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Numerical solution of a convection diffusion problem with Robin boundary conditions[☆]

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Abstract

We consider a one-dimensional steady-state convection dominated convection–diffusion problem with Robin boundary conditions. We show, both theoretically and with numerical experiments, that numerical solutions obtained using an upwind finite difference scheme on Shishkin meshes are uniformly convergent with respect to the diffusion coefficient.

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1. Introduction

We will consider the convection–diffusion equation

$$L_\varepsilon u_\varepsilon \equiv \varepsilon u_\varepsilon'' + a(x)u_\varepsilon' = f(x) \quad (1)$$

with the Robin boundary conditions

$$\beta_1 u_\varepsilon(0) - \beta_2 \varepsilon u_\varepsilon'(0) = A, \quad \gamma_1 u_\varepsilon(1) + \gamma_2 u_\varepsilon'(1) = B, \quad (2)$$

where

$$a, f \in C^2(\Omega), \quad a(x) \geq \alpha > 0, \quad x \in \bar{\Omega},$$

$$\beta_1, \beta_2 \geq 0, \quad \beta_1 + \beta_2 > 0, \quad \gamma_2 \geq 0 \quad \text{and} \quad \gamma_1 > 0.$$

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We seek a numerical method, which is uniformly convergent with respect to the diffusion coefficient ε .

A good review of the literature on numerical methods for singular perturbation problems may be found in [11]. In particular, various fitted operator methods were developed in [1,4,6,7,10], while methods using exponentially graded meshes were introduced in [3] and other graded meshes in [8,15]. Simpler piecewise-uniform meshes were proposed in [13]. Such meshes together with appropriate difference schemes give solutions that are robust with respect to the diffusion coefficient, while the simplicity of the construction of the meshes affords the flexibility to tackle problems in higher dimensions. Results related to Shishkin meshes may be found in [9,5,14].

In [5], problem (1) and (2) is analysed separately in the Dirichlet case: $\beta_1 = 1$, $\beta_2 = 0$, $\gamma_1 = 1$ and $\gamma_2 = 0$, and in the Neumann case: $\beta_1 = 0$, $\beta_2 = -1$, $\gamma_1 = 1$ and $\gamma_2 = 0$. For each case it is established that a numerical method comprising a finite difference operator, using central differencing for the second-order derivative and upwinding for the first-order derivative on Shishkin meshes, is uniformly convergent in ε for the solution and for appropriately scaled discrete derivatives. Ref. [5] will form the basis of the approach adopted here.

Note that the presence of ε multiplying the derivative term in the condition at $x = 0$ amplifies the significance of the boundary layer. In the absence of ε , the layer is sufficiently weak that uniformly convergent numerical solutions may be obtained using uniform meshes together with the same upwind method. However, to obtain uniformly accurate approximations of the derivative of the solution, a uniform mesh will not suffice, e.g., Ref. [16] uses an exponentially graded mesh for this purpose.

Doolan et al. [4] obtain a uniform in ε result for problem (1) and (2), using a fitted operator method on uniform meshes. Andreyev and Savin [2] solve a similar problem which has a Robin boundary condition only at the left boundary. The numerical method comprises a modified Samarskii scheme [12] with Shishkin meshes. Using a Greens function approach, Andreyev and Savin [2] prove ε -uniformly convergence of the numerical solution with order $(N^{-2} \ln^2 N)$.

Note that the L^∞ norm,

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

is employed for all error estimates.

2. The continuous problem

We first establish a priori bounds for the solution and its derivatives. The differential operator L_ε satisfies the following minimum principle:

Theorem 2.1. *Let L_ε be the differential operator defined in (1) and $v \in C^2(\bar{\Omega})$. If $\beta_1 v(0) - \beta_2 \varepsilon v'(0) \geq 0$, $\gamma_1 v(1) + \gamma_2 v'(1) \geq 0$ and $L_\varepsilon v \leq 0$ for all $x \in \Omega$, then $v(x) \geq 0$ for all $x \in \bar{\Omega}$.*

Proof. We start with the assumption that there is some point $p \in \bar{\Omega}$ such that $v(p) = \min_{\bar{\Omega}} v(x) < 0$. Note that our hypotheses imply that $p \neq 1$.

We consider three cases: $\gamma_2 = 0$, $\gamma_1/\gamma_2 > \alpha/2\varepsilon$ and $\gamma_1/\gamma_2 \leq \alpha/\varepsilon$.

If $\gamma_2 = 0$ the condition at $x = 1$ becomes $v(1) \geq 0$. We define the auxiliary function $w(x) = v(x)e^{xx/2\varepsilon}$. We choose the point $q \in \bar{\Omega}$, such that $w(q) = \min_{\bar{\Omega}} w(x) < 0$ and note that $q \neq 1$. Now suppose that

$q \in \Omega$; then $w'(q) = 0$ and $w''(q) \geq 0$, and thus $L_\varepsilon v(q) > 0$, which contradicts our hypotheses. The only possibility then is that $q = 0$, which means that $w'(0) \geq 0$ and $w(0) < 0$. Using the auxiliary function we conclude that $v(0) < 0$, and $v'(0) > 0$, which is also a contradiction.

Next, if $\gamma_1/\gamma_2 > \alpha/2\varepsilon$, consider the auxiliary function $w(x) = v(x)e^{\alpha x/2\varepsilon}$ again. We choose q to minimise w again and note that $q \neq 1$ since if $q = 1$ then the minimum of $w(x)$ occurs at $x = 1$; thus $w(1) < 0$ and $w'(1) \leq 0$. From this it follows that $v'(1) \leq -(\alpha/2\varepsilon)v(1)$ and since $\gamma_1/\gamma_2 > \alpha/2\varepsilon$ we have $v'(1) < -(\gamma_1/\gamma_2)v(1)$ which violates our hypothesis. If $q \in \Omega$ then $L_\varepsilon v(q) > 0$ which contradicts our hypotheses, so once again the only possibility is that w attain its minimum at the end-point $x = 0$. Analysis analogous to that of the first case leads to a contradiction.

Finally, if $\gamma_1/\gamma_2 \leq \alpha/\varepsilon$, define the auxiliary function $w(x) = v(x)e^{\gamma_1 x/2\gamma_2}$. We choose q as before and again note that, as in the last case, $q \neq 1$. If $q \in \Omega$; then $L_\varepsilon v(q) > 0$ which contradicts our hypotheses. Thus the only remaining possibility is that $q = 0$, which can be excluded as before. \square

We now calculate a priori bounds on the exact solution and its derivatives for problem (1) and (2).

Lemma 2.2. *The solution u_ε of problem (1) and (2) satisfies the bound*

$$\|u_\varepsilon\| \leq \frac{1}{\alpha} \left(1 + \frac{\gamma_2}{\gamma_1}\right) \|f\| + \frac{C}{\beta_1 + \beta_2\alpha} |\beta_1 u_\varepsilon(0) - \beta_2 \varepsilon u'_\varepsilon(0)| + \frac{1}{\gamma_1} |\gamma_1 u_\varepsilon(1) + \gamma_2 u'_\varepsilon(1)|.$$

Proof. Consider the barrier functions

$$\begin{aligned} \psi^\pm(x) = & \frac{|\beta_1 u_\varepsilon(0) - \beta_2 \varepsilon u'_\varepsilon(0)|}{(\beta_1 + \beta_2\alpha) - \beta_1(1 - \gamma_2\alpha/\gamma_1\varepsilon)e^{-\alpha/\varepsilon}} \left(e^{-\alpha x/\varepsilon} - \left(1 - \frac{\gamma_2\alpha}{\gamma_1\varepsilon}\right) e^{-\alpha/\varepsilon} \right) \\ & + \frac{1}{\gamma_1} |\gamma_1 u_\varepsilon(1) + \gamma_2 u'_\varepsilon(1)| + \frac{1}{\alpha} \|f\| \left(1 + \frac{\gamma_2}{\gamma_1} - x\right) \pm u_\varepsilon(x). \end{aligned}$$

We note here that $\beta_1 \psi^\pm(0) - \beta_2 \varepsilon (\psi^\pm)'(0) \geq 0$ and $\gamma_1 \psi^\pm(1) + \gamma_2 (\psi^\pm)'(1) \geq 0$ and furthermore that, for $x \in \Omega$, we have $L_\varepsilon \psi^\pm(x) \leq 0$. The minimum principle in Theorem 2.1 now applies and we have $\psi^\pm(x) \geq 0$ for all $x \in \Omega$, from which we have the required result. \square

Bounds for the derivatives are given in the following lemma.

Lemma 2.3. *The derivatives $u_\varepsilon^{(k)}$ of the solution u_ε of problem (1) and (2) satisfy the bounds*

$$\|u_\varepsilon^{(k)}\| \leq C\varepsilon^{-k} \max\{\|f\|, \|u_\varepsilon\|\}, \quad k = 1, 2,$$

$$\|u_\varepsilon^{(3)}\| \leq C\varepsilon^{-3} \max\{\|f\|, \|f'\|, \|u_\varepsilon\|\},$$

where C depends only on $\|a\|$ and $\|a'\|$.

Proof. The proof here is analogous to that in [5]. We start by noting that

$$\left| \int_0^x (f - au'_\varepsilon)(t) dt \right| \leq \|f\| + C\|u_\varepsilon\|, \quad (3)$$

where C depends on $\|a\|$ and $\|a'\|$. From the mean value theorem, there exists a point $z \in (0, \varepsilon)$ such that

$$|\varepsilon u'_\varepsilon(z)| \leq 2\|u_\varepsilon\|. \quad (4)$$

Integrating the differential equation (1) gives, for all $x \in \Omega$

$$\varepsilon u'_\varepsilon(x) - \varepsilon u'_\varepsilon(0) = \int_0^x (f - au'_\varepsilon)(t) dt. \quad (5)$$

Using (5) with $x = z$, and combining with (3), it follows that

$$|\varepsilon u'_\varepsilon(0)| \leq \|f\| + C\|u_\varepsilon\|.$$

From (5) we have

$$|\varepsilon u'_\varepsilon(x)| \leq \|f\| + C\|u_\varepsilon\| \quad \forall x \in \Omega,$$

which gives the required result for $k = 1$. Again from the differential equation (1) we have

$$\varepsilon u''_\varepsilon = f - au'_\varepsilon \quad \text{and} \quad \varepsilon u'''_\varepsilon = (f - au'_\varepsilon)',$$

which gives successively the required bounds on the second and third derivatives. \square

Our objective is to derive ε -uniform error estimates, for which we require sharper bounds on the derivatives of the solution. This is achieved by employing the following decomposition of the solution into smooth and singular components

$$u_\varepsilon = v_\varepsilon + w_\varepsilon,$$

where v_ε is the solution of the problem

$$L_\varepsilon v_\varepsilon = f \quad (6a)$$

with boundary conditions

$$\begin{aligned} \beta_1 v_\varepsilon(0) - \beta_2 \varepsilon v'_\varepsilon(0) &= \beta_1 v_0(0) - \beta_2 \varepsilon v'_0(0) + \varepsilon(\beta_1 v_1(0) - \beta_2 \varepsilon v'_1(0)), \\ \gamma_1 v_\varepsilon(1) + \gamma_2 v'_\varepsilon(1) &= \gamma_1 u_\varepsilon(1) + \gamma_2 u'_\varepsilon(1), \end{aligned} \quad (6b)$$

where

$$v_\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 \quad (6c)$$

and v_0 , v_1 and v_2 are defined, respectively, to be the solutions of the problems:

$$\begin{aligned} av'_0 &= f, \quad \gamma_1 v_0(1) + \gamma_2 v'_0(1) = \gamma_1 u_\varepsilon(1) + \gamma_2 u'_\varepsilon(1), \\ av'_1 &= -v''_0, \quad \gamma_1 v_1(1) + \gamma_2 v'_1(1) = 0, \\ L_\varepsilon v_2 &= -v''_1, \quad \beta_1 v_2(0) - \beta_2 \varepsilon v'_2(0) = 0, \quad \gamma_1 v_2(1) + \gamma_2 v'_2(1) = 0. \end{aligned} \quad (6d)$$

We note that v_0 , and v_1 are independent of ε , and v_2 is the solution of a problem similar to that defining u_ε ; also we have $v''_1 \in C^0(\Omega)$. The singular component w_ε is the solution of the homogeneous

problem

$$L_\varepsilon w_\varepsilon = 0, \quad (7a)$$

$$\begin{aligned} \beta_1 w_\varepsilon(0) - \beta_2 \varepsilon w'_\varepsilon(0) &= \beta_1 u_\varepsilon(0) - \beta_2 \varepsilon u'_\varepsilon(0) - (\beta_1 v_\varepsilon(0) - \beta_2 \varepsilon v'_\varepsilon(0)), \\ \gamma_1 w_\varepsilon(1) + \gamma_2 w'_\varepsilon(1) &= 0. \end{aligned} \quad (7b)$$

In the following lemma, we obtain bounds for the components of the solution and their respective derivatives.

Lemma 2.4. *The solution u_ε of problem (1)–(2) can be represented as*

$$u_\varepsilon = v_\varepsilon + w_\varepsilon,$$

where v_ε and w_ε are as defined in (6a) and (6b) and (7a) and (7b), respectively. In addition, the components $v_\varepsilon, w_\varepsilon$ and their derivatives satisfy the bounds

$$\|v_\varepsilon^{(k)}\| \leq C(1 + \varepsilon^{2-k}), \quad k = 0, 1, 2, 3,$$

$$|w_\varepsilon^{(k)}(x)| \leq C\varepsilon^{-k} e^{-\alpha x/\varepsilon} \quad \text{for all } x \in \bar{\Omega}, \quad k = 0, 1, 2, 3.$$

Proof. Noting the definitions in (6c) and (6d) we have $\|v_2\| \leq C$, and so

$$\|v_\varepsilon\| \leq C(1 + \varepsilon^2). \quad (8)$$

Note that v_0 and v_1 are independent of ε as mentioned earlier. Hence to get a bound on the derivatives of v_ε , we need to consider v_2 . Since v_2 is a problem similar to u_ε , its derivative bounds follow from Lemma 2.3. Using this with the bound (8) we therefore have

$$\|v_\varepsilon^{(k)}\| \leq C(1 + \varepsilon^{2-k}), \quad k = 0, 1, 2.$$

For $k = 3$ we make use of the differential equation (6a) giving us

$$\varepsilon v''_\varepsilon = f - av'_\varepsilon \quad \text{and} \quad \varepsilon v'''_\varepsilon = (f - av'_\varepsilon)',$$

which leads to the bound $\|v_\varepsilon^{(3)}\| \leq C\varepsilon^{-1}$. Our proof for v_ε and its derivatives is thus complete.

For the bounds on w_ε and its derivatives we consider the barrier functions

$$\psi^\pm(x) = C \left[\frac{e^{-\alpha x/\varepsilon} - (1 - \gamma_2 \alpha / \gamma_1 \varepsilon) e^{-\alpha/\varepsilon}}{1 - (1 - \gamma_2 \alpha / \gamma_1 \varepsilon) e^{-\alpha/\varepsilon}} \right] \pm w_\varepsilon(x).$$

We note that $\beta_1 \psi^\pm(0) - \beta_2 \varepsilon (\psi^\pm)'(0) \geq 0$, $\gamma_1 \psi^\pm(1) + \gamma_2 (\psi^\pm)'(1) = 0$ and $L_\varepsilon \psi^\pm \leq 0$, thus applying the minimum principle in Theorem 2.1 we have $\psi^\pm \geq 0$ and the required result follows, i.e.,

$$|w_\varepsilon(x)| \leq C e^{-\alpha x/\varepsilon} \quad \text{for all } x \in \bar{\Omega} \quad (9)$$

Bounds on the derivatives of w_ε are established as in [5]. \square

3. The discretised problem

We now consider the discretisation of problem (1) and (2). We employ an upwind finite difference operator

$$\begin{aligned} L_\varepsilon^N U_\varepsilon &\equiv \varepsilon \delta^2 U_\varepsilon + a(x_i) D^+ U_\varepsilon = f(x_i), \quad x_i \in \Omega_\varepsilon^N, \\ \beta_1 U_\varepsilon(x_0) - \beta_2 \varepsilon D^+ U_\varepsilon(x_0) &= \beta_1 u_\varepsilon(0) - \beta_2 \varepsilon u'_\varepsilon(0), \\ \gamma_1 U_\varepsilon(x_N) + \gamma_2 D^- U_\varepsilon(x_N) &= \gamma_1 u_\varepsilon(1) + \gamma_2 u'_\varepsilon(1), \end{aligned} \quad (10)$$

where

$$\begin{aligned} D^+ V(x_i) &\equiv \frac{V(x_{i+1}) - V(x_i)}{h_{i+1}}, \\ \delta^2 V(x_i) &\equiv \frac{1}{\bar{h}_i} \left(\frac{V(x_{i+1}) - V(x_i)}{h_{i+1}} - \frac{V(x_i) - V(x_{i-1}))}{h_i} \right), \\ h_{i+1} &= x_{i+1} - x_i, \quad h_i = x_i - x_{i-1}, \quad \bar{h}_i = \frac{h_i + h_{i+1}}{2} \end{aligned}$$

for any mesh function V , and where the Shishkin mesh Ω_ε^N , is defined as

$$\Omega_\varepsilon^N = x_i | x_i = \begin{cases} 2i\sigma/N & \text{for } i \leq N/2, \\ x_{i-1} + 2(1-\sigma)/N & \text{for } i > N/2 \end{cases} \quad (11)$$

with

$$\sigma = \min \left\{ 0.5, \frac{1}{\alpha} \varepsilon \ln N \right\}.$$

As there are only two different mesh steps, for convenience we set

$$h_i = \begin{cases} h, & 1 \leq i \leq N/2, \\ H, & N/2 + 1 \leq i \leq N. \end{cases}$$

Thus h is the fine, and H the coarse, mesh step.

We now state a discrete minimum principle for the difference operator defined in (10).

Theorem 3.1. *Let L_ε^N be the upwind finite difference operator defined in (10) and let Ω^N be an arbitrary mesh of $N + 1$ mesh points. If V is any mesh function defined on this mesh such that*

$$\beta_1 V(x_0) - \beta_2 \varepsilon D^+ V(x_0) \geq 0, \quad \gamma_1 V(x_N) + \gamma_2 D^- V(x_N) \geq 0 \quad \text{and} \quad L_\varepsilon^N V \leq 0,$$

then

$$V(x_i) \geq 0 \quad \forall x_i \in \Omega^N.$$

Proof. We start by considering $\beta_2 = 0$, thus the condition on the left becomes $V(x_0) \geq 0$. Let $V_k = \min_i \{V_i\} < 0$ then $k \neq 0$ since this violates the hypothesis. Hence V_k is the minimum value, $D^+ V_k \geq 0$ and $\delta^2 V_k \geq 0$. To avoid a contradiction we must have $V_k = V_{k-1} = V_{k+1} < 0$. Repeating this argument leads to $V_0 < 0$ which is a contradiction.

Next consider the situation when $\beta_2 \neq 0$. Let $W_i = (1 + h_i\beta_1/\varepsilon\beta_2)V(x_i)$. Furthermore, let $W_k = \min_i \{W_i\}$. Now assume that $V_k < 0$ and hence $W_k < 0$. Note that, if $k=N$, then $D^-V_N \leq 0$; also $V_N < 0$ which violates our hypothesis. Hence, $k \neq N$. We thus need to consider two remaining possibilities:

First, suppose that $k=0$ so that $W_0 = \min_i \{W_i\} < 0$. Using the condition on the left we have $V(x_1) \leq W_0$, and noting that $W_0 < 0$ we have $W_1 \leq W_0$. Thus, since $W_0 = \min_i \{W_i\} < 0$, the only possibility is that $W_1 = W_0$; now we use the condition on the difference operator $L_e^N V_1 \leq 0$ and so $L_e^N W_1 \leq 0$, which leads to $D^+W_1 \leq 0$, and $W_2 \leq W_1$, which in turn implies that $W_2 = W_1 = W_0$. Applying the same argument repeatedly we eventually conclude that $W_N = W_0$ which means that $V(x_N) < 0$. From the above analysis it follows that $D^+V(x_1) = 0$. Repeated applications of this process leads to $D^+V(x_{N-1}) = 0$, which in turn implies that $D^-V(x_N) = 0$. Combining these results we have $\gamma_1 V(x_N) + \gamma_2 D^-V(x_N) < 0$, which contradicts the hypothesis.

Finally, suppose $0 < k < N$. Let $V_k = \min_i \{V_i\}$ and that $V_k < 0$. Since V_k is the minimum, then $D^+V_k \geq 0$ and $\delta^2 V_k \geq 0$, which together with $L_e^N V \leq 0$, implies that $V_{k-1} = V_k = V_{k+1}$. This means that $D^+V_k = 0$. Since these results are true for $0 \leq k \leq N$, therefore $\gamma_1 V(x_N) - \gamma_2 D^-V(x_N) < 0$ which is clearly a contradiction. The proof is thus complete. \square

The following lemma will be useful later.

Lemma 3.2. *The solution of the constant coefficient problem*

$$\varepsilon \delta^2 \Phi_i + \omega D^+ \Phi_i = 0, \quad 1 \leq i \leq N-1, \quad (12a)$$

where $\omega > 0$, with boundary conditions

$$\beta_1 \Phi_0 - \beta_2 \varepsilon D^+ \Phi_0 = 1, \quad \gamma_1 \Phi_N + \gamma_2 D^- \Phi_N = 0 \quad (12b)$$

on a uniform mesh or the Shishkin mesh Ω_e^N satisfies

$$D^+ \Phi_i \leq 0 \quad \text{for all } 1 \leq i \leq N-1.$$

Proof. We need to consider separately the cases $\sigma = \frac{1}{2}$ and $\sigma = \varepsilon/\omega \ln N$. With $\sigma = \frac{1}{2}$, i.e., the uniform mesh case, we have

$$\Phi_i = \frac{\lambda^{N-i} + (\omega\gamma_2/\gamma_1\varepsilon) - 1}{[\beta_1(\lambda^N + (\omega\gamma_2/\gamma_1\varepsilon) - 1) + \beta_2\omega\lambda^{N-1}]}, \quad \lambda = 1 + \frac{\omega h}{\varepsilon} \quad (13)$$

and therefore

$$D^+ \Phi_i = -\frac{\omega\lambda^{N-i-1}}{\varepsilon[\beta_1(\lambda^N + (\omega\gamma_2/\gamma_1\varepsilon) - 1) + \beta_2\omega\lambda^{N-1}]} \leq 0. \quad (14)$$

For the second case $\sigma = \varepsilon/\omega \ln N$, we start by noting that solution of problem (12a) and (12b) is

$$\Phi_i = \begin{cases} \Phi_{N/2} + (1 - \beta_1 \Phi_{N/2})\chi_i & \text{if } i \leq N/2, \\ \Phi_{N/2}\zeta_i & \text{if } i \geq N/2, \end{cases} \quad (15a)$$

where

$$\chi_i = \frac{\lambda^{N/2-i} - 1}{\beta_1(\lambda^{N/2} - 1) + \beta_2\omega\lambda^{N/2-1}}, \quad \lambda = 1 + \frac{\omega h}{\varepsilon}, \quad (15b)$$

$$\zeta_i = \frac{A^{N-i} + (\gamma_2\omega/\gamma_1\varepsilon) - 1}{A^{N/2} + (\gamma_2\omega/\gamma_1\varepsilon) - 1}, \quad A = 1 + \frac{\omega H}{\varepsilon} \quad (15c)$$

and $\Phi_{N/2}$ satisfies

$$(\varepsilon\delta^2 + \omega D^+)\Phi_{N/2} = 0. \quad (15d)$$

Note that since $A^{N/2} > A^{N/2-1}$, we know from (15c) that

$$\zeta_{N/2+1} < 1. \quad (16)$$

Also we note that

$$\chi_{N/2-1} = \frac{\omega h}{\varepsilon[\beta_1(\lambda^{N/2} - 1) + \beta_2\omega\lambda^{N/2-1}]} \geq 0. \quad (17)$$

From (15d) and noting (16) and (17) we have

$$\Phi_{N/2} = \frac{\varepsilon N \chi_{N/2-1}}{(h/H)(\varepsilon N + \omega)(1 - \zeta_{N/2+1}) + \varepsilon N \beta_1 \chi_{N/2-1}} \geq 0. \quad (18)$$

Also,

$$1 - \beta_1 \Phi_{N/2} = \frac{(h/H)(\varepsilon N + \omega)(1 - \zeta_{N/2+1})}{(h/H)(\varepsilon N + \omega)(1 - \zeta_{N/2+1}) + \varepsilon N \beta_1 \chi_{N/2-1}} \geq 0. \quad (19)$$

Applying the forward difference operator, D^+ , to (15b) we have

$$D^+ \chi_i = -\frac{\lambda^{N/2-i-1}\omega}{\varepsilon[\beta_1(\lambda^{N/2} - 1) + \beta_2\omega\lambda^{N/2-1}]} \leq 0, \quad 1 \leq i < N/2. \quad (20)$$

In addition, we note that

$$D^- \chi_{N/2} = -\frac{\lambda^{-1}\omega}{\varepsilon[\beta_1(\lambda^{N/2} - 1) + \beta_2\omega\lambda^{N/2-1}]} \leq 0. \quad (21)$$

Furthermore, applying the forward difference operator, D^+ , to (15c) we get

$$D^+ \zeta_i = -\frac{\omega A^{N-i-1}}{\varepsilon[A^{N/2} + (\gamma_2\omega/\gamma_1\varepsilon) - 1]} \leq 0, \quad i \geq N/2. \quad (22)$$

Combining (18)–(22) we have the desired result. \square

Corollary 3.3. *The solution of the constant coefficient problem (12a) and (12b) is bounded as*

$$|\Phi_i| \leq C.$$

where C is a constant independent of ε .

The discrete minimum principle in Theorem 3.1 leads to the following bounds for the solution U_ε of the discretised form of problem (1).

Lemma 3.4. *The solution U_ε obtained by applying the numerical method (10) and (11) to problem (1) and (2) is bounded as*

$$|U_\varepsilon(x_i)| \leq \frac{1}{\alpha} \left(1 + \frac{\gamma_2}{\gamma_1} \right) \|f\| + C |\beta_1 U_\varepsilon(0) - \beta_2 \varepsilon D^+ U_\varepsilon(0)| + \frac{1}{\gamma_1} |\gamma_1 U_\varepsilon(1) + \gamma_2 D^- U_\varepsilon(1)|.$$

Proof. Consider the two mesh functions

$$\begin{aligned} \Psi^\pm(x_i) &= \frac{1}{\alpha} \left(1 + \frac{\gamma_2}{\gamma_1} - x_i \right) \|f\| + |\beta_1 U_\varepsilon(0) - \beta_2 \varepsilon D^+ U_\varepsilon(0)| \Phi_i \\ &\quad + \frac{1}{\gamma_1} |\gamma_1 U_\varepsilon(1) + \gamma_2 D^- U_\varepsilon(1)| \pm U_\varepsilon(x_i), \end{aligned}$$

where Φ_i is the solution of the constant coefficient problem

$$\varepsilon \delta^2 \Phi_i + \alpha D^+ \Phi_i = 0,$$

with boundary conditions

$$\beta_1 \Phi_0 - \beta_2 \varepsilon D^+ \Phi_0 = 1, \quad \gamma_1 \Phi_N + \gamma_2 D^- \Phi_N = 0.$$

Using Lemma 3.2 and recalling that $a \geq \alpha$, we have $L_\varepsilon^N \Psi^\pm(x_i) \leq 0$. Furthermore, $\beta_1 \Psi^\pm(0) - \beta_2 \varepsilon D^+ \Psi^\pm(0) \geq 0$ and $\gamma_1 \Psi^\pm(1) + \gamma_2 D^- \Psi^\pm(1) \geq 0$. Thus the minimum principle now applies and we have the desired result. \square

Analogous to the continuous case, the discrete solution U_ε can be decomposed into the sum

$$U_\varepsilon = V_\varepsilon + W_\varepsilon,$$

where V_ε and W_ε are, respectively, the solutions of the problems

$$\begin{aligned} L_\varepsilon^N V_\varepsilon &= f(x_i), \quad x_i \in \Omega_\varepsilon^N, \\ \beta_1 V_\varepsilon(0) - \beta_2 \varepsilon D^+ V_\varepsilon(0) &= \beta_1 v_\varepsilon(0) - \beta_2 \varepsilon v'_\varepsilon(0), \end{aligned} \tag{23}$$

$$\gamma_1 V_\varepsilon(1) + \gamma_2 D^- V_\varepsilon(1) = \gamma_1 v_\varepsilon(1) + \gamma_2 v'_\varepsilon(1)$$

$$\begin{aligned} L_\varepsilon^N W_\varepsilon &= 0, \quad x_i \in \Omega_\varepsilon^N, \\ \beta_1 W_\varepsilon(0) - \beta_2 \varepsilon D^+ W_\varepsilon(0) &= \beta_1 w_\varepsilon(0) - \beta_2 \varepsilon w'_\varepsilon(0), \end{aligned} \tag{24}$$

$$\gamma_1 W_\varepsilon(1) + \gamma_2 D^- W_\varepsilon(1) = 0.$$

4. Error estimates for the solution

We obtain separate errors estimates for each component of the numerical solution.

Lemma 4.1. *The error in the smooth component of the numerical solution is bounded as*

$$|(V_\varepsilon - v_\varepsilon)(x_i)| \leq CN^{-1} \text{ for all } x_i \in \bar{\Omega}_\varepsilon^N,$$

where v_ε is the solution of (6a) and (6b) and V_ε is the solution of (23).

Proof. Consider the local truncation error

$$L_\varepsilon^N(V_\varepsilon - v_\varepsilon) = (L_\varepsilon - L_\varepsilon^N)v_\varepsilon = \varepsilon \left(\frac{d^2}{dx^2} - \delta^2 \right) v_\varepsilon + a \left(\frac{d}{dx} - D^+ \right) v_\varepsilon.$$

Then, by standard local truncation error estimates [9] and Lemma 2.4 we have

$$|L_\varepsilon^N(V_\varepsilon - v_\varepsilon)(x_i)| \leq \frac{\varepsilon}{3} (x_{i+1} - x_{i-1}) \|v_\varepsilon^{(3)}\| + \frac{a(x_i)}{2} (x_{i+1} - x_i) \|v_\varepsilon^{(2)}\| \leq CN^{-1}.$$

We use the comparison functions

$$\Psi^\pm(x_i) = CN^{-1} \left(\Phi_i + \frac{1}{\gamma_1} \right) \pm (V_\varepsilon - v_\varepsilon)(x_i),$$

where Φ_i is the solution of the constant coefficient problem

$$\varepsilon \delta^2 \Phi_i + \alpha D^+ \Phi_i = 0, \quad \beta_1 \Phi_0 - \beta_2 \varepsilon D^+ \Phi_0 = 1, \quad \gamma_1 \Phi_N + \gamma_2 D^- \Phi_N = 0.$$

Note the inequalities

$$|\beta_1(V_\varepsilon - v_\varepsilon)(0) - \beta_2 \varepsilon D^+(V_\varepsilon - v_\varepsilon)(0)| \leq CN^{-1},$$

$$|\gamma_1(V_\varepsilon - v_\varepsilon)(1) + \gamma_2 D^-(V_\varepsilon - v_\varepsilon)(1)| \leq CN^{-1},$$

which follow immediately from (23) and Lemma 2.4.

Thus, employing Lemma 3.2 with $\omega = \alpha$ we can choose C large enough such that $\beta_1 \Psi_0^\pm - \beta_2 \varepsilon D^+ \Psi_0^\pm \geq 0$, $\gamma_1 \Psi_N^\pm + \gamma_2 D^- \Psi_N^\pm \geq 0$ and $L_\varepsilon^N \Psi^\pm(x_i) \leq 0$. Therefore the minimum principle applies and the result follows. \square

Lemma 4.2. *The error in the singular component of the numerical solution is bounded as*

$$|(W_\varepsilon - w_\varepsilon)(x_i)| \leq CN^{-1} \ln N, \text{ for all } x_i \in \bar{\Omega}_\varepsilon^N,$$

where w_ε is the solution of (7a) and (7b) and W_ε is the solution of (24).

Proof. First, note the inequalities

$$|\beta_1(W_\varepsilon - w_\varepsilon)(0) - \beta_2 \varepsilon D^+(W_\varepsilon - w_\varepsilon)(0)| \leq CN^{-1} \ln N, \quad (25)$$

$$|\gamma_1(W_\varepsilon - w_\varepsilon)(1) + \gamma_2 D^-(W_\varepsilon - w_\varepsilon)(1)| \leq CN^{-1}. \quad (26)$$

Eq. (26) follows immediately from (24) and Lemma 2.4, while (25) follows since, again using (24) and Lemma 2.4,

$$\begin{aligned} |\beta_1(W_\varepsilon - w_\varepsilon)(0) - \beta_2 \varepsilon D^+(W_\varepsilon - w_\varepsilon)(0)| &= |\beta_2 \varepsilon (D^+ w_\varepsilon - w'_\varepsilon)(0)| \\ &\leq \left| \frac{\varepsilon}{h} \int_0^h (s-h) w''_\varepsilon(s) ds \right| \\ &\leq Ch/\varepsilon \leq CN^{-1} \ln N. \end{aligned}$$

We first consider the uniform mesh case, when $\sigma = \frac{1}{2}$, and so $\varepsilon^{-1} \leq C \ln N$ and $h = H = N^{-1}$. Using the standard bound for the local truncation error [9] and Lemma 2.4 we have

$$|L_\varepsilon^N(W_\varepsilon - w_\varepsilon)(x_i)| \leq C\varepsilon^{-2}(x_{i+1} - x_{i-1})e^{-\alpha x_{i-1}/\varepsilon} \leq C\varepsilon^{-2}N^{-1}e^{-\alpha x_{i-1}/\varepsilon}. \quad (27)$$

We employ the mesh functions

$$\Psi^\pm(x_i) = \frac{C e^{2\tau h/\varepsilon}}{\tau(\alpha - \tau)} \varepsilon^{-1} N^{-1} \left(Y_i + \frac{1}{\gamma_1} \right) \pm (W_\varepsilon - w_\varepsilon)(x_i),$$

where τ is a constant with $0 < \tau < \alpha$ and Y_i is the solution of the constant coefficient problem

$$(\varepsilon \delta^2 + \tau D^+) Y_i = 0, \quad \beta_1 Y_0 - \varepsilon \beta_2 D^+ Y_0 = 1, \quad \gamma_1 Y_N + \gamma_2 D^- Y_N = 0. \quad (28)$$

Using Lemma 3.2 with $\omega = \tau$ we can choose C large enough such that $L_\varepsilon^N \Psi_i^\pm \leq 0$, and also $\beta_1 \Psi_0^\pm - \beta_2 \varepsilon D^+ \Psi_0^\pm \geq 0$ and $\gamma_1 \Psi_N^\pm + \gamma_2 D^- \Psi_N^\pm \geq 0$. Thus, applying the minimum principle for L_ε^N , we conclude that $\Psi_i^\pm \geq 0$. Therefore, using this result and noting that $0 \leq Y_i \leq 1$ we have for all $x_i \in \bar{\Omega}_\varepsilon^N$,

$$|(W_\varepsilon - w_\varepsilon)(x_i)| \leq \frac{C e^{2\tau h/\varepsilon}}{\tau(\alpha - \tau)} \varepsilon^{-1} N^{-1} Y_i \leq CN^{-1} \ln N.$$

We now consider the case $\sigma = (\varepsilon/\alpha) \ln N$. Here we need to take account of the fine and coarse meshes separately. First, suppose that $x_i \in [\sigma, 1]$. Using the triangle inequality we have

$$|(W_\varepsilon - w_\varepsilon)(x_i)| \leq |W_\varepsilon(x_i)| + |w_\varepsilon(x_i)|.$$

Using Lemma 2.4 we have

$$|w_\varepsilon(x_i)| \leq C e^{-\alpha \sigma/\varepsilon} = CN^{-1}.$$

The bound for $|W_\varepsilon(x_i)|$ is established by considering the function Y_i which is the solution of the constant coefficient discretised problem

$$(\varepsilon \delta^2 + \alpha D^+) Y_i = 0, \quad 1 \leq i \leq N-1,$$

$$Y_0 = 1, \quad \gamma_1 Y_N + \gamma_2 D^- Y_N = 0.$$

This problem has the following solution

$$Y_i = \begin{cases} 1 + (Y_{N/2} - 1) \ell_i & \text{if } i \leq N/2, \\ Y_{N/2} r_i & \text{if } i \geq N/2, \end{cases} \quad (29a)$$

where

$$\ell_i = \frac{1 - \lambda^{-i}}{1 - \lambda^{-N/2}}, \quad \lambda = 1 + \frac{\alpha h}{\varepsilon} \quad (29b)$$

$$r_i = \frac{A^{N-i} + (\gamma_2 \alpha / \gamma_1 \varepsilon) - 1}{A^{N/2} + (\gamma_2 \alpha / \gamma_1 \varepsilon) - 1}, \quad A = 1 + \frac{\alpha H}{\varepsilon} \quad (29c)$$

and $Y_{N/2}$ satisfies

$$(\varepsilon \delta^2 + \alpha D^+) Y_{N/2} = 0. \quad (30)$$

We start by noting that

$$\left(1 + \frac{2 \ln N}{N}\right)^{-N/2} \leq 2N^{-1} \quad \forall N \geq 1, \quad (31)$$

from which we have

$$\lambda^{-N/2} \leq 2N^{-1}. \quad (32)$$

Using the fact that $H = 2(1 - \sigma)/N$, we observe that

$$\frac{\alpha}{\varepsilon} A^{-N/2} \leq 2. \quad (33)$$

Noting the definition of ℓ_i in (29b), and combining it with (31) and (32) we get

$$0 \leq \frac{\varepsilon}{\alpha} D^- \ell_{N/2} = \frac{\lambda^{-N/2}}{1 - \lambda^{-N/2}} \leq \frac{2N^{-1}}{1 - 2N^{-1}} \leq 4N^{-1}. \quad (34)$$

Now, from the definition of r in (29c) we have

$$-\frac{\varepsilon}{\alpha} D^+ r_{N/2} = \frac{A^{N/2-1}}{A^{N/2} + (\gamma_2 \alpha / \gamma_1 \varepsilon) - 1} \geq \frac{1}{A} \left[\frac{1}{1 + (\gamma_2 \alpha / \gamma_1 \varepsilon) A^{-N/2}} \right]$$

and employing (33) we have

$$-\frac{\varepsilon}{\alpha} D^+ r_{N/2} \geq \frac{1}{A} \left[\frac{\gamma_1}{\gamma_1 + 2\gamma_2} \right]. \quad (35)$$

We next need to calculate the value of $Y_{N/2}$, which we do by combining (29a) and (30) to get

$$Y_{N/2} = \frac{D^- \ell_{N/2}}{D^- \ell_{N/2} - (1/2)(\lambda + A) D^+ r_{N/2}} \geq 0. \quad (36)$$

Using (34) and (35) with (36) we obtain

$$Y_{N/2} \leq \frac{4N^{-1}}{(1/2)(\lambda + A)1/A \left[\frac{\gamma_1}{\gamma_1 + 2\gamma_2} \right]} \leq \frac{8N^{-1} A(\gamma_1 + 2\gamma_2)}{\gamma_1(\lambda + A)} \leq 8N^{-1} \left(1 + \frac{2\gamma_2}{\gamma_1} \right).$$

Furthermore, we note that

$$D^+ \ell_i \leq 0 \quad \text{and} \quad D^+ r_i \leq 0,$$

and noting the results at $i = N/2$ from (34)–(36) we have

$$D^+ Y_i \leq 0, \quad 0 \leq i \leq N-1.$$

We now consider the mesh functions

$$\Psi_i^\pm = |\beta_1 W_\varepsilon(0) - \beta_2 \varepsilon D^+ W_\varepsilon(0)| Y_i \pm W_\varepsilon(x_i).$$

Then

$$L_\varepsilon^N \Psi_i^\pm = |\beta_1 W_\varepsilon(0) - \beta_2 \varepsilon D^+ W_\varepsilon(0)| (a(x_i) - \alpha) D^+ Y_i \leq 0,$$

and, in addition, $\beta_1 \Psi_0^\pm - \beta_2 \varepsilon D^+ \Psi_0^\pm \geq 0$ and $\gamma_1 \Psi_N^\pm + \gamma_2 D^- \Psi_N^\pm = 0$. Thus, applying the minimum principle in Theorem 3.1, $\Psi_i^\pm \geq 0$ and, employing Lemma 2.4, we have for all $x_i \in [\sigma, 1]$

$$|W_\varepsilon(x_i)| \leq |\beta_1 W_\varepsilon(0) - \beta_2 \varepsilon D^+ W_\varepsilon(0)| Y_i \leq |\beta_1 w_\varepsilon(0) - \beta_2 \varepsilon w'_\varepsilon(0)| Y_{N/2} \leq CN^{-1}.$$

We next need to prove the result for $x_i \in [0, \sigma)$. The proof follows on similar lines to the case $\sigma = \frac{1}{2}$, except that we use the discrete minimum principle on $[0, \sigma]$. We will also need to use the bound $|W_\varepsilon(x_{N/2})| \leq CN^{-1}$ from Lemma 2.4. We have in this case

$$|L_\varepsilon^N (W_\varepsilon - w_\varepsilon)(x_i)| \leq C \sigma \varepsilon^{-2} N^{-1} e^{-\alpha x_{i-1}/\varepsilon} \quad \text{for all } 0 \leq i \leq N/2. \quad (37)$$

Analogously to the earlier case, we introduce the mesh functions

$$\Psi^\pm(x_i) = \frac{C e^{2\tau h/\varepsilon}}{\tau(\alpha - \tau)} \sigma \varepsilon^{-1} N^{-1} Z_i + C' N^{-1} \pm (W_\varepsilon - w_\varepsilon)(x_i),$$

where τ is a constant with $0 < \tau < \alpha$ and Z_i is the solution of the constant coefficient problem

$$(\varepsilon \delta^2 + \tau D^+) Z_i = 0, \quad \beta_1 Z_0 - \varepsilon \beta_2 D^+ Z_0 = 1, \quad \gamma_1 Z_{N/2} + \gamma_2 D^- Z_{N/2} = 0. \quad (38)$$

Thus

$$Z_i = \frac{\lambda^{N/2-i} + (\gamma_2 \tau / \gamma_1 \varepsilon) - 1}{\lambda^{N/2} + (\gamma_2 \tau / \gamma_1 \varepsilon) - 1}, \quad \lambda = 1 + (\tau h / \varepsilon).$$

We can now see that

$$D^+ Z_i = -\frac{\tau \lambda^{N/2-i-1}}{\varepsilon [\lambda^{N/2} + (\gamma_2 \tau / \gamma_1 \varepsilon) - 1]} \leq 0. \quad (39)$$

Now, $\beta_1 \Psi_0^\pm - \beta_2 \varepsilon D^+ \Psi_0^\pm \geq 0$, $\gamma_1 \Psi_{N/2}^\pm + \gamma_2 D^- \Psi_{N/2}^\pm \geq 0$ and $L_\varepsilon^N \Psi_i^\pm \leq 0$; therefore, applying the minimum principle in Theorem 3.1, $\Psi_i^\pm \geq 0$. Hence, for $x_i \in [0, \sigma)$, we have

$$|(W_\varepsilon - w_\varepsilon)(x_i)| \leq CN^{-1} \ln N$$

as required. \square

The above error estimates for the individual components of the numerical solution now lead to the following theorem on the error estimate for the numerical solution U_ε , which is obtained by combining them using the triangle inequality.

Theorem 4.3. *If u_ε is the solution of problem (1) and (2) and if U_ε is the corresponding numerical solution using the method outlined in (10), then we have*

$$\sup_{0 < \varepsilon \leq 1} \|U_\varepsilon - u_\varepsilon\|_{\tilde{\Omega}_\varepsilon^N} \leq CN^{-1} \ln N \quad \forall N \geq 4.$$

where the constant C is independent of ε and N .

As in [5], we can easily extend the preceding nodal ε -uniform error result to a global ε -uniform result.

Theorem 4.4. *If u_ε is the solution of problem (1) and (2) and U_ε is the corresponding numerical solution computed using the method outlined in (10), then we have*

$$\sup_{0 < \varepsilon \leq 1} \|\tilde{U}_\varepsilon - u_\varepsilon\|_{\tilde{\Omega}_\varepsilon^N} \leq CN^{-1} \ln N,$$

where \tilde{U}_ε is the piecewise linear interpolant of U_ε on $\tilde{\Omega}$ and C is a constant independent of N and ε .

This particular result is not dependent on the boundary conditions and as such we refer the reader to the analogous proof in [5].

5. Numerical experiments

We now look at computational results a sample problem which is an example of problem (1) and (2). The problem considered is

$$\varepsilon u_\varepsilon'' + \frac{1}{1+x} u_\varepsilon' = x + 1 \quad (40)$$

with boundary conditions

$$u_\varepsilon(0) - \varepsilon u_\varepsilon'(0) = 1, \quad u_\varepsilon(1) + u_\varepsilon'(1) = 1. \quad (41)$$

The solution is

$$u = \frac{(x+1)^3}{3(2\varepsilon+1)} + D \left[\frac{(x+1)^{1-1/\varepsilon}}{\varepsilon-1} - \left(\frac{2^{1-1/\varepsilon}}{\varepsilon-1} + \frac{2^{-1/\varepsilon}}{\varepsilon} \right) \right] + \left[1 - \frac{20}{3(2\varepsilon+1)} \right], \quad (42)$$

where

$$D = \frac{(19+3\varepsilon)/(3(2\varepsilon+1))}{((1-2^{1-1/\varepsilon})/(\varepsilon-1) - 2^{-1/\varepsilon}/\varepsilon) - 1}.$$

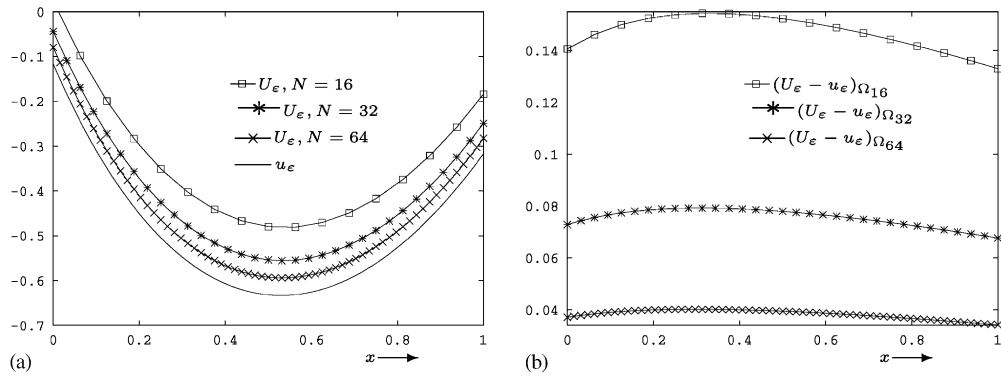


Fig. 1. Plots illustrating the convergence of the numerical solution (U_ε) to the exact solution (u_ε) are shown in (a). In (b) the error in the numerical solution w.r.t. the exact solution is shown. Both plots are on Ω_ε^N ; $\varepsilon = 2^{-1}$, $N = 16, 32$ and 64 .

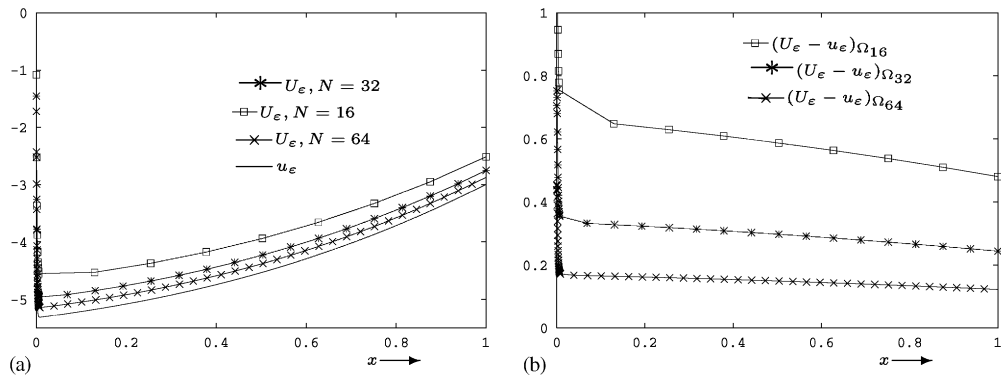


Fig. 2. Plots illustrating the convergence of the numerical solution (U_ε) to the exact solution (u_ε) are shown in (a). In (b) the error in the numerical solution w.r.t. the exact solution is shown. Both plots are on Ω_ε^N ; $\varepsilon = 2^{-10}$, $N = 16, 32$ and 64 .

We solve this problem using the numerical method (10) with the constant α used in the definition of the transition point σ set to 0.5.

In Fig. 1(a), the numerical solution U_ε with $N = 16, 32$ and 64 for $\varepsilon = 0.5$, and the exact solution u_ε are shown for $\varepsilon = 0.5$. The numerical solutions clearly converge to the exact solution as $N \rightarrow \infty$. Fig. 1(b) shows the errors in the numerical solution for $N = 16, 32$ and 64 . Fig. 2 shows a similar set of graphs for $\varepsilon = 2^{-10}$.

Fig. 3 shows the numerical solution U_ε and the error $|U_\varepsilon - u_\varepsilon|_{\Omega_N}$ for the restricted interval $x \in [0, \varepsilon]$, with $\varepsilon = 2^{-20}$. Table 1 gives the errors $E_\varepsilon^N = |U_\varepsilon - u_\varepsilon|_{\Omega_N}$ for various values of ε and N and also $E_{\max}^N = \max_\varepsilon E_\varepsilon^N$. Clearly, for each ε , as N increases, the error reduces, while, as ε decreases, the errors for any particular N stabilise. Table 2 gives the estimated convergence rates $R_{N,ep}$ calculated from the errors in Table 1, using the formula

$$R_{N,ep} = \log_2 \frac{E_\varepsilon^N}{E_\varepsilon^{2N}}.$$

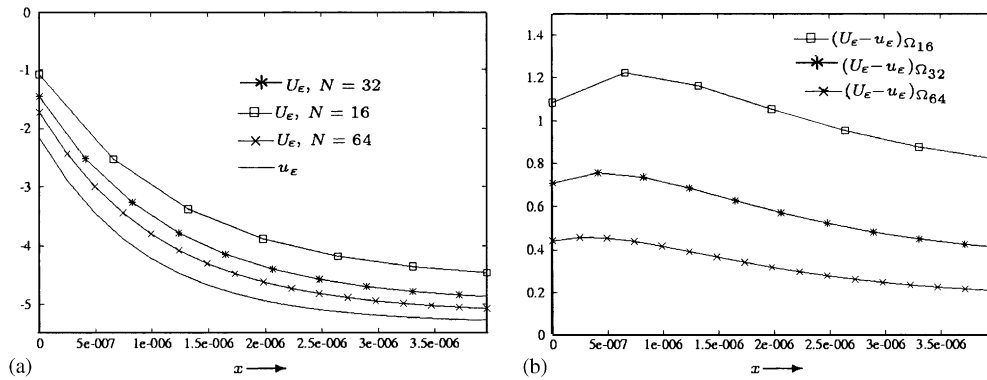


Fig. 3. Plots illustrating the convergence of the numerical solution (U_ε) to the exact solution (u_ε) are shown in (a). In (b) the error in the numerical solution w.r.t. the exact solution is shown. Both plots are on $\Omega_\varepsilon^N \cap [0, \sigma]$; $\varepsilon = 2^{-20}$, $N = 16, 32$ and 64 .

Table 1
Maximum pointwise errors ($U_\varepsilon - u_\varepsilon$)

ε	Number of intervals N							
	32	64	128	256	512	1024	2048	4096
2^{-1}	0.793D - 01	0.402D - 01	0.202D - 01	0.101D - 01	0.508D - 02	0.254D - 02	0.127D - 02	0.636D - 03
2^{-2}	0.186D + 00	0.953D - 01	0.483D - 01	0.243D - 01	0.122D - 01	0.610D - 02	0.305D - 02	0.153D - 02
2^{-3}	0.367D + 00	0.192D + 00	0.983D - 01	0.498D - 01	0.250D - 01	0.126D - 01	0.629D - 02	0.315D - 02
2^{-4}	0.590D + 00	0.354D + 00	0.186D + 00	0.956D - 01	0.485D - 01	0.244D - 01	0.123D - 01	0.614D - 02
2^{-5}	0.662D + 00	0.404D + 00	0.238D + 00	0.137D + 00	0.772D - 01	0.428D - 01	0.234D - 01	0.123D - 01
2^{-6}	0.706D + 00	0.429D + 00	0.252D + 00	0.145D + 00	0.816D - 01	0.452D - 01	0.247D - 01	0.134D - 01
2^{-7}	0.730D + 00	0.442D + 00	0.260D + 00	0.150D + 00	0.841D - 01	0.465D - 01	0.254D - 01	0.138D - 01
2^{-8}	0.743D + 00	0.449D + 00	0.265D + 00	0.152D + 00	0.854D - 01	0.472D - 01	0.258D - 01	0.140D - 01
2^{-9}	0.749D + 00	0.453D + 00	0.267D + 00	0.153D + 00	0.860D - 01	0.476D - 01	0.260D - 01	0.141D - 01
2^{-10}	0.753D + 00	0.455D + 00	0.268D + 00	0.154D + 00	0.864D - 01	0.478D - 01	0.261D - 01	0.142D - 01
2^{-11}	0.754D + 00	0.456D + 00	0.268D + 00	0.154D + 00	0.865D - 01	0.479D - 01	0.262D - 01	0.142D - 01
2^{-12}	0.755D + 00	0.456D + 00	0.269D + 00	0.154D + 00	0.866D - 01	0.479D - 01	0.262D - 01	0.142D - 01
2^{-13}	0.755D + 00	0.457D + 00	0.269D + 00	0.154D + 00	0.867D - 01	0.479D - 01	0.262D - 01	0.142D - 01
2^{-14}	0.756D + 00	0.457D + 00	0.269D + 00	0.154D + 00	0.867D - 01	0.480D - 01	0.262D - 01	0.142D - 01
2^{-15}	0.756D + 00	0.457D + 00	0.269D + 00	0.154D + 00	0.867D - 01	0.480D - 01	0.262D - 01	0.142D - 01
E_{\max}^N	0.756D + 00	0.457D + 00	0.269D + 00	0.154D + 00	0.867D - 01	0.480D - 01	0.262D - 01	0.142D - 01

These rates are increasing as N increases for any fixed ε and eventually stabilise for any fixed N . Also given are the estimated uniform convergence rates for each N :

$$R_{\text{unif}}^N = \log_2 \frac{E_{\max}^N}{E_{\max}^{2N}}.$$

Table 2

Rates of convergence R_ε^N and R_{unif}^N for U_ε

ε	Number of intervals N						
	32	64	128	256	512	1024	2048
2^{-1}	0.98	0.99	0.99	1.00	1.00	1.00	1.00
2^{-2}	0.96	0.98	0.99	1.00	1.00	1.00	1.00
2^{-3}	0.93	0.97	0.98	0.99	1.00	1.00	1.00
2^{-4}	0.74	0.93	0.96	0.98	0.99	0.99	1.00
2^{-5}	0.71	0.76	0.80	0.83	0.85	0.87	0.93
2^{-6}	0.72	0.76	0.80	0.83	0.85	0.87	0.88
2^{-7}	0.72	0.76	0.80	0.83	0.85	0.87	0.88
2^{-8}	0.72	0.76	0.80	0.83	0.85	0.87	0.88
2^{-9}	0.73	0.76	0.80	0.83	0.85	0.87	0.88
2^{-10}	0.73	0.76	0.80	0.83	0.85	0.87	0.88
2^{-11}	0.73	0.76	0.80	0.83	0.85	0.87	0.88
2^{-12}	0.73	0.76	0.80	0.83	0.85	0.87	0.88
2^{-13}	0.73	0.76	0.80	0.83	0.85	0.87	0.88
2^{-14}	0.73	0.76	0.80	0.83	0.85	0.87	0.88
2^{-15}	0.73	0.76	0.80	0.83	0.85	0.87	0.88
R_{unif}^N	0.73	0.76	0.80	0.83	0.85	0.87	0.88

These tables thus verify the ε -uniform convergence of the numerical solutions and the computed rates are in agreement with Theorem 4.3.

6. Summary

Our objective was to solve the convection diffusion problem (1) and (2), using a method that is robust with respect to the perturbation parameter. We solved this problem by employing standard upwind finite difference operators on a Shishkin meshes. We have shown that the method displays robustness with respect to the perturbation parameter for numerical approximation of the solution.

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