}{\gamma}\left\|\frac{\partial^2 q_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2} - q_0^L - \nabla_y \mathfrak{q}_1^L\right\|_{H_1}^2 + c\gamma\left\|\frac{\partial p_0^L}{\partial t} + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}\right\|_{H_1}^2
\end{aligned}$$

for a constant $\gamma > 0$. Integrating both sides on $(0,t)$ for $0 < t < T$, and using the coercivity of the matrices a and b, we have

$$\begin{aligned}
&\left\|\frac{\partial p_0^L}{\partial t}(t) + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}(t)\right\|_{H_1}^2 + \left\|\operatorname{curl} p_0^L(t) + \operatorname{curl}_y p_1^L(t)\right\|_{H_1}^2 \\
&\quad \le \frac{c}{\gamma}\left\|\frac{\partial^2 q_0^L}{\partial t^2} + \nabla_y \frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2} - q_0^L - \nabla_y \mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2 + c\gamma T \sup_{t\in[0,T]}\left\|\frac{\partial p_0^L}{\partial t}(t) + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}(t)\right\|_{H_1}^2 \\
&\qquad + c\left\|\frac{\partial p_0^L}{\partial t}(0) + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}(0)\right\|_{H_1}^2 + c\left\|\operatorname{curl} p_0^L(0) + \operatorname{curl}_y p_1^L(0)\right\|_{H_1}^2.
\end{aligned}$$

As this holds for all $t \in [0, T]$, we have

$$\sup_{t\in[0,T]}\left\|\frac{\partial p_0^L}{\partial t}(t)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(t)\right\|_{H_1}^2 \le \frac{c}{\gamma}\left\|\frac{\partial^2 q_0^L}{\partial t^2}+\nabla_y\frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2}-q_0^L-\nabla_y\mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2$$
$$+c\gamma T\sup_{t\in[0,T]}\left\|\frac{\partial p_0^L}{\partial t}(t)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(t)\right\|_{H_1}^2+c\left\|\frac{\partial p_0^L}{\partial t}(0)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(0)\right\|_{H_1}^2+c\left\|\operatorname{curl}p_0^L(0)+\operatorname{curl}_y p_1^L(0)\right\|_{H_1}^2.$$

Choosing γ such that $c\gamma T = 1/2$, there is a constant c depending on T so that

$$\sup_{t\in[0,T]}\left\|\frac{\partial p_0^L}{\partial t}(t)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(t)\right\|_{H_1}^2$$
$$\le c\left\|\frac{\partial^2 q_0^L}{\partial t^2}+\nabla_y\frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2}-q_0^L-\nabla_y\mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2+c\left\|\frac{\partial p_0^L}{\partial t}(0)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(0)\right\|_{H_1}^2+c\left\|\operatorname{curl}p_0^L(0)+\operatorname{curl}_y p_1^L(0)\right\|_{H_1}^2.$$

Thus for all $t \in (0, T)$

$$\left\|\frac{\partial p_0^L}{\partial t}(t)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(t)\right\|_{H_1}^2+\left\|\operatorname{curl}p_0^L(t)+\operatorname{curl}_y p_1^L(t)\right\|_{H_1}^2$$
$$\le c\left[\left\|\frac{\partial^2 q_0^L}{\partial t^2}+\nabla_y\frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2}-q_0^L-\nabla_y\mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2+\left\|\frac{\partial p_0^L}{\partial t}(0)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(0)\right\|_{H_1}^2+\left\|\operatorname{curl}p_0^L(0)+\operatorname{curl}_y p_1^L(0)\right\|_{H_1}^2\right].$$

Consider Eq. (3.5) for $t = 0$. Let $v_0^L = 0$, $\mathfrak{v}_1^L = 0$ and $v_1^L = p_1^L$. We then have

$$\int_D\int_Y a(x,y)\left(\operatorname{curl}p_0^L(0)+\operatorname{curl}_y p_1^L(0)\right)\cdot\operatorname{curl}_y p_1^L(0)dydx = 0,$$

i.e.,

$$\int_D\int_Y a(x,y)\operatorname{curl}_y p_1^L(0)\cdot\operatorname{curl}_y p_1^L(0)dydx = -\int_D\int_Y a(x,y)\operatorname{curl}p_0^L(0)\cdot\operatorname{curl}_y p_1^L(0)dydx.$$

Using (2.1), we deduce that $\left\|\operatorname{curl}_y p_1^L(0)\right\|_{H_1}\le c\left\|\operatorname{curl}p_0^L(0)\right\|_H$. We then get the conclusion. □

Proposition 3.5. *Assume that* $\frac{\partial^2 \boldsymbol{u}}{\partial t^2}\in L^2((0,T),\mathbf{W})$, *and that*

$$\lim_{L\to\infty}\|g_0^L-g_0\|_W = 0 \quad \text{and} \quad \lim_{L\to\infty}\|g_1^L-g_1\|_H = 0. \tag{3.6}$$

Then

$$\lim_{L\to\infty}\left\{\left\|\frac{\partial(u_0^L-u_0)}{\partial t}\right\|_{L^\infty((0,T),H)}+\left\|\nabla_y\frac{\partial(\mathfrak{u}_1^L-\mathfrak{u}_1)}{\partial t}\right\|_{L^\infty((0,T),H_1)}\right.$$
$$\left.+\left\|\operatorname{curl}(u_0^L-u_0)\right\|_{L^\infty((0,T),H)}+\left\|\operatorname{curl}_y(u_1^L-u_1)\right\|_{L^\infty((0,T),H_1)}\right\}=0.$$

Proof. From Proposition 3.4, as $\boldsymbol{u}^L-\boldsymbol{u}=\boldsymbol{p}^L+\boldsymbol{q}^L$, we have

$$\left\|\frac{\partial(u_0^L-u_0)}{\partial t}+\nabla_y\frac{\partial(\mathfrak{u}_1^L-\mathfrak{u}_1)}{\partial t}\right\|_{L^\infty((0,T),H_1)}^2+\left\|\operatorname{curl}(u_0^L-u_0)+\operatorname{curl}_y(u_1^L-u_1)\right\|_{L^\infty((0,T),H_1)}^2$$
$$\le c\left[\left\|\frac{\partial^2 q_0^L}{\partial t^2}+\nabla_y\frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2}-q_0^L-\nabla_y\mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2+\left\|\frac{\partial p_0^L}{\partial t}(0)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(0)\right\|_{H_1}^2+\left\|\operatorname{curl}p_0^L(0)\right\|_H^2\right]$$
$$+\left\|\frac{\partial q_0^L}{\partial t}+\nabla_y\frac{\partial \mathfrak{q}_1^L}{\partial t}\right\|_{L^\infty((0,T),H_1)}^2+\|\boldsymbol{q}^L\|_{L^\infty((0,T),\mathbf{W})}^2. \tag{3.7}$$

We show that $\lim_{L\to\infty}\|\boldsymbol{q}^L\|_{L^\infty((0,T),\mathbf{W})}=0$. As $\boldsymbol{u}\in C([0,T],\mathbf{W})$, $\boldsymbol{u}$ is uniformly continuous as a function from $[0,T]$ to $\mathbf{W}$. For $\delta>0$, there is a piecewise constant (with respect to t) function $\tilde{\boldsymbol{u}}\in L^\infty((0,T),\mathbf{W})$ such that $\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^\infty((0,T),\mathbf{W})}<\delta$. As $\tilde{\boldsymbol{u}}(t)$ obtains only a finite number of $\mathbf{W}$-values, when L is sufficiently large, there is $\boldsymbol{v}^L\in L^\infty((0,T),\mathbf{W}^L)$ such that $\|\tilde{\boldsymbol{u}}-\boldsymbol{v}^L\|_{L^\infty((0,T),\mathbf{W})}<\delta$. Thus

$$\lim_{L\to\infty}\sup_{t\in(0,T)}\inf_{\boldsymbol{v}^L\in\mathbf{W}^L}\|\boldsymbol{u}(t)-\boldsymbol{v}^L\|_{\mathbf{W}}=0.$$

We then apply Lemma 3.2. Similarly, we have from Lemmas 3.2 and 3.3

$$\lim_{L\to\infty}\left\|\frac{\partial^2 q_0^L}{\partial t^2}+\nabla_y\frac{\partial^2 \mathfrak{q}_1^L}{\partial t^2}-q_0^L-\nabla_y \mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}=0,\quad\text{and}\quad \lim_{L\to\infty}\left\|\frac{\partial q_0^L}{\partial t}+\nabla_y\frac{\partial \mathfrak{q}_1^L}{\partial t}\right\|_{L^\infty((0,T),H_1)}=0.$$

Furthermore, we have that $\left\|\operatorname{curl} p_0^L(0)\right\|_H\le\left\|\operatorname{curl} u_0^L(0)-\operatorname{curl} u_0(0)\right\|_H+\left\|\operatorname{curl} u_0(0)-\operatorname{curl} w_0^L(0)\right\|_H$, which converges to 0 due to (3.6) and Lemma 3.2. Similarly, we have

$$\lim_{L\to\infty}\left\|\frac{\partial p_0^L}{\partial t}(0)+\nabla_y\frac{\partial \mathfrak{p}_1^L}{\partial t}(0)\right\|_{H_1}=0.$$

We then get the conclusion. □

3.2. Fully discrete problem

Following the scheme of Dupont [20], we discretize problem (3.1) in both spatial and temporal variables. Let $\Delta t=\frac{T}{M}$ where M is a positive integer. Let $t_m=m\Delta t$. We employ the following notations of Dupont for a function $r\in C([0,T],X)$ where X is a Banach space and $r_m=r(t_m,\cdot)$

$$\begin{aligned} r_{m+1/2}&=\frac{1}{2}(r_{m+1}+r_m), & r_{m,\theta}&=\theta r_{m+1}+(1-2\theta)r_m+\theta r_{m-1},\\ \partial_t r_{m+1/2}&=(r_{m+1}-r_m)/\Delta t, & \partial_t^2 r_m&=(r_{m+1}-2r_m+r_{m-1})/(\Delta t)^2,\\ \delta_t r_m&=(r_{m+1}-r_{m-1})/(2\Delta t). \end{aligned}$$

We consider the following fully discrete problem:
For $m=1,\ldots,M$ find $\boldsymbol{u}_m^L=(u_{0,m}^L,u_{1,m}^L,\mathfrak{u}_{1,m}^L)\in\mathbf{W}^L$ such that for $m=1,\ldots,M-1$

$$\begin{aligned}&\int_D\int_Y\Big[b(x,y)\left(\partial_t^2 u_{0,m}^L+\nabla_y\partial_t^2\mathfrak{u}_{1,m}^L\right)\cdot\left(v_0^L+\nabla_y\mathfrak{v}_1^L\right)+\\ &a(x,y)\left(\operatorname{curl} u_{0,m,1/4}^L+\operatorname{curl}_y u_{1,m,1/4}^L\right)\cdot\left(\operatorname{curl} v_0^L+\operatorname{curl}_y v_1^L\right)\Big]\,dydx=\int_D f_{m,1/4}(t,x)\cdot v_0^L(x)dx,\end{aligned}\tag{3.8}$$

for all $\boldsymbol{v}^L=(v_0^L,v_1^L,\mathfrak{v}_1^L)\in\mathbf{W}^L$. For continuous functions $r:[0,T]\to X$, let

$$\|r\|_{\tilde{L}^\infty((0,T),X)}:=\max_{0\le m<M}\|r_{m+1/2}\|_X,\quad \|\partial_t r\|_{\tilde{L}^\infty((0,T),X)}:=\max_{0\le m<M}\|\partial_t r_{m+1/2}\|_X.$$

Let $\boldsymbol{p}_m^L:=\boldsymbol{u}_m^L-\boldsymbol{w}_m^L$.

Lemma 3.6. *Assume that $\boldsymbol{u}\in H^2((0,T),\mathbf{W})$, $\frac{\partial^2 q_0^L}{\partial t^2}\in L^2((0,T),H)$, $\frac{\partial^2}{\partial t^2}\nabla_y\mathfrak{q}_1^L\in L^2((0,T),H_1)$. If $\frac{\partial^3 u_0}{\partial t^3}\in L^2((0,T),H)$ and $\frac{\partial^3}{\partial t^3}\nabla_y\mathfrak{u}_1\in L^2((0,T),H_1)$, then there exists a constant c independent of Δt and $\boldsymbol{u}$ such that for each $j=1,2,\ldots,M-1$*

$$\begin{aligned}&\|\partial_t p_{0,j+1/2}^L\|_H^2+\|\partial_t\nabla_y\mathfrak{p}_{1,j+1/2}^L\|_{H_1}^2+\|\operatorname{curl} p_{0,j+1/2}^L\|_H^2+\|\operatorname{curl}_y p_{1,j+1/2}^L\|_{H_1}^2\\ &\quad\le c(\Delta t)^2\left(\left\|\frac{\partial^3 u_0}{\partial t^3}\right\|_H^2+\left\|\frac{\partial^3\nabla_y\mathfrak{u}_1}{\partial t^3}\right\|_{H_1}^2\right)+c\left(\left\|\frac{\partial^2 q_0^L}{\partial t^2}\right\|_{L^2((0,T),H)}^2+\left\|\frac{\partial^2}{\partial t^2}\nabla_y\mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2+\|q_0^L\|_{L^\infty((0,T),H)}^2\right.\\ &\quad\left.+\|\nabla_y\mathfrak{q}_1^L\|_{L^\infty((0,T),H_1)}^2+\|\partial_t p_{0,1/2}^L\|_H^2+\|\partial_t\nabla_y\mathfrak{p}_{1,1/2}^L\|_{H_1}^2+\|\operatorname{curl} p_{0,1/2}^L\|_H^2+\|\operatorname{curl}_y p_{1,1/2}^L\|_{H_1}^2\right).\end{aligned}$$

Further, if $\frac{\partial^4 u_0}{\partial t^4}\in L^2((0,T),H)$ and $\frac{\partial^4}{\partial t^4}\nabla_y\mathfrak{u}_1\in L^2((0,T),H_1)$, then there exists a constant c independent of Δt and $\boldsymbol{u}$ such that for each $j=1,2,\ldots,M-1$

$$\begin{aligned}&\|\partial_t p_{0,j+1/2}^L\|_H^2+\|\partial_t\nabla_y\mathfrak{p}_{1,j+1/2}^L\|_{H_1}^2+\|\operatorname{curl} p_{0,j+1/2}^L\|_H^2+\|\operatorname{curl}_y p_{1,j+1/2}^L\|_{H_1}^2\\ &\le c(\Delta t)^4\left(\left\|\frac{\partial^4 u_0}{\partial t^4}\right\|_H^2+\left\|\frac{\partial^4\nabla_y\mathfrak{u}_1}{\partial t^4}\right\|_{H_1}^2\right)+c\left(\left\|\frac{\partial^2 q_0^L}{\partial t^2}\right\|_{L^2((0,T),H)}^2+\left\|\frac{\partial^2}{\partial t^2}\nabla_y\mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2+\|q_0^L\|_{L^\infty((0,T),H)}^2\right.\\ &\left.+\|\nabla_y\mathfrak{q}_1^L\|_{L^\infty((0,T),H_1)}^2+\|\partial_t p_{0,1/2}^L\|_H^2+\|\partial_t\nabla_y\mathfrak{p}_{1,1/2}^L\|_{H_1}^2+\|\operatorname{curl} p_{0,1/2}^L\|_H^2+\|\operatorname{curl}_y p_{1,1/2}^L\|_{H_1}^2\right).\end{aligned}$$

We prove this Lemma in Appendix B. We then have the following error estimates.

Proposition 3.7. *Assume that $\boldsymbol{u} \in H^2((0,T),\mathbf{W})$. If $\frac{\partial^3 u_0}{\partial t^3} \in L^2((0,T),H)$ and $\frac{\partial^3 \nabla_y \mathfrak{u}_1}{\partial t^3} \in L^2((0,T),H_1)$, then there is a constant c such that*

$$
\begin{aligned}
&\|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty((0,T),H)} + \|\partial_t \nabla_y \mathfrak{u}_1^L - \partial_t \nabla_y \mathfrak{u}_1\|_{\tilde{L}^\infty((0,T),H_1)} \\
&\quad + \|\operatorname{curl} u_0^L - \operatorname{curl} u_0\|_{\tilde{L}^\infty((0,T),H)} + \|\operatorname{curl}_y u_1^L - \operatorname{curl}_y u_1\|_{\tilde{L}^\infty((0,T),H_1)} \\
&\quad \le c \left[\Delta t \left\| \frac{\partial^3 u_0}{\partial t^3} \right\|_{L^2((0,T),H)} + \Delta t \left\| \frac{\partial^3 \nabla_y \mathfrak{u}_1}{\partial t^3} \right\|_{L^2((0,T),H_1)} + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2((0,T),H)} \right. \\
&\quad \left. + \left\| \frac{\partial^2}{\partial t^2} \nabla_y \mathfrak{q}_1^L \right\|_{L^2((0,T),H_1)} + \|q_0^L\|_{L^\infty((0,T),H)} + \|\nabla_y \mathfrak{q}_1^L\|_{L^\infty((0,T),H_1)} \right] \\
&\quad + c \left(\|\partial_t p_{0,1/2}^L\|_H + \|\partial_t \nabla_y \mathfrak{p}_{1,1/2}^L\|_{H_1} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \|\operatorname{curl}_y p_{1,1/2}^L\|_{H_1} \right. \\
&\quad \left. + \|\partial_t q_0^L\|_{\tilde{L}^\infty((0,T),H)} + \|\operatorname{curl} q_0^L\|_{\tilde{L}^\infty((0,T),H)} + \left(\|\partial_t \nabla_y \mathfrak{q}_1^L\|_{\tilde{L}^\infty((0,T),H_1)} + \|\operatorname{curl}_y q_1^L\|_{\tilde{L}^\infty((0,T),H_1)} \right) \right).
\end{aligned}
$$

If $\frac{\partial^4 u_0}{\partial t^4} \in L^2((0,T),H)$ and $\frac{\partial^4 \nabla_y \mathfrak{u}_1}{\partial t^4} \in L^2((0,T),H_1)$, then there is a constant c such that

$$
\begin{aligned}
&\|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty((0,T),H)} + \|\partial_t \nabla_y \mathfrak{u}_1^L - \partial_t \nabla_y \mathfrak{u}_1\|_{\tilde{L}^\infty((0,T),H_1)} \\
&\quad + \|\operatorname{curl} u_0^L - \operatorname{curl} u_0\|_{\tilde{L}^\infty((0,T),H)} + \|\operatorname{curl}_y u_1^L - \operatorname{curl}_y u_1\|_{\tilde{L}^\infty((0,T),H_1)} \\
&\quad \le c \left[(\Delta t)^2 \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_{L^2((0,T),H)} + (\Delta t)^2 \left\| \frac{\partial^4 \nabla_y \mathfrak{u}_1}{\partial t^4} \right\|_{L^2((0,T),H_1)} + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2((0,T),H)} \right. \\
&\quad \left. + \left\| \frac{\partial^2}{\partial t^2} \nabla_y \mathfrak{q}_1^L \right\|_{L^2((0,T),H_1)} + \|q_0^L\|_{L^\infty((0,T),H)} + \|\nabla_y \mathfrak{q}_1^L\|_{L^\infty((0,T),H_1)} \right] \\
&\quad + c \left(\|\partial_t p_{0,1/2}^L\|_H + \|\partial_t \nabla_y \mathfrak{p}_{1,1/2}^L\|_{H_1} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \|\operatorname{curl}_y p_{1,1/2}^L\|_{H_1} \right. \\
&\quad \left. + \|\partial_t q_0^L\|_{\tilde{L}^\infty((0,T),H)} + \|\operatorname{curl} q_0^L\|_{\tilde{L}^\infty((0,T),H)} + \left(\|\partial_t \nabla_y \mathfrak{q}_1^L\|_{\tilde{L}^\infty((0,T),H_1)} + \|\operatorname{curl}_y q_1^L\|_{\tilde{L}^\infty((0,T),H_1)} \right) \right).
\end{aligned}
$$

Proof. We note that $\boldsymbol{u}^L - \boldsymbol{u} = \boldsymbol{p}^L + \boldsymbol{q}^L$. The conclusions follow from Lemma 3.6. □

From this, we deduce

Proposition 3.8. *If $\boldsymbol{u} \in H^2((0,T),\mathbf{W})$, $\frac{\partial^3 u_0}{\partial t^3} \in L^2((0,T),H)$ and $\frac{\partial^3 \nabla_y \mathfrak{u}_1}{\partial t^3} \in L^2((0,T),H_1)$, and if we choose $\boldsymbol{u}_0^L = (u_{0,0}^L, u_{1,0}^L, \mathfrak{u}_{1,0}^L)$ and $\boldsymbol{u}_1^L = (u_{0,1}^L, u_{1,1}^L, \mathfrak{u}_{1,1}^L)$ such that*

$$
\begin{aligned}
\lim_{L\to 0} \Big(& \|\partial_t u_{0,1/2}^L - \partial_t w_{0,1/2}^L\|_H + \|\partial_t \nabla_y \mathfrak{u}_{1,1/2}^L - \partial_t \nabla_y \mathfrak{w}_{1,1/2}^L\|_{H_1} + \|\operatorname{curl} u_{0,1/2}^L - \operatorname{curl} w_{0,1/2}^L\|_H \\
& + \|\operatorname{curl}_y u_{1,1/2}^L - \operatorname{curl}_y w_{1,1/2}^L\|_{H_1} \Big) = 0,
\end{aligned}
$$

i.e

$$
\lim_{L\to 0} \left(\|\partial_t p_{0,1/2}^L\|_H + \|\partial_t \nabla_y \mathfrak{p}_{1,1/2}^L\|_{H_1} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \|\operatorname{curl}_y p_{1,1/2}^L\|_{H_1} \right) = 0,
$$

then

$$
\begin{aligned}
\lim_{L\to\infty} \Big(& \|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty((0,T),H)} + \|\partial_t \nabla_y \mathfrak{u}_1^L - \partial_t \nabla_y \mathfrak{u}_1\|_{\tilde{L}^\infty((0,T),H_1)} \\
& + \|\operatorname{curl} u_0^L - \operatorname{curl} u_0\|_{\tilde{L}^\infty((0,T),H)} + \|\operatorname{curl}_y u_1^L - \operatorname{curl}_y u_1\|_{\tilde{L}^\infty((0,T),H_1)} \Big) = 0.
\end{aligned}
$$

Proof. From the hypothesis and Lemma 3.3, we have that

$$
\lim_{L\to\infty} \left\| \frac{\partial^2 \boldsymbol{q}^L}{\partial t^2} \right\|_{L^2((0,T),\mathbf{W})} = 0.
$$

As $\boldsymbol{u} \in C([0,T],\mathbf{W})$, from the proof of Proposition 3.5 $\lim_{L\to\infty} \|\boldsymbol{q}^L\|_{L^\infty((0,T),\mathbf{W})} = 0$. We have that

$$
\|\boldsymbol{q}^L\|_{\tilde{L}^\infty((0,T),\mathbf{W})} \le \|\boldsymbol{q}^L\|_{L^\infty((0,T),\mathbf{W})}
$$

so

$$\lim_{L\to\infty} \|\boldsymbol{q}^L\|_{\tilde{L}^\infty((0,T),\mathbf{W})} = 0.$$

Further, from (3.4), we have that

$$B(\partial_t \boldsymbol{w}^L_{m+1/2} - \partial_t \boldsymbol{u}_{m+1/2}, \boldsymbol{v}^L) + A(\partial_t \boldsymbol{w}^L_{m+1/2} - \partial_t \boldsymbol{u}_{m+1/2}, \boldsymbol{v}^L) = 0$$

for all $\boldsymbol{v}^L \in \mathbf{W}^L$. We thus have

$$\|\partial_t \boldsymbol{w}^L_{m+1/2} - \partial_t \boldsymbol{u}_{m+1/2}\|_{\mathbf{W}} \le c \inf_{\boldsymbol{v}^L\in\mathbf{W}^L} \|\boldsymbol{v}^L - \partial_t \boldsymbol{u}_{m+1/2}\|_{\mathbf{W}}.$$

As $\partial_t \boldsymbol{u}_{m+1/2} = \frac{\partial \boldsymbol{u}}{\partial t}(\xi)$ for $\xi \in (0, T)$, we deduce that

$$\|\partial_t \boldsymbol{q}^L\|_{\tilde{L}^\infty((0,T),\mathbf{W})} \le c \sup_{t\in(0,T)} \inf_{\boldsymbol{v}^L\in\mathbf{W}^L} \|\boldsymbol{v}^L - \frac{\partial \boldsymbol{u}}{\partial t}(t)\|_{\mathbf{W}}.$$

As $\frac{\partial \boldsymbol{u}}{\partial t} \in C([0,T],\mathbf{W})$, a proof identical to that for $\|\boldsymbol{q}^L\|_{L^\infty((0,T),\mathbf{W})}$ in Proposition 3.5 shows that

$$\lim_{L\to\infty} \|\partial_t \boldsymbol{q}^L\|_{\tilde{L}^\infty((0,T),\mathbf{W})} = 0.$$

We thus get the conclusion. □

4. Regularity of the solution

To derive an explicit error estimate for the full and sparse tensor product FE approximating problems in the next section, we now establish the regularity of u_0, u_1 and $\nabla_y \mathrm{u}_1$ with respect to t. The functions u_1 and u_1 can be written in terms of u_0 from the solution of the cell problems. Let $w^k \in V_1$ be the solution of the cell problem

$$\nabla_y \cdot (b(x,y)(e^k + \nabla_y w^k)) = 0 \tag{4.1}$$

where e^k is the kth unit vector with every component equals 0, except the kth component which equals 1. The homogenized coefficient b^0 is defined as

$$b^0_{pq}(x,y) = \int_y b_{kl}(x,y)\left(\delta_{ql} + \frac{\partial w^q}{\partial y_l}\right)\left(\delta_{pk} + \frac{\partial w^p}{\partial y_k}\right) dy. \tag{4.2}$$

Let $N^k \in W_1$ be the solution of the cell problem

$$\mathrm{curl}_y(a(x,y)(e^k + \mathrm{curl}_y N^k)) = 0. \tag{4.3}$$

The coefficient a^0 is defined as

$$a^0_{pq}(x,y) = \int_Y a_{kl}(x,y)\left(\delta_{ql} + (\mathrm{curl}_y N^q)_l\right)\left(\delta_{pk} + (\mathrm{curl}_y N^p)_k\right) dy. \tag{4.4}$$

From (2.7), u_0 satisfies the homogenized equation

$$b^0(x)\frac{\partial^2 u_0}{\partial t^2}(t,x) + \mathrm{curl}\,(a^0(x)\mathrm{curl}\, u_0(t,x)) = f(t,x); \tag{4.5}$$

and u_1 and u_1 can be written as

$$u_1 = N^r(\mathrm{curl}\, u_0)_r, \quad \nabla_y \mathrm{u}_1 = u_{0r}\nabla_y w^r - g_{1r_0}\nabla_y w^r t - g_{0r}\nabla_y w^r. \tag{4.6}$$

We make the following assumption.

Assumption 4.1. The matrix functions a and b belong to $C^1(\bar{D}, C^2(\bar{Y}))^{d\times d}$.

With this assumption, we have

Proposition 4.2. *Under Assumption 4.1, for all $r = 1,\dots,d$, $\mathrm{curl}_y N^r \in C^1(\bar{D}, H^2(Y))^3 \subset C^1(\bar{D}, C(\bar{Y}))^3$ and $w^r \in C^1(\bar{D}, H^3(Y))^3$.*

We refer to [18] for a proof of this proposition. We have the following regularity results for the solution u_0 of the homogenized equation (4.5).

Proposition 4.3. *Under Assumption 4.1, assume*

$$\begin{cases} f \in H^2((0,T),H), \\ g_1 \in W, \\ (b^0)^{-1}[f(0) - \mathrm{curl}\,(a^0(x)\mathrm{curl}\, g_0)] \in W, \\ (b^0)^{-1}[\frac{\partial f}{\partial t}(0) - \mathrm{curl}\,(a^0(x)\mathrm{curl}\, g_1)] \in H, \end{cases} \tag{4.7}$$

then

$$\frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T),W), \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty((0,T),H), \quad \textit{and} \quad \frac{\partial^3}{\partial t^3}\nabla_y \mathrm{u}_1 \in L^\infty((0,T),H_1). \tag{4.8}$$

Further, if

$$\begin{cases} f \in H^3((0,T),H), \\ g_1 \in W, \\ (b^0)^{-1}[f(0) - \mathrm{curl}\,(a^0(x)\mathrm{curl}\, g_0)] \in W, \\ (b^0)^{-1}[\frac{\partial f}{\partial t}(0) - \mathrm{curl}\,(a^0(x)\mathrm{curl}\, g_1)] \in W, \\ (b^0)^{-1}[\frac{\partial^2 f}{\partial t^2}(0) - \mathrm{curl}\,(a^0(x)\mathrm{curl}\,((b_0)^{-1}(f(0) - \mathrm{curl}\,(a^0(x)\mathrm{curl}\, g_0))))] \in H, \end{cases} \tag{4.9}$$

then

$$\frac{\partial^3 u_0}{\partial t^3} \in L^\infty((0,T),W), \quad \frac{\partial^4 u_0}{\partial t^4} \in L^\infty((0,T),H), \quad \textit{and} \quad \frac{\partial^4}{\partial t^4}\nabla_y \mathrm{u}_1 \in L^\infty((0,T),H_1). \tag{4.10}$$

Proof. We use the regularity theory of general hyperbolic equations (see, e.g., Wloka [21], Chapter 5). From (4.7) we have

$$b^0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial t}\right) + \mathrm{curl}\left(a^0 \mathrm{curl}\,\frac{\partial u_0}{\partial t}\right) = \frac{\partial f}{\partial t} \tag{4.11}$$

with compatibility initial conditions

$$\frac{\partial u_0}{\partial t}(0) = g_1 \in W, \quad \frac{\partial}{\partial t}\frac{\partial u_0}{\partial t}(0) = (b^0)^{-1}[f(0) - \mathrm{curl}\,(a^0 \mathrm{curl}\, g_0)] \in W$$

and

$$b^0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u_0}{\partial t^2}\right) + \mathrm{curl}\left(a^0 \mathrm{curl}\,\frac{\partial^2 u_0}{\partial t^2}\right) = \frac{\partial^2 f}{\partial t^2}, \tag{4.12}$$

with compatibility initial conditions

$$\frac{\partial^2 u_0}{\partial t^2}(0) = (b^0)^{-1}[f(0) - \mathrm{curl}\,\left(a^0 \mathrm{curl}\, g_0\right)] \in W \ \text{ and } \ \frac{\partial}{\partial t}\frac{\partial^2 u_0}{\partial t^2}(0) = (b^0)^{-1}[\frac{\partial f}{\partial t}(0) - \mathrm{curl}\,(a^0 \mathrm{curl}\, g_1)] \in H.$$

We thus deduce that

$$\frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T),W) \ \text{ and } \ \frac{\partial^3 u_0}{\partial t^3} \in L^\infty((0,T),H).$$

From (4.6) and Proposition 4.2, we deduce that

$$\frac{\partial^3}{\partial t^3}\nabla_y \mathrm{u}_1 \in L^\infty((0,T),H_1).$$

Similarly, we deduce regularity (4.10) from (4.9). □

To derive explicitly the rate of convergence for the full and sparse tensor FE approximations in the next section, we define the following regularity spaces. Let $\bar{\mathcal{H}}^1$ be the space of functions belonging to $L^2(D, H^1_\#(\mathrm{curl}, Y)) \bigcap H^1(D, \tilde{H}_\#(\mathrm{curl}, Y))$. For $\frac{1}{2} < s < 1$, by interpolation, we define the space $\bar{\mathcal{H}}^s$ which consists of functions w that belong to $L^2(D, H^s_\#(\mathrm{curl}, Y)) \bigcap H^s(D, \tilde{H}_\#(\mathrm{curl}, Y))$. We equip $\bar{\mathcal{H}}^s$ with the norm

$$\|w\|_{\bar{\mathcal{H}}^s} = \|w\|_{L^2(D, H^s_\#(\mathrm{curl}, Y))} + \|w\|_{H^s(D, \tilde{H}_\#(\mathrm{curl}, Y))}.$$

We define $\bar{\mathfrak{H}}^s$ as the space of functions $w \in L^2(D, H^{1+s}_\#(Y)) \bigcap H^s(D, H^1_\#(Y))$. We equip this space with the norm

$$\|w\|_{\bar{\mathfrak{H}}^s} = \|w\|_{L^2(D, H^{1+s}_\#(Y))} + \|w\|_{H^s(D, H^1_\#(Y))}.$$

We define the regularity space $\bar{\boldsymbol{\mathcal{H}}}^s$ as

$$\bar{\boldsymbol{\mathcal{H}}}^s = H^s(\mathrm{curl}, D) \times \bar{\mathcal{H}}^s \times \bar{\mathfrak{H}}^s.$$

We define $\hat{\mathcal{H}}^1$ as the space of functions $w \in L^2(D, H^1_\#(\mathrm{curl}, Y))$ which are periodic with respect to y with the period being Y such that for any $\alpha_0 \in \mathbb{R}^d$ with $|\alpha_0| \le 1$,

$$\frac{\partial^{|\alpha_0|}}{\partial x^{\alpha_0}} w \in L^2(D, H^1_\#(\mathrm{curl}, Y)).$$

We equip $\hat{\mathcal{H}}^1$ with the norm

$$\|w\|_{\hat{\mathcal{H}}^1} = \sum_{\alpha_0 \in \mathbb{R}^d, |\alpha_0| \le 1} \left\| \frac{\partial^{|\alpha_0|}}{\partial x^{\alpha_0}} w \right\|_{L^2(D, H^1_\#(\mathrm{curl}, Y))}.$$

We can write $\hat{\mathcal{H}}^1$ as $H^1(D, H^1_\#(\mathrm{curl}, Y))$. By interpolation, we define $\hat{\mathcal{H}}^s = H^s(D, H^s_\#(\mathrm{curl}, Y))$ for $\frac{1}{2} < s < 1$.

We define $\hat{\mathfrak{H}}^1$ as the space of functions $w \in L^2(D, H^2_\#(Y))$ that are periodic with respect to y with the period being Y such that for $\alpha_0 \in \mathbb{R}^d$with $|\alpha_0| \le 1$,

$$\frac{\partial^{|\alpha_0|}}{\partial x^{\alpha_0}} w \in L^2(D, H^2_\#(Y)).$$

The space $\hat{\mathfrak{H}}^1$ is equipped with the norm

$$\|w\|_{\hat{\mathfrak{H}}^1} = \sum_{\alpha_0 \in \mathbb{R}^d, |\alpha_0| \le 1} \left\| \frac{\partial^{|\alpha_0|}}{\partial x^{\alpha_0}} w \right\|_{L^2(D, H^2_\#(Y))}.$$

We can write $\hat{\mathfrak{H}}^1$ as $H^1(D, H^2_\#(Y))$. By interpolation, we define the space $\hat{\mathfrak{H}}^s := H^s(D, H^{1+s}_\#(Y))$. The regularity space $\hat{\boldsymbol{\mathcal{H}}}^s$ is defined as

$$\hat{\boldsymbol{\mathcal{H}}}^s = H^s(\mathrm{curl}, D) \times \hat{\mathcal{H}}^s \times \hat{\mathfrak{H}}^s.$$

For the regularity of u_0, we have the following result.

Proposition 4.4. *Under Assumption 4.1, if D is a Lipschitz polyhedral domain, $f \in H^1((0,T), H)$, $g_0 \in H^1(\mathrm{curl}, D)$ and $g_1 \in W$, $\mathrm{div} f \in L^\infty((0,T), L^2(D))$, $\mathrm{div}(b^0 g_0) \in L^2(D)$ and $\mathrm{div}(b^0 g_1) \in L^2(D)$, there is a constant $s \in (\frac{1}{2}, 1]$ such that $u_0 \in L^\infty((0,T), H^s(\mathrm{curl}, D))$.*

Proof. Using Proposition 4.2, Eqs. (4.4) and (4.2), we have that $a^0, b^0 \in C^1(\bar{D})^{d\times d}$. As $f \in H^1((0,T), H)$ and $g_0 \in H^1(\mathrm{curl}, D)$, we have that $(b^0)^{-1}[f - \mathrm{curl}\,(a^0 \mathrm{curl}\, g^0)] \in H$. The compatibility initial conditions hold so that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T), H)$. Thus

$$\mathrm{curl}\,(a^0 \mathrm{curl}\, u_0) = f - b^0 \frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T), H).$$

Let $U(t) = a^0 \mathrm{curl}\, u_0(t)$. As $\mathrm{div}((a^0)^{-1} U(t)) = 0$ and $(a^0)^{-1} U(t) \cdot \nu = 0$, there is a constant c and a constant $s \in (\frac{1}{2}, 1]$ which depend on a^0 and the domain D so that

$$\|U(t)\|_{H^s(D)^3} \le c(\|\mathrm{curl}\, U(t)\|_{L^2(D)^3} + \|U(t)\|_{L^2(D)^3})$$

so $U \in L^\infty((0,T), H^s(D)^3)$. As $\mathrm{curl}\, u_0(t) = (a^0)^{-1} U(t)$ and $(a^0)^{-1} \in C^1(\bar{D})^{d\times d}$, $\mathrm{curl}\, u_0 \in L^\infty((0,T), H^s(D))$. We note that

$$\mathrm{div}\left(b^0 \frac{\partial^2 u_0}{\partial t^2} \right) = \mathrm{div} f,$$

so

$$\mathrm{div}(b^0 u_0(t)) = \int_0^t \int_0^s \mathrm{div} f(r) dr ds + t\,\mathrm{div}(b^0 g_1) + \mathrm{div}(b^0 g_0) \in L^\infty((0,T), L^2(D)).$$

From Theorem 4.1 of Hiptmair [26], we deduce that there is a constant $s \in (\frac{1}{2}, 1]$ (we take it as the same constant as above), so that

$$\|u_0(t)\|_{H^s(D)^3} \le c(\|u_0(t)\|_{H(\mathrm{curl}, D)} + \|\mathrm{div}(b^0 u_0(t))\|_{L^2(D)}).$$

Thus $u_0 \in L^\infty((0,T), H^s(\mathrm{curl}, D))$. □

Similarly, we can deduce the regularity for $\frac{\partial^2 u_0}{\partial t^2}$.

Proposition 4.5. *Under Assumption 4.1, if D is a Lipschitz polyhedral domain, if the compatibility conditions (4.9) hold, and if $\mathrm{div} f \in L^\infty((0,T), L^2(D))$, then there is a constant $s \in (\frac{1}{2}, 1]$ such that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T), H^s(\mathrm{curl}, D))$.*

Proof. From Eq. (4.12), we have

$$\operatorname{curl}\left(a^0 \operatorname{curl}\frac{\partial^2 u_0}{\partial t^2}\right) = \frac{\partial^2 f}{\partial t^2} - b^0\frac{\partial^4 u_0}{\partial t^4} \in L^\infty((0,T),H)$$

as $\frac{\partial^4 u_0}{\partial t^4} \in L^\infty((0,T),H)$ due to (4.9). Following a similar argument as in the proof of Proposition 4.4 we deduce that $\operatorname{curl}\frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T),H^s(D)^3)$. We note that

$$\operatorname{div} b^0\frac{\partial^2 u_0}{\partial t^2} = \operatorname{div} f \in L^\infty((0,T),L^2(D)).$$

From Theorem 4.1 of [26], we deduce that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T),H^s(D)^3)$. □

From these we deduce

Proposition 4.6. *Under Assumption 4.1, and the hypothesis of Proposition 4.4, there is a constant $s \in (\frac{1}{2},1]$ so that $\boldsymbol{u} \in L^\infty((0,T),\hat{\boldsymbol{\mathcal{H}}}^s)$.*

Proof. From Proposition 4.2, we have $\operatorname{curl} N_1^r$ belongs to $C^1(\bar{D},H^2_\#(Y))$. Together with $u_0 \in L^\infty((0,T),H^s(\operatorname{curl},D))$, this implies $u_1 \in L^\infty((0,T),\hat{\mathcal{H}}^s)$. Similarly, we have $u_1 \in L^\infty((0,T),\hat{\mathfrak{H}}_1^s)$. □

Similarly, we have:

Proposition 4.7. *Under Assumption 4.1, and the hypothesis of Proposition 4.5, there is a constant $s \in (\frac{1}{2},1]$ so that $\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^\infty((0,T),\hat{\boldsymbol{\mathcal{H}}}^s)$.*

Remark 4.8. We have

$$\operatorname{curl}\frac{\partial u_0(t)}{\partial t} = \int_0^t \operatorname{curl}\frac{\partial^2 u_0}{\partial t^2}(s)ds + \operatorname{curl} g_1, \quad \text{and} \quad \frac{\partial u_0(t)}{\partial t} = \int_0^t \frac{\partial^2 u_0}{\partial t^2}(s)ds + g_1,$$

$$\operatorname{curl} u_0(t) = \int_0^t\int_0^s \operatorname{curl}\frac{\partial^2 u_0}{\partial t^2}(r)drds + t\operatorname{curl} g_1 + \operatorname{curl} g_0, \quad \text{and} \quad u_0(t) = \int_0^t\int_0^s \frac{\partial^2 u_0}{\partial t^2}(r)drds + tg_1 + g_0.$$

Thus with the hypothesis of Proposition 4.5, together with $g_0 \in H^s(\operatorname{curl},D)$ and $g_1 \in H^s(\operatorname{curl},D)$, we deduce that $\frac{\partial u_0}{\partial t} \in L^\infty((0,T),H^s(\operatorname{curl},D))$ and $u_0 \in L^\infty((0,T),H^s(\operatorname{curl},D))$. This implies also that $\boldsymbol{u} \in L^\infty((0,T),\hat{\boldsymbol{\mathcal{H}}}^s)$.

5. Full and sparse tensor product approximations

We consider the approximations of problem (2.7) using the full and sparse tensor product FEs. For a polyhedral domain D in $\mathbb{R}^3$, let $\mathcal{T}^l$ ($l = 0,1,\dots$) be the sets of regular simplices in D with mesh size $h_l = O(2^{-l})$ which are determined recursively where $\mathcal{T}^{l+1}$ is obtained from $\mathcal{T}^l$ by dividing each simplex in $\mathcal{T}^l$ into 8 tedrahedra. For a tedrahedron $T \in \mathcal{T}^l$, we consider the edge FE space

$$R(T) = \{v: \quad v = \alpha + \beta\times x, \quad \alpha,\beta \in \mathbb{R}^3\}.$$

When D is a polygon in $\mathbb{R}^2$, $\mathcal{T}^{l+1}$ is obtained from $\mathcal{T}^l$ by dividing each simplex in $\mathcal{T}^l$ into 4 congruent triangles. For each triangle $T \in \mathcal{T}^l$, we consider the edge FE space

$$R(T) = \left\{v: \quad v = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} + \beta\begin{pmatrix}x_2\\ -x_1\end{pmatrix}, \ \alpha_1,\alpha_2,\beta \in \mathbb{R}\right\}.$$

Alternatively, when D is partitioned into cubic meshes, we can use edge FE on cubic mesh instead (see [19]). For each simplex $T \in \mathcal{T}^l$, we denote by $\mathcal{P}_1(T)$ the set of linear polynomials in $\mathcal{T}$. In the following, we only present the analysis for the three dimensional case as the two dimensional case is similar.

We define the FE spaces

$$W^l = \{v \in H_0(\operatorname{curl},D),\ v|_T \in R(T)\ \forall T \in \mathcal{T}^l\}, \quad V^l = \{v \in H^1(D),\ v|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}^l\}.$$

For the cube Y, we consider a hierarchy of simplices $\mathcal{T}^l_\#$ that are distributed periodically. We consider the space of functions

$$W^l_\# = \{v \in H_\#(\operatorname{curl},Y),\ v|_T \in R(T)\ \forall T \in \mathcal{T}^l_\#\}, \quad V^l_\# = \{v \in H^1_\#(Y),\ v|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}^l_\#\}.$$

We then have the following standard estimates (see Monk [19] and Ciarlet [27]). For $\frac{1}{2} < s \le 1$

$$\inf_{v_l \in W^l} \|v - v_l\|_{H(\mathrm{curl},D)} \le c h_l^s (\|v\|_{H^s(D)^d} + \|\mathrm{curl}\, v\|_{H^s(D)^d}), \quad \forall v \in H_0(\mathrm{curl}, D) \bigcap H^s(\mathrm{curl}, D),$$

$$\inf_{v_l \in W^l_\#} \|v - v_l\|_{H_\#(\mathrm{curl},Y)} \le c h_l^s (\|v\|_{H^s(Y)^d} + \|\mathrm{curl}\, v\|_{H^s(Y)^d}), \quad \forall v \in H_\#(\mathrm{curl}, Y) \bigcap H^s(\mathrm{curl}, Y).$$

For $0 \le s \le 1$

$$\inf_{v_l \in V^l} \|v - v_l\|_{L^2(D)} \le c h_l^s \|v\|_{H^s(D)}, \quad \forall v \in H^s(D).$$

5.1. Full tensor product FEs

As $L^2(D, \tilde{H}_\#(\mathrm{curl}, Y)) \cong L^2(D) \otimes \tilde{H}_\#(\mathrm{curl}, Y)$ we use the tensor product FE space $\bar{W}^l_1 = V^l \otimes W^l_\#$ to approximate u_1. Similarly, as $\mathrm{u}_1 \in L^2(D, H^1_\#(Y))$, we use the FE space $\bar{V}^l_1 = V^l \otimes V^l_\#$ to approximate u_1. We define the space

$$\bar{\mathbf{W}}^L = W^L \otimes \bar{W}^L_1 \otimes \bar{V}^L_1.$$

The spatially semidiscrete full tensor product FE approximating problem is: Find $\bar{\boldsymbol{u}}^L(t) \in \bar{\mathbf{W}}^L$ so that for all $\bar{\boldsymbol{v}}^L \in \bar{\mathbf{W}}^L$:

$$\int_D \int_Y b(x, y) \Bigg[\left(\frac{\partial^2 \bar{u}^L_0}{\partial t^2}(t) + \frac{\partial^2}{\partial t^2} \nabla_y \bar{\mathrm{u}}^L_1(t) \right) \cdot \left(\bar{v}^L_0 + \nabla_y \bar{\mathrm{v}}^L_1 \right) dydx$$
$$+ a(x, y) \left(\mathrm{curl}\, \bar{u}^L_0(t) + \mathrm{curl}_y \bar{u}^L_1(t) \right) \cdot \left(\mathrm{curl}\, \bar{v}^L_0 + \mathrm{curl}_y \bar{v}^L_1 \right) \Bigg] dydx = \int_D f(t, x) \cdot \bar{v}^L_0(x) dx. \tag{5.1}$$

To deduce an error estimate for the full tensor product approximation of (2.7), we note the following approximations

Lemma 5.1. *For $w \in \bar{\mathcal{H}}^s$ with $s \in (\frac{1}{2}, 1]$,*

$$\inf_{w^l \in \bar{W}^l_1} \|w - w^l\|_{L^2(D, \tilde{H}_\#(\mathrm{curl},Y))} \le c h_l^s \|w\|_{\bar{\mathcal{H}}^s}.$$

For $w \in \bar{\mathfrak{H}}^s$, with $s \in [0, 1]$

$$\inf_{w^l \in \bar{V}^l_1} \|w - w^l\|_{L^2(D, H^1_\#(Y))} \le c h_l^s \|w\|_{\bar{\mathfrak{H}}^s}.$$

The proofs of these results are similar to those for full tensor product FEs in [11] and [28], using orthogonal projection. We refer to [11] and [28] for details. From this we deduce that for $\boldsymbol{w} \in \bar{\boldsymbol{\mathcal{H}}}^s$ for $s \in (\frac{1}{2}, 1]$

$$\inf_{\boldsymbol{w}^L \in \bar{\mathbf{W}}^L} \|\boldsymbol{w} - \boldsymbol{w}^L\|_{\mathbf{W}} \le c h_L^s \|\boldsymbol{w}\|_{\bar{\boldsymbol{\mathcal{H}}}^s}.$$

We then have the following result for the spatially semidiscrete approximation.

Proposition 5.2. *Assume that condition (4.9) and Assumption 4.1 hold, D is a Lipschitz polyhedral domain, $\mathrm{div} f \in L^\infty((0, T), L^2(D))$ and g_0, g_1 belong to $H^s(\mathrm{curl}, D)$. If g^L_0 and g^L_1 are chosen so that*

$$\|g^L_0 - g_0\|_W \le c h^s_L \text{ and } \|g^L_1 - g_1\|_H \le c h^s_L, \tag{5.2}$$

where $s \in (\frac{1}{2}, 1]$ is the constant in Proposition 4.5. Then

$$\left\| \frac{\partial (\bar{u}^L_0 - u_0)}{\partial t} \right\|_{L^\infty((0,T),H)} + \left\| \nabla_y \frac{\partial (\bar{\mathrm{u}}^L_1 - \mathrm{u}_1)}{\partial t} \right\|_{L^\infty((0,T),H_1)}$$
$$+ \|\mathrm{curl}(\bar{u}^L_0 - u_0)\|_{L^\infty((0,T),H)} + \|\mathrm{curl}_y(\bar{u}^L_1 - u_1)\|_{L^\infty((0,T),H_1)} \le c h^s_L.$$

Proof. From Proposition 4.7 and Remark 4.8, we deduce that $\boldsymbol{u} \in L^\infty((0, T), \bar{\boldsymbol{\mathcal{H}}}^s)$, $\frac{\partial \boldsymbol{u}}{\partial t} \in L^\infty((0, T), \bar{\boldsymbol{\mathcal{H}}}^s)$ and $\frac{\partial^2 \boldsymbol{u}}{\partial t^2} \in L^\infty((0, T), \bar{\boldsymbol{\mathcal{H}}}^s)$. From Lemmas 3.2 and 3.3, we have

$$\|\boldsymbol{q}^L\|_{L^\infty((0,T),\mathbf{W})} \le c h^s_L, \quad \left\| \frac{\partial \boldsymbol{q}^L}{\partial t} \right\|_{L^\infty((0,T),\mathbf{W})} \le c h^s_L, \quad \text{and} \quad \left\| \frac{\partial^2 \boldsymbol{q}^L}{\partial t^2} \right\|_{L^2((0,T),\mathbf{W})} \le c h^s_L. \tag{5.3}$$

These together with

$$\frac{\partial p^L_0}{\partial t}(0) = \frac{\partial}{\partial t}(u^L_0(0) - u_0(0)) - \frac{\partial q^L_0}{\partial t}, \quad \nabla_y \frac{\partial \mathrm{p}^L_1}{\partial t}(0) = \nabla_y \frac{\partial}{\partial t}(\mathrm{u}^L_1(0) - \mathrm{u}_1(0)) - \nabla_y \frac{\partial \mathrm{q}^L_1}{\partial t}(0),$$
$$\mathrm{curl}\, p^L_0(0) = \mathrm{curl}\,(u^L_0(0) - u_0(0)) - \mathrm{curl}\, q^L_0(0),$$

and (5.2), we have that

$$\left\| \frac{\partial p_0^L}{\partial t}(0) + \nabla_y \frac{\partial \mathfrak{p}_1^L}{\partial t}(0) \right\|_{H_1} \le c h_L^s,$$

and $\|\mathrm{curl}\, p_0^L(0)\|_H \le c h_L^s$. Thus the right hand side of (3.7) is not more than $c h_L^{2s}$. We thus get the conclusion. □

The fully discrete problem now becomes: For $m = 1, \ldots, M$ find $\bar{\boldsymbol{u}}_m^L = (\bar{u}_{0,m}^L, \bar{u}_{1,m}^L, \bar{\mathfrak{u}}_{1,m}^L) \in \bar{\mathbf{W}}^L$ such that for $m = 1, \ldots, M-1$

$$\begin{aligned} &\int_D \int_Y \left[b(x,y) \left(\partial_t^2 \bar{u}_{0,m}^L + \nabla_y \partial_t^2 \bar{\mathfrak{u}}_{1,m}^L \right) \cdot \left(\bar{v}_0^L + \nabla_y \bar{\mathfrak{v}}_1^L \right) \right. \\ &\quad \left. + a(x,y) \left(\mathrm{curl}\, \bar{u}_{0,m,1/4}^L + \mathrm{curl}_y \bar{u}_{1,m,1/4}^L \right) \cdot \left(\mathrm{curl}\, \bar{v}_0^L + \mathrm{curl}_y \bar{v}_1^L \right) \right] dy dx = \int_D f_{m,1/4}(x) \cdot \bar{v}_0^L(x) dx \end{aligned} \tag{5.4}$$

for all $\bar{\boldsymbol{v}}^L = (\bar{v}_0^L, \bar{v}_1^L, \bar{\mathfrak{v}}_1^L) \in \bar{\mathbf{W}}^L$.

Proposition 5.3. *Assume that condition (4.9) and Assumption 4.1 hold, D is a Lipschitz polyhedral domain, $\mathrm{div} f \in L^\infty((0,T), L^2(D))$ and g_0, g_1 belong to $H^s(\mathrm{curl}, D)$ where $s \in (\frac{1}{2}, 1]$ is the constant in Proposition 4.5. If the initial value $\bar{\boldsymbol{u}}_1^L$ is chosen so that*

$$\|\partial_t p_{0,1/2}^L\|_H + \|\partial_t \nabla_y \mathfrak{p}_{1,1/2}^L\|_{H_1} + \|\mathrm{curl}\, p_{0,1/2}^L\|_H + \|\mathrm{curl}_y p_{1,1/2}^L\|_{H_1} \le c((\Delta t)^2 + h_L^s),$$

then

$$\begin{aligned} &\|\partial_t \bar{u}_0^L - \partial_t u_0\|_{\tilde{L}^\infty((0,T),H)} + \left\| \partial_t \nabla_y (\bar{\mathfrak{u}}_1^L - \mathfrak{u}_1) \right\|_{\tilde{L}^\infty((0,T),H_1)} \\ &\quad + \|\mathrm{curl}\, (\bar{u}_0^L - u_0)\|_{\tilde{L}^\infty((0,T),H)} + \|\mathrm{curl}_y (\bar{u}_1^L - u_1)\|_{\tilde{L}^\infty((0,T),H_1)} \le c((\Delta t)^2 + h_L^s). \end{aligned}$$

Proof. The proof is similar to that of Proposition 5.2. We note that

$$\|\partial_t q_{0,m+1/2}^L\|_H = \left\| \frac{q_{0,m+1}^L - q_{0,m}^L}{\Delta t} \right\|_H \le \sup_{t \in (0,T)} \left\| \frac{\partial q_0^L}{\partial t}(t) \right\|_H \le c h_L^s$$

due to (5.3). Similarly $\|\partial_t \nabla_y \mathfrak{q}_{1,m+1/2}^L\|_{H_1} \le c h_L^s$. We then get the conclusion. □

5.2. Sparse tensor product FEs

The dimension of the full tensor product FE space $\bar{\mathbf{W}}^L$ is $O(2^{2dL})$ which is prohibitively large when L is large. We construct below the sparse tensor product FE space with dimension $O(L2^{dL})$ which produces an essentially equal accuracy as for the full tensor product FE spaces, i.e. the FE error differs from that obtained from the full tensor product FE space by only a logarithmic multiplying factor. For the sparse tensor product FE approximation, we require more regularity for u_1 and $\mathfrak{u}_1$ than for the full tensor product FEs, in particular, we require that these functions possess the necessary regularity with respect to x and y at the same time. We note that for the construction of the full tensor product FEs above, we take the tensor product of all the basis functions of V^L and $W_\#^L$ to construct the FE space $\bar{W}_1^L$ for u_1, and similarly for the space $\bar{V}_1^L$. For the sparse tensor product FEs, each basis function in V^L is multiplied by the basis functions of only a linear subspace of $W_\#^L$, thus reducing substantially the dimension. To achieve this goal, we employ the following orthogonal projection $P^l : L^2(D) \to V^l$, with the convention $P^{-1} = 0$. The detail spaces are defined as

$$\mathcal{V}^l = (P^l - P^{(l-1)}) V^l.$$

We note that $V^l = \bigoplus_{0 \le l_0 \le l} \mathcal{V}^{l_0}$. Therefore the full tensor product FE spaces $\bar{W}_1^L$ and $\bar{V}_1^L$ can be written as

$$\bar{W}_1^L = \left(\bigoplus_{0 \le l_0 \le L} \mathcal{V}^{l_0} \right) \otimes W_\#^L, \quad \text{and} \quad \bar{V}_1^L = \left(\bigoplus_{0 \le l_0 \le L} \mathcal{V}^{l_0} \right) \otimes V_\#^L.$$

As introduced above, for the sparse tensor product FEs, each basis function in V^l is multiplied by the basis functions of only a subspace of $W_\#^L$, which is determined by the space $\mathcal{V}^{l_0}$ that the basis function belongs to. We define the sparse tensor product FE spaces as

$$\hat{W}_1^L = \bigoplus_{0 \le l_0 \le L} \mathcal{V}^{l_0} \otimes W_\#^{L-l_0} \quad \text{and} \quad \hat{V}_1^L = \bigoplus_{0 \le l_0 \le L} \mathcal{V}^{l_0} \otimes V_\#^{L-l_0}; \tag{5.5}$$

and

$$\hat{\mathbf{W}}^L = W^L \otimes \hat{W}_1^L \otimes \hat{V}_1^L.$$

The space $\hat{\mathbf{W}}^L$ is a subspace of $\mathbf{W}^L$ with the dimension $O(L2^{dL})$ which is far less than the dimension of $\mathbf{W}^L$ which is $O(2^{2dL})$. We refer to references such as [28] for a detailed and extensive introduction on sparse tensor product spaces.

The spatially semidiscrete sparse tensor product FE approximating problem is: Find $\hat{\boldsymbol{u}}^L(t) \in \hat{\mathbf{W}}^L$ such that :

$$\int_D \int_Y \Bigg[b(x,y) \left(\frac{\partial^2 \hat{u}_0^L}{\partial t^2}(t) + \nabla_y \frac{\partial^2 \hat{\mathfrak{u}}_1^L}{\partial t^2}(t) \right) \cdot \left(\hat{v}_0^L + \nabla_y \hat{\mathfrak{v}}_1^L \right) \\ + a(x,y) \left(\operatorname{curl} \hat{u}_0^L(t) + \operatorname{curl}_y \hat{u}_1^L(t) \right) \cdot \left(\operatorname{curl} \hat{v}_0^L + \operatorname{curl}_y \hat{v}_1^L \right) \Bigg] dydx = \int_D f(x) \cdot \hat{v}_0^L(x) dx \tag{5.6}$$

for all $\hat{\boldsymbol{v}}^L = (\hat{v}_0^L, \hat{v}_1^L, \hat{\mathfrak{v}}_1^L) \in \hat{\mathbf{W}}^L$. To find an error estimate for the sparse tensor product FE approximation we note the following results

Lemma 5.4. *For $w \in \hat{\mathcal{H}}^s$ with $s \in (\frac{1}{2}, 1]$*

$$\inf_{w^L \in \hat{W}_1^L} \| w - w^L \|_{L^2(D, H_\#(\operatorname{curl}, Y))} \le c L^{1/2} h_L^s \| w \|_{\hat{\mathcal{H}}^s};$$

for $w \in \hat{\mathfrak{H}}^s$ with $s \in [0, 1]$,

$$\inf_{w^L \in \hat{V}_1^L} \| w - w^L \|_{L^2(D, H^1_\#(Y))} \le c L^{1/2} h_L^s \| w \|_{\hat{\mathfrak{H}}^s}.$$

The proof of these results follows from that for sparse tensor product approximation in [28] and [11]. Therefore, for $\boldsymbol{w} \in \hat{\boldsymbol{\mathcal{H}}}^s$

$$\inf_{\boldsymbol{w}^L \in \hat{\mathbf{W}}^L} \| \boldsymbol{w} - \boldsymbol{w}^L \|_{\mathbf{W}} \le c L^{1/2} h_L^s \| \boldsymbol{w} \|_{\hat{\boldsymbol{\mathcal{H}}}^s}.$$

We then have the following result.

Proposition 5.5. *Assume that condition (4.9) and Assumption 4.1 hold, D is a Lipschitz polyhedral domain, $\operatorname{div} f \in L^\infty((0,T), L^2(D))$ and g_0, g_1 belong to $H^s(\operatorname{curl}, D)$. If g_0^L and g_1^L are chosen so that*

$$\| g_0^L - g_0 \|_V \le c L^{1/2} h_L^s \text{ and } \| g_1^L - g_1 \|_H \le c L^{1/2} h_L^s,$$

where $s \in (\frac{1}{2}, 1]$ is the constant in Proposition 4.5, then the solution of the spatially semidiscrete approximating problem (5.6) satisfies

$$\left\| \frac{\partial (\hat{u}_0^L - u_0)}{\partial t} \right\|_{L^\infty((0,T),H)} + \left\| \nabla_y \frac{\partial (\hat{\mathfrak{u}}_1^L - \mathfrak{u}_1)}{\partial t} \right\|_{L^\infty((0,T),H_1)} \\ + \| \operatorname{curl} (\hat{u}_0^L - u_0) \|_{L^\infty((0,T),H)} + \| \operatorname{curl}_y (\hat{u}_1^L - u_1) \|_{L^\infty((0,T),H_1)} \le c L^{1/2} h_L^s.$$

The proof of this proposition is identical to that of Proposition 5.2.

The fully discrete sparse tensor FE product problem is: For $m = 1, \ldots, M$ find $\hat{\boldsymbol{u}}_m^L = (\hat{u}_{0,m}^L, \hat{u}_{1,m}^L, \hat{\mathfrak{u}}_{1,m}^L) \in \hat{\mathbf{W}}^L$ such that

$$\int_D \int_Y \Bigg[b(x,y) \left(\partial_t^2 \hat{u}_{0,m}^L + \nabla_y \partial_t^2 \hat{\mathfrak{u}}_{1,m}^L \right) \cdot \left(\hat{v}_0^L + \nabla_y \hat{\mathfrak{v}}_1^L \right) \\ + a(x,y) \left(\operatorname{curl} \hat{u}_{0,m,1/4}^L + \operatorname{curl}_y \hat{u}_{1,m,1/4}^L \right) \cdot \left(\operatorname{curl} \hat{v}_0^L + \operatorname{curl}_y \hat{v}_1^L \right) \Bigg] dydx = \int_D f_{m,1/4}(x) \cdot \hat{v}_0^L(x) dx \tag{5.7}$$

for all $\hat{\boldsymbol{v}}^L = (\hat{v}_0^L, \hat{v}_1^L, \hat{\mathfrak{v}}_1^L) \in \hat{\mathbf{W}}^L$.

For the fully discrete problem, we have the following result

Proposition 5.6. *Assume that condition (4.9) and Assumption 4.1 hold, D is a Lipschitz polyhedral domain, $\operatorname{div} f \in L^\infty((0,T), L^2(D))$ and g_0, g_1 belong to $H^s(\operatorname{curl}, D)$ where $s \in (\frac{1}{2}, 1]$ is the constant in Proposition 4.5. If the initial value $\hat{\boldsymbol{u}}_1^L$ is chosen so that*

$$\| \partial_t p_{0,1/2}^L \|_{L^2(D)} + \| \partial_t \nabla_y \mathfrak{p}_{1,1/2}^L \|_{H_1} + \| \operatorname{curl} p_{0,1/2}^L \|_{L^2(D)} + \| \operatorname{curl}_y p_{1,1/2}^L \|_{H_1} \le c((\Delta t)^2 + L^{1/2} h_L^s),$$

then

$$\| \partial_t \hat{u}_0^L - \partial_t u_0 \|_{\tilde{L}^\infty((0,T),H)} + \| \nabla_y (\hat{\mathfrak{u}}_1^L - \mathfrak{u}_1) \|_{\tilde{L}^\infty((0,T),H_1)} + \| \operatorname{curl} (\hat{u}_0^L - u_0) \|_{\tilde{L}^\infty((0,T),H)} \\ + \| \operatorname{curl}_y (\hat{u}_1^L - u_1) \|_{\tilde{L}^\infty((0,T),H_1)} \le c((\Delta t)^2 + L^{1/2} h_L^s).$$

The proof is identical to that of Proposition 5.3.

6. Numerical correctors

We construct numerical correctors in this section.

6.1. Analytic homogenization error

We have the following homogenization error. This result generalizes the well known $O(\varepsilon^{1/2})$ homogenization error in [29] and [30] to the case where the solution u_0 of the homogenized equation possesses low regularity. We present the proof in Appendix C.

Proposition 6.1. *Assume that $g_0 = 0$, $g_1 \in H^1(D)\bigcap W$, $f \in H^1((0,T),H)$, u_0, $\frac{\partial u_0}{\partial t}$ and $\frac{\partial^2 u_0}{\partial t^2}$ belong to $L^\infty((0,T),H^s(\mathrm{curl}\,,D))$ for $0 < s \le 1$, $N^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\mathrm{curl}\,_y N^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $w^r \in C^1(\bar{D}, C(\bar{Y}))$ for all $r = 1, 2, 3$. There exists a constant c that does not depend on ε such that*

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \left[\frac{\partial u_0}{\partial t} + \nabla_y \frac{\partial u_1}{\partial t}\left(\cdot,\cdot,\frac{\cdot}{\varepsilon}\right)\right]\right\|_{L^\infty((0,T),H)} + \left\|\mathrm{curl}\, u^\varepsilon - \left[\mathrm{curl}\, u_0 + \mathrm{curl}\,_y u_1\left(\cdot,\cdot,\frac{\cdot}{\varepsilon}\right)\right]\right\|_{L^\infty((0,T),H)} \le c\varepsilon^{\frac{s}{1+s}}.$$

Remark 6.2. Generally, the energy of a two-scale wave equation does not always converge to the energy of the homogenized wave equation when $g_0 \neq 0$. We therefore restrict our consideration to the case where $g_0 = 0$. As shown in [31], the corrector of a general two scale wave equation involves the solution of another two-scale equation in the domain D. However, the scale interacting terms in (2.7) always form a part of the corrector.

6.2. Numerical correctors

We now establish numerical correctors with an explicit error estimate. We define the operator $\mathcal{U}^\varepsilon : L^1(D\times Y) \to L^1(D)$ as

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon t, \left\{\frac{x}{\varepsilon}\right\}\right)dt$$

for all functions $\Phi \in L^1(D\times Y)$. We note the following property.

Lemma 6.3. *For each function $\Phi \in L^1(D\times Y)$ we have*

$$\int_{\tilde{D}^\varepsilon} \mathcal{U}^\varepsilon(\Phi)dx = \int_D\int_Y \Phi(x,y)dydx, \tag{6.1}$$

where $\tilde{D}^\varepsilon$ is the 2ε neighbourhood of D.

We refer to [32] for a proof. We first note the following result.

Lemma 6.4. *Assume that $\frac{\partial u_0}{\partial t} \in L^\infty((0,T),H^s(D)^3)$ and $\mathrm{curl}\, u_0 \in L^\infty((0,T),H^s(D)^3)$, $N^r \in C^1(\bar{D}, C^1_\#(\bar{Y}))^3$ and $w^r \in C^1(\bar{D}, C^1_\#(\bar{Y}))$, $r = 1, 2, 3$. Then*

$$\sup_{t\in[0,T]}\int_D \left|\mathrm{curl}_y u_1\left(t,x,\frac{x}{\varepsilon}\right) - \mathcal{U}^\varepsilon(\mathrm{curl}_y u_1(t,\cdot,\cdot))(x)\right|^2 dx \le c\varepsilon^{2s}$$

and

$$\sup_{t\in[0,T]}\int_D \left|\frac{\partial}{\partial t}\nabla_y u_1\left(t,x,\frac{x}{\varepsilon}\right) - \mathcal{U}^\varepsilon\left(\frac{\partial}{\partial t}\nabla_y u_1(t,\cdot,\cdot)\right)(x)\right|^2 dx \le c\varepsilon^{2s}.$$

The proof of this result is similar to that for the time independent case in Appendix B of [18], which utilizes the ideas of the proof of Lemma 5.5 in [33]. We then have the following numerical corrector results.

Theorem 6.5. *Assume that condition (4.9) and Assumption 4.1 hold, with $g_0 = 0$ and $\mathrm{div} f \in L^\infty((0,T),L^2(D))$, D is a Lipschitz polyhedral domain, and that g_1^L is chosen so that $\|g_1^L - g_1\|_H \le ch_L^s$ where $s \in (\frac{1}{2}, 1]$ is the constant in Proposition 4.5. Then for the solution of the semidiscrete problem (5.1) using the full tensor product FEs, we have*

$$\left\|\frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \bar{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon\left(\frac{\partial}{\partial t}\nabla_y \bar{u}_1^L\right)\right)\right\|_{L^\infty((0,T),H)}$$
$$+ \left\|\mathrm{curl}\, u^\varepsilon - \left(\mathrm{curl}\,\bar{u}_0^L + \mathcal{U}^\varepsilon\left(\mathrm{curl}_y \bar{u}_1^L\right)\right)\right\|_{L^\infty((0,T),H)} \le c\left(h_L^s + \varepsilon^{\frac{s}{s+1}}\right)$$

where c is independent of ε and the meshsize h_L. For the semidiscrete problem (5.6) using the sparse tensor product FEs, if $\|g_1^L - g_1\|_H \le cL^{1/2}h_L^s$, we have

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \hat{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \hat{u}_1^L \right) \right) \right\|_{L^\infty((0,T),H)} + \left\| \operatorname{curl} u^\varepsilon - \left(\operatorname{curl} \hat{u}_0^L + \mathcal{U}^\varepsilon \left(\operatorname{curl}_y \hat{u}_1^L \right) \right) \right\|_{L^\infty((0,T),H)} \le c \left(L^{1/2} h_L^s + \varepsilon^{\frac{s}{s+1}} \right)$$

where c is independent of ε and the meshsize h_L.

Proof. With the hypothesis of the theorem, from Propositions 4.2 and 4.5, the conditions of Proposition 6.1 hold. We then have from (6.1)

$$\left\| \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y u_1(t) - \frac{\partial}{\partial t} \nabla_y \bar{u}_1^L(t) \right) \right\|_H \le \left\| \frac{\partial}{\partial t} \nabla_y u_1(t) - \frac{\partial}{\partial t} \nabla_y \bar{u}_1^L(t) \right\|_{H_1}$$

and

$$\left\| \mathcal{U}^\varepsilon \left(\operatorname{curl} u_0 + \operatorname{curl}_y u_1 - \operatorname{curl} \bar{u}_0^L - \operatorname{curl}_y \bar{u}_1^L \right) \right\|_H \le \left\| \operatorname{curl} u_0 + \operatorname{curl}_y u_1 - \operatorname{curl} \bar{u}_0^L - \operatorname{curl}_y \bar{u}_1^L \right\|_{H_1}.$$

We note that

$$\begin{aligned} \left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \bar{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{u}_1^L \right) \right) \right\|_{L^\infty((0,T),H)} &\le \left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial u_0}{\partial t} + \frac{\partial}{\partial t} \nabla_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon}) \right) \right\|_{L^\infty((0,T),H)} \\ &+ \left\| \frac{\partial u_0}{\partial t} - \frac{\partial \bar{u}_0^L}{\partial t} \right\|_{L^\infty((0,T),H)} + \left\| \frac{\partial}{\partial t} \nabla_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon}) - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y u_1 \right) \right\|_{L^\infty((0,T),H)} \\ &+ \left\| \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y u_1 \right) - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{u}_1^L \right) \right\|_{L^\infty((0,T),H)} \end{aligned}$$

From Proposition 6.1, we have

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial u_0}{\partial t} + \frac{\partial}{\partial t} \nabla_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon}) \right) \right\|_{L^\infty((0,T),H)} \le c \varepsilon^{\frac{s}{s+1}}.$$

From Proposition 5.2, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \bar{u}_0^L}{\partial t} \right\|_{L^\infty((0,T),H)} \le c h_L^s.$$

From Lemma 6.4, we have

$$\left\| \frac{\partial}{\partial t} \nabla_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon}) - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y u_1 \right) \right\|_{L^\infty((0,T),H)} \le c \varepsilon^s.$$

We note that

$$\left\| \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y u_1 \right) - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{u}_1^L \right) \right\|_{L^\infty((0,T),H)} \le \left\| \frac{\partial}{\partial t} \nabla_y u_1 - \frac{\partial}{\partial t} \nabla_y \bar{u}_1^L \right\|_{L^\infty((0,T),H_1)} \le c h_L^s.$$

Thus

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \bar{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{u}_1^L \right) \right) \right\|_{L^\infty((0,T),H)} \le c \varepsilon^{\frac{s}{s+1}} + c h_L^s + c \varepsilon^s + c h_L^s \le c (h_L^s + \varepsilon^{\frac{s}{s+1}}).$$

Similarly,

$$\begin{aligned} &\left\| \operatorname{curl} u^\varepsilon - \left(\operatorname{curl} \bar{u}_0^L + \mathcal{U}^\varepsilon \left(\operatorname{curl}_y \bar{u}_1^L \right) \right) \right\|_{L^\infty((0,T),H)} \\ &\le \| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0 - \operatorname{curl}_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon}) \|_{L^\infty((0,T),H)} + \left\| \operatorname{curl} u_0 - \operatorname{curl} \bar{u}_0^L \right\|_{L^\infty((0,T),H)} \\ &\quad + \left\| \operatorname{curl}_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon}) - \mathcal{U}^\varepsilon \left(\operatorname{curl}_y u_1 \right) \right\|_{L^\infty((0,T),H)} + \left\| \mathcal{U}^\varepsilon \left(\operatorname{curl}_y u_1 \right) - \mathcal{U}^\varepsilon \left(\operatorname{curl}_y \bar{u}_1^L \right) \right\|_{L^\infty((0,T),H)} \\ &\le c \varepsilon^{\frac{s}{s+1}} + c h_L^s + c \varepsilon^s + c h_L^s \le c (h_L^s + \varepsilon^{\frac{s}{s+1}}). \end{aligned}$$

We then have the desired estimate.

The proof for the semidiscrete sparse tensor FE solution is similar. □

For fully discrete problems, we have the following results.

Theorem 6.6. *Assume that condition (4.9) and Assumption 4.1 hold, with $g_0 = 0$, $g_1 \in H^s(\text{curl}, D)$ and $\text{div} f \in L^\infty((0,T), L^2(D))$, D is a Lipschitz polyhedral domain ($s \in (\frac{1}{2}, 1]$ is the constant in Proposition 4.5). For the fully discrete full tensor product FE problem (5.4), assume that $\bar{\boldsymbol{u}}_1^L$ is chosen so that*

$$\left\|\partial_t p^L_{0,1/2}\right\|_H + \left\|\partial_t \nabla_y \mathrm{p}^L_{1,1/2}\right\|_{H_1} + \left\|\text{curl}\, p^L_{0,1/2}\right\|_H + \left\|\text{curl}_y p^L_{1,1/2}\right\|_{H_1} \le c((\Delta t)^2 + h_L^s),$$

then

$$\Delta t \max_{0\le m<M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \bar{u}^L_{0,m+1/2} - \mathcal{U}^\varepsilon(\partial_t \nabla_y \bar{\mathrm{u}}^L_{1,m+1/2}) \right\|_H + \left\|\text{curl}\, u^\varepsilon - \text{curl}\, \bar{u}^L_0 - \mathcal{U}^\varepsilon(\text{curl}_y \bar{u}^L_1)\right\|_{\tilde{L}^\infty((0,T),H)} \le c\left((\Delta t)^2 + h_L^s + \varepsilon^{\frac{s}{s+1}}\right).$$

For the sparse tensor product FE problem (5.7), if $\hat{\boldsymbol{u}}_1^L$ is chosen so that

$$\|\partial_t p^L_{0,1/2}\|_H + \|\partial_t \nabla_y \mathrm{p}^L_{1,1/2}\|_{H_1} + \|\text{curl}\, p^L_{0,1/2}\|_H + \|\text{curl}_y p^L_{1,1/2}\|_{H_1} \le c\left((\Delta t)^2 + L^{1/2} h_L^s\right),$$

then

$$\Delta t \max_{0\le m<M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \hat{u}^L_{0,m+1/2} - \mathcal{U}^\varepsilon(\partial_t \nabla_y \hat{\mathrm{u}}^L_{1,m+1/2}) \right\|_H + \left\|\text{curl}\, u^\varepsilon - \text{curl}\, \hat{u}^L_0 - \mathcal{U}^\varepsilon(\text{curl}_y \hat{u}^L_1)\right\|_{\tilde{L}^\infty((0,T),H)} \le c\left((\Delta t)^2 + L^{1/2} h_L^s + \varepsilon^{\frac{s}{s+1}}\right).$$

Proof. From the compatibility condition $\frac{\partial^4 u_0}{\partial t^4} \in L^\infty((0,T),H)$ so $\frac{\partial^2 u_0}{\partial t^2} \in C([0,T],H)$. To use the homogenization error in Proposition 6.1, we estimate

$$\frac{1}{\Delta t}(u_{0,m+1} - u_{0,m}) - \frac{\partial u_0}{\partial t}(t_m) = \frac{\partial u_0}{\partial t}(\tau) - \frac{\partial u_0}{\partial t}(t_m) = \int_{t_m}^{\tau} \frac{\partial^2 u_0}{\partial t^2}(\sigma) d\sigma,$$

for a value $t_m \le \tau \le t_{m+1}$. With the compatibility condition (4.9), we have that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty((0,T),H)$. Thus

$$\sup_{0\le m<M} \left\| \partial_t u_{0,m+1/2} - \frac{\partial u_0}{\partial t}(t_m) \right\|_H \le c\Delta t.$$

Similarly, using the smoothness of N^r and w^r for $r = 1, 2, 3$, we have that $\frac{\partial^2}{\partial t^2}\nabla \mathrm{u}_1 \in L^\infty((0,T),H_1)$. We note that

$$\partial_t \nabla_y \mathrm{u}_{1,m+1/2} - \frac{\partial}{\partial t}\nabla_y \mathrm{u}_1(t_m) = \frac{\nabla_y \mathrm{u}_{1,m+1} - \nabla_y \mathrm{u}_{1,m}}{\Delta t} - \frac{\partial}{\partial t}\nabla_y \mathrm{u}_1(t_m) = \frac{\partial}{\partial t}\nabla_y \mathrm{u}_1(\tau) - \frac{\partial}{\partial t}\nabla_y \mathrm{u}_1(t_m) = \int_{t_m}^{\tau} \frac{\partial^2}{\partial t^2}\nabla_y \mathrm{u}_1(\sigma) d\sigma,$$

for a value $t_m \le \tau \le t_{m+1}$. Thus

$$\sup_{0\le m<M} \left\| \partial_t \nabla_y \mathrm{u}_{1,m+1/2} - \frac{\partial}{\partial t}\nabla_y \mathrm{u}_1(t_m) \right\|_{H_1} \le c\Delta t.$$

We then get the result from Propositions 5.6 and 6.1. □

7. Numerical results

We present in this section some numerical examples for two scale problems that confirm our analysis. To identify the detail spaces defined in Section 5.2, we employ Riesz basis and define the equivalent norms in the space $L^2(D)$. The Riesz basis functions satisfy:

Assumption 7.1. For all vectors $j \in \mathbb{N}_0^d$, there exist an index set $I^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi^{jk} \in L^2(D)$ for $k \in I^j$, such that $V^l = \text{span}\left\{\phi^{jk} : |j|_\infty \le l, k \in I^j\right\}$. For all $\phi = \sum_{|j|_\infty \le l, k \in I^j} \phi^{jk} c_{jk} \in V^l$

$$c_1 \sum_{\substack{|j|_\infty \le l \\ k\in I^j}} |c_{jk}|^2 \le \|\phi\|^2_{L^2(D)} \le c_2 \sum_{\substack{|j|_\infty \le l \\ k\in I^j}} |c_{jk}|^2,$$

where $c_1 > 0$ and $c_2 > 0$ are independent of ϕ and l.

From this assumption, ϕ^{jk} for all j and $k \in I^j$ form a basis for $L^2(D)$; and $(\sum_{|j|_\infty=1}^{\infty} \sum_{k\in I^j} c_{jk}^2)^{1/2}$ is an equivalent norm of $\|\phi\|_{L^2(D)}$ for $\phi = \sum_{|j|_\infty=1}^{\infty} \sum_{k\in I^j} c_{jk}\phi^{jk}$. With this equivalent norm, the projection P^l can be defined as $P^l\phi = \sum_{|j|_\infty \le l, k\in I^j} \phi^{jk} c_{jk}$; and therefore the detail spaces are defined as $\mathcal{V}^l = \mathrm{span}\{\phi^{jk} : |j|_\infty = l, k \in I^j\}$. Having identified the basis functions in each of the spaces $\mathcal{V}^l$, we construct the sparse tensor product FE spaces as in (5.5). We refer to, e.g., [13] for details.

Example. For the space $L^2(0,1)$, a Riesz basis can be constructed as follows. Level 0 contains three piecewise linear basis functions: ψ^{01} obtains values $(1,0)$ at $(0,1/2)$ and is 0 in $(1/2,1)$, ψ^{02} obtains values $(0,1,0)$ at $(0,1/2,1)$, and ψ^{03} obtains values $(0,1)$ at $(1/2,1)$ and is 0 in $(0,1/2)$. For other levels, the basis functions are constructed from the function ψ that takes values $(0,-1,2,-1,0)$ at $(0,1/2,1,3/2,2)$, the left boundary function ψ^{left} taking values $(-2,2,-1,0)$ at $(0,1/2,1,3/2)$, and the right boundary function ψ^{right} taking values $(0,-1,2,-2)$ at $(1/2,1,3/2,2)$. For levels $j \ge 1$ with $I^j = \{1,2,\dots,2^j\}$, the basis functions are $\psi^{j1}(x) = 2^{j/2}\psi^{left}(2^j x)$, $\psi^{jk}(x) = 2^{j/2}\psi(2^j x - k + 3/2)$ for $k = 2,\dots,2^j - 1$ and $\psi^{j2^j} = 2^{j/2}\psi^{right}(2^j x - 2^j + 2)$. This basis satisfies Assumption 7.1(i).

A Riesz basis for the space $L^2((0,1)^d)$ can be constructed by taking the tensor products of the basis functions in $(0,1)$ with an appropriate scaling, see [34].

Remark 7.2. We note that the norm equivalence above are not necessary for the approximations in Lemma 5.4 to hold, as explained in [12] and [18].

We consider a two scale Maxwell wave equation in the two dimension domain $D = (0,1)^2$

$$b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} + \mathrm{curl}\,(a^\varepsilon \mathrm{curl}\, u^\varepsilon) = f(t,x), \quad \text{in } (0,T)\times D$$

$$u^\varepsilon(t,\cdot)\times\nu = 0, \text{ on } \partial D, \quad u^\varepsilon(0,x) = 0, \quad u_t^\varepsilon(0,x) = 0.$$

In the first example, the coefficients are

$$a(x,y) = \frac{1}{(1+x_1)(1+x_2)(1+\cos^2 2\pi y_1)(1+\cos^2 2\pi y_2)}, \quad \text{and} \quad b(x,y) = \frac{(1+x_1)(1+x_2)}{(1+\cos^2 2\pi y_1)(1+\cos^2 2\pi y_2)}.$$

The exact homogenized coefficients are

$$a^0(x) = \frac{4}{9(1+x_1)(1+x_2)} \quad \text{and} \quad b^0(x) = \frac{\sqrt{2}(1+x_1)(1+x_2)}{3}.$$

We choose

$$f(x_1,x_2) = \begin{pmatrix} 2\sqrt{2}(1+x_1)(1+x_2)x_1x_2(1-x_2)t + \dfrac{4t^3}{9(1+x_2)^2} \\ 2\sqrt{2}(1+x_1)(1+x_2)x_1x_2(1-x_1)t + \dfrac{4t^3}{9(1+x_1)^2} \end{pmatrix}$$

so that the solution to the homogenized equation is

$$u_0 = \begin{pmatrix} x_1x_2(1-x_2)t^3 \\ x_1x_2(1-x_1)t^3 \end{pmatrix}.$$

From the relation (4.6), we compute the solution $\mathrm{curl}_y u_1$ exactly as

$$\mathrm{curl}_y u_1 = \left(\frac{4(1+\cos^2 2\pi y_1)(1+\cos^2 2\pi y_2)}{9} - 1\right)(x_2 - x_1)t^3.$$

In Fig. 1 we plot the errors $\|u_0 - \hat{u}_0^L\|_{H(\mathrm{curl},D)}$ and $\|\mathrm{curl}\, u_1 - \mathrm{curl}\,\hat{u}_1^L\|_{L^2(D)}$ versus the mesh size for the sparse tensor product FEs for $(\Delta t, h) = (1/4,1/4), (1/6,1/8), (1/8,1/12)$ and $(1/16,1/32)$. The result confirms our analysis.

In the second example, we choose

$$a(x,y) = \frac{(1+x_1)(1+x_2)}{(1+\cos^2 2\pi y_1)(1+\cos^2 2\pi y_2)}, \quad \text{and} \quad b(x,y) = \frac{1}{(1+x_1)(1+x_2)(1+\cos^2 2\pi y_1)(1+\cos^2 2\pi y_2)}.$$

In this case, the homogenized coefficients are

$$a^0(x) = \frac{4(1+x_1)(1+x_2)}{9} \quad \text{and} \quad b^0(x) = \frac{\sqrt{2}}{3(1+x_1)(1+x_2)}.$$

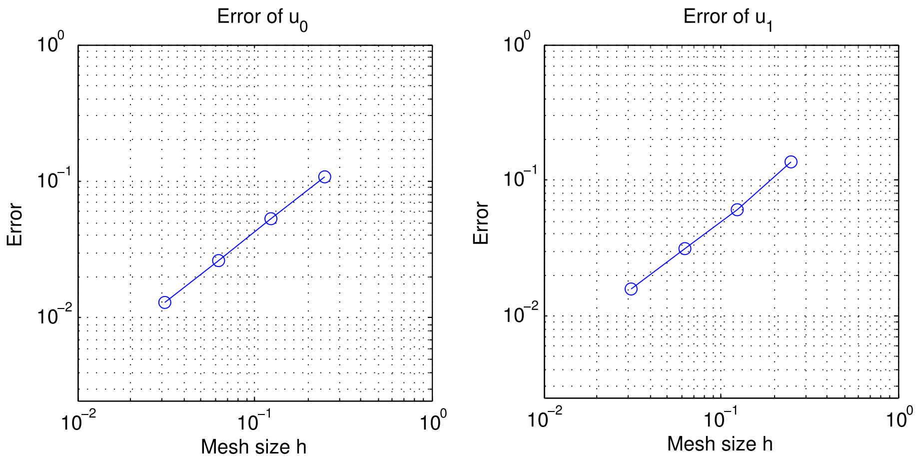

Fig. 1. The sparse tensor errors $\|u_0 - \hat{u}_0^L\|_{H(\mathrm{curl},D)}$ and $\|\mathrm{curl}_y u_1 - \mathrm{curl}_y \hat{u}_1^L\|_{L^2(D)}$.

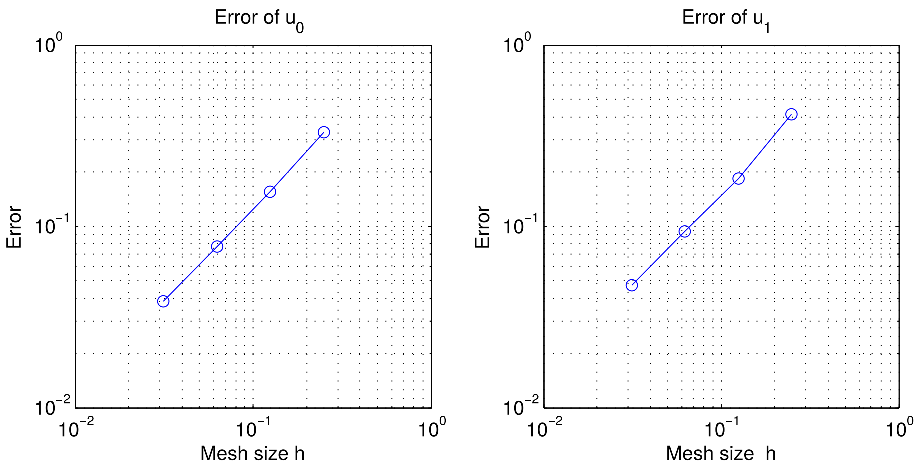

Fig. 2. The sparse tensor errors $\|u_0 - \hat{u}_0^L\|_{H(\mathrm{curl},D)}$ and $\|\mathrm{curl}_y u_1 - \mathrm{curl}_y \hat{u}_1^L\|_{L^2(D)}$.

We choose

$$f(x_1, x_2) = \begin{pmatrix} \dfrac{2\sqrt{2}x_2(1-x_2)t}{(1+x_2)} + \dfrac{4t^3(1+x_1)(2x_2 - x_1 + 1)}{3} \\ \dfrac{2\sqrt{2}x_1(1-x_1)t}{(1+x_1)} + \dfrac{4t^3(1+x_1)(2x_1 - x_2 + 1)}{3} \end{pmatrix}$$

so that the solution to the homogenized problem is

$$u_0 = \begin{pmatrix} (1+x_1)x_2(1-x_2)t^3 \\ (1+x_2)x_1(1-x_1)t^3 \end{pmatrix}$$

and

$$\mathrm{curl}_y u_1 = \left(\frac{4(1+\cos^2 2\pi y_1)(1+\cos^2 2\pi y_2)}{3} - 1 \right)(x_2 - x_1)t^3.$$

In Fig. 2 we plot the errors $\|u_0 - \hat{u}_0^L\|_{H(\mathrm{curl},D)}$ and $\|\mathrm{curl}_y u_1 - \mathrm{curl}_y \hat{u}_1^L\|_{L^2(D)}$ versus the mesh size for the sparse tensor product FEs for $(\Delta t, h) = (1/4, 1/4), (1/6, 1/8), (1/8, 1/12)$ and $(1/16, 1/32)$. The result once more confirms our analysis.

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Appendix A

We prove Propositions 2.4 and 2.5 in this appendix.

Proof of Proposition 2.4. Let $q \in \mathcal{D}(0,T)$. Let $v_0 \in \mathcal{D}(D)$, $v_1 \in \mathcal{D}(D, C^\infty_\#(Y))^3$ and $\mathfrak{v}_1 \in \mathcal{D}(D, C^\infty_\#(Y))$. Choosing a test function of the form $\phi(t,x) = \left(v_0(x) + \varepsilon v_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla \mathfrak{v}_1(x, \frac{x}{\varepsilon})\right) q(t)$ we obtain

$$\int_0^T \int_D b(x, \frac{x}{\varepsilon}) u^\varepsilon(t,x) \cdot \left(v_0(x) + \varepsilon v_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_x \mathfrak{v}_1(x, \frac{x}{\varepsilon}) + \nabla_y \mathfrak{v}_1(x, \frac{x}{\varepsilon})\right) q''(t) dxdt$$
$$+ \int_0^T \int_D a(x, \frac{x}{\varepsilon}) \operatorname{curl} u^\varepsilon(t,x) \cdot \left(\operatorname{curl} v_0(x) + \varepsilon \operatorname{curl}_x v_1(x, \frac{x}{\varepsilon}) + \operatorname{curl}_y v_1(x, \frac{x}{\varepsilon})\right) q(t) dxdt$$
$$= \int_0^T \int_D f(t,x) \cdot \left(v_0(x) + \varepsilon v_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_x \mathfrak{v}_1(x, \frac{x}{\varepsilon}) + \nabla_y \mathfrak{v}_1(x, \frac{x}{\varepsilon})\right) q(t) dxdt.$$

Passing to the two scale limit, we have

$$\int_0^T \int_D \int_Y b(x,y) \left(u_0(t,x) + \nabla_y \mathfrak{u}_1(t,x,y)\right) \cdot \left(v_0(x) + \nabla_y \mathfrak{v}_1(x,y)\right) q''(t) dydxdt$$
$$+ \int_0^T \int_D \int_Y a(x,y) \left(\operatorname{curl} u_0(t,x) + \operatorname{curl}_y u_1(t,x,y)\right) \cdot \left(\operatorname{curl} v_0(x) + \operatorname{curl}_y v_1(x,y)\right) q(t) dydxdt$$
$$= \int_0^T \int_D \int_Y f(t,x) \cdot \left(v_0(x) + \nabla_y \mathfrak{v}_1(x,y)\right) q(t) dydxdt = \int_0^T \int_D f(t,x) \cdot v_0(x) q(t) dxdt.$$

Using a density argument, we find that this equation holds for all $(v_0, v_1, \mathfrak{v}_1) \in \mathbf{W}$. We now establish the initial conditions. Let $\phi \in C^\infty([0,T] \times D)$ with $\phi = 0$ when $t = T$. We have

$$\int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \cdot \phi(t,x) dxdt = -\int_D u^\varepsilon(0,x) \cdot \phi(0,x) - \int_0^T \int_D u^\varepsilon(t,x) \cdot \frac{\partial \phi}{\partial t}(t,x) dxdt$$
$$\to -\int_D g_0(x) \cdot \phi(0,x) dx - \int_0^T \int_D u_0(t,x) \cdot \frac{\partial \phi}{\partial t}(t,x) dxdt.$$

On the other hand

$$\int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \cdot \phi(t,x) dxdt \to \int_0^T \int_D \frac{\partial u_0}{\partial t}(t,x) \cdot \phi(t,x) = -\int_D u_0(0,x) \cdot \phi(0,x) - \int_0^T \int_D u_0(t,x) \cdot \frac{\partial \phi}{\partial t}(t,x) dxdt.$$

Thus $u_0(0,x) = g_0$. As $\{\frac{\partial u^\varepsilon}{\partial t}\}_\varepsilon$ is bounded in $L^2((0,T), H)$ so there is a subsequence that two-scale converges. Let $\xi \in L^2((0,T), H_1)$ be the two-scale limit. Let $\phi(t,x,y) \in C^\infty_0((0,T), C^\infty_0(D, C^\infty_\#(Y)))$. We have that

$$\lim_{\varepsilon \to 0} \int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t}(t,x) \cdot \phi(t,x,\frac{x}{\varepsilon}) dxdt \to \int_0^T \int_D \int_Y \xi(t,x,y) \cdot \phi(t,x,y) dydxdt.$$

On the other hand

$$\int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t}(t,x) \cdot \phi(t,x,\frac{x}{\varepsilon}) dxdt = -\int_0^T \int_D u^\varepsilon(t,x) \cdot \frac{\partial \phi}{\partial t}(t,x,\frac{x}{\varepsilon}) dxdt$$
$$\to -\int_0^T \int_D \int_Y (u_0 + \nabla_y \mathfrak{u}_1) \cdot \frac{\partial \phi}{\partial t}(t,x,y) dydxdt \quad \text{when} \quad \varepsilon \to 0.$$

Thus $\xi(t,x,y) = \frac{\partial}{\partial t}(u_0 + \nabla_{y_1} \mathfrak{u}_1)$ so $\frac{\partial}{\partial t} \nabla_{y_1} \mathfrak{u}_1 = \xi(t,x,y) - \frac{\partial u_0}{\partial t} \in L^2((0,T), H_1)$. Now we choose $\phi(t,x,y)$ such that it equals 0 only when $t = T$. Then

$$\int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \cdot \phi(t,x,\frac{x}{\varepsilon}) dxdt = -\int_D u^\varepsilon(0,x) \cdot \phi(0,x,\frac{x}{\varepsilon}) dx - \int_0^T \int_D u^\varepsilon(t,x) \cdot \frac{\partial \phi}{\partial t}(t,x,\frac{x}{\varepsilon}) dxdt$$
$$\to -\int_D \int_Y g_0(x) \cdot \phi(0,x,y) dydx - \int_0^T \int_D \int_Y (u_0 + \nabla_y \mathfrak{u}_1) \cdot \frac{\partial \phi}{\partial t}(t,x,y) dydxdt.$$

On the other hand,

$$\int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t}(t,x) \cdot \phi(t,x,\frac{x}{\varepsilon}) dxdt \to \int_0^T \int_D \int_Y (\frac{\partial u_0}{\partial t} + \frac{\partial}{\partial t} \nabla_y \mathfrak{u}_1) \cdot \phi(t,x,y) dydxdt$$
$$= -\int_D \int_Y (u_0(0,x) + \nabla_y \mathfrak{u}_1(0,x,y)) \cdot \phi(0,x,y) dydx - \int_0^T \int_D \int_Y (u_0 + \nabla_y \mathfrak{u}_1) \cdot \frac{\partial \phi}{\partial t}(t,x,y) dydxdt.$$

We thus deduce that $\nabla_y \mathfrak{u}_1(0,x,y) = 0$.

Let $q \in C^\infty([0,T])$ with $q(T)=0$. Let $\phi(x)=v_0(x)+\varepsilon v_1(x,\frac{x}{\varepsilon})+\varepsilon\nabla\mathfrak{v}_1(x,\frac{x}{\varepsilon})$ as above. We have

$$\int_0^T \left\langle b^\varepsilon(x)\frac{\partial^2 u^\varepsilon}{\partial t^2},\phi\right\rangle q(t)dt = \int_0^T \frac{\partial}{\partial t}\left(\left\langle b^\varepsilon\frac{\partial u^\varepsilon}{\partial t},\phi\right\rangle q(t)\right)dt - \int_0^T\left\langle b^\varepsilon\frac{\partial u^\varepsilon}{\partial t},\phi\right\rangle\frac{dq(t)}{dt}dt$$
$$= -\left\langle b^\varepsilon\frac{\partial u^\varepsilon}{\partial t}(0),\phi\right\rangle q(0) - \int_0^T\left\langle b^\varepsilon\frac{\partial u^\varepsilon}{\partial t},\phi\right\rangle\frac{dq(t)}{dt}dt = -\int_D b^\varepsilon g_1\cdot\phi q(0)dx - \int_0^T\left\langle b^\varepsilon\frac{\partial u^\varepsilon}{\partial t},\phi\right\rangle\frac{dq(t)}{dt}dt$$
$$\to -\int_D\int_Y b(x,y)g_1(x)\cdot(v_0(x)+\nabla_y\mathfrak{v}_1(x,y))q(0)dydx -$$
$$\int_0^T\int_D\int_Y b(x,y)\left(\frac{\partial u_0}{\partial t}(x)+\frac{\partial}{\partial t}\nabla_y\mathfrak{u}_1(t,x,y)\right)\cdot\left(v_0(x)+\nabla_y\mathfrak{v}_1(x,y)\right)\frac{dq(t)}{dt}dydxdt \tag{A.1}$$

when $\varepsilon\to 0$. On the other hand, let q_n be a sequence in $C_0^\infty(0,T)$ that converges to $q(t)$ in $L^2(0,T)$ when $n\to\infty$. As $b^\varepsilon\frac{\partial^2}{\partial t^2}u^\varepsilon$ is bounded in $L^2((0,T),W')$ so there is a constant $c>0$ such that

$$\left|\int_0^T\langle b^\varepsilon\frac{\partial^2u^\varepsilon}{\partial t^2},\phi\rangle q_n(t)dt - \int_0^T\langle b^\varepsilon\frac{\partial^2u^\varepsilon}{\partial t^2},\phi\rangle q(t)dt\right| \le c\|q_n-q\|_{L^2(0,T)}.$$

As $q_n\in C_0^\infty(0,T)$, when $\varepsilon\to 0$,

$$\lim_{\varepsilon\to0}\int_0^T\langle b^\varepsilon\frac{\partial^2u^\varepsilon}{\partial t^2},\phi\rangle q_n(t)dt =$$
$$\int_0^T\left(\int_D\int_Y b(x,y)(u_0(t,x)+\nabla_y\mathfrak{u}_1(t,x,y))\cdot(v_0(x)+\nabla_y\mathfrak{v}_1(x,y))dydx\right)q_n''(t)dt$$
$$=\int_0^T\frac{\partial^2}{\partial t^2}\left(\int_D\int_Y b(x,y)(u_0(t,x)+\nabla_y\mathfrak{u}_1(t,x,y))\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx\right)q_n(t)dt$$

Passing to the limit when $n\to\infty$, we have

$$\lim_{\varepsilon\to0}\int_0^T\langle b^\varepsilon\frac{\partial^2u^\varepsilon}{\partial t^2},\phi\rangle q(t)dt = \int_0^T\frac{\partial^2}{\partial t^2}\left(\int_D\int_Y b(x,y)(u_0(t,x)+\nabla_y\mathfrak{u}_1(t,x,y))\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx\right)q(t)dt. \tag{A.2}$$

The right hand side of (A.2) can be written as

$$\int_0^T\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\int_D\int_Y b(x,y)(u_0(t,x)+\nabla_y\mathfrak{u}_1(t,x,y))\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx\right)q(t)dt -$$
$$\int_0^T\frac{\partial}{\partial t}\left(\int_D\int_Y b(x,y)(u_0(t,x)+\nabla_y\mathfrak{u}_1(t,x,y))\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx\right)\frac{dq(t)}{dt}dt$$
$$=-\frac{\partial}{\partial t}\int_D\int_Y b(x,y)(u_0+\nabla_y\mathfrak{u}_1)\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx\Big|_{t=0}q(0) -$$
$$\int_0^T\frac{\partial}{\partial t}\left(\int_D\int_Y b(x,y)(u_0(t,x)+\nabla_y\mathfrak{u}_1(t,x,y))\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx\right)\frac{dq(t)}{dt}dt \tag{A.3}$$

Comparing (A.1) and (A.3), we have

$$\frac{\partial}{\partial t}\int_D\int_Y b(x,y)(u_0(t,x)+\nabla_y\mathfrak{u}_1(t,x,y))\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx\Big|_{t=0} = \int_D\int_Y b(x,y)g_1(x)\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx.$$

Proof of Proposition 2.5. We show that when $f=0$, $g_0=0$ and $g_1=0$, the solution of (2.7) is $u_0=0$, $u_1=0$ and $\mathfrak{u}_1=0$.

Following the procedure in [21] Theorem 19.1 for showing the uniqueness of a solution of a wave equation, fixing $s\in(0,T)$, we define

$$w_0(t)=\begin{cases}-\int_t^s u_0(\sigma)d\sigma, & t<s\\ 0, & t\ge s\end{cases};\quad \mathfrak{w}_1(t)=\begin{cases}-\int_t^s \mathfrak{u}_1(\sigma)d\sigma, & t<s\\ 0, & t\ge s\end{cases};\quad w_1(t)=\begin{cases}-\int_t^s u_1(\sigma)d\sigma, & t<s\\ 0, & t\ge s.\end{cases}$$

We denote by $b(u_0+\nabla_y\mathfrak{u}_1)$ the functional in $(W\times V_1)'$ which maps $(v_0,\mathfrak{v}_1)\in W\times V_1$ to

$$\int_D\int_Y b(u_0+\nabla_y\mathfrak{u}_1)\cdot(v_0+\nabla_y\mathfrak{v}_1)dydx.$$

From (2.7), this functional belongs to $H^2((0,T),(W\times V_1)')$ so $\frac{\partial}{\partial t}b(u_0+\nabla_y\mathfrak{u}_1)\in H^1((0,T),(W\times V_1)')\subset C([0,T],(W\times V_1)')$. From (2.7), we have

$$\frac{\partial}{\partial t}\left\langle\frac{\partial}{\partial t}b(u_0+\nabla_y\mathfrak{u}_1),(w_0,\mathfrak{w}_1)\right\rangle_{(W\times V_1)',W\times V_1}$$
$$=-\int_D\int_Y a(x,y)\left(\operatorname{curl}u_0(t,x)+\operatorname{curl}_y u_1(t,x,y)\right)\cdot\left(\operatorname{curl}w_0(t,x)+\operatorname{curl}_y w_1(t,x,y)\right)dydx$$
$$+\int_D\int_Y b(x,y)\left(\frac{\partial u_0}{\partial t}+\frac{\partial}{\partial t}\nabla_y\mathfrak{u}_1\right)\cdot\left(\frac{\partial w_0}{\partial t}+\frac{\partial}{\partial t}\nabla_y\mathfrak{w}_1\right)dydx.$$

Integrating over $(0,s)$, we get

$$\left\langle\frac{\partial}{\partial t}b(u_0+\nabla_y\mathfrak{u}_1)\Big|_{t=s},(w_0(s),\mathfrak{w}_1(s))\right\rangle_{(W\times V_1)',W\times V_1}-\left\langle\frac{\partial}{\partial t}b(u_0+\nabla_y\mathfrak{u}_1)\Big|_{t=0},(w_0(0),\mathfrak{w}_1(0))\right\rangle_{(W\times V_1)',W\times V_1}$$
$$=\int_0^s\int_D\int_Y b(x,y)\left(\frac{\partial u_0}{\partial t}+\frac{\partial}{\partial t}\nabla_y\mathfrak{u}_1\right)\cdot\left(\frac{\partial w_0}{\partial t}+\frac{\partial}{\partial t}\nabla_y\mathfrak{w}_1\right)dydxdt$$
$$-\int_0^s\int_D\int_Y a(x,y)\left(\operatorname{curl}u_0+\operatorname{curl}_y u_1\right)\cdot\left(\operatorname{curl}w_0+\operatorname{curl}_y w_1\right)dydxdt$$
$$=\int_0^s\frac{1}{2}\frac{d}{dt}\int_D\int_Y b(x,y)(u_0+\nabla_y\mathfrak{u}_1)\cdot(u_0+\nabla_y\mathfrak{u}_1)dydxdt-$$
$$\int_0^s\frac{1}{2}\frac{d}{dt}\int_D\int_Y a(x,y)(\operatorname{curl}w_0+\operatorname{curl}_y w_1)\cdot(\operatorname{curl}w_0+\operatorname{curl}_y w_1)dydxdt$$

due to $u_0=\frac{\partial w_0}{\partial t}$, $u_1=\frac{\partial w_1}{\partial t}$ and $\mathfrak{u}_1=\frac{\partial\mathfrak{w}_1}{\partial t}$. Using (2.9), we have

$$0=\frac{1}{2}\int_D\int_Y b(x,y)(u_0(s,x)+\nabla_y\mathfrak{u}_1(s,x,y))\cdot(u_0(s,x)+\nabla_y\mathfrak{u}_1(s,x,y))dydx+$$
$$\frac{1}{2}\int_D\int_Y a(x,y)(\operatorname{curl}w_0(0,x)+\operatorname{curl}_y w_1(0,x,y))\cdot(\operatorname{curl}w_0(0,x)+\operatorname{curl}_y w_1(0,x,y))dydx.$$

We thus deduce that $u_0(s,\cdot)=0$, $\nabla_y\mathfrak{u}_1(s,\cdot)=0\ \forall s$, $\operatorname{curl}w_0(0,\cdot)=0$ and $\operatorname{curl}_y w_1(0,\cdot)=0$. This means that

$$\int_0^s\operatorname{curl}u_0(\sigma,\cdot)d\sigma=0\ \text{ and }\ \int_0^s\operatorname{curl}_y u_1(\sigma,\cdot)d\sigma=0$$

for all s. Thus for all σ, $\operatorname{curl}u_0(\sigma,\cdot)=0$ and $\operatorname{curl}_y(\sigma,\cdot)=0$. □

Appendix B

We prove Lemma 3.6 in this appendix. From (3.4) and (3.8), we have

$$A(\boldsymbol{w}^L,\boldsymbol{v}^L)=A(\boldsymbol{u},\boldsymbol{v}^L)-B(\boldsymbol{w}^L-\boldsymbol{u},\boldsymbol{v}^L)=\int_D f(t,x)\cdot v_0^L(x)dx$$
$$-\int_D\int_Y b(x,y)\left(\frac{\partial^2u_0}{\partial t^2}+\frac{\partial^2}{\partial t^2}\nabla_y\mathfrak{u}_1\right)\cdot\left(v_0^L+\nabla_y\mathfrak{v}_1^L\right)dydx-B(\boldsymbol{q}^L,\boldsymbol{v}^L).$$

Averaging this equation at t_{m+1}, t_m and t_{m-1} with weights $\frac14,\frac12,\frac14$ respectively, and using (3.8), we get

$$\int_D\int_Y b(x,y)\left(\partial_t^2u_{0,m}^L+\nabla_y\partial_t^2\mathfrak{u}_{1,m}^L\right)\cdot\left(v_0^L+\nabla_y\mathfrak{v}_1^L\right)dydx+A(\boldsymbol{p}_{m,1/4}^L,\boldsymbol{v}^L)$$
$$=\int_D\int_Y b(x,y)\left(\frac{\partial^2u_{0,m,1/4}}{\partial t^2}+\frac{\partial^2}{\partial t^2}\nabla_y\mathfrak{u}_{1,m,1/4}\right)\cdot\left(v_0^L+\nabla_y\mathfrak{v}_1^L\right)dydx+B(\boldsymbol{q}_{m,1/4}^L,\boldsymbol{v}^L).$$

Thus

$$\int_D\int_Y b(x,y)\left(\partial_t^2p_{0,m}^L+\nabla_y\partial_t^2\mathfrak{p}_{1,m}^L\right)\cdot\left(v_0^L+\nabla_y\mathfrak{v}_1^L\right)dydx+A(\boldsymbol{p}_{m,1/4}^L,\boldsymbol{v}^L)$$
$$=\int_D\int_Y b(x,y)\left(\frac{\partial^2u_{0,m,1/4}}{\partial t^2}-\partial_t^2u_{0,m}+\left(\frac{\partial^2}{\partial t^2}\nabla_y\mathfrak{u}_{1,m,1/4}-\nabla_y\partial_t^2\mathfrak{u}_{1,m}\right)\right)\cdot\left(v_0^L+\nabla_y\mathfrak{v}_1^L\right)dydx$$
$$-\int_D\int_Y b(x,y)\left(\partial_t^2q_{0,m}^L+\nabla_y\partial_t^2q_{1,m}^L\right)\cdot\left(v_0^L+\nabla_y\mathfrak{v}_1^L\right)dydx+B(\boldsymbol{q}_{m,1/4}^L,\boldsymbol{v}^L).$$

We denote

$$s_{0,m}=\frac{\partial^2 u_{0,m,1/4}}{\partial t^2}-\partial_t^2 u_{0,m},\quad s_{1,m}=\frac{\partial^2}{\partial t^2}\nabla_y \mathfrak{u}_{1,m,1/4}-\partial_t^2\nabla_y \mathfrak{u}_{1,m}.$$

Let $\boldsymbol{v}^L=\delta_t\boldsymbol{p}^L_m$. Using the following relationships:

$$\partial_t^2 r_m=\frac{1}{\Delta t}(\partial_t r_{m+1/2}-\partial_t r_{m-1/2}),\qquad r_{m,1/4}=\frac{1}{2}(r_{m+1/2}+r_{m-1/2})$$
$$\delta_t r_m=\frac{1}{2}(\partial_t r_{m+1/2}+\partial_t r_{m-1/2})=\frac{1}{\Delta t}(r_{m+1/2}-r_{m-1/2}),$$

we have

$$\begin{aligned}
&\frac{1}{2\Delta t}\int_D\int_Y b(x,y)\left(\partial_t p^L_{0,m+1/2}-\partial_t p^L_{0,m-1/2}+\nabla_y\left(\partial_t \mathfrak{p}^L_{1,m+1/2}-\partial_t \mathfrak{p}^L_{1,m-1/2}\right)\right)\\
&\qquad\cdot\left(\partial_t p^L_{0,m+1/2}+\partial_t p^L_{0,m-1/2}+\nabla_y\left(\partial_t \mathfrak{p}^L_{1,m+1/2}+\partial_t \mathfrak{p}^L_{1,m-1/2}\right)\right)dydx\\
&+\frac{1}{2\Delta t}\int_D\int_Y a(x,y)\left(\operatorname{curl}\left(p^L_{0,m+1/2}+p^L_{0,m-1/2}\right)+\operatorname{curl}_y\left(p^L_{1,m+1/2}+p^L_{1,m-1/2}\right)\right)\\
&\qquad\cdot\left(\operatorname{curl}\left(p^L_{0,m+1/2}-p^L_{0,m-1/2}\right)+\operatorname{curl}_y\left(p^L_{1,m+1/2}-p^L_{1,m-1/2}\right)\right)dydx\\
&=\frac{1}{2}\int_D\int_Y b(x,y)\left(s_{0,m}-\partial_t^2 q^L_{0,m}+q^L_{0,m,1/4}+s_{1,m}-\nabla_y\partial_t^2\mathfrak{q}^L_{1,m}+\nabla_y\mathfrak{q}^L_{1,m,1/4}\right)\\
&\qquad\cdot\left(\partial_t p^L_{0,m+1/2}+\partial_t p^L_{0,m-1/2}+\nabla_y\partial_t\mathfrak{p}^L_{1,m+1/2}+\nabla_y\partial_t\mathfrak{p}^L_{1,m-1/2}\right)dydx.
\end{aligned}$$

We thus have

$$\begin{aligned}
&\frac{1}{2\Delta t}\left[B\left(\partial_t\boldsymbol{p}^L_{m+1/2},\partial_t\boldsymbol{p}^L_{m+1/2}\right)-B\left(\partial_t\boldsymbol{p}^L_{m-1/2},\partial_t\boldsymbol{p}^L_{m-1/2}\right)+A\left(\boldsymbol{p}^L_{m+1/2},\boldsymbol{p}^L_{m+1/2}\right)-A\left(\boldsymbol{p}^L_{m-1/2},\boldsymbol{p}^L_{m-1/2}\right)\right]\\
&\quad\le c\|s_{0,m}-\partial_t^2 q^L_{0,m}+q^L_{0,m,1/4}+s_{1,m}-\nabla_y\partial_t^2\mathfrak{q}^L_{1,m}+\nabla_y\mathfrak{q}^L_{1,m,1/4}\|_{H_1}\\
&\qquad\cdot\left\|\partial_t p^L_{0,m+1/2}+\partial_t p^L_{0,m-1/2}+\nabla_y\partial_t\mathfrak{p}^L_{1,m+1/2}+\nabla_y\partial_t\mathfrak{p}^L_{1,m-1/2}\right\|_{H_1}\\
&\quad\le\frac{c}{\gamma}\left(\|s_{0,m}\|_H^2+\|s_{1,m}\|_{H_1}^2+\|\partial_t^2 q^L_{0,m}\|_H^2+\|\nabla_y\partial_t^2\mathfrak{q}^L_{1,m}\|_{H_1}^2+\|q^L_{0,m,1/4}\|_H^2+\|\nabla_y\mathfrak{q}^L_{1,m,1/4}\|_{H_1}^2\right)\\
&\qquad+c\gamma\Big(\|\partial_t p^L_{0,m+1/2}\|_H^2+\|\partial_t p^L_{0,m-1/2}\|_H^2+\|\nabla_y\partial_t\mathfrak{p}^L_{1,m+1/2}\|_{H_1}^2+\|\nabla_y\partial_t\mathfrak{p}^L_{1,m-1/2}\|_{H_1}^2\Big).
\end{aligned}$$

For $1\le j<M$, summing this up for all $m=1,\dots,j$, we deduce

$$\begin{aligned}
&B(\partial_t\boldsymbol{p}^L_{j+1/2},\partial_t\boldsymbol{p}^L_{j+1/2})-B(\partial_t\boldsymbol{p}^L_{1/2},\partial_t\boldsymbol{p}^L_{1/2})+A(\boldsymbol{p}^L_{j+1/2},\boldsymbol{p}^L_{j+1/2})-A(\boldsymbol{p}^L_{1/2},\boldsymbol{p}^L_{1/2})\\
&\le\frac{c}{\gamma}2\Delta t\sum_{m=1}^{M-1}\left(\|s_{0,m}\|_H^2+\|s_{1,m}\|_{H_1}^2+\|\partial_t^2 q^L_{0,m}\|_H^2+\|\nabla_y\partial_t^2\mathfrak{q}^L_{1,m}\|_{H_1}^2+\|q^L_{0,m,1/4}\|_H^2+\|\nabla_y\mathfrak{q}^L_{1,m,1/4}\|_{H_1}^2\right)\\
&+c\gamma 4\Delta t M\left(\max_{1\le m<M}\|\partial_t p^L_{0,m+1/2}\|_H^2+\max_{1\le m<M}\|\partial_t\nabla_y\mathfrak{p}^L_{1,m+1/2}\|_{H_1}^2\right)+c\gamma 2\Delta t\left(\|\partial_t p^L_{0,1/2}\|_H^2+\|\partial_t\nabla_y\mathfrak{p}^L_{1,1/2}\|_{H_1}^2\right).
\end{aligned}$$

From (2.1), we have

$$\begin{aligned}
&\|\partial_t p^L_{0,j+1/2}\|_H^2+\|\partial_t\nabla_y\mathfrak{p}^L_{1,j+1/2}\|_{H_1}^2+\|\operatorname{curl}p^L_{0,j+1/2}\|_H^2+\|\operatorname{curl}_y p^L_{1,j+1/2}\|_{H_1}^2\\
&\quad\le\frac{c}{\gamma}2\Delta t\sum_{m=1}^{M-1}\left(\|s_{0,m}\|_H^2+\|s_{1,m}\|_{H_1}^2+\|\partial_t^2 q^L_{0,m}\|_H^2+\|\nabla_y\partial_t^2\mathfrak{q}^L_{1,m}\|_{H_1}^2+\|q^L_{0,m,1/4}\|_H^2+\|\nabla_y\mathfrak{q}^L_{1,m,1/4}\|_{H_1}^2\right)\\
&\qquad+c\gamma 4\Delta t M\left(\max_{1\le m<M}\|\partial_t p^L_{0,m+1/2}\|_H^2+\max_{1\le m<M}\|\partial_t\nabla_y\mathfrak{p}^L_{1,m+1/2}\|_{H_1}^2\right)\\
&\qquad+c\left(\|\partial_t p^L_{0,1/2}\|_H^2+\|\partial_t\nabla_y\mathfrak{p}^L_{1,1/2}\|_{H_1}^2+\|\operatorname{curl}p^L_{0,1/2}\|_H^2+\|\operatorname{curl}_y p^L_{1,1/2}\|_{H_1}^2\right).
\end{aligned}$$

Arguing as in the proof of Proposition 3.4, choosing γ sufficiently small, we deduce that

$$\begin{aligned}
&\|\partial_t p^L_{0,j+1/2}\|_H^2+\|\partial_t\nabla_y\mathfrak{p}^L_{1,j+1/2}\|_{H_1}^2+\|\operatorname{curl}p^L_{0,j+1/2}\|_H^2+\|\operatorname{curl}_y p^L_{1,j+1/2}\|_{H_1}^2\\
&\le c2\Delta t\sum_{m=1}^{M-1}\left(\|s_{0,m}\|_H^2+\|s_{1,m}\|_{H_1}^2+\|\partial_t^2 q^L_{0,m}\|_H^2+\|\nabla_y\partial_t^2\mathfrak{q}^L_{1,m}\|_{H_1}^2+\|q^L_{0,m,1/4}\|_H^2+\|\nabla_y\mathfrak{q}^L_{1,m,1/4}\|_{H_1}^2\right)\\
&+c\left(\|\partial_t p^L_{0,1/2}\|_H^2+\|\partial_t\nabla_y\mathfrak{p}^L_{1,1/2}\|_{H_1}^2+\|\operatorname{curl}p^L_{0,1/2}\|_H^2+\|\operatorname{curl}_y p^L_{1,1/2}\|_{H_1}^2\right).
\end{aligned}$$

Following Dupont [20], using the integral formula of the remainder of Taylor expansion, we have,

$$\partial_t^2 q_{0,m}^L = (\Delta t)^{-2}\int_{-\Delta t}^{\Delta t}(\Delta t-|\tau|)\frac{\partial^2 q_0^L}{\partial t^2}(t_m+\tau)d\tau,$$

and similarly,

$$\partial_t^2(\nabla_y \mathfrak{q}_{1,m}^L) = (\Delta t)^{-2}\int_{-\Delta t}^{\Delta t}(\Delta t-|\tau|)\frac{\partial^2 \nabla_y \mathfrak{q}_1^L}{\partial t^2}(t_m+\tau)d\tau.$$

Using Cauchy–Schwarz inequality, we have

$$\sum_{m=1}^{M-1}\|\partial_t^2 q_{0,m}^L\|_H^2\Delta t \le \frac{4}{3}\left\|\frac{\partial^2 q_0^L}{\partial t^2}\right\|_{L^2((0,T),H)}^2,$$

and similarly, we have

$$\sum_{m=1}^{M}\|\partial_t^2\nabla_y \mathfrak{q}_{1,m}^L\|_{H_1}^2\Delta t \le \frac{4}{3}\left\|\frac{\partial^2}{\partial t^2}\nabla_y \mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2.$$

We write

$$s_{0,m} = \frac{1}{4}\int_0^{\Delta t}\left(1-2\left(1-\frac{|\tau|}{\Delta t}\right)^2\right)\frac{\partial^3 u_0}{\partial t^3}(t_m+\tau)d\tau - \frac{1}{4}\int_{-\Delta t}^{0}\left(1-2\left(1-\frac{|\tau|}{\Delta t}\right)^2\right)\frac{\partial^3 u_0}{\partial t^3}(t_m+\tau)d\tau$$

and

$$s_{1,m} = \frac{1}{4}\int_0^{\Delta t}\left(1-2\left(1-\frac{|\tau|}{\Delta t}\right)^2\right)\frac{\partial^3 \nabla_y \mathfrak{u}_1}{\partial t^3}(t_m+\tau)d\tau - \frac{1}{4}\int_{-\Delta t}^{0}\left(1-2\left(1-\frac{|\tau|}{\Delta t}\right)^2\right)\frac{\partial^3 \nabla_y \mathfrak{u}_1}{\partial t^3}(t_m+\tau)d\tau.$$

Therefore

$$\|s_{0,m}\|_H^2 \le c\Delta t\int_{t_{m-1}}^{t_{m+1}}\left\|\frac{\partial^3 u_0}{\partial t^3}(\tau)\right\|_H^2 d\tau,\quad \|s_{1,m}\|_{H_1}^2 \le c\Delta t\int_{t_{m-1}}^{t_{m+1}}\left\|\frac{\partial^3 \nabla_y \mathfrak{u}_1}{\partial t^3}(\tau)\right\|_{H_1}^2 d\tau.$$

We also have

$$\|q_{0,m,1/4}^L\|_H \le \max_{t\in[0,T]}\|q_0^L(t)\|_H \text{ and } \|\nabla_y \mathfrak{q}_{1,m,1/4}^L\|_{H_1} \le \max_{t\in[0,T]}\|\nabla_y \mathfrak{q}_1^L(t)\|_{H_1}.$$

We thus deduce

$$\begin{aligned}
&\|\partial_t p_{0,j+1/2}^L\|_H^2 + \|\partial_t\nabla_y \mathfrak{p}_{1,j+1/2}^L\|_{H_1}^2 + \|\mathrm{curl}\, p_{0,j+1/2}^L\|_H^2 + \|\mathrm{curl}_y \mathfrak{p}_{1,j+1/2}^L\|_{H_1}^2\\
&\quad\le c\Bigg[(\Delta t)^2\left\|\frac{\partial^3 u_0}{\partial t^3}\right\|_{L^2((0,T),H)} + (\Delta t)^2\left\|\frac{\partial^3\nabla_y \mathfrak{u}_1}{\partial t^3}\right\|_{L^2((0,T),H_1)}^2 + \left\|\frac{\partial^2 q_0^L}{\partial t^2}\right\|_{L^2((0,T),H)}^2\\
&\qquad+\left\|\frac{\partial^2}{\partial t^2}\nabla_y \mathfrak{q}_1^L\right\|_{L^2((0,T),H_1)}^2 + \|q_0^L\|_{L^\infty((0,T),H)}^2 + \|\nabla_y \mathfrak{q}_1^L\|_{L^\infty((0,T),H_1)}^2\Bigg]\\
&\qquad+c\left(\|\partial_t p_{0,1/2}^L\|_H^2 + \|\partial_t\nabla_y \mathfrak{p}_{1,1/2}^L\|_{H_1}^2 + \|\mathrm{curl}\, p_{0,1/2}^L\|_H^2 + \|\mathrm{curl}_y \mathfrak{p}_{1,1/2}^L\|_{H_1}^2\right).
\end{aligned}$$

When

$$\frac{\partial^4 u_0}{\partial t^4}\in L^2((0,T),H) \text{ and } \frac{\partial^4}{\partial t^4}\nabla_y \mathfrak{u}_1 \in L^2((0,T),H_1),$$

we have

$$s_{0,m} = \frac{1}{12}\int_{-\Delta t}^{\Delta t}(\Delta t-|\tau|)\left(3-2\left(1-\frac{|\tau|}{\Delta t}\right)^2\right)\frac{\partial^4 u_0}{\partial t^4}(t_m+\tau)d\tau,$$

and

$$s_{1,m} = \frac{1}{12}\int_{-\Delta t}^{\Delta t}(\Delta t-|\tau|)\left(3-2\left(1-\frac{|\tau|}{\Delta t}\right)^2\right)\frac{\partial^4 \nabla_y \mathfrak{u}_1}{\partial t^4}(t_m+\tau)d\tau.$$

Therefore

$$\|s_{0,m}\|_H^2 \le c(\Delta t)^3\int_{t_{m-1}}^{t_{m+1}}\left\|\frac{\partial^4 u_0}{\partial t^4}(\tau)\right\|_H^2 d\tau \text{ and } \|s_{1,m}\|_{H_1}^2 \le c(\Delta t)^3\int_{t_{m-1}}^{t_{m+1}}\left\|\frac{\partial^4 \nabla_y \mathfrak{u}_1}{\partial t^4}(\tau)\right\|_{H_1}^2 d\tau.$$

Thus we have

$$\|\partial_t p^L_{0,j+1/2}\|^2_H + \|\partial_t \nabla_y \mathfrak{p}^L_{1,j+1/2}\|^2_{H_1} + \|\mathrm{curl}\, p^L_{0,j+1/2}\|^2_H + \|\mathrm{curl}_y p^L_{1,j+1/2}\|^2_{H_1}$$
$$\leq c\left[(\Delta t)^4 \left\|\frac{\partial^4 u_0}{\partial t^4}\right\|^2_{L^2((0,T),H)} + (\Delta t)^4 \left\|\frac{\partial^4 \nabla_y \mathfrak{u}_1}{\partial t^4}\right\|^2_{L^2((0,T),H_1)} + \left\|\frac{\partial^2 q^L_0}{\partial t^2}\right\|^2_{L^2((0,T),H)}\right.$$
$$\left. + \left\|\frac{\partial^2}{\partial t^2}\nabla_y \mathfrak{q}^L_1\right\|^2_{L^2((0,T),H_1)} + \|q^L_0\|^2_{L^\infty((0,T),H)} + \|\nabla_y \mathfrak{q}^L_1\|^2_{L^\infty((0,T),H_1)}\right]$$
$$+ c\left(\|\partial_t p^L_{0,1/2}\|^2_H + \|\partial_t \nabla_y \mathfrak{p}^L_{1,1/2}\|^2_{H_1} + \|\mathrm{curl}\, p^L_{0,1/2}\|^2_H + \|\mathrm{curl}_y p^L_{1,1/2}\|^2_{H_1}\right). \quad \square$$

Appendix C

We prove Proposition 6.1 in this appendix. We note that $\frac{\partial u^\varepsilon}{\partial t}$ satisfies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial u^\varepsilon}{\partial t}\right) + \mathrm{curl}\left(a^\varepsilon \mathrm{curl}\,\frac{\partial u^\varepsilon}{\partial t}\right) = \frac{\partial f}{\partial t} \tag{C.1}$$

with the initial condition $\frac{\partial u^\varepsilon}{\partial t}(0) = g_1 \in W$ and $\frac{\partial^2 u^\varepsilon}{\partial t^2}(0) = f(0) \in H$ (due to $g_0 = 0$). We therefore deduce that

$$\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|_{L^\infty((0,T),W)} \leq c\left(\left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T),H)} + \|g_1\|_V + \|f(0)\|_H\right)$$

where c only depends on the constants α and β in (2.1). Thus $\frac{\partial u^\varepsilon}{\partial t}$ is uniformly bounded in $L^\infty((0,T),W)$ for all ε. We consider a set of M open cubes Q_i ($i = 1,\ldots,M$) of size ε^{s_1}, where $s_1 = 1/(1+s)$, such that $D \subset \bigcup_{i=1}^M Q_i$ and $Q_i \cap D \neq \emptyset$. Each cube Q_i intersects with only a finite number, which does not depend on ε, of other cubes. We consider a partition of unity that consists of M functions ρ_i such that ρ_i has support in Q_i, $\sum_{i=1}^M \rho_i(x) = 1$ for all $x \in D$ and $|\nabla \rho_i| \leq c\varepsilon^{-s_1}$ for all x. For $r = 1,2,3$ and $i = 1,\ldots,M$, we denote

$$U_i^r(t) = \frac{1}{|Q_i|}\int_{Q_i} \mathrm{curl}\, u_0(t,x)_r dx, \quad \text{and} \quad V_i^r(t) = \frac{1}{|Q_i|}\int_{Q_i} u_0(t,x)_r dx$$

(as $u_0 \in H^s(D)^3$ and $\mathrm{curl}\, u_0 \in H^s(D)^3$, for the Lipschitz domain D, we can extend each of them, separately, continuously outside D and understand u_0 and curl u_0 as these extensions (see Wloka [21] Theorem 5.6)). Let U_i and V_i denote the vector (U_i^1, U_i^2, U_i^3) and (V_i^1, V_i^2, V_i^3) respectively.

We consider the function

$$u_1^\varepsilon(t,x) = u_0(t,x) + \varepsilon N^r\left(x,\frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x) + \varepsilon\nabla\left[w^r\left(x,\frac{x}{\varepsilon}\right)\left(V_j^r(t)\rho_j(x) - g_1^r(x)t\right)\right].$$

The same argument as in the proof of equation (A.5) in [18] shows that for $s_1 = \frac{1}{s+1}$ we have

$$\left\|b^\varepsilon \frac{\partial^2 u_1^\varepsilon}{\partial t^2} - b^0 \frac{\partial^2 u_0}{\partial t^2}\right\|_{L^\infty((0,T),W')} + \left\|\mathrm{curl}\,(a^\varepsilon \mathrm{curl}\, u_1^\varepsilon) - \mathrm{curl}\,(a^0 \mathrm{curl}\, u_0)\right\|_{L^\infty((0,T),W')} \leq c\varepsilon^{\frac{s}{s+1}} \tag{C.2}$$

and

$$\|\mathrm{curl}\,(a^\varepsilon \mathrm{curl}\,\frac{\partial}{\partial t}u_1^\varepsilon) - \mathrm{curl}\,(a^0 \mathrm{curl}\,\frac{\partial}{\partial t}u_0)\|_{L^\infty((0,T),W')} \leq c\varepsilon^{\frac{s}{(s+1)}}. \tag{C.3}$$

Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ such that $\tau^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\sup_{x\in D}\varepsilon|\nabla\tau^\varepsilon(x)| < c$ where c is independent of ε. Let $D^\varepsilon \subset D$ be the ε neighbourhood of ∂D. Let

$$w_1^\varepsilon(t,x) = u_0(t,x) + \varepsilon\tau^\varepsilon(x)N^r\left(x,\frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x) + \varepsilon\nabla\left[\tau^\varepsilon(x)w^r\left(x,\frac{x}{\varepsilon}\right)\left(V_j^r(t)\rho_j(x) - g_1^r(x)t\right)\right]. \tag{C.4}$$

The same argument as in Appendix A page 266 of [18] shows that

$$\|\mathrm{curl}\,(u_1^\varepsilon - w_1^\varepsilon)\|_{L^\infty((0,T),H)} \leq c\varepsilon^{\frac{s}{1+s}}, \quad \|\frac{\partial u_1^\varepsilon}{\partial t} - \frac{\partial w_1^\varepsilon}{\partial t}\|_{L^\infty((0,T),H)} \leq c\varepsilon^{\frac{s}{1+s}}.$$

From this and (C.2), we have

$$\|b^\varepsilon \frac{\partial^2 w_1^\varepsilon}{\partial t^2} - b^0 \frac{\partial^2 u_0}{\partial t^2}\|_{L^\infty((0,T),W')} \leq c\varepsilon^{\frac{s}{1+s}}.$$

Using

$$b^\varepsilon \frac{\partial u^\varepsilon}{\partial t^2} + \mathrm{curl}\,(a^\varepsilon \mathrm{curl}\, u^\varepsilon) = b^0 \frac{\partial^2 u_0}{\partial t^2} + \mathrm{curl}\,(a^0 \mathrm{curl}\, u_0)$$

we have

$$b^\varepsilon \frac{\partial^2(u^\varepsilon - w_1^\varepsilon)}{\partial t^2} + \mathrm{curl}\,(a^\varepsilon \mathrm{curl}\,(u^\varepsilon - w_1^\varepsilon))$$
$$= b^0 \frac{\partial^2 u_0}{\partial t^2} - b^\varepsilon \frac{\partial^2 w_1^\varepsilon}{\partial t^2} + \mathrm{curl}\,(a^\varepsilon \mathrm{curl}\,(u_1^\varepsilon - w_1^\varepsilon)) + \mathrm{curl}\,(a^0 \mathrm{curl}\,u_0) - \mathrm{curl}\,(a^\varepsilon \mathrm{curl}\,u_1^\varepsilon).$$

As shown above, $\frac{\partial u^\varepsilon}{\partial t}$ is uniformly bounded in $L^\infty((0,T),W)$ with respect to ε. For w_1^ε, we have

$$\frac{\partial w_1^\varepsilon}{\partial t}(t,x) = \frac{\partial u_0}{\partial t}(t,x) + \varepsilon\tau^\varepsilon(x) N^r\left(x,\frac{x}{\varepsilon}\right)\frac{\partial U_j^r}{\partial t}(t)\rho_j(x)$$
$$+\left(\varepsilon\nabla\tau^\varepsilon(x) w^r\left(x,\frac{x}{\varepsilon}\right) + \varepsilon\tau^\varepsilon(x)\nabla_x w^r\left(x,\frac{x}{\varepsilon}\right) + \tau^\varepsilon(x)\nabla_y w^r\left(x,\frac{x}{\varepsilon}\right)\right)\left(\frac{\partial V_j^r}{\partial t}(t)\rho_j(x) - g_{1r}\right)$$
$$+\varepsilon\tau^\varepsilon(x) w^r\left(x,\frac{x}{\varepsilon}\right)\left(\frac{\partial V_j^r}{\partial t}(t)\nabla\rho_j(x) - \nabla g_{1r}(x)\right)$$

and

$$\mathrm{curl}\,\frac{\partial w_1^\varepsilon}{\partial t}(t,x) = \mathrm{curl}\,\frac{\partial u_0}{\partial t}(t,x) + \tau^\varepsilon(x)\frac{\partial U_j^r}{\partial t}(t)\rho_j(x)\left(\varepsilon\,\mathrm{curl}_x N^r\left(x,\frac{x}{\varepsilon}\right) + \mathrm{curl}_y N^r\left(x,\frac{x}{\varepsilon}\right)\right)$$
$$+\frac{\partial U_j^r}{\partial t}(t)\left(\varepsilon\nabla\tau^\varepsilon(x)\rho_j(x) + \varepsilon\tau^\varepsilon(x)\nabla\rho_j(x)\right)\times N^r\left(x,\frac{x}{\varepsilon}\right).$$

As $\frac{\partial u_0}{\partial t} \in L^\infty((0,T),H(\mathrm{curl},D))$, $\|\frac{\partial U_j^r}{\partial t}(t)\rho_j\|_{L^2(D)} \le c$, $\|\frac{\partial V_j^r}{\partial t}(t)\rho_j\|_{L^2(D)} \le c$, $\|\frac{\partial U_j^r}{\partial t}(t)\nabla\rho_j\|_{L^2(D)} \le c\varepsilon^{-s_1}$ and $\|\frac{\partial V_j^r}{\partial t}(t)\nabla\rho_j\|_{L^2(D)} \le c\varepsilon^{-s_1}$, curl $\frac{\partial w_1^\varepsilon}{\partial t}(t)$ and $\frac{\partial w_1^\varepsilon}{\partial t}(t)$ are uniformly bounded in H with respect to ε, i.e. $\frac{\partial w_1^\varepsilon}{\partial t}$ is uniformly bounded in $L^\infty((0,T),H_0(\mathrm{curl},D))$. We have

$$\int_0^t \left\langle b^0\frac{\partial^2 u_0}{\partial t^2}(s) - b^\varepsilon\frac{\partial^2 w_1^\varepsilon}{\partial t^2}(s), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds \le c\varepsilon^{\frac{s}{1+s}}\sup_{0\le s\le T}\left\|\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\|_H dt.$$

We also have

$$\int_0^t \left\langle \mathrm{curl}\,(a^0\mathrm{curl}\,u_0(s)) - \mathrm{curl}\,(a^\varepsilon\mathrm{curl}\,u_1^\varepsilon(s)), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds =$$
$$\int_0^t \frac{\partial}{\partial s}\left\langle \mathrm{curl}\,(a^0\mathrm{curl}\,u_0(s)) - \mathrm{curl}\,(a^\varepsilon\mathrm{curl}\,u_1^\varepsilon(s)), (u^\varepsilon - w_1^\varepsilon)(s)\right\rangle ds -$$
$$\int_0^t \left\langle \mathrm{curl}\,(a^0\mathrm{curl}\,\frac{\partial u_0}{\partial s}(s)) - \mathrm{curl}\,(a^\varepsilon\mathrm{curl}\,\frac{\partial u_1^\varepsilon}{\partial s}(s)), (u^\varepsilon - w_1^\varepsilon)(s)\right\rangle ds.$$

Since $u_0(0) = 0$ and $u_1^\varepsilon(0) = 0$, together with (C.3) we have that

$$\left|\int_0^t \left\langle \mathrm{curl}\,(a^0\mathrm{curl}\,u_0(s)) - \mathrm{curl}\,(a^\varepsilon\mathrm{curl}\,u_1^\varepsilon(s)), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds\right|$$
$$\le \left|\left\langle \mathrm{curl}\,(a^0\mathrm{curl}\,u_0(t)) - \mathrm{curl}\,(a^\varepsilon\mathrm{curl}\,u_1^\varepsilon(t)), (u^\varepsilon - w_1^\varepsilon)(t)\right\rangle\right| + c\int_0^t \varepsilon^{\frac{s}{s+1}}\|(u^\varepsilon - w_1^\varepsilon)(s)\|_W$$
$$\le c\varepsilon^{\frac{s}{s+1}}\sup_{0\le t\le T}\|(u^\varepsilon - w_1^\varepsilon)(t)\|_W.$$

Now we estimate

$$\int_0^t \left\langle \mathrm{curl}\,(a^\varepsilon\mathrm{curl}\,(u_1^\varepsilon(s) - w_1^\varepsilon(s))), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds.$$

From [18] page 266, as u_0 and $\frac{\partial u_0}{\partial t}$ belong to $L^\infty((0,T),H^s(D))$, we have that

$$\|U_j^r(t)\rho_j\|_{L^2(D^\varepsilon)} \le c\varepsilon^{(1-s_1+s_1 s)/2},\quad \|U_j^r(t)\nabla\rho_j\|_{L^2(D^\varepsilon)} \le c\varepsilon^{(1-3s_1+ss_1)/2},\quad \|\frac{\partial U_j^r}{\partial t}(t)\rho_j\|_{L^2(D^\varepsilon)} \le c\varepsilon^{(1-s_1+s_1 s)/2}.$$

We note that

$$\mathrm{curl}\,(u_1^\varepsilon - w_1^\varepsilon) = \varepsilon\,\mathrm{curl}_x N^r\left(x,\frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x)(1-\tau^\varepsilon(x)) + \mathrm{curl}_y N^r\left(x,\frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x)(1-\tau^\varepsilon(x))$$
$$-\varepsilon U_j^r(t)\rho_j(x)\nabla\tau^\varepsilon(x)\times N^r\left(x,\frac{x}{\varepsilon}\right) + \varepsilon(1-\tau^\varepsilon(x))U_j^r(t)\nabla\rho_j(x)\times N^r\left(x,\frac{x}{\varepsilon}\right). \quad \text{(C.5)}$$

We have

$$\left|\int_0^t \left\langle \varepsilon \mathrm{curl}\, \left(a^\varepsilon \mathrm{curl}_x N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(t)\rho_j(x)(1-\tau^\varepsilon(x))\right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds\right|$$
$$\le c\varepsilon \int_0^t \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\|_W ds \le c\varepsilon;$$

and

$$\left|\int_0^t \left\langle \varepsilon \mathrm{curl}\, \left(a^\varepsilon(1-\tau^\varepsilon(s))U_j^r(t)\nabla\rho_j(x)\times N^r\left(x, \frac{x}{\varepsilon}\right)\right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle dx\right|$$
$$\le c\varepsilon \int_0^t \|U_j^r(s)\nabla\rho_j\|_{L^2(D^\varepsilon)^3}\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\|_W \le c\varepsilon\varepsilon^{\frac{1-3s_1+ss_1}{2}} = c\varepsilon^{\frac{2s}{s+1}}$$

where we have used the fact that $\|U_j^r(x)\nabla\rho_j\|_{L^2(D^\varepsilon)^3} \le c\varepsilon^{\frac{1-3s_1+ss_1}{2}}$ (see [18] page 266). For the other two terms in (C.5), we have

$$\int_0^t \left\langle \mathrm{curl}\, \left(a^\varepsilon \mathrm{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(s)\rho_j(x)(1-\tau^\varepsilon(x))\right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds$$
$$= \int_0^t \frac{\partial}{\partial s}\int_D a^\varepsilon(x)\mathrm{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(s)\rho_j(x)(1-\tau^\varepsilon(x))\cdot \mathrm{curl}\,(u^\varepsilon(s) - w_1^\varepsilon(s))dxds -$$
$$\int_0^t\int_D a^\varepsilon(x)\mathrm{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right)\frac{\partial U_j^r}{\partial s}(s)\rho_j(x)(1-\tau^\varepsilon(x))\cdot \mathrm{curl}\,(u^\varepsilon(s) - w_1^\varepsilon(s))ds$$
$$= \int_D a^\varepsilon \mathrm{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(t)\rho_j(x)(1-\tau^\varepsilon(x))\mathrm{curl}\,(u^\varepsilon(t) - w_1^\varepsilon(t)) -$$
$$\int_D a^\varepsilon \mathrm{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(0)\rho_j(x)(1-\tau^\varepsilon(x))\mathrm{curl}\,(u^\varepsilon(0) - w_1^\varepsilon(0)) -$$
$$\int_0^t\int_D a^\varepsilon(x)\mathrm{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right)\frac{\partial U_j^r}{\partial s}(s)\rho_j(x)(1-\tau^\varepsilon(x))\cdot \mathrm{curl}\,(u^\varepsilon(s) - w_1^\varepsilon(s))ds.$$

We then have

$$\left|\int_0^t \left\langle \mathrm{curl}\, \left(a^\varepsilon \mathrm{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(s)\rho_j(x)(1-\tau^\varepsilon(x))\right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds\right|$$
$$\le \|U_j^r(t)\rho_j\|_{L^2(D^\varepsilon)}\|\mathrm{curl}\,(u^\varepsilon(t) - w_1^\varepsilon(t))\|_{L^2(D)^3} + c\int_0^t \left\|\frac{\partial U_j^r(s)}{\partial s}\rho_j\right\|_{L^2(D^\varepsilon)}\|\mathrm{curl}\,(u^\varepsilon(s) - w_1^\varepsilon(s))\|_{L^2(D)^3}ds$$
$$\le c\varepsilon^{\frac{s}{1+s}}\sup_{0\le t\le T}\|\mathrm{curl}\,(u^\varepsilon(t) - w_1^\varepsilon(t))\|_{L^2(D)^3}.$$

Similarly, we have

$$\left|\int_0^t \left\langle \varepsilon \mathrm{curl}\, \left(a^\varepsilon U_j^r(s)\rho_j(x)\nabla\tau^\varepsilon(x)\times N^r\left(x, \frac{x}{\varepsilon}\right)\right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds\right|$$
$$\le c\varepsilon^{\frac{s}{1+s}}\sup_{0\le t\le T}\|\mathrm{curl}\,(u^\varepsilon(t) - w_1^\varepsilon(t))\|_{L^2(D)^3}.$$

Therefore,

$$\left|\int_0^t \left\langle \mathrm{curl}\,(a^\varepsilon \mathrm{curl}\,(u_1^\varepsilon(s) - w_1^\varepsilon(s))), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle\right| \le c\varepsilon^{\frac{s}{s+1}}\sup_{0\le t\le T}\|\mathrm{curl}\,(u^\varepsilon(t) - w_1^\varepsilon(t))\|_{L^2(D)^3}.$$

We then deduce

$$\int_0^t \left\langle b^\varepsilon\frac{\partial^2(u^\varepsilon - w_1^\varepsilon)}{\partial s^2}(s) + \mathrm{curl}\,(a^\varepsilon \mathrm{curl}\,(u^\varepsilon - w_1^\varepsilon))(s), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial s}(s)\right\rangle ds$$
$$\le c\varepsilon + c\varepsilon^{\frac{2s}{s+1}} + c\varepsilon^{\frac{s}{1+s}}\sup_{0\le t\le T}\left\|\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t)\right\|_{L^2(D)^3} + c\varepsilon^{\frac{s}{1+s}}\sup_{0\le t\le T}\left\|\mathrm{curl}\,(u^\varepsilon(t) - w_1^\varepsilon(t))\right\|_{L^2(D)^3}.$$

Therefore

$$\frac{1}{2}\int_D b^\varepsilon(x)\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t)\cdot\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t)dx + \frac{1}{2}\int_D a^\varepsilon(x)\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t)\cdot\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t)dx$$
$$\leq c\varepsilon^{\frac{2s}{s+1}} + c\varepsilon^{\frac{s}{1+s}}\sup_{0\leq t\leq T}\left\|\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t)\right\|_{L^2(D)^3} + c\varepsilon^{\frac{s}{1+s}}\sup_{0\leq t\leq T}\left\|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\right\|_{L^2(D)^3}$$
$$+\int_D b^\varepsilon(x)\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0)\cdot\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0)dx$$

(note that $u^\varepsilon(0) = w_1^\varepsilon(0) = 0$). We have

$$\left\|\frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0)\right\|_H$$
$$= \left\|\frac{\partial u_0}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0)\right\|_H$$
$$= \left\|\varepsilon\tau^\varepsilon N^r\left(x, \frac{x}{\varepsilon}\right)\frac{\partial U_j^r}{\partial t}(0)\rho_j(x) + \varepsilon\nabla\left(\tau^\varepsilon w^r\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial V_j^r}{\partial t}(0)\rho_j(x) - g_1^r(x)\right)\right)\right\|_H$$
$$= \Bigg\|\varepsilon\tau^\varepsilon N^r\left(x, \frac{x}{\varepsilon}\right)\frac{\partial U_j^r}{\partial t}(0)\rho_j(x)$$
$$+\left(\varepsilon\nabla\tau^\varepsilon w^r\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\tau^\varepsilon\nabla_x w^r\left(x, \frac{x}{\varepsilon}\right) + \tau^\varepsilon\nabla_y w^r\left(x, \frac{x}{\varepsilon}\right)\right)\left(\frac{\partial V_j^r}{\partial t}(0)\rho_j(x) - g_{1r}(x)\right)$$
$$+\varepsilon\tau^\varepsilon w^r\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial V_j^r}{\partial t}(0)\nabla\rho_j(x) - \nabla g_{1r}(x)\right)\Bigg\|_H$$
$$\leq c\varepsilon + c\left\|\frac{\partial V_j^r}{\partial t}(0)\rho_j(x) - g_{1r}(x)\right\|_H + c\varepsilon\left\|\frac{\partial V_j^r}{\partial t}(0)\nabla\rho_j - \nabla g_{1r}\right\|_H.$$

As $g_1 \in H^1(D)^3$, the proof of equation (A4) in [18] shows that

$$\left\|\frac{\partial V_j^r}{\partial t}(0)\rho_j(x) - g_{1r}(x)\right\|_H \leq \varepsilon^{s_1} < c\varepsilon^{\frac{s}{1+s}}.$$

Further the argument in [18] page 266 gives

$$\left\|\frac{\partial V_j^r}{\partial t}(0)\nabla\rho_j(x)\right\|_{L^2(D)^3} \leq c\varepsilon^{-s_1}.$$

Thus

$$\left\|\frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0)\right\|_H \leq c\varepsilon^{\frac{s}{1+s}} + c\varepsilon^{1-s_1} \leq c\varepsilon^{\frac{s}{s+1}}.$$

Using (2.1) we get

$$\left\|\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t)\right\|_H^2 + \left\|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\right\|_H^2$$
$$\leq c\varepsilon^{\frac{2s}{s+1}} + c\varepsilon^{\frac{s}{s+1}}\max_{0\leq t\leq T}\left\|\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}\right\|_H + c\varepsilon^{\frac{s}{s+1}}\max_{0\leq t\leq T}\|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H.$$

From this we deduce that for all $t \in (0, T)$

$$\left\|\frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t)\right\|_H + \left\|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\right\|_H \leq c\varepsilon^{\frac{s}{s+1}}. \tag{C.6}$$

We have

$$\frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t}(t) = \varepsilon(1 - \tau^\varepsilon(x))N^r\left(x, \frac{x}{\varepsilon}\right)\frac{\partial U_j^r}{\partial t}(t)\rho_j(x)$$
$$+\left[-\varepsilon\nabla\tau^\varepsilon(x)w^r\left(x, \frac{x}{\varepsilon}\right) + \varepsilon(1 - \tau^\varepsilon(x))\nabla_x w^r\left(x, \frac{x}{\varepsilon}\right) + (1 - \tau^\varepsilon(x))\nabla_y w^r\left(x, \frac{x}{\varepsilon}\right)\right]\left(\frac{\partial V_j^r}{\partial t}(t)\rho_j(x) - g_{1r}\right)$$
$$+\varepsilon(1 - \tau^\varepsilon(x))w^r\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial V_j^r}{\partial t}(t)\nabla\rho_j(x) - \nabla g_{1r}\right).$$

Therefore, using $g_1 \in H^1(D)^3$ we get

$$\left\|\frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t}(t)\right\|_H \le c\varepsilon \left\|\frac{\partial U_j^r}{\partial t}(t)\rho_j\right\|_{L^2(D^\varepsilon)^3} + \left\|\frac{\partial V_j^r}{\partial t}(t)\rho_j\right\|_{L^2(D^\varepsilon)^3} + c\|g_1\|_{L^2(D^\varepsilon)^3} + c\varepsilon\left\|\frac{\partial V_j^r}{\partial t}(t)\nabla\rho_j - \nabla g_1^r\right\|_{L^2(D^\varepsilon)^3}.$$

As $\frac{\partial}{\partial t}\operatorname{curl} u_0 \in L^\infty((0,T), H^s(D)^3)$, $\frac{\partial}{\partial t}u_0 \in L^\infty((0,T), H^s(D)^3)$ and $g_1 \in H^1(D)^3$, the same argument of [18] page 266 shows that

$$\left\|\frac{\partial U_j^r}{\partial t}(t)\rho_j\right\|_{L^2(D^\varepsilon)^3} \le c, \quad \left\|\frac{\partial V_j^r}{\partial t}(t)\rho_j\right\|_{L^2(D^\varepsilon)^3} \le c\varepsilon^{\frac{1-s_1+ss_1}{2}},$$

$$\left\|\frac{\partial V_j^r}{\partial t}(t)\nabla\rho_j\right\|_{L^2(D^\varepsilon)^3} \le c\varepsilon^{\frac{1-3s_1+ss_1}{2}}, \quad \|g_1\|_{L^2(D^\varepsilon)^3} \le c\varepsilon^{1/2}.$$

Therefore

$$\left\|\frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t}(t)\right\|_H \le c\varepsilon^{\frac{1-s_1+ss_1}{2}} + c\varepsilon\varepsilon^{\frac{1-3s_1+ss_1}{2}} + c\varepsilon^{1/2} \le c\varepsilon^{\frac{s}{s+1}}. \tag{C.7}$$

Thus

$$\left\|\frac{\partial(u^\varepsilon - u_1^\varepsilon)}{\partial t}(t)\right\|_H + \left\|\operatorname{curl}(u^\varepsilon(t) - u_1^\varepsilon(t))\right\|_H \le c\varepsilon^{\frac{s}{s+1}}. \tag{C.8}$$

We note that

$$\left\|\varepsilon \operatorname{curl}_x N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r\rho_j\right\|_H \le c\varepsilon, \quad \text{and} \quad \left\|\varepsilon N^r\left(x, \frac{x}{\varepsilon}\right) \times (U_j^r\nabla\rho_j)\right\|_H \le c\varepsilon\varepsilon^{-s_1} = c\varepsilon^{\frac{s}{s+1}}$$

so from

$$\begin{aligned}\operatorname{curl} u_1^\varepsilon = \operatorname{curl} u_0 &+ \varepsilon \operatorname{curl}_x N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(t)\rho_j(x) + \varepsilon U_j^r(t)\nabla\rho_j(x) \times N^r\left(x, \frac{x}{\varepsilon}\right) + \\ &\operatorname{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r(t)\rho_j(x),\end{aligned}$$

$$\|\operatorname{curl} u_1^\varepsilon - [\operatorname{curl} u_0 + \operatorname{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) U_j^r\rho_j]\|_{L^\infty((0,T),H)} \le c\varepsilon^{\frac{s}{s+1}}.$$

As

$$\|(\operatorname{curl} u_0)_r - (U_j^r\rho_j)\|_{L^\infty((0,T),H)} \le c^{\frac{s}{1+s}},$$

(equation (A4) of [18]), we deduce that

$$\|\operatorname{curl} u_1^\varepsilon - [\operatorname{curl} u_0 + \operatorname{curl}_y N^r\left(x, \frac{x}{\varepsilon}\right) \operatorname{curl} u_0(x)_r]\|_H \le c\varepsilon^{\frac{s}{s+1}}. \tag{C.9}$$

We further have

$$\begin{aligned}\frac{\partial u_1^\varepsilon}{\partial t}(t) = \frac{\partial u_0}{\partial t}(t) &+ \varepsilon N^r\left(x, \frac{x}{\varepsilon}\right)\frac{\partial U_j^r}{\partial t}(t)\rho_j(x) + \left(\varepsilon\nabla_x w^r\left(x, \frac{x}{\varepsilon}\right) + \nabla_y w^r\left(x, \frac{x}{\varepsilon}\right)\right)\left(\frac{\partial V_j^r}{\partial t}\rho_j(x) - g_{1r}\right) + \\ &\varepsilon w^r\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial V_j^r(t)}{\partial t}\nabla\rho_j(x) - \nabla g_{1r}(x)\right).\end{aligned}$$

As

$$\left\|N^r\left(\cdot, \frac{\cdot}{\varepsilon}\right)\frac{\partial U_j^r}{\partial t}(t)\rho_j\right\|_H \le c, \quad \left\|\frac{\partial V_j^r}{\partial t}\rho_j\right\|_H \le c, \quad \left\|\frac{\partial V_j^r(t)}{\partial t}\nabla\rho_j\right\|_H \le c\varepsilon^{-s_1}$$

we have that

$$\left\|\frac{\partial u_1^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y w^r\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial V_j^r}{\partial t}(t)\rho_j - g_{1r}\right)\right\|_H \le c\varepsilon^{\frac{s}{s+1}}.$$

Using

$$\left\|\frac{\partial V_j^r}{\partial t}(t)\rho_j(x) - \frac{\partial u_{0r}}{\partial t}(t)\right\|_H \le c\varepsilon^{ss_1} = c\varepsilon^{\frac{s}{s+1}}$$

we deduce that

$$\left\|\frac{\partial u_1^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y w^r\left(x, \frac{x}{\varepsilon}\right)\left(\frac{\partial u_{0r}}{\partial t}(t) - g_{1r}(x)\right)\right\|_H \le c\varepsilon^{\frac{s}{s+1}}. \tag{C.10}$$

From (C.8), (C.9) and (C.10), we get

$$\left\| \frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y w^r\left(\cdot, \frac{\cdot}{\varepsilon}\right)\left(\frac{\partial u_{0r}}{\partial t}(t) - g_{1r}\right)\right\|_H + \\ \|\mathrm{curl}\, u^\varepsilon - \mathrm{curl}\, u_0 - \mathrm{curl}_y N^r\left(\cdot, \frac{\cdot}{\varepsilon}\right)\mathrm{curl}\, u_0(\cdot)_r\|_H \le c\varepsilon^{\frac{s}{s+1}}. \quad \square$$

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