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Journal of Computational and Applied Mathematics 189 (2006) 387–411
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Exact solutions for some nonlinear evolution equations which describe pseudo-spherical surfaces

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Received 28 September 2004; received in revised form 22 August 2005

Abstract

For some nonlinear evolution equations which describe pseudo-spherical surfaces two new exact solution classes are generated from known solutions: either the seed solution is constant or a traveling wave. Exact traveling wave and solitary wave solutions for Burgers' equation and also for a fifth-order evolution equation containing the fifth-order Korteweg–de Vries equation and also the Sawada–Kotera equation, are obtained by using an improved sine–cosine method and the Wu's elimination method. We also apply Bäcklund transformations to these solutions and generate new soliton solution classes.

MSC: 35; 53C; 58J; 58Z05

Keywords: Nonlinear evolution equations; Bäcklund transformations; Pseudo-spherical surfaces

1. Introduction

Bäcklund transformations (BTs) have been useful in the calculation of soliton solutions of certain nonlinear evolution equations (NLEEs) of physical significance [9,37,26,36] restricted to one space variable x and a time coordinate t. The classical treatment of the surface transformations which provide the origin of Bäcklund theory was developed in [12]. BTs are local geometric transformations, which construct from a given surface of constant Gaussian curvature -1 a two parameter family of such surfaces. To find such transformations, one needs to solve a system of compatible ordinary differential equations (ODEs) [10].

Khater et al. [19,20] used the notion of a differential equation (DE) for a function $u(x,t)$ that describes a pseudo-spherical surface (pss), and they derived some BTs for NLEEs which are the integrability condition of $sl(2,R)$-valued linear problems [37,14–18,22–24].

The method of characteristics is used (see discussion after Eq. (86)) and BTs are employed to generate new solutions from a known solution for some NLEEs (see, for example [37]). Previously, Konno and Wadati [25], for example, had derived some BTs for NLEEs of the AKNS class. These BTs explicitly express the new solutions in terms of the

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doi:10.1016/j.cam.2005.10.007

known solutions of the NLEEs and corresponding wave functions which are solutions of the associated Ablowitz, Kaup, Newell, Segur (AKNS) system [49,1].

Wahlquist and Eastbrook [41,42] have also devised a method which can be applied to the construction of BTs and Lax formulations. The Wahlquist–Eastbrook procedure has the advantage that it highlights the roles of various Lie groups associated with DEs that admit BTs. Moreover, the method has a natural formulation within a jet-bundle context. The latter paper [41] was concerned with obtaining the prolongation structure of the Korteweg–de Vries (KdV) equation, and illustrating its relation to the many known techniques for treating this equation. It also serves to relate the KdV equation to a number of other nonlinear partial differential equations (NLPDEs). A BT for the KdV and the potential KdV equations were derived by Wahlquist and Eastbrook [40]. Among the numerous BTs found after the work [40] (see, for example [37,36] for detailed discussions), a BT for the modified Korteweg–de Vries (mKdV) equation was also found by Lamb [26].

In the present paper, we use BTs derived in [19,20] in the construction of exact soliton solutions for some NLEEs describing pss [4–8,32,34,2,13,38] which are beyond the AKNS class: the $(-)$ and $(+)$ Beals–Rabelo–Tenenblat (BRT) equations [2] (which include the sine-Gordon, sinh-Gordon and Liouville' equations [see, Eqs. (15) and (33)]), the Cavalcante–Tenenblat (CT) equation [3] (the first example for a nongeneric case in the classification given in [4] [see, Eq. (21)]), mKdV and Burgers' equations. In addition, we use other solution techniques (the sine–cosine and Wu's elimination methods) to obtain new soliton solutions for Burgers and the generalized fifth-order Korteweg–de Vries (gfKdV) equations.

The soliton phenomena and integrable NLEEs represent an important and well-established field of modern physics, mathematical physics and applied mathematics. Solitons are found in various areas of physics: hydrodynamics and plasma physics, nonlinear optics, solid-state physics, field theory and gravitation.

In [30,35], Reyes showed that if an equation describes nontrivial one-parameter families of pss, its conservation laws, nonlocal symmetries, and BTs, can be studied by geometrical means. The special case of the Camassa–Holm equation is in [33]. It is also shown in [35] that there exist correspondences between arbitrary solutions (which satisfy genericity conditions) of any two equations describing pss. The relation between the above papers and the present paper is in the use of the geometrical properties of NLEEs which describe pss to obtain an analytic information, in our case explicit solutions and BTs.

We also point out that in [39], Schiff obtained a unified and simplified understanding of symmetries of KdV equation. A direct connection between nonlocal symmetries and BTs is made by him. He explained how the known symmetries, including BTs, of KdV equation arise from simple, field independent, actions on the loop group. A variety of issues in understanding the algebraic structure of BTs are thus resolved.

The paper is organized as follows: in Section 2, the AKNS system and the general form of the BTs for some NLEEs are illustrated. In Section 3, a new exact soliton solution class from a known constant solution is obtained for the $(-)$ and $(+)$ BRT equations, the CT equation, mKdV and Burgers' equations. In Section 4, in the case that the known solution is a traveling wave, we construct new traveling wave solutions for the mKdV, the $(+)$ BRT, sine-Gordon, sinh-Gordon and Liouville's equations. On the other hand, exact traveling wave and solitary wave solutions for Burgers' equation are obtained by using an improved sine–cosine method, the Wu's elimination method and BTs to generate a new soliton solution class. The latter method is used to obtain traveling wave solutions for the gfKdV equation. Finally, we give some conclusions in Section 5.

2. The AKNS system for some NLEEs which describe pss

We recall the definition [4,38] of a DE that describes a pss. Let M^2 be a two-dimensional differentiable manifold with coordinates (x, t). A DE for a real function $u(x, t)$ describes a pss if it is a necessary and sufficient condition for the existence of differentiable functions

$$f_{ij}, \quad 1 \leqslant i \leqslant 3, \quad 1 \leqslant j \leqslant 2, \tag{1}$$

depending on u and its derivatives such that the one-forms

$$\omega_1 = f_{11}\,\mathrm{d}x + f_{12}\,\mathrm{d}t, \quad \omega_2 = f_{21}\,\mathrm{d}x + f_{22}\,\mathrm{d}t, \quad \omega_3 = f_{31}\,\mathrm{d}x + f_{32}\,\mathrm{d}t, \tag{2}$$

satisfy the structure equations of a pss, i.e.,

$$\mathrm{d}\omega_1 = \omega_3 \wedge \omega_2, \quad \mathrm{d}\omega_2 = \omega_1 \wedge \omega_3, \quad \mathrm{d}\omega_3 = \omega_2 \wedge \omega_1. \tag{3}$$

As a consequence, each solution of the DE provides a local metric on M^2, whose Gaussian curvature is constant, equal to -1. Moreover, the above definition is equivalent to saying that DE for u is the integrability condition for the problem [41,42,2,38]:

$$\mathrm{d}\phi = \Omega\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{4}$$

where d denotes exterior differentiation, ϕ is a column vector and the 2×2 matrix Ω (Ω_{ij}, i, $j = 1, 2$) is traceless

$$\Omega = \frac{1}{2} \begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}.$$

Take

$$\Omega = \begin{pmatrix} \frac{\eta}{2}\,\mathrm{d}x + A\,\mathrm{d}t & q\,\mathrm{d}x + B\,\mathrm{d}t \\ r\,\mathrm{d}x + C\,\mathrm{d}t & -\frac{\eta}{2}\,\mathrm{d}x - A\,\mathrm{d}t \end{pmatrix} = P\,\mathrm{d}x + Q\,\mathrm{d}t \tag{5}$$

from Eqs. (4) and (5), we obtain

$$\phi_x = P\phi, \quad \phi_t = Q\phi, \tag{6}$$

where P and Q are two 2×2 null-trace matrices

$$P = \begin{pmatrix} \frac{\eta}{2} & q \\ r & -\frac{\eta}{2} \end{pmatrix}, \tag{7}$$

$$Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{8}$$

Here η is a parameter, independent of x and t, while q and r are functions of x and t. Now

$$0 = \mathrm{d}^2\phi = \mathrm{d}\Omega\phi - \Omega \wedge \mathrm{d}\phi = (\mathrm{d}\Omega - \Omega \wedge \Omega)\phi,$$

which requires the vanishing of the two form

$$\Theta \equiv \mathrm{d}\Omega - \Omega \wedge \Omega = 0 \tag{9}$$

or in component form

$$A_x = qC - rB, \quad q_t - 2Aq - B_x + \eta B = 0, \quad C_x = r_t + 2Ar - \eta C, \tag{10}$$

Chern and Tenenblat [4] obtained Eq. (10) directly from the structure equations (3). By suitably choosing r, A, B and C in (10), we shall obtain various NLEEs which q must satisfy. Konno and Wadati introduced the function [25]

$$\Gamma = \frac{\phi_1}{\phi_2}, \tag{11}$$

this function first appeared and explained the geometric context of pss equations in [31,29], see also the classical papers by Sasaki [38] and Chern–Tenenblat [4]. Then Eq. (6) is reduced to the Riccati equations:

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma - r\Gamma^2 + q, \tag{12}$$

$$\frac{\partial \Gamma}{\partial t} = 2A\Gamma - C\Gamma^2 + B \tag{13}$$

and the BT we obtain is of the form

$$u' = u + f(\Gamma, \eta). \tag{14}$$

Chern and Tenenblat [4] introduced several examples of (14) for pss equations. For use in the sequel, we list the NLEEs and their corresponding BT in the following.

2.1. The (−) BRT equation

Beals, Rabelo and Tenenblat (BRT) [2] introduced the family of equations as follows:

$$[u_t - (\alpha g(u) + \beta)u_x]_x = -g'(u), \quad \text{where } g'' + \mu g = \theta, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}u}, \tag{15}$$

where $g(u)$ is a differentiable function of $u(x,t)$, with α, β, μ and θ are real constants, such that $\zeta^2 = \alpha\eta^2 + \mu$. This family includes the sine-Gordon, sinh-Gordon and Liouville' equations. For any solution $u(x,t)$ of the (−) BRT equation (15), the matrices P and Q are [19]

$$P = \begin{pmatrix} \dfrac{\eta}{2} & -\zeta\dfrac{u_x}{2} \\ \zeta\dfrac{u_x}{2} & -\dfrac{\eta}{2} \end{pmatrix}, \tag{16}$$

$$Q = \begin{pmatrix} \dfrac{1}{2}\left(\dfrac{\zeta^2 g - \theta}{\eta} + \beta\eta\right) & \dfrac{1}{2}\left[\dfrac{\zeta}{\eta}g' + \zeta(\alpha g + \beta)u_x\right] \\ \dfrac{1}{2}\left[-\dfrac{\zeta}{\eta}g' + \zeta(\alpha g + \beta)u_x\right] & -\dfrac{1}{2}\left(\dfrac{\zeta^2 g - \theta}{\eta} + \beta\eta\right) \end{pmatrix}, \tag{17}$$

the above matrices P, Q satisfy Eq. (10). Then Eq. (12) becomes

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma - \frac{\zeta}{2}u_x(1 + \Gamma^2). \tag{18}$$

If we choose Γ' and u' as [20]

$$\Gamma' = \frac{1}{\Gamma}, \tag{19}$$

$$u' = u + \frac{4}{\zeta}\tan^{-1}\Gamma, \tag{20}$$

then Γ' and u' satisfy Eq. (18) and u' is a new solution of Eq. (15).

2.2. The CT equation

Cavalcante and Tenenblat [35] introduced a third-order evolution equation as follows:

$$u_t = (u_x^{-1/2})_{xx} + u_x^{3/2}, \tag{21}$$

which is the first example for a nongeneric case in the classification for autonomous evolution equations of pseudo-spherical type $u_t = F(u, u_x, \ldots, u_{x^k})$ $(u_{x^k} = \partial^k u/\partial x^k)$ [4]. For any solution $u(x,t)$ of the CT equation (21), the matrices P and Q are [19]

$$P = \begin{pmatrix} \dfrac{\eta}{2} & -\dfrac{1}{2}\eta \mathrm{e}^{-u} \\ \dfrac{1}{2}\eta \mathrm{e}^{u} & -\dfrac{\eta}{2} \end{pmatrix}, \tag{22}$$

$$Q = \begin{pmatrix} -\dfrac{1}{2}\eta^2 u_x^{-1/2} & \dfrac{1}{2}\eta \mathrm{e}^{-u}[(u_x^{-1/2})_x - u_x^{1/2} + \eta u_x^{-1/2}] \\ \dfrac{1}{2}\eta \mathrm{e}^{u}[(u_x^{-1/2})_x + u_x^{1/2} - \eta u_x^{-1/2}] & \dfrac{1}{2}\eta^2 u_x^{-1/2} \end{pmatrix}, \tag{23}$$

the above matrices P, Q satisfy Eq. (10). Then Eq. (12) becomes

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma - \frac{-\eta}{2}e^{-u} - \frac{\eta}{2}e^{u}\Gamma^2. \tag{24}$$

If we choose Γ' and u' as [20]

$$\Gamma' = \Gamma, \tag{25}$$

$$u' = -u - 2\ln\Gamma, \tag{26}$$

then Γ' and u' satisfy Eq. (24) and u' is a new solution to Eq. (21).

2.3. A mKdV equation

Consider a mKdV equation in the form [11]

$$u_t = u_{xxx} + (a + u^2)u_x, \quad \text{where } a \text{ is a constant.} \tag{27}$$

For any solution $u(x,t)$ of a mKdV Eq. (27), the matrices P and Q are [19]

$$P = \begin{pmatrix} \frac{\eta}{2} & -\frac{u}{\sqrt{6}} \\ \frac{u}{\sqrt{6}} & -\frac{\eta}{2} \end{pmatrix}, \tag{28}$$

$$Q = \begin{pmatrix} \frac{1}{2}\left(\eta^3 + \frac{\eta u^2}{3} + a\eta\right) & \frac{1}{\sqrt{6}}\left(-\eta u_x - u_{xx} - \frac{u^3}{3} - \eta^2 u - au\right) \\ \frac{1}{\sqrt{6}}\left(-\eta u_x + u_{xx} + \frac{u^3}{3} + \eta^2 u + au\right) & -\frac{1}{2}\left(\eta^3 + \frac{\eta u^2}{3} + a\eta\right) \end{pmatrix}, \tag{29}$$

the above matrices P, Q satisfy Eqs. (10). Then Eq. (12) becomes

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma - \sqrt{\frac{1}{6}}u(1 + \Gamma^2). \tag{30}$$

If we choose Γ' and u' as [20]

$$\Gamma' = \frac{1}{\Gamma}, \tag{31}$$

$$u' = u + 2\sqrt{6}\frac{\partial}{\partial x}\tan^{-1}\Gamma, \tag{32}$$

then Γ' and u' satisfy Eq. (30) and u' is a new solution to Eq. (27).

2.4. The (+) BRT equation

Beals, Rabelo and Tenenblat [2] introduced the family of equations as follows:

$$[u_t - (\alpha g(u) + \beta)u_x]_x = g'(u), \quad \text{where } g'' + \mu g = \theta, \tag{33}$$

where $g' = \mathrm{d}g/\mathrm{d}u$, $\zeta^2 = \alpha\eta^2 - \mu$, also this family includes the sine-Gordon, sinh-Gordon and Liouville' equations. For any solution $u(x,t)$ of the $(+)$ BRT equation (33), the matrices P and Q are [19]

$$P = \begin{pmatrix} \frac{\eta}{2} & \zeta\frac{u_x}{2} \\ \zeta\frac{u_x}{2} & -\frac{\eta}{2} \end{pmatrix}, \tag{34}$$

$$Q = \begin{pmatrix} \frac{1}{2}\left(\frac{\zeta^2 g - \theta}{\eta} + \beta\eta\right) & \frac{1}{2}\left[\frac{\zeta}{\eta}g' + \zeta(\alpha g + \beta)u_x\right] \\ \frac{1}{2}\left[-\frac{\zeta}{\eta}g' + \zeta(\alpha g + \beta)u_x\right] & -\frac{1}{2}\left(\frac{\zeta^2 g - \theta}{\eta} + \beta\eta\right) \end{pmatrix}, \tag{35}$$

the above matrices P, Q satisfy Eq. (10). Then Eq. (12) becomes

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma + \frac{\zeta}{2}u_x(1 - \Gamma^2). \tag{36}$$

If we choose Γ' and u' as [20]

$$\Gamma' = \frac{1}{\Gamma}, \tag{37}$$

$$u' = -u + \frac{4}{\zeta}\tanh^{-1}\Gamma, \tag{38}$$

then Γ' and u' satisfy Eq. (36) and u' is a new solution to Eq. (33).

2.5. Burgers' equation

Consider the Burgers' equation as follows:

$$2u_t - 2uu_x - u_{xx} = 0 \tag{39}$$

for any solution $u(x,t)$ of the Burgers' equation (39), the matrices P and Q are [19]

$$P = \begin{pmatrix} \frac{\eta}{2} & \frac{u}{2} + \frac{\eta}{2} \\ \frac{u}{2} - \frac{\eta}{2} & -\frac{\eta}{2} \end{pmatrix}, \tag{40}$$

$$Q = \begin{pmatrix} \frac{u\eta}{4} & \frac{u\eta}{4} + \frac{u^2}{4} + \frac{u_x}{4} \\ \frac{-u\eta}{4} + \frac{u^2}{4} + \frac{u_x}{4} & -\frac{u\eta}{4} \end{pmatrix}, \tag{41}$$

the above matrices P, Q satisfy Eq. (10). Then Eq. (12) becomes

$$\frac{\partial \Gamma}{\partial x} = \frac{\eta}{2}(1 + 2\Gamma + \Gamma^2) + \frac{u}{2}(1 - \Gamma^2). \tag{42}$$

If we choose Γ' and u' as [20]

$$\Gamma' = \frac{1}{\Gamma}, \tag{43}$$

$$u' = -u + 4\frac{\partial}{\partial x}\tanh^{-1}\Gamma, \tag{44}$$

then Γ' and u' satisfy Eq. (42) and u' is a new solution to Eq. (39).

Now we shall choose some known solutions of the above NLEEs and substitute these solutions into the corresponding matrices P and Q. Next, we solve Eqs. (6) for ϕ_1 and ϕ_2. Then, by (11) and the corresponding BT we shall obtain the new solutions for the NLEEs.

3. The known solution is a constant u_0

3.1. The $(-)$ BRT equation

Substitute $u = u_0$ into the matrices P and Q in (16) and (17), then by (6) we have

$$d\phi = \phi_x\,dx + \phi_t\,dt = P\phi\,d\rho, \tag{45}$$

where

$$P = \begin{pmatrix} \frac{\eta}{2} & 0 \\ 0 & -\frac{\eta}{2} \end{pmatrix}, \tag{46}$$

$$\rho = x + kt, \quad k = \frac{\zeta^2 g(u_0) - \theta}{\eta^2} + \beta. \tag{47}$$

The solution of Eq. (45) is

$$\phi = e^{\rho P}\phi_0 = \left(I + \rho P + \frac{\rho^2 P^2}{2!} + \frac{\rho^3 P^3}{3!} + \cdots\right)\phi_0, \tag{48}$$

where ϕ_0 is a constant column vector. The solution of Eq. (48) is

$$\phi = \begin{pmatrix} \cosh\frac{\eta}{2}\rho + \sinh\frac{\eta}{2}\rho & 0 \\ 0 & \cosh\frac{\eta}{2}\rho - \sinh\frac{\eta}{2}\rho \end{pmatrix}\phi_0. \tag{49}$$

Now, we choose $\phi_0 = (1, 1)^{\mathrm{T}}$ in (49), then we have

$$\phi = \begin{pmatrix} e^{\frac{\eta\rho}{2}} \\ e^{-\frac{\eta\rho}{2}} \end{pmatrix}. \tag{50}$$

Substitute (50) into (11), then by (20), we obtain the new solutions of the $(-)$ BRT equation (15)

$$u' = u_0 + \frac{4}{\zeta}\tan^{-1}(e^{\eta\rho}). \tag{51}$$

In this case, we observe that taking $\alpha = \beta = \theta = 0$, the sine-Gordon and sinh-Gordon equations are obtained from (15) by considering appropriate values for μ.

(i) When $\mu = 1$, $g(u) = \cos u$, we obtain the sine-Gordon equation

$$u_{xt} = \sin u \tag{52}$$

and its solution is

$$u' = u_0 \pm 4\tan^{-1}(e^{\eta\rho}), \quad \rho = x + \frac{\cos u_0}{\eta^2}t. \tag{53}$$

(ii) When $\mu = -1$, $g(u) = -\cosh u$, we obtain the sinh-Gordon equation

$$u_{xt} = \sinh u \tag{54}$$

and its solution is

$$u' = u_0 \pm \frac{4}{i}\tan^{-1}(e^{\eta\rho}), \quad \rho = x + \frac{\cosh u_0}{\eta^2}t. \tag{55}$$

3.2. The CT equation

Substitute $u = u_0$ into the matrices P and Q in (22) and (23), then by (6) we have

$$\mathrm{d}\phi = \phi_x\,\mathrm{d}x + \phi_t\,\mathrm{d}t = P\phi\,\mathrm{d}x, \tag{56}$$

where

$$P = \begin{pmatrix} \frac{\eta}{2} & -\frac{\eta}{2}\mathrm{e}^{-u_0} \\ \frac{\eta}{2}\mathrm{e}^{u_0} & -\frac{\eta}{2} \end{pmatrix}. \tag{57}$$

The solution of Eq. (56) is

$$\phi = \mathrm{e}^{xP}\phi_0 = \left(I + xP + \frac{x^2P^2}{2!} + \frac{x^3P^3}{3!} + \cdots\right)\phi_0, \tag{58}$$

then

$$\phi = \begin{pmatrix} 1 + \frac{x\eta}{2} & -\frac{x\eta}{2}\mathrm{e}^{-u_0} \\ \frac{x\eta}{2}\mathrm{e}^{u_0} & 1 - \frac{x\eta}{2} \end{pmatrix}\phi_0. \tag{59}$$

Choosing $\phi_0 = (1, 0)^{\mathrm{T}}$ in (59), then, by (11) and (26), we obtain the new solutions of the CT equation which reads

$$u' = u_0 + \tfrac{1}{2}\ln x\eta. \tag{60}$$

3.3. A mKdV equation

Substitute $u = u_0$ into the matrices P and Q in (28) and (29), then by (6) we have

$$\mathrm{d}\phi = \phi_x\,\mathrm{d}x + \phi_t\,\mathrm{d}t = P\phi\,\mathrm{d}\rho, \tag{61}$$

where

$$P = \begin{pmatrix} \frac{\eta}{2} & -\frac{u_0}{\sqrt{6}} \\ \frac{u_0}{\sqrt{6}} & -\frac{\eta}{2} \end{pmatrix}, \tag{62}$$

$$\rho = x + kt, \quad k = (\eta^2 + \tfrac{1}{3}u_0^2 + a). \tag{63}$$

The solution of Eq. (61) is

$$\phi = \mathrm{e}^{\rho P}\phi_0 = \left(I + \rho P + \frac{\rho^2P^2}{2!} + \frac{\rho^3P^3}{3!} + \cdots\right)\phi_0, \tag{64}$$

where $\phi_0 = (1, 0)^{\mathrm{T}}$. According to the sign of the quantity $\eta^2/4 - u_0^2/6$, the solution (64) may take the following three forms:

(i) $\frac{\eta^2}{4} - \frac{u_0^2}{6} > 0$, $\alpha^2 = \frac{\eta^2}{4} - \frac{u_0^2}{6}$,

$$\phi = \begin{pmatrix} \cosh\alpha\rho + \frac{\eta}{2\alpha}\sinh\alpha\rho & -\frac{u_0}{\alpha\sqrt{6}}\sinh\alpha\rho \\ \frac{u_0}{\alpha\sqrt{6}}\sinh\alpha\rho & \cosh\alpha\rho - \frac{\eta}{2\alpha}\sinh\alpha\rho \end{pmatrix}\phi_0, \tag{65}$$

$$u' = u_0 + 2\sqrt{6}\frac{\partial}{\partial x}\left[\tan^{-1}\frac{\sqrt{6}}{u_0}\left(\alpha\coth\alpha\rho + \frac{\eta}{2}\right)\right]. \tag{66}$$

(ii) $\frac{\eta^2}{4}-\frac{u_0^2}{6}<0$, $\alpha^2=\frac{u_0^2}{6}-\frac{\eta^2}{4}$,

$$\phi=\begin{pmatrix}\cos\alpha\rho+\dfrac{\eta}{2\alpha}\sin\alpha\rho & -\dfrac{u_0}{\alpha\sqrt{6}}\sin\alpha\rho\\ \dfrac{u_0}{\alpha\sqrt{6}}\sin\alpha\rho & \cos\alpha\rho-\dfrac{\eta}{2\alpha}\sin\alpha\rho\end{pmatrix}\phi_0, \tag{67}$$

$$u'=u_0+2\sqrt{6}\frac{\partial}{\partial x}\left[\tan^{-1}\frac{\sqrt{6}}{u_0}\left(\alpha\cot\alpha\rho+\frac{\eta}{2}\right)\right]. \tag{68}$$

(iii) $\frac{\eta^2}{4}-\frac{u_0^2}{6}=0$, $\rho=x+(u_0^2+a)t$,

$$\phi=\begin{pmatrix}1+\dfrac{\rho u_0}{\sqrt{6}} & -\dfrac{\rho u_0}{\sqrt{6}}\\ \dfrac{\rho u_0}{\sqrt{6}} & 1-\dfrac{\rho u_0}{\sqrt{6}}\end{pmatrix}\phi_0, \tag{69}$$

$$u'=u_0+2\sqrt{6}\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{\sqrt{6}}{\rho u_0}+1\right)\right]. \tag{70}$$

3.4. The (+) BRT equation

Proceeding as for the (−) BRT equation we obtain

$$u'=-u_0+\frac{4}{\zeta}\tanh^{-1}(e^{\eta\rho}). \tag{71}$$

In this case, we observe that taking $\alpha=\beta=\theta=0$, the sine-Gordon and sinh-Gordon equations are obtained from (33) by considering appropriate values for μ.

(i) When $\mu=1$, $g(u)=-\cos u$, we obtain the sine-Gordon equation

$$u_{xt}=\sin u \tag{72}$$

and its solution is

$$u'=-u_0\mp\frac{4}{i}\tanh^{-1}(e^{\eta\rho}),\quad \rho=x+\frac{\cos u_0}{\eta^2}t. \tag{73}$$

(ii) When $\mu=-1$, $g(u)=\cosh u$, we obtain the sinh-Gordon equation

$$u_{xt}=\sinh u, \tag{74}$$

and its solution is

$$u'=-u_0\mp 4\tanh^{-1}e^{\eta\rho},\quad \rho=x+\frac{\cosh u_0}{\eta^2}t. \tag{75}$$

3.5. Burgers' equation

Substitute $u=u_0$ into the matrices P and Q in (40) and (41), then by (6) we have

$$d\phi=\phi_x\,dx+\phi_t\,dt=P\phi\,d\rho, \tag{76}$$

where

$$P=\begin{pmatrix}\dfrac{\eta}{2} & \dfrac{u_0}{2}+\dfrac{\eta}{2}\\ \dfrac{u_0}{2}-\dfrac{\eta}{2} & -\dfrac{\eta}{2}\end{pmatrix}, \tag{77}$$

$$\rho=x+\alpha t,\quad \alpha=\frac{u_0}{2}. \tag{78}$$

The solution of Eq. (76) is

$$\phi=(\exp\rho P)\phi_0=\left(I+\rho P+\frac{\rho^2P^2}{2!}+\frac{\rho^3P^3}{3!}+\cdots\right)\phi_0, \tag{79}$$

where ϕ_0 is a constant column vector. The solution (79) takes the following form:

$$\phi=\begin{bmatrix}\cosh\alpha\rho+\dfrac{\eta}{2\alpha}\sinh\alpha\rho & \left(1+\dfrac{\eta}{2\alpha}\right)\sinh\alpha\rho\\ \left(1-\dfrac{\eta}{2\alpha}\right)\sinh\alpha\rho & \cosh\alpha\rho-\dfrac{\eta}{2\alpha}\sinh\alpha\rho\end{bmatrix}\phi_0. \tag{80}$$

Now, we choose $\phi_0=(1,0)^{\mathrm{T}}$ in (80) and use (6) and the BT (44); we obtain the new solution class of the Burgers' equation (39) corresponding to the known constant solution u_0 as follows:

$$u=-u_0+\frac{2\alpha(2\alpha-\eta)}{2\alpha\sinh^2(\alpha\rho/2)+\eta}\tanh\frac{\alpha\rho}{2}. \tag{81}$$

If we choose $\phi_0=(0,1)^{\mathrm{T}}$ in (80), we shall obtain another solution. Obviously all of these solutions are traveling waves with velocity $\alpha=u_0/2$.

4. The known solution is a traveling wave

In this section, we shall find new traveling wave solutions $u'(x,t)$ of the mKdV equation, the (+) BRT equation, sine-Gordon, sinh-Gordon, Liouville, Burgers and gfKdV equations.

4.1. A mKdV equation

$$u_t=u_{xxx}+(a+u^2)u_x,\quad \text{where } a \text{ is a constant} \tag{82}$$

we shall choose a known solution of the above mKdV Eq. (82) as a traveling wave $u(x,t)$ and substitute these solution into the corresponding matrices P and Q. Next we solve Eq. (6) for ϕ_1 and ϕ_2. Then by (11) and the corresponding BTs (32) we shall obtain the new solution classes. By direct calculation we take

$$u=-2\sqrt{6}\eta\,\mathrm{sech}\,2\eta\rho, \tag{83}$$

as a traveling wave solution class of the mKdV Eq. (82). The traveling wave known solution of the mKdV equation takes the form

$$u=u(\rho),\quad \rho=x-kt,\quad k=-(4\eta^2+a). \tag{84}$$

In this case the AKNS system (6)–(8) has a general solution. Let us consider the more general case. Suppose that the components of the matrices P and Q are function of ρ only [21,18]. Under these assumptions, the following result holds, which is crucial in the subsequent exact solution. The quantity

$$\beta_1=(A+\tfrac{1}{2}k\eta)^2+(B+kq)(C+kr), \tag{85}$$

is constant.

From (6)–(8), after inserting the known solution $u(x, t)$ of the NLEE into the corresponding matrices P and Q, we will have the following system of PDEs for the unknowns ϕ_1 and ϕ_2:

$$\begin{aligned}
\phi_{1x} &= \frac{\eta}{2}\phi_1 + q\phi_2,\\
\phi_{2x} &= r\phi_1 - \frac{\eta}{2}\phi_2,\\
\phi_{1t} &= A\phi_1 + B\phi_2,\\
\phi_{2t} &= C\phi_1 - A\phi_2.
\end{aligned} \tag{86}$$

Solve (86) for ϕ_1 giving

$$\phi_1 = \frac{1}{r}\left(\phi_{2x} + \frac{\eta}{2}\phi_2\right). \tag{87}$$

Substituting this ϕ_1 into (86) together with (10) we get

$$C\phi_{2x} - r\phi_{2t} = \tfrac{1}{2}(C_x - r_t)\phi_2. \tag{88}$$

The PDE (88) possesses the following characteristic equations:

$$\frac{\mathrm{d}t}{-r} = \frac{\mathrm{d}x}{C} = \frac{2\,\mathrm{d}\phi_2}{(C_x - r_t)\phi_2}. \tag{89}$$

Using (84) and $C = C(\rho)$, $r = r(\rho)$, we have

$$C_x - r_t = \frac{\mathrm{d}C}{\mathrm{d}\rho} + k\frac{\mathrm{d}r}{\mathrm{d}\rho} = \frac{\mathrm{d}}{\mathrm{d}\rho}(C + kr), \tag{90}$$

substituting (90) into (89) gives

$$\frac{\mathrm{d}t}{-r} = \frac{\mathrm{d}\rho}{(C + kr)} = \frac{2\,\mathrm{d}\phi_2}{(C + kr)'_\rho\phi_2}. \tag{91}$$

These equations yield the following system of ODE's:

$$\frac{\mathrm{d}(\ln\phi_2)}{\mathrm{d}\rho} = \frac{(C + kr)'_\rho}{2(C + kr)}, \tag{92}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{-(C + kr)}{r}. \tag{93}$$

Integrating Eq. (92) leads to

$$\phi_2 = k_2(C + kr)^{1/2}, \tag{94}$$

where k_2 is an integration constant. Integrating (93) we get

$$-t + k_1 = \int \frac{r\,\mathrm{d}\rho}{(C + kr)}, \tag{95}$$

where k_1 is another integration constant. From (94) and (95), we obtain the general solution of Eq. (88):

$$\phi_2 = (C + kr)^{1/2}F(\xi), \tag{96}$$

where

$$\xi = t + \int \frac{r\,\mathrm{d}\rho}{(C + kr)} \tag{97}$$

and $F(\xi)$ is a differentiable function of ξ. Substituting (96) into (87) gives the general solution for ϕ_1: this can be done by the following direct calculation:

$$\phi_1 = \frac{1}{r}\left(\phi_{2x} + \frac{\eta}{2}\phi_2\right)$$

from (96), we obtain

$$\phi_{2x} = \tfrac{1}{2}(C+kr)^{-1/2}(C+kr)'_\rho F(\xi) + (C+kr)^{1/2}F'_\xi \xi_x,$$

where $\xi_x = r/(C+kr)$, then

$$\phi_1 = (C+kr)^{-1/2}\left[F'_\xi + \left(\frac{1}{2r}(C+kr)'_\rho + \frac{\eta}{2r}(C+kr)\right)F(\xi)\right]. \tag{98}$$

Now from (90) and (10), we obtain

$$(C+kr)'_\rho = C_x - r_t = 2rA - \eta C,$$

then Eq. (98) take the form

$$\phi_1 = (C+kr)^{-1/2}(F'_\xi + (A + \tfrac{1}{2}k\eta)F). \tag{99}$$

To determine the function F, Eqs. (96) and (99) are substituted into (86) ($\phi_{1x} = (\eta/2)\phi_1 + q\phi_2$), by the same of direct calculation we find that F must satisfy the following second-order ODE:

$$F''_{\xi\xi} - \beta_1 F = 0, \tag{100}$$

where β_1 is a constant defined in (85). According to the sign of β_1, Eq. (100) will have the following three different solutions:

$$F = c_1\xi + c_2 \quad \text{when } \beta_1 = 0, \tag{101}$$

$$F = c_1 \sinh\omega(\xi + c_2) \quad \text{when } \beta_1 > 0, \omega^2 = \beta_1, \tag{102}$$

$$F = c_1 \sin\omega(\xi + c_2) \quad \text{when } \beta_1 < 0, \omega^2 = -\beta_1, \tag{103}$$

where c_1 and c_2 are integration constants. Substituting these solutions into (99) and (96), respectively, we obtain the corresponding solutions of the AKNS system (6)–(8) are:

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}(C+kr)^{-1/2}[(A+\frac{1}{2}k\eta)(c_1\xi + c_2) + c_1]\\ (C+kr)^{1/2}(c_1\xi + c_2)\end{bmatrix} \quad \text{when } \beta_1 = 0, \tag{104}$$

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}c_1(C+kr)^{-1/2}[(A+\frac{1}{2}k\eta)\sinh\omega(\xi+c_2) + \omega\cosh\omega(\xi+c_2)]\\ c_1(C+kr)^{1/2}\sinh\omega(\xi+c_2)\end{bmatrix} \quad \text{when } \beta_1 > 0,\ \omega^2 = \beta_1 \tag{105}$$

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}c_1(C+kr)^{-1/2}[(A+\frac{1}{2}k\eta)\sin\omega(\xi+c_2) + \omega\cos\omega(\xi+c_2)]\\ c_1(C+kr)^{1/2}\sin\omega(\xi+c_2)\end{bmatrix} \quad \text{when } \beta_1 < 0. \tag{106}$$

Now applying the results obtained here and the known traveling wave solutions for the mKdV equation, respectively, to construct new solution class of the corresponding mKdV equation by means of the BTs. The constant β_1 and ξ defined by (85), (97) can be determined by using (83)

$$\xi = \frac{1}{3\eta^3}[\eta\rho - \ln 3/\eta\rho + 3\eta^3(t + c_3)], \ \text{where } c_3 \text{ is an integration constant.} \tag{107}$$

Consequently, we obtain Γ from (105) for $\beta_1 > 0$

$$\Gamma = (C + kr)^{-1}[(A + \tfrac{1}{2}k\eta) + \omega \coth \omega(\xi + c_2)], \tag{108}$$

then substituting this Γ into the BTs (32) and using (83), we arrive at the new solution u' of the mKdV Eq. (82) corresponding to the known traveling wave solution class (83), then

$$u' = -2\sqrt{6}\eta \operatorname{sech} 2\eta\rho + 2\sqrt{6}\frac{\partial}{\partial x} \tan^{-1} \Gamma. \tag{109}$$

Now we shall find a traveling wave solution class $u'(x, t)$ of the (+) BRT equation (33). From the known traveling wave solutions of the (+) BRT equation obtained in Section 3, we construct a new solution of the latter Eq. (33) by means of the BTs. In this case, the constant β_1 defined by (104) is zero and therefore the corresponding solution of the AKNS system (6)–(8) is (104). By substituting (104) into (11) we get a common expression of Γ for Eq. (33) (with respect to $\rho = x \pm kt$):

$$\Gamma = (C + kr)^{-1}[(A + \tfrac{1}{2}k\eta) + 1/(\xi + c_0)], \quad c_0 = \frac{c_2}{c_1}. \tag{110}$$

In the following, we omit some tedious calculations but only list the main results of the (+) BRT, sine-Gordon, sinh-Gordon and Liouville's equations.

4.2. The (+) BRT equation

$$[u_t - (\alpha g(u) + \beta)u_x]_x = g'(u), \quad \text{where } g'' + \mu g = \theta, \tag{111}$$

where $g' = \mathrm{d}g/\mathrm{d}u$, $\zeta^2 = \alpha\eta^2 - \mu$, and the solution is

$$u = -u_0 + \frac{4}{\zeta} \tanh^{-1} \mathrm{e}^{\eta\rho}, \quad \rho = x + kt, \quad k = \frac{\zeta^2 g(u_0) - \theta}{\eta^2} + \beta, \tag{112}$$

$$\xi = t + \int \frac{1}{\frac{\zeta}{2\eta^2} g'(u) \sinh \eta\rho + (\alpha g(u) + k + \beta)} \, \mathrm{d}\rho, \tag{113}$$

$$\Gamma = (C + kr)^{-1}[(A + \tfrac{1}{2}k\eta) + 1/(\xi + c_0)], \quad c_0 = \frac{c_2}{c_1}, \tag{114}$$

$$u' = -u + \frac{4}{\zeta} \tanh^{-1} \Gamma. \tag{115}$$

In this case, we observe that taking $\alpha = \beta = \theta = 0$, the sine-Gordon, sinh-Gordon and Liouville's equations are obtained from (111) by considering appropriate values for μ.

4.3. Sine-Gordon equation

When $\mu = 1$, $g(u) = -\cos u$, we obtain the sine-Gordon equation

$$u_{xt} = \sin u. \tag{116}$$

From a known traveling wave solutions of sine-Gordon equation obtained in Section 3 and choosing the constant solution u_0 of Eq. (116) as $u_0 = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, then the solution of Eq. (116) takes the form

$$u = -n\pi \mp \frac{4}{\mathrm{i}} \tanh^{-1} \mathrm{e}^{\eta\rho}, \quad \rho = x + \frac{(-1)^n}{\eta^2} t, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{117}$$

we can calculate ξ from (97) as follows:

$$\xi = t + \int \frac{1}{(\mathrm{i}/2\eta^2) \sin u \sinh \eta\rho + ((-1)^n)/\eta^2} \, \mathrm{d}\rho = t - \tfrac{1}{8}\eta[2\eta\rho + \mathrm{e}^{-2\eta\rho}], \tag{118}$$

consequently we obtain Γ from (104) for $\beta_1 = 0$

$$\Gamma = \frac{-2\eta \sinh \eta\rho}{\mathrm{i} \sin u \sinh \eta\rho - 2(-1)^n} \left[\left(\frac{\cos u}{2\eta} + \frac{\eta}{2}(-1)^n \right) + 1/(\xi + c_0) \right], \tag{119}$$

then substituting this Γ into the BTs (38) and using (117), we arrive at the new solution u' of the sine-Gordon Eq. (116) corresponding to the known traveling wave solution class (117), then

$$u' = -u \pm 4\mathrm{i} \tanh^{-1} \Gamma. \tag{120}$$

4.4. Sinh-Gordon equation

When $\mu = -1$, $g(u) = \cosh u$, we obtain the sinh-Gordon equation

$$u_{xt} = \sinh u. \tag{121}$$

From a known traveling wave solutions of sinh-Gordon equation obtained in Section 3 and choosing the constant solution u_0 of Eq. (121) as $u_0 = n\pi\mathrm{i}$, $n = 0, \pm 1, \pm 2, \ldots$, then the solution is

$$u = -n\pi\mathrm{i} \mp 4 \tanh^{-1} \mathrm{e}^{\eta\rho}, \quad \rho = x + \frac{(-1)^n}{\eta^2} t, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{122}$$

we can calculate ξ from (97) as follows:

$$\xi = t + \int \frac{1}{(1/2\eta^2) \sinh u \sinh \eta\rho + ((-1)^n)/\eta^2} \mathrm{d}\rho = t + \tfrac{1}{8}\eta[2\eta\rho + \cosh 2\eta\rho - \sinh 2\eta\rho], \tag{123}$$

consequently we obtain Γ from (104) for $\beta_1 = 0$

$$\Gamma = \frac{-2\eta \sinh \eta\rho}{\sinh u \sinh \eta\rho - 2(-1)^n} \left[\left(\frac{\cosh u}{2\eta} + \frac{\eta}{2}(-1)^n \right) + 1/(\xi + c_0) \right], \tag{124}$$

then substituting this Γ into the BTs (38) and using (122), we arrive at the new solution u' of the sinh-Gordon Eq. (121) corresponding to the known traveling wave solution class (122), then

$$u' = -u \mp 4 \tanh^{-1} \Gamma. \tag{125}$$

4.5. Liouville's equation

When $\mu = -1$, $g(u) = \mathrm{e}^u$, we obtain the Liouville's equation

$$u_{xt} = \mathrm{e}^u. \tag{126}$$

We shall choose a known solution of the above Liouville's equation (126) as a traveling wave $u(x, t)$ and substitute these solution into the corresponding matrices P and Q. Next we solve Eq. (6) for ϕ_1 and ϕ_2. Then by (11) and the corresponding BTs (38) we shall obtain the new solution classes. Liouville's equation has not a constant solution, but by direct calculation we take (proceeding as for mKdV equation in Section 4.1)

$$u = \ln \frac{2k}{\rho^2}, \quad \rho = x + kt. \tag{127}$$

We can calculate ξ from (97) as follows:

$$\xi = t + \int \frac{\rho}{k\rho + (k/\eta)} \mathrm{d}\rho = t + \frac{1}{k}\rho - \frac{1}{\eta k} \ln(\eta\rho - 1), \tag{128}$$

consequently we obtain Γ from (104) for $\beta_1 = 0$

$$\Gamma = \frac{-2\eta\rho^2}{1+\eta\rho}\left[\left(\frac{1}{\eta\rho^2}+\frac{\eta}{2}\right)+1/\left(\rho-\frac{1}{\eta}\ln(\eta\rho-1)+k(t+c_0)\right)\right], \tag{129}$$

then substituting this Γ into the BTs (38) and using (127), we arrive at the new solution u' of the Liouville's equation (126) corresponding to the known traveling wave solution: class (127), then

$$u' = -\ln\frac{2k}{\rho^2} \mp 4\tanh^{-1}\Gamma. \tag{130}$$

4.6. Burgers' equation

Now we shall find a traveling wave solutions $u'(x,t)$ for Burgers' equation

$$2u_t - 2uu_x - u_{xx} = 0. \tag{131}$$

Wang [43] has found some exact solutions for compound KdV–Burgers equations by using the homogenous balance method. In this section, we obtain traveling wave solution class for Burgers' equation by using an improved sine–cosine method [47,48] and Wu's elimination method [45]. The main idea of the algorithm is as follows. Given a PDE of the form

$$f(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{132}$$

where f is a polynomial. By assuming traveling wave solutions of the form

$$u(x,t) = \phi(\rho), \quad \rho = \eta(x-kt+c), \tag{133}$$

where k, η are constant parameters to be determined, and c is an arbitrary constant, from the two Eqs. (132) and (133) we obtain an ODE

$$f(\phi', \phi'', \phi''', \ldots) = 0, \tag{134}$$

where $\phi' = \mathrm{d}\phi/\mathrm{d}\rho$. According to the sine–cosine method [44,43,45,46,48,47,50], we suppose that Eq. (134) has the following formal traveling wave solution:

$$\phi(\rho) = \sum_{i=1}^{n} \sin^{i-1}\psi(B_i\sin\psi + A_i\cos\psi) + A_0 \tag{135}$$

and

$$\frac{\mathrm{d}\psi}{\mathrm{d}\rho} = \sin\psi \quad \text{or} \quad \frac{\mathrm{d}\psi}{\mathrm{d}\rho} = \cos\psi, \tag{136}$$

where $A_0, \ldots, A_n$ and $B_0, \ldots, B_n$ are constants to be determined. Then we proceed as follows:

(i) Equating the highest-order nonlinear term and highest-order linear partial derivative in (134), yield the value of n.
(ii) Substituting Eqs. (135), (136) into (134), we obtain a polynomial equation involving $\cos\psi\sin^i\psi$, $\sin^i\psi$ for $i = 0, 1, 2, \ldots, n$ (with n being positive integer)
(iii) Setting the constant term and coefficients of $\sin\psi$, $\cos\psi$, $\sin\psi\cos\psi$, $\sin^2\psi, \ldots$, in the equation obtained in (ii) to zero, we obtain a system of algebraic equations about the unknown numbers k, η, B_0, A_i, B_i for $i = 0, 1, 2, \ldots, n$.
(iv) Using the Mathematica and the Wu's elimination methods, the algebraic equations in (iii) can be solved.

These yield the solitary wave solutions for the system (134). We remark that the above method yield solutions that includes terms $\operatorname{sech}\rho$ or $\tanh\rho$, as well as their combinations. There are different forms of those obtained by other methods, such as the homogeneous balance method [44,43,45,46,48,47,50]. We assume formal solutions of the form

$$u(x,t)=\phi(\rho), \quad \rho=\eta(x-kt+c), \tag{137}$$

where k, η are constant parameters to be determined later, and c is an arbitrary constant. Substituting from (137) and (131), we obtain an ODE

$$k\phi' + \tfrac{1}{2}\phi\phi' + \tfrac{1}{2}\eta\phi'' = 0. \tag{138}$$

(i) We suppose that Eq. (138) has the following formal solutions:

$$\phi(\rho) = A_0 + A_1\sin\psi + A_2\cos\psi \tag{139}$$

and

$$\frac{d\psi}{d\rho} = \sin\psi. \tag{140}$$

(ii) From two Eqs. (139) and (140), we get

$$\begin{aligned} k\phi' + \tfrac{1}{2}\phi\phi' + \tfrac{1}{2}\eta\phi'' = {} & \tfrac{1}{2}(\eta A_1 + A_1A_2)\sin\psi + (kA_1 + \tfrac{1}{2}A_0A_1)\sin\psi\cos\psi \\ & - (\tfrac{1}{2}A_0A_2 + kA_2)\sin^2\psi + [-\eta A_2 - \tfrac{1}{2}(A_2^2 - A_1^2)] \\ & \times \cos\psi\sin^2\psi - (A_1A_2 + \eta A_1)\sin^3\psi = 0. \end{aligned} \tag{141}$$

(iii) Setting the coefficients of $\sin^j\psi\cos^i\psi$ for $i=0,1$ and $j=1,2,3$ to zero, we have the following set of over determined equations in the unknowns A_0, A_1, A_2, η and k:

$$\begin{aligned} &(kA_1 + \tfrac{1}{2}A_0A_1) = 0, \quad (\tfrac{1}{2}A_0A_2 + kA_2) = 0, \quad [\eta A_2 + \tfrac{1}{2}(A_2^2 - A_1^2)] = 0, \\ &(A_1A_2 + \eta A_1) = 0, \quad (\eta A_1 + A_1A_2) = 0. \end{aligned} \tag{142}$$

(iv) We now solve the above set of equations by using Mathematica and the Wu's elimination method, and obtain the following solution:

$$A_1 = 0, \quad A_2 = -2\eta, \quad k = -\frac{A_0}{2}, \tag{143}$$

by integrating (140) and taking the integration constant equal zero, we obtain

$$\sin\psi = \operatorname{sech}\rho, \quad \cos\psi = \pm\tanh\rho. \tag{144}$$

Substituting (143) and (144) into (139), we obtain

$$u(x,t) = A_0 \pm 2\eta\tanh\rho, \quad \rho = \eta\left(x + \frac{A_0}{2}t + c\right). \tag{145}$$

Now we shall find a new traveling wave solutions $u'(x,t)$ of the Burgers' equation (131) and substitute these solution into the corresponding matrices P and Q. Next we solve Eq. (6) for ϕ_1 and ϕ_2. Then by (11) and the corresponding BTs (44) we shall obtain the new solution for Burgers' equations.

We use a known traveling wave solutions for the Burgers' equation to generate a new solution for Burgers' equation by means of the BTs. In this case, the constant β_1 defined by (104) is zero, we get the common expression of Γ for the Burgers' equation (with respect to $\rho = \eta(x \pm kt + c)$):

$$\Gamma = \frac{1}{\eta}(C + kr)^{-1}[(A + k\eta^2) + 1/(\xi + c_0)], \quad \text{where } c_0 \text{ is a constant.} \tag{146}$$

In the following, we omit some tedious calculations but only list the main results of the Burgers' equation

$$\xi = t + \frac{1}{2(\eta^2 - A_0^2)} \ln[A_0^2 - 2A_0\eta + \eta^2 + 3\eta A_0(\tanh\rho) - 2\eta^2 \tanh\rho + \eta^2 \tanh^2\rho]$$
$$+ \frac{\eta A_0}{(\eta^2 - A_0^2)\sqrt{(4\eta^3 A_0 - 5\eta^2 A_0^2)}} \tanh^{-1}\left(\frac{3\eta A_0 - 2\eta^2 + 2\eta^2 \tanh\rho}{\sqrt{(4\eta^3 A_0 - 5\eta^2 A_0^2)}}\right)$$
$$- \frac{2\eta^2}{(\eta^2 - A_0^2)\sqrt{(4\eta^3 A_0 - 5\eta^2 A_0^2)}} \tan^{-1}\left(\frac{3\eta A_0 - 2\eta^2 + 2\eta^2 \tanh\rho}{\sqrt{(4\eta^3 A_0 - 5\eta^2 A_0^2)}}\right)$$
$$- \frac{1}{2\eta(\eta + A_0)} \ln(\tanh\rho - 1) + \frac{1}{2\eta(A_0 - \eta)} \ln(\tanh\rho + 1), \tag{147}$$

$$\Gamma = \frac{[((\eta/2)A_0 + \eta^2 \tanh\rho) + (\eta^2/2)A_0 + 1/(\xi + c_0)]}{\eta[((A_0^2/2) + \eta A_0 + (\eta^2/2)) + ((3/2)\eta A_0 - \eta^2)\tanh\rho + (\eta^2/2)\tanh^2\rho]}, \tag{148}$$

$$u' = -A_0 - 2\eta\tanh\rho + 4\frac{\partial}{\partial x}\tanh^{-1}\Gamma. \tag{149}$$

4.7. A gfKdV equation

Now we shall find a traveling wave solutions $u(x, t)$ for the gfKdV equation which can be shown in the form

$$u_t + au^2u_x + bu_xu_{xx} + huu_{xxx} + \mathrm{d}u_{xxxxx} = 0, \tag{150}$$

where a, b, h and d are constants. This equation has been known as the general form of the fifth-order KdV equation. Eq. (150) is known as Lax's fifth-order KdV equation with $a = 30,\ b = 30,\ h = 10,\ d = 1$ and the Sawada–Kotera equation with $a = 45,\ b = 15,\ h = 15,\ d = 1$ [28,27]. We assume formal solution of Eq. (150) of the form (137). Substituting from (137) and (150), we obtain an ODE

$$-\eta k\phi' + a\eta\phi^2\phi' + b\eta^3\phi'\phi'' + c\eta^3\phi\phi''' + d\eta^5\phi''''' = 0. \tag{151}$$

(i) Equating the highest-order nonlinear term and highest-order linear partial derivative in (151), yield $n = 3$. Then Eq. (151) has the following formal solutions:

$$\phi(\rho) = A_0 + B_1\sin\psi + A_1\cos\psi + \sin\psi(B_2\sin\psi + A_2\cos\psi) + \sin^2\psi(B_3\sin\psi + A_3\cos\psi) \tag{152}$$

and

$$\frac{\mathrm{d}\psi}{\mathrm{d}\rho} = \sin\psi. \tag{153}$$

(ii) With the aid of Mathematica, substituting Eqs. (152), (153) into (151), then we obtain a polynomial equation involving $\cos^i\psi\sin^j\psi$ for $i = 0, 1,\ \ j = 1, 2, \ldots, 10$.

$$\begin{aligned} &- \eta k\phi' + a\eta\phi^2\phi' + b\eta^3\phi'\phi'' + c\eta^3\phi\phi''' + d\eta^5\phi''''' \\ &\quad = (-\eta kA_2 + 2aA_0B_1A_1\eta + a\eta A_0^2A_2 + h\eta^3A_0A_1 + a\eta A_1^2A_2 + dB_1\eta^5 + h\eta^3A_1B_1)\sin\psi \\ &\qquad + (-\eta kB_1 + a\eta A_0^2B_1 + a\eta B_1A_1^2 + 2a\eta A_0A_1A_2 + h\eta^3B_1A_0 + h\eta^3A_1A_2 + dA_2\eta^5)\sin\psi\cos\psi \\ &\qquad + \cdots + a\eta B_3(-9A_3^2 + 3B_3^2)\cos\psi\sin^9\psi + a\eta A_3(-9B_3^2 + 3A_3^2)\sin^{10}\psi = 0. \end{aligned} \tag{154}$$

(iii) Setting the constant term and coefficients of $\sin\psi$, $\sin\psi\cos\psi, \ldots, \sin^{10}\psi$, in the equation obtained in (ii) to zero, we obtain a system of algebraic equations about the unknown numbers A_i, B_j for $i=0,1,2,3$, $j=1,2,3$.

$$\begin{aligned}
&-\eta k A_2 + 2aA_0B_1A_1\eta + a\eta A_0^2A_2 + h\eta^3A_0A_2 + a\eta A_1^2A_2 + dB_1\eta^5 + h\eta^3A_1B_1 = 0,\\
&-\eta k B_1 + a\eta A_0^2B_1 + a\eta B_1A_1^2 + 2a\eta A_0A_1A_2 + h\eta^3B_1A_0 + h\eta^3A_1A_2 + dA_2\eta^5 = 0,\\
&\qquad\vdots\\
&a\eta B_3(-9A_3^2 + 3B_3^2) = 0,\\
&a\eta A_3(-9B_3^2 + 3A_3^2) = 0
\end{aligned} \tag{155}$$

from Eq. (155), we obtain $A_3 = B_3 = 0$. Then Eq. (151) has the following formal solutions:

$$\phi(\rho) = A_0 + B_1\sin\psi + A_1\cos\psi + \sin\psi(B_2\sin\psi + A_2\cos\psi) \quad \text{and} \quad \frac{d\psi}{d\rho} = \sin\psi. \tag{156}$$

By means of the same procedures above (ii) and (iii), we obtain a system of algebraic equations in the unknown numbers k, η, A_i, B_j for $i=0,1,2$, $j=1,2$.

$$\begin{aligned}
&-\eta k A_2 + 2a\eta A_0B_1A_1 + a\eta A_0^2A_2 + h\eta^3A_0A_2 + a\eta A_1^2A_2 + dA_2\eta^5 + h\eta^3A_1B_1 = 0,\\
&-\eta k B_1 + a\eta A_0^2B_1 + a\eta B_1A_1^2 + 2a\eta A_0A_1A_2 + h\eta^3B_1A_0 + h\eta^3A_1A_2 + dB_1\eta^5 = 0,\\
&\eta kA_1 + a\eta A_0^2A_1 + 2a\eta A_1B_1^2 + -a\eta A_1^3 + 4a\eta A_0A_1B_2 + 4a\eta A_0B_1A_2 + 2a\eta A_1A_2^2\\
&\quad + 2b\eta^3B_1A_2 - 16d\eta^5A_1 - 4h\eta^3A_0A_1 + 8h\eta^3A_1B_2 + 2h\eta^3B_1A_2 = 0,\\
&\quad - 2\eta kB_2 + 2a\eta A_0B_1^2 - 2a\eta A_0A_1^2 + 2a\eta A_0^2B_2 + 2a\eta A_1^2B_2 + 4a\eta B_1A_1A_2 + 2a\eta A_0A_2^2\\
&\quad + b\eta^3B_1^2 + b\eta^3A_2^2 + 32d\eta^5B_2 + h\eta^3B_1^2 - 4h\eta^3A_1^2 + 8h\eta^3A_0B_2 + h\eta^3A_2^2 = 0,\\
&4a\eta B_1^2B_2 - 4a\eta B_2A_1^2 + 4a\eta A_0B_2^2 - 8a\eta B_1A_1A_2 - 4a\eta A_0A_2^2 + 4a\eta B_2A_2^2 - 2b\eta^3B_1^2\\
&\quad + 2b\eta^3A_1^2 + 8b\eta^3B_2^2 - 8b\eta^3A_2^2 - 480d\eta^5B_2 - 6h\eta^3B_1^2 + 6h\eta^3A_1^2\\
&\quad - 24h\eta^3A_0B_2 + 8h\eta^3B_2^2 - 20h\eta^3A_2^2 = 0,\\
&2a\eta B_2^3 - 6a\eta B_2A_2^2 - 12b\eta^3B_2^2 + 12b\eta^3A_2^2 + 720d\eta^5B_2 - 24h\eta^3B_2^2 + 24h\eta^3A_2^2 = 0,\\
&\quad - 6a\eta A_2B_2^2 + 2a\eta A_2^3 + 24b\eta^3B_2A_2 - 720d\eta^5B_2 + 48h\eta^3B_2A_2 = 0.
\end{aligned} \tag{157}$$

(iv) Now we solve the above set of Eqs. (157) by using Mathematica and the Wu's elimination method, we obtain the following solutions that include two different cases:

Case 1:

$$\begin{aligned}
&A_1 = B_1 = A_2 = 0, \quad B_2 = \frac{3\eta^2}{a}\left[(b+2h) \pm \sqrt{(b+2h)^2 - 40da}\right],\\
&A_0 = \frac{120d\eta^4 - 2\eta^2(b+h)B_2}{aB_2 - 6h\eta^2}, \quad k = (aA_0^2 + 4hA_0\eta^2 + 16d\eta^4).
\end{aligned} \tag{158}$$

Case 2:

$$\begin{aligned}
&A_1 = B_1 = A_0 = 0, \quad B_2 = \frac{15d\eta^2(2b-h)}{(h+b)^2 - 15da},\\
&A_2^2 = \frac{450d^2\eta^4(h-2b)}{(h+b)^3 - 15da(h+b)}, \quad k = d\eta^4,
\end{aligned} \tag{159}$$

by integrating (153) and taking the integration constant equal zero, we obtain

$$\sin\psi = \operatorname{sech}\rho, \quad \cos\psi = \pm\tanh\rho. \tag{160}$$

Then, we find the following two types of traveling wave solutions for the gfKdV equation (150):

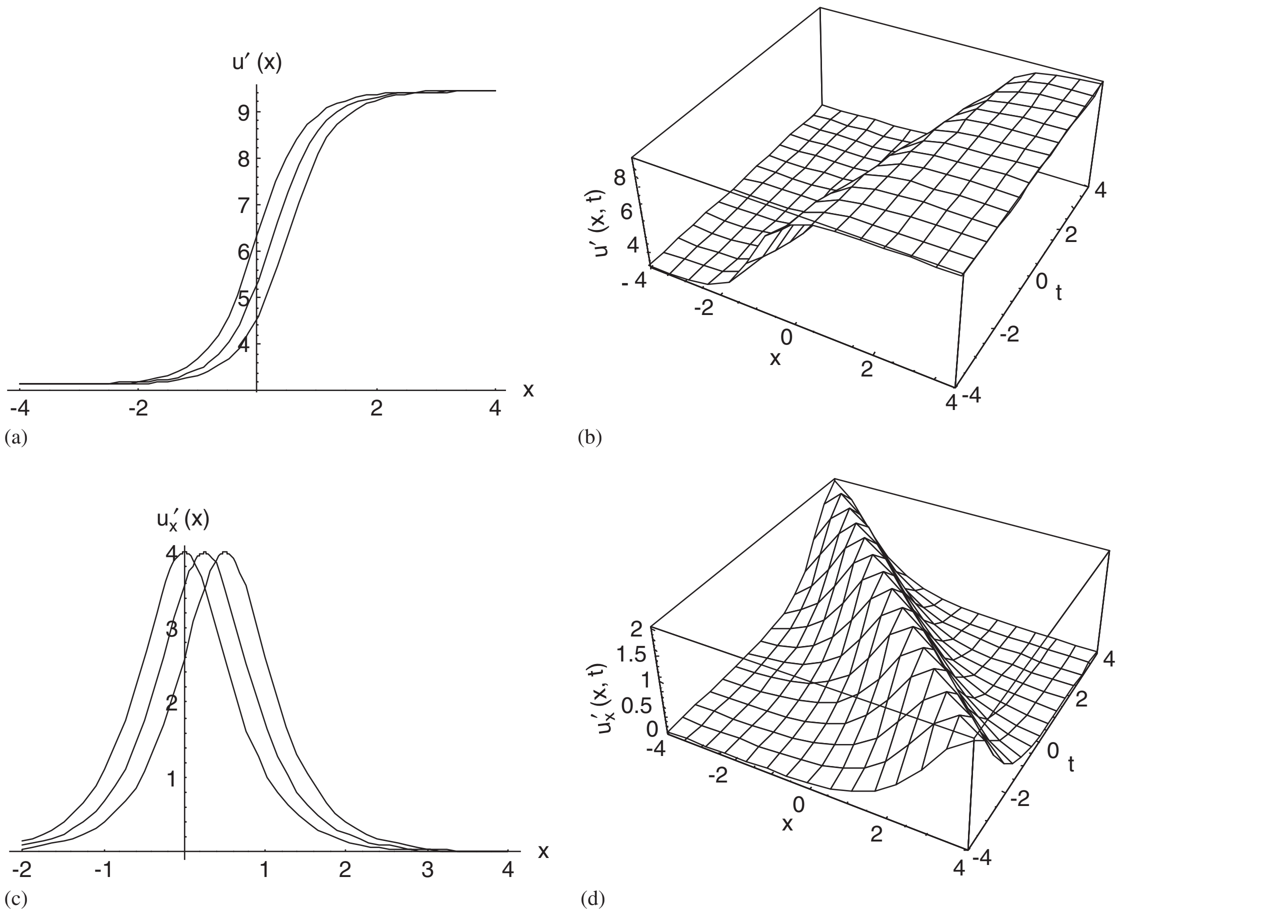

Fig. 1. The single-kink solution (53) in the (a) plane at different times $t = 0, 1, 2$ and (b) surface with $u_0 = \pi$ and $\eta = 2$. The single-soliton solution of the derivative of Eq. (53) with respect to x in the (c) plane at different times $t = 0, 1, 2$ and (d) surface with $u_0 = \pi$ and $\eta = 2$.

Type 1:

$$u_1(x,t) = A_0 + B_2 \operatorname{sech}^2 \rho, \quad \text{where } \rho = \eta(x - kt + c), \ A_0 = \frac{120 d\eta^4 - 2\eta^2(b+h)B_2}{aB_2 - 6h\eta^2},$$

$$B_2 = \frac{3\eta^2}{a}\left[(b+2h) \pm \sqrt{(b+2h)^2 - 40da}\right], \quad k = (aA_0^2 + 4hA_0\eta^2 + 16d\eta^4). \tag{161}$$

Type 2:

$$u_2(x,t) = \operatorname{sech}^2 \rho \left[\frac{15d\eta^2(2b-h)}{(h+b)^2 - 15da} \pm \sqrt{\frac{450d^2\eta^4(h-2b)}{(h+b)^3 - 15da(h+b)}} \sinh \rho\right], \quad \rho = \eta(x - d\eta^4 t + c). \tag{162}$$

By means of the same procedures as above, we obtain solutions of fKdV and Sawada–Kotera equations as follows:

(1) For the fKdV equation

$$u_t + 30u^2 u_x + 30u_x u_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \tag{163}$$

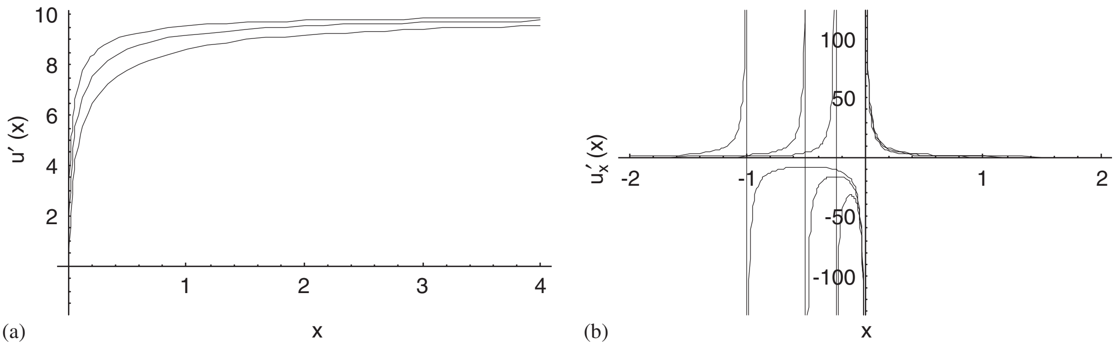

Fig. 2. (a) The single-kink solution (60) in the plane at different parameter $\eta = 2, 4, 8$ with $u_0 = 10$. (b) The single-soliton solution of the derivative of Eq. (60) with respect to x in the plane at different parameter $\eta = 2, 4, 8$ with $u_0 = 10$.

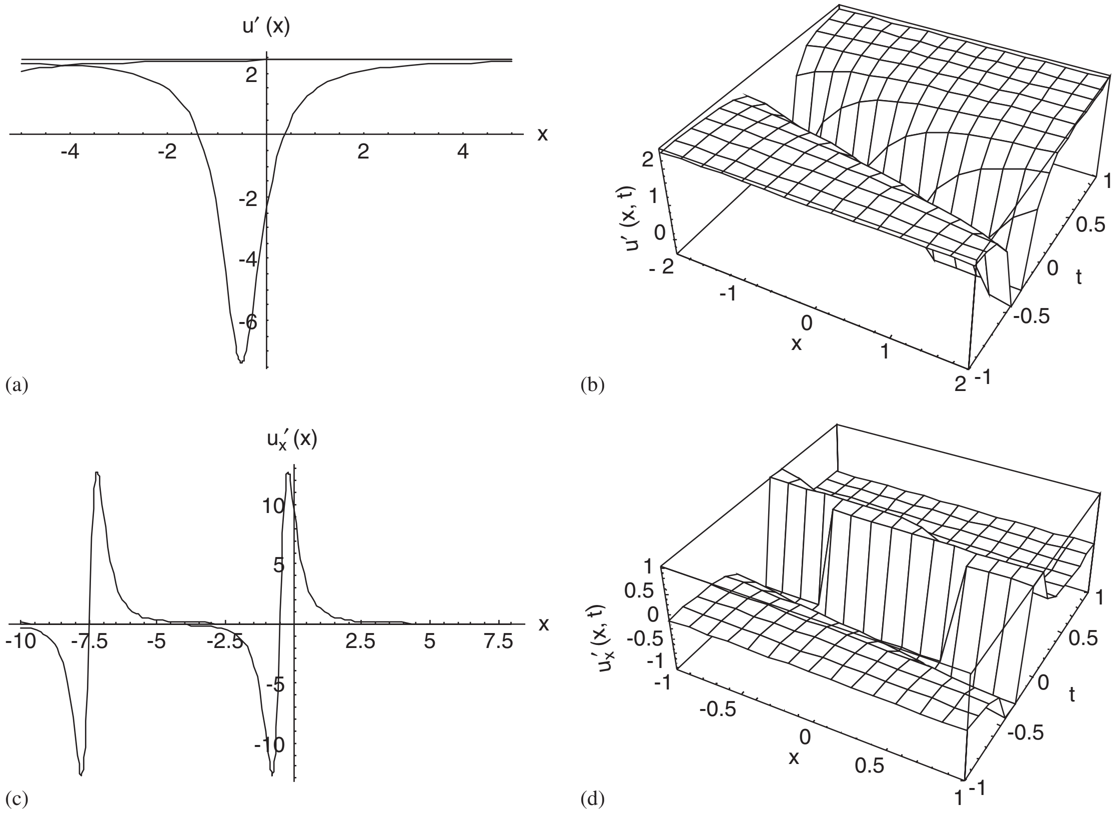

Fig. 3. The single-antikink solution (70) in the (a) plane at different times $t = 0, 1, 2$ and (b) surface with $u_0 = \sqrt{6}$, $a = 1$, $\eta = 2$. The single-soliton solution of the derivative of Eq. (70) with respect to x in the (c) plane at different times $t = 0, 1, 2$ and (d) surface with $u_0 = \sqrt{6}$, $a = 1$, $\eta = 2$.

we have the following formal solitary wave solutions:

$$u_3(x,t) = A_0 + B_2 \operatorname{sech}^2 \rho, \quad \text{where } \rho = \eta(x - kt + c),\ A_0 = \eta^2 \left[\frac{12 - 8(5 \pm \sqrt{13})}{3\left(5 \pm \sqrt{13}\right) - 6} \right],$$

$$B_2 = \eta^2 \left(5 \pm \sqrt{13}\right), \quad k = (30A_0^2 + 40A_0\eta^2 + 16\eta^4), \tag{164}$$

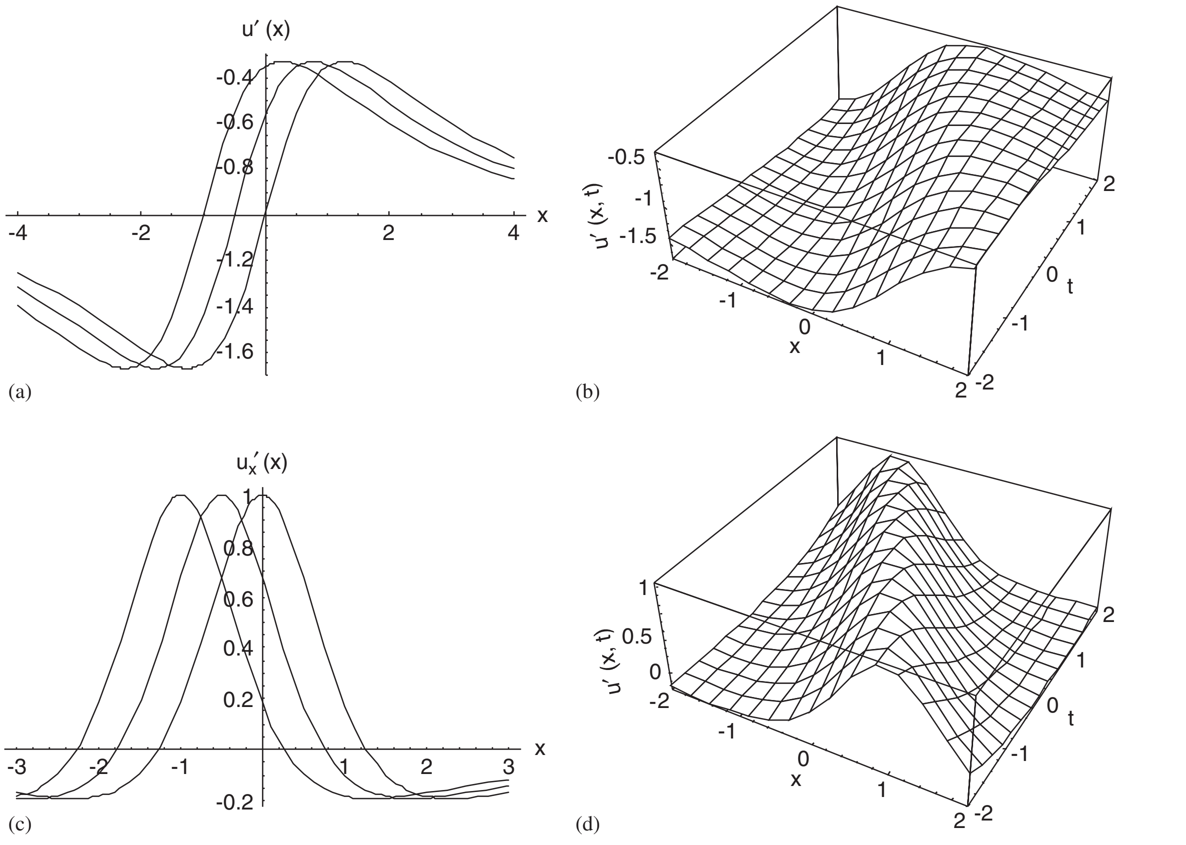

Fig. 4. The single-kink solution (81) in the (a) plane at different times $t = 0, 1, 2$ and (b) surface with $u_0 = 1, \alpha = \frac{1}{2}, \eta = \frac{1}{2}$. The single-soliton solution of the derivative of Eq. (81) with respect to x in the (c) plane at different times $t = 0, 1, 2$ and (d) surface with $u_0 = 1, \alpha = \frac{1}{2}, \eta = \frac{1}{2}$.

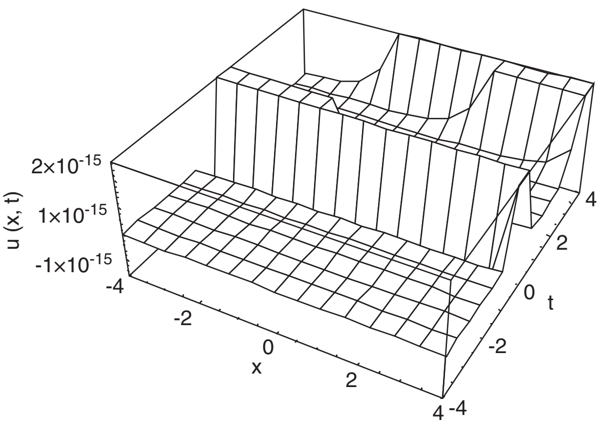

Fig. 5. The double-kink solution (109) in the surface with $k = -17, a = 1, \eta = 2$.

$$u_4(x,t) = 3\eta^2 \operatorname{sech}^2 \rho \left[\frac{5}{23} \pm \frac{\mathrm{i}}{2}\sqrt{\frac{5}{23}} \sinh \rho \right], \quad \rho = \eta(x - \eta^4 t + c). \tag{165}$$

(2) For the Sawada–Kotera equation

$$u_t + 45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx} = 0, \tag{166}$$

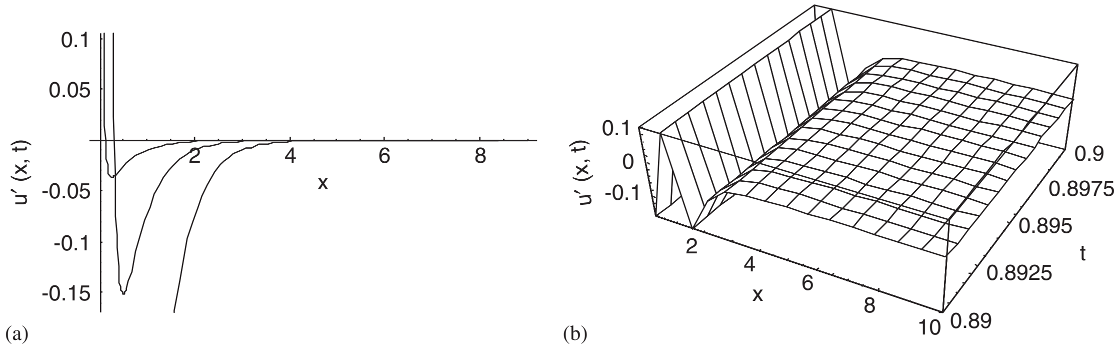

Fig. 6. The double-kink solution (125) in the (a) plane at different times $t = 0, 1, 2$ and (b) surface with $n = 1$, $\eta = 2$.

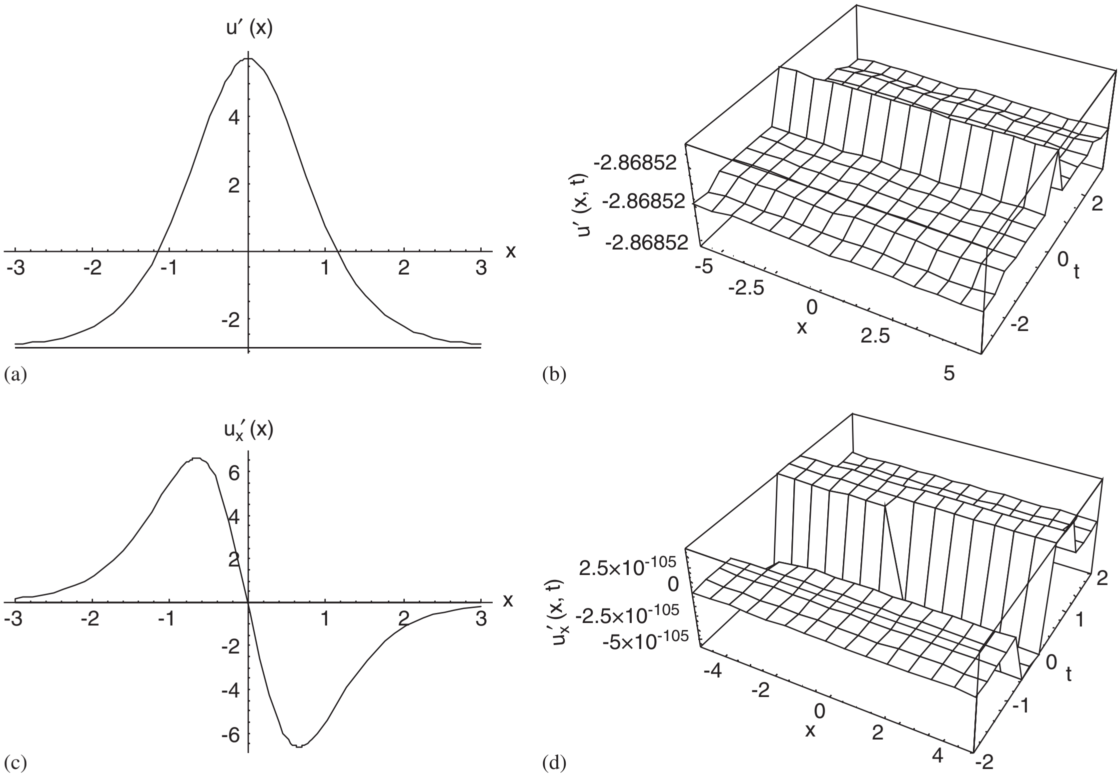

Fig. 7. The single-soliton solution (164) in the (a) plane at different times $t = 0, 1, 2$ and (b) surface with $c = 0$ and $\eta = 1$. The double-soliton solution of the derivative of Eq. (164) with respect to x in the (c) plane at different times $t = 0, 1, 2$ and (d) surface with $c = 0$ and $\eta = 1$.

we have the following formal solitary wave solutions:

$$u_5(x,t) = 2\eta^2 \operatorname{sech}^2 \rho, \quad \rho = \eta(x - 16\eta^4 t + c), \tag{167}$$

$$u_6(x,t) = -4\eta^2/3 + 4\eta^2 \operatorname{sech}^2 \rho, \quad \rho = \eta(x - 26\eta^4 t + c), \tag{168}$$

$$u_7(x,t) = \eta^2 \operatorname{sech}^2 \rho[1 \pm \mathrm{i} \sinh \rho], \quad \rho = \eta(x - \eta^4 t + c). \tag{169}$$

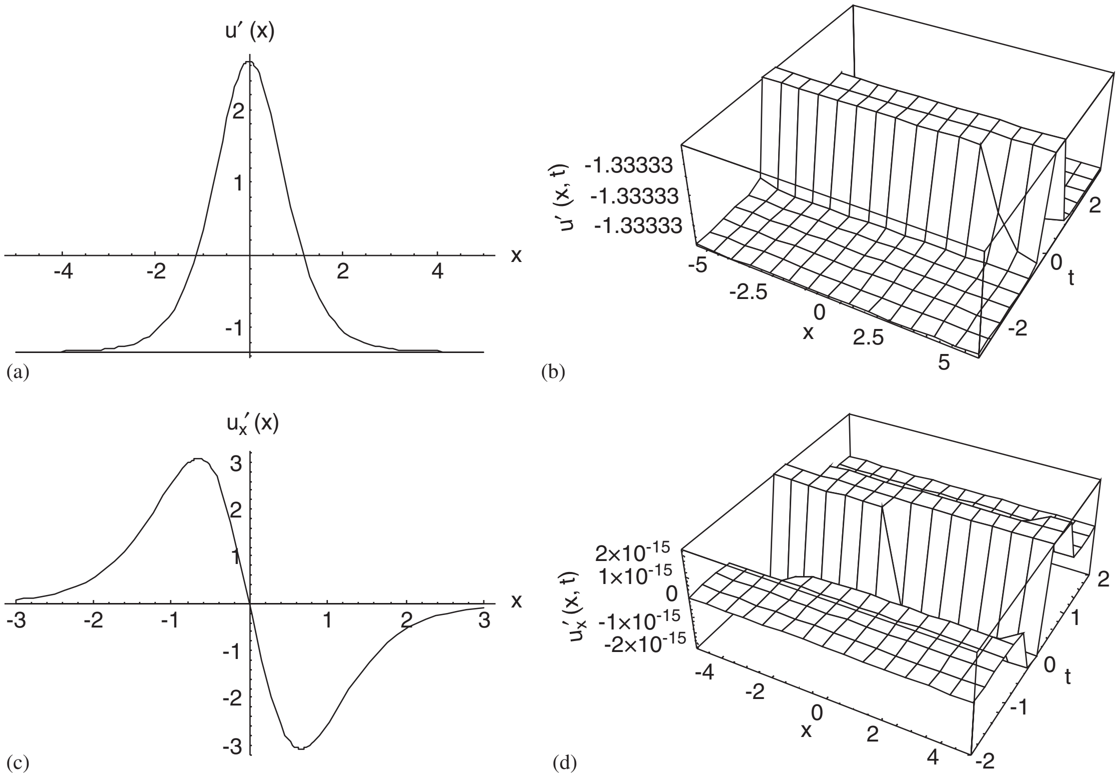

Fig. 8. The single-soliton solution (168) in the (a) plane at different times $t = 0, 1, 2$ and (b) surface with $c = 0$ and $\eta = 1$. The double-soliton solution of the derivative of Eq. (168) with respect to x in the (c) plane at different times $t = 0, 1, 2$ and (d) surface with $c = 0$ and $\eta = 1$.

5. Conclusions

In this paper, we considered the construction of exact solutions to some NLEEs (the $(-)$ and $(+)$ BRT equations, the CT equation and mKdV equations) of the pseudo-spherical class. It has been shown that the implementation of certain BTs for a class of NLEE requires the solution of the underlying linear differential equation whose coefficients depend on the known solution $u(x, t)$ of the NLEE. We obtain traveling wave solutions for Burgers and gfKdV equations by using an improved sine–cosine method and Wu's elimination method. Employing BTs involving explicitly the wave solutions, new solutions are generated for Burgers' equation.

The solutions (53), (60), (70) and (81) are the single-kink solutions [see Figs. 1(a), 1(b), 1(c), 2(a), 3(a), 3(b), 4(a), 4(b)] corresponding to the eigenvalue η. Associated with the single-kink solutions are the single-soliton solutions given by the derivative of Eqs. (53), (60), (70) and (81) with respect to x [see Figs. 1(c), 1(d), 2(b), 3(c), 3(d), 4(c), 4(d)]. The double-kink solutions (109) and (125) are characterized by the eigenvalue η [see Figs. 5(a), 6 (a), 6(b)]. The solutions (164) and (168) are the single-soliton solutions [see Figs. 7(a), 7(b), 8(a), 8(b)] corresponding to the eigenvalue η. Associated with the single-soliton solutions are the double-soliton solutions given by the derivative of Eqs. (164) and (168) with respect to x [see Figs. 7(c), 7(d), 8(c), 8(d)].

Acknowledgements

We are really grateful to the referees for their critical comments.

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