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A dynamic IS-LM business cycle model with two time delays in capital accumulation equation

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ABSTRACT

In this paper, we analyze a augmented IS-LM business cycle model with the capital accumulation equation that two time delays are considered in investment processes according to Kalecki's idea. Applying stability switch criteria and Hopf bifurcation theory, we prove that time delays cause the equilibrium to lose or gain stability and Hopf bifurcation occurs.

1. Introduction

Since time delay was first considered in the investment processes in [13], lots of literature such as [9,12,15,28,29,23–26,6,7] have incorporated time lag into the dynamic economics and considered the impacts of delayed time on the whole economic system.

Furthermore, time delay has also been introduced into the extended IS-LM (the related IS-LM model with some extension, e.g. [27,21,22,2,16,17,20]), for example, De cassare and Sportelli [5] investigated the equilibrium point's stability and the existence of the limit cycle of a IS-LM model by introducing a fixed time delay into the tax revenue, Cai [3] and Zhou and Li [30] both discussed an IS-LM model with a time lag in the capital accumulation equation although with a little different extension in the latter, Neamtu [18] presented an IS-LM model with the same lag into the tax revenues and the capital accumulation equation, they all analyzed the qualitative behavior of the model via Hopf bifurcation or stability switch criteria, while Fanti and Manfredi [8] considered an IS-LM model with distributed tax collection lag and showed that it could display from stability to stable oscillations, and finally to chaotic motion.

In this paper, we consider the generalized IS-LM model with time delay proposed in [3], but with two distinct time lags in the capital accumulation equation and use the analytical approaches presented in [19] to analyze the qualitative behavior of the model. To our knowledge, there are few papers discussing the dynamic IS-LM model with two time delays in investment processes.

Our aim is to show that the time delays in the capital accumulation processes could cause the equilibrium to lose or gain stability and cycles in dynamic macroeconomics.

The outline of the paper is as follows, in the next section, we construct the IS-LM model and analyze its qualitative behavior, finally we conclude the paper in Section 3.

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2. The model

2.1. Assumptions of the model

As a combination of the standard IS-LM model (Torre [27])

$$\begin{cases}\dot{Y} = \alpha(I(Y,r) - S(Y,r)),\\ \dot{r} = \beta(L(Y,r) - \overline{M}),\end{cases} \tag{1}$$

and the Kaldor model

$$\begin{cases}\dot{Y} = \alpha(I(Y,K) - S(Y,K)),\\ \dot{K} = I(Y,K) - \delta K,\end{cases} \tag{2}$$

Gabisch and Lorenz [10], Boldrin [2] investigated the augmented IS-LM model

$$\begin{cases}\dot{Y} = \alpha(I(Y,K,r) - S(Y,r)),\\ \dot{r} = \beta(L(Y,r) - \overline{M}),\\ \dot{K} = I(Y,K,r) - \delta K,\end{cases} \tag{3}$$

furthermore, Cai [3] studied the following augmented IS-LM with Kalecki's time lag in the capital accumulation, which assumed that saved part of profit is invested and capital growth is due to past investment decisions:

$$\begin{cases}\dot{Y} = \alpha(I(Y,K,r) - S(Y,r)),\\ \dot{r} = \beta(L(Y,r) - \overline{M}),\\ \dot{K} = I(Y(t-T),K,r) - \delta K,\end{cases} \tag{4}$$

where $I, S, L, K, Y, r, \overline{M}, T$ respectively represents investment, savings, liquidity preference (demand for money), capital stock, gross product, interest rate, constant money supply, time delay. $\alpha > 0, \beta > 0$ are respectively the adjustment coefficients in the markets of goods and money, $\delta > 0$ means the depreciation of the capital stock, $\dot{Y}(t), Y'(t)$ denote the derivative $\frac{dY(t)}{dt}$.

Actually in sys. (4), investment in capital accumulation equation depends on the income at the time investment decisions are made and on the capital stock at the time investment is finished, while in Section 3.1 of Zak [28], Zak investigated the Solow growth model with time lag, and considered that investment depended only on the capital stock at the past time and that the capital stock depreciated at the same gestation period, which it takes to produce and install capital goods (i.e. 'time-to-build' models, see [14,23,24,29]):

$$\dot{K}(t) = sf(K(t-r)) - \delta K(t-r), \tag{5}$$

with f as production function, s constant saving rate, the saved part of neoclassical product function sf is invested.

Here, we would assume that the investment function in capital accumulation depends on the income and the capital stock both at the past time, and also at the different gestation period, i.e.investment function,

$$I(Y,K,r) = I_1(Y,r) + I_2(K) = I_1(Y(t-\tau_1), r(t)) + I_2(K(t-\tau_2)) = I_1(Y(t-\tau_1), r(t)) + \beta_1 K(t-\tau_2),$$

where $-1 < \beta_1 < 0$ is propensities to investment $I_2(K)$ with respect to capital stock, $0 < S_Y = s_1 < 1$ is saving rate, τ_1, τ_2 are time delays,
and capital accumulation equation,

$$\dot{K}(t) = I_1(Y(t-\tau_1), r(t)) - (\delta - \beta_1)K(t-\tau_2),$$

so the fixed price disequilibrium augmented IS-LM model we discuss would be

$$\begin{cases}\dot{Y}(t) = \alpha[I_1(Y(t), r(t)) + \beta_1 K(t) - s_1 Y(t)],\\ \dot{r}(t) = \beta[L(Y(t), r(t)) - \overline{M}],\\ \dot{K}(t) = I_1(Y(t-\tau_1), r(t)) - (\delta - \beta_1)K(t-\tau_2),\end{cases} \tag{6}$$

clearly, when $\tau_2 = 0$, system (6) is the same as system (4) except that we assume the different investment function and saving function.

2.2. Qualitative behavior of the model

2.2.1. The equilibrium point

It's easy to know that the equilibrium point (Y^*, r^*, K^*) is the solution of

$$\begin{cases}I_1(Y^*, r^*) = \dfrac{(\delta - \beta_1)s_1}{\delta} Y^*,\\ L(Y^*, r^*) = \overline{M},\\ K^* = \dfrac{s_1}{\delta} Y^*.\end{cases}$$

2.2.2. Linear stability analysis

The linearization of the system in a neighborhood of the equilibrium (Y^*, r^*, K^*) yields:

$$\begin{pmatrix} Y(t) \\ r(t) \\ K(t) \end{pmatrix}' = \eta_1 \begin{pmatrix} Y(t)-Y^* \\ r(t)-r^* \\ K(t)-K^* \end{pmatrix} + \eta_2 \begin{pmatrix} Y(t-\tau_1)-Y^* \\ r(t-\tau_1)-r^* \\ K(t-\tau_1)-K^* \end{pmatrix} + \eta_3 \begin{pmatrix} Y(t-\tau_2)-Y^* \\ r(t-\tau_2)-r^* \\ K(t-\tau_2)-K^* \end{pmatrix}, \tag{7}$$

$$\eta_1 = \begin{pmatrix} \alpha(I_Y - s_1) & \alpha I_r & \alpha\beta_1 \\ \beta L_Y & \beta L_r & 0 \\ 0 & I_r & 0 \end{pmatrix}, \qquad \eta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_Y & 0 & 0 \end{pmatrix}, \qquad \eta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(\delta-\beta_1) \end{pmatrix},$$

where $I_Y = \frac{\partial I_1}{\partial Y}(Y^*, r^*) > 0, I_r = \frac{\partial I_1}{\partial r}(Y^*, r^*) < 0, L_Y = \frac{\partial L}{\partial Y}(Y^*, r^*) > 0, L_r = \frac{\partial L}{\partial r}(Y^*, r^*) < 0$. So the characteristic equation of system (7) is $\det(\lambda I - \eta_1 - \eta_2 e^{-\tau_1\lambda} - \eta_3 e^{-\tau_2\lambda}) = 0$, which leads to

$$D(\lambda, \tau_1, \tau_2) = R(\lambda)e^{-\lambda\tau_2} + Q(\lambda)e^{-\lambda\tau_1} + P(\lambda) = 0, \tag{8}$$

where

$$\begin{aligned} &R(\lambda) = (\delta-\beta_1)[\lambda^2 - (\alpha(I_Y - s_1) + \beta L_r)\lambda + \alpha\beta L_r(I_Y - s_1) - \alpha\beta L_Y I_r] = r_2\lambda^2 + r_1\lambda + r_0, r_2 > 0, \\ &Q(\lambda) = -\alpha\beta_1 I_Y \lambda + \alpha\beta_1\beta I_Y L_r = q_1\lambda + q_0, \quad q_1 > 0, q_0 > 0, \\ &P(\lambda) = \lambda^3 - [\alpha(I_Y - s_1) + \beta L_r]\lambda^2 + [\alpha(I_Y - s_1)\beta L_r - \alpha\beta I_r L_Y]\lambda - \alpha\beta_1\beta I_r L_Y = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0, \quad p_0 < 0. \end{aligned}$$

2.2.3. The case $\tau_1 = \tau_2 = 0$

So the characteristic polynomial is

$$D(\lambda, 0, 0) = R(\lambda) + Q(\lambda) + P(\lambda) = \lambda^3 + (p_2 + r_2)\lambda^2 + (p_1 + q_1 + r_1)\lambda + (p_0 + q_0 + r_0) = 0, \tag{9}$$

according to the Routh-Hurwitz criterion, the equilibrium point is stable if and only if

$$(H_1) \quad p_2 + r_2 > 0, \qquad (p_2 + r_2)(p_1 + q_1 + r_1) - (p_0 + q_0 + r_0) > 0.$$

2.2.4. The case $\tau_1 = 0, \tau_2 \neq 0$

Let $\tau_1 = 0$ in (8), the characteristic equation becomes

$$\begin{aligned} D(\lambda, 0, \tau_2) &= R(\lambda)e^{-\lambda\tau_2} + Q(\lambda) + P(\lambda) \\ &= (r_2\lambda^2 + r_1\lambda + r_0)e^{-\lambda\tau_2} + \lambda^3 + p_2\lambda^2 + (p_1 + q_1)\lambda + p_0 + q_0 \\ &= 0. \end{aligned} \tag{10}$$

According to the works of Beretta and Kuang [1] we should find the imaginary solutions of Eq. (10). Let $\lambda = \omega i$ be these solutions. As $D(0, 0, 0) = p_0 + q_0 + r_0 \neq 0$ under assumption (H_1), i.e. the imaginary axis cannot be crossed by real values. We suppose that $\omega > 0$. We find roots $\lambda = \omega i$ of (10), because $\lambda = -\omega i$ is also a root of (10). Substituting it into (10) and separating the real and imaginary parts, we have

$$\begin{cases} \cos(\omega\tau_2) = -\mathrm{Re}\left(\dfrac{P(\omega i) + Q(\omega i)}{R(\omega i)}\right) = -\dfrac{(-p_2\omega^2 + p_0 + q_0)(-r_2\omega^2 + r_0) + [(p_1 + q_1)\omega - \omega^3]r_1\omega}{(-r_2\omega^2 + r_0)^2 + (r_1\omega)^2}, \\ \sin(\omega\tau_2) = \mathrm{Im}\left(\dfrac{P(\omega i) + Q(\omega i)}{R(\omega i)}\right) = \dfrac{-(-p_2\omega^2 + p_0 + q_0)r_1\omega + [(p_1 + q_1)\omega - \omega^3](-r_2\omega^2 + r_0)}{(-r_2\omega^2 + r_0)^2 + (r_1\omega)^2}. \end{cases} \tag{11}$$

A necessary condition to have ω as a solution of (10) is that ω must be a root of the following equation:

$$\begin{aligned} F(\omega) &= |P(\omega i) + Q(\omega i)|^2 - |R(\omega i)|^2 \\ &= \omega^6 + [p_2^2 - 2(p_1 + q_1) - r_2^2]\omega^4 + [(p_1 + q_1)^2 - r_1^2 + 2r_0r_2 - 2p_2(p_0 + q_0)]\omega^2 + (p_0 + q_0)^2 - r_0^2 \\ &= 0. \end{aligned} \tag{12}$$

Let $u = w^2$. Then

$$F(u) = u^3 + Au^2 + Bu + C = 0, \tag{13}$$

where $A = p_2^2 - 2(p_1 + q_1) - r_2^2, B = (p_1 + q_1)^2 - r_1^2 + 2r_0r_2 - 2p_2(p_0 + q_0), C = (p_0 + q_0)^2 - r_0^2$.
Using the results of [5] we assume that

(H_2) anyone of $A \geq 0, C < 0$; $B \leq 0, C < 0$; $A < 0, B > 0, C < 0, \triangle > 0$; $A < 0, B = 0, C = 0$; $B < 0, C = 0$;
(H_3) anyone of $A < 0, B > 0, C > 0, \triangle < 0$; $A < 0, B > 0, C = 0, A^2 > 4B$;
(H_4) $A < 0, B > 0, C < 0, \triangle < 0$;

(H_5) anyone of $A \geq 0, B \geq 0, C \geq 0$; $C > 0, \triangle > 0$;

where the discriminant $\triangle = \frac{[F(k)]^2}{4} + \frac{[F'(k)]^3}{9}$, $k = -\frac{A}{3}$ for the reduced form (13), and obtain that

Lemma 2.1. *For Eq. (13), it's well known that*

i. *if (H_2) holds, Eq. (13) has only one positive real root ω_1;*
ii. *if (H_3) holds, Eq. (13) has two distinct real positive root ω_2, ω_3 (setting $\omega_2 < \omega_3$);*
iii. *if (H_4) holds, Eq. (13) has three distinct positive real root $\omega_1 < \omega_2 < \omega_3$;*
iv. *if (H_5) holds, Eq. (13) has no positive real root.*

Remark. To the simplicity, we assume $\omega_1, \omega_2, \omega_3$ in the above three different cases only for the same monotonicity of F around the points but not the same roots of Eq. (13).

We should know that a solution ω of Eq. (13) is also a solution of the characteristic equation (11) if and only if $\tau_2^* = \frac{\phi(\omega^*)+2n\pi}{\omega^*}$ $n \in N, \phi \in [0, 2\pi]$. From the Lemma 2.1 and Lemma in Cooke and Grossman [4] or Ruan and Wei [19], we could easily get

Lemma 2.2. *For Eq. (10),we have*

i. *if (H_2) holds and $\tau_2 = \tau_{2,n}^1$, then Eq. (10) has only one pair of purely imaginary roots $\pm\omega_1$i.*
ii. *if (H_3) holds and $\tau_2 = \tau_{2,n}^2, \tau_{2,n}^3$, then Eq. (10) has two pairs of purely imaginary roots $\pm\omega_2 i, \pm\omega_3$i.*
iii. *if (H_4) holds and $\tau_2 = \tau_{2,n}^1, \tau_{2,n}^2, \tau_{2,n}^3$, then Eq. (10) has three pairs of purely imaginary roots $\pm\omega_1$i, $\pm\omega_2 i, \pm\omega_3$i.*
iv. *if (H_5) holds, then Eq. (10) has no purely imaginary roots where*

$$\tau_{2,n}^1 = \frac{1}{\omega_1}\cos^{-1}\left\{-\frac{(-p_2\omega_1^2+p_0+q_0)(-r_2\omega_1^2+r_0)+[(p_1+q_1)\omega_1-\omega_1^3]r_1\omega_1}{(-r_2\omega_1^2+r_0)^2+(r_1\omega_1)^2}\right\}+\frac{2n\pi}{\omega_1},$$
$$\tau_{2,n}^2 = \frac{1}{\omega_2}\cos^{-1}\left\{-\frac{(-p_2\omega_2^2+p_0+q_0)(-r_2\omega_2^2+r_0)+[(p_1+q_1)\omega_2-\omega_2^3]r_1\omega_2}{(-r_2\omega_2^2+r_0)^2+(r_1\omega_2)^2}\right\}+\frac{2n\pi}{\omega_2},$$
$$\tau_{2,n}^3 = \frac{1}{\omega_3}\cos^{-1}\left\{-\frac{(-p_2\omega_3^2+p_0+q_0)(-r_2\omega_3^2+r_0)+[(p_1+q_1)\omega_3-\omega_3^3]r_1\omega_3}{(-r_2\omega_3^2+r_0)^2+(r_1\omega_3)^3}\right\}+\frac{2n\pi}{\omega_3}.$$

Lemma 2.3. *Let τ_2^* denote an element of either the sequence $\{\tau_{2,n}^1\}$ or $\{\tau_{2,n}^2\}$ or $\{\tau_{2,n}^3\}$, then the following transversality condition are satisfied:*

$$\operatorname{sign}\left\{\left.\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau_2}\right|_{\tau_2=\tau_2^*}\right\} = \operatorname{sign} F'(\omega^2) = \operatorname{sign}(3\omega^4+2A\omega^2+B),$$
$$\operatorname{sign}\left\{\left.\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau_2}\right|_{\tau_2=\tau_{2,n}^1}\right\} > 0, \qquad \operatorname{sign}\left\{\left.\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau_2}\right|_{\tau_2=\tau_{2,n}^2}\right\} < 0, \qquad \operatorname{sign}\left\{\left.\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau_2}\right|_{\tau_2=\tau_{2,n}^3}\right\} > 0. \tag{14}$$

Proof. From [5], we know that the results of the first line is true, and also $F(\omega)$ is increasing at ω_1, ω_3, decreasing at ω_2, therefore (14) is set up.

Lemma 2.4. *For Eq. (10), we have the following*

i. *If (H_1) and (H_2) hold, then when $\tau_2 \in [0, \tau_{2,0}^1)$ all roots of Eq. (10) have negative real parts, and when $\tau_2 > \tau_{2,0}^1$ Eq. (10) has at least one root with positive real part.*
ii. *If (H_1) and (H_3) hold, then there exist k witches from stability to instability when the parameters such that $\tau_{2,0}^3 < \tau_{2,0}^2 < \tau_{2,1}^3 <, \cdots, < \tau_{2,k-2}^3 < \tau_{2,k-2}^2 < \tau_{2,k-1}^3 < \tau_{2,k}^3 < \tau_{2,k-1}^2$, all roots of Eq. (10) have negative real parts when $\tau_2 \in (\tau_{2,n}^2, \tau_{2,n+1}^3), \tau_{2,-1}^2 = 0, n = -1, 0, \ldots, k-1$. When $\tau_2 \in [\tau_{2,n}^3, \tau_{2,n}^2), n = 0, 1, \ldots, k-1$,and $\tau_2 > \tau_{2,k}^3$ Eq. (10) has at least one root with positive real parts.*
iii. *If (H_1) not hold but (H_3) hold, when the parameters such that $\tau_{2,0}^2 < \tau_{2,0}^3 < \tau_{2,1}^2 <, \ldots, < \tau_{2,k-1}^2 < \tau_{2,k-1}^3 < \tau_{2,k}^3 < \tau_{2,k}^2$, there may exist k switches from instability to stability, that is when $\tau_2 \in [\tau_{2,n}^3, \tau_{2,n+1}^2)$ and $\tau_2 > \tau_{2,k-1}^3, \tau_{2,-1}^3 = 0, n = -1, 0, \ldots, k-2$ Eq. (10) has at least one root with positive real parts. When $\tau_2 \in [\tau_{2,n}^2, \tau_{2,n}^3), n = 0, 1, \ldots, k-1$, all roots of Eq. (10) have negative real parts.*
iv. *If (H_1) and (H_4) hold, then there exists at least one stability switch.*

Proof. i. As (H_1) and (H_2) hold, then the equilibrium of the Eq. (10) is stable and the Eq. (10) has complex root with negative real parts for $\tau_2 = 0$, and also for $\tau_2 = \tau_{2,0}^1$ Eq. (10) has purely imaginary roots and the real parts of the root changes continuously increasing with increased τ_2 because of sign$\{\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau_2}|_{\tau_2=\tau_{2,n}^1}\} > 0$, then for $\tau_2 \in [0, \tau_{2,0}^1)$ all roots of Eq. (10) have negative real parts and Eq. (10) has at least one root with positive real parts when $\tau_2 > \tau_{2,0}^1$.

ii. Because $\omega_3 > \omega_2$ and $\tau_{2,n}^i = \frac{\phi(\omega_i)+2n\pi}{\omega_i}, i = 2, 3$ are the two corresponding sequences for the time delay τ_2, then $\exists k, \tau_{2,k}^3 - \tau_{2,k-1}^3 = \frac{2\pi}{w_3} < \frac{2\pi}{w_2} = \tau_{2,k-1}^2 - \tau_{2,k-2}^2$, and also (H_1) and (H_2) hold, then the equilibrium is stable for $\tau_2 = 0$. Therefore necessarily $\tau_{2,0}^3 < \tau_{2,0}^2$ (or the multiplicities of roots with negative real parts could become two for (14) when τ_2 is decreased, it's impossible. c.f. Cooke and Grossman [4]), when the parameters are such that $\tau_{2,0}^3 < \tau_{2,0}^2 < \tau_{2,1}^3 <, \dots, < \tau_{2,k-2}^2 < \tau_{2,k-1}^3 < \tau_{2,k}^3 < \tau_{2,k-1}^2$,we know that there exist a lot of stability switches and the stability of the equilibrium of change a finite of times at most, eventually it becomes unstable.

iii. the proof is the similar as ii.

iv. If (H_1) and (H_4) hold, then the equilibrium of the Eq. (10) is stable and Eq. (10) has three pairs of purely imaginary roots $\omega_1, \omega_2, \omega_3$, and (14) is hold, so there exist at least one stability switch at $\tau_{2,0}^1$ or $\tau_{2,0}^3$. □

According to the above analysis and the theorem in Hale [11], we obtain the following results:

Theorem 2.1. *When the conditions corresponding to Lemma 2.4(i., ii., iii.) are satisfied, then*

i. *The equilibrium (Y^*, K^*) is locally asymptotically stable when $\tau_2 \in [0, \tau_{2,0}^1)$ and a Hopf bifurcation occurs at (Y^*, K^*) when $\tau_2 = \tau_{2,0}^1$.*

ii. *The equilibrium (Y^*, K^*) is locally asymptotically stable when $\tau_2 \in \{0\} \cup (\tau_{2,n}^2, \tau_{2,n+1}^3), \tau_{2,-1}^2 = 0, n = -1, 0, \dots, k-2$, a Hopf bifurcation occurs when $\tau_2 = \tau_{2,n}^m \cup \tau_{2,k-1}^3, m = 2, 3, n = 0, 1, \dots, k-2$.*

iii. *The equilibrium (Y^*, K^*) is locally asymptotically stable when $\tau_2 \in [\tau_{2,n}^2, \tau_{2,n}^3), n = 0, 1, \dots, k-1$, a Hopf bifurcation occurs when $\tau_2 = \tau_{2,n}^m, m = 2, 3, n = 0, 1, \dots, k-1$.*

2.2.5. The case $\tau_1 \neq 0, \tau_2 \neq 0$

Next, we return to the Eq. (8) with $\tau_1 > 0$ and τ_2 in stable regions. Regard τ_1 as a parameter, following Ruan and Wei [19], we have

Lemma 2.5. *If all roots of Eq. (10) have negative real parts for $\tau_2 > 0$, then there exists a $\tau_1^*(\tau_2) > 0$, such that when $0 \le \tau_1 < \tau_1^*(\tau_2)$ all roots of Eq. (8) have negative real parts.*

Proof. Since the left hand side of Eq. (8) is analytic in λ and τ_1, following the Theorem 2.1 of Ruan and Wei [19], when τ_1 varies, the sum of the multiplicities of zeros of the left hand side of Eq. (8) in the open right half-plane can change only if a zero on or cross the imaginary axis. □

Theorem 2.2. *Assume (H_1) holds true.*

i. *If (H_2) holds, then for any $\tau_2 \in [0, \tau_{2,0}^1)$, there exists a $\tau_1^*(\tau_2)$ such that the equilibrium of system (6) is locally asymptotically stable when $\tau_1 \in [0, \tau_1^*(\tau_2))$.*

ii. *If (H_3) holds, then for any $\tau_2 \in (\tau_{2,n}^2, \tau_{2,n+1}^3), \tau_{2,-1}^2 = 0, n = -1, 0, \dots, k-1$, there exists a $\tau_1^*(\tau_2)$ such that the equilibrium of system (6) is locally asymptotically stable when $\tau_1 \in [0, \tau_1^*(\tau_2))$.*

iii. *If (H_5) holds, then for any $\tau_2 \ge 0$, there exists a $\tau_1^*(\tau_2)$ such that the equilibrium of system (6) is locally asymptotically stable when $\tau_1 \in [0, \tau_1^*(\tau_2))$.*

Proof. i. and ii. According to i. and ii. of Lemmas 2.4 and 2.5, we could easily get the results.

iii. If (H_1) and (H_5) hold, then we know that for any $\tau_2 \ge 0$ all roots of Eq. (10) have negative real parts, so following the Lemma 2.5, there exists a $\tau_1^*(\tau_2) > 0$ such that all roots of Eq. (8) have negative real parts for $0 \le \tau_1 < \tau_1^*(\tau_2)$. □

It's clear that Hopf bifurcation occurs at $\tau_1^*(\tau_2)$ if it holds the conditions of Lemma 2.5 or Theorem 2.2. And also there may exist a lot of stability switches. If we let τ_2 in unstable region, then there may exist no $\tau_1^*(\tau_2)$ such that when the system (6) is unstable in $0 \le \tau_1 < \tau_1^*(\tau_2)$, it's stable in $\tau_1^*(\tau_2) < \tau_1$.

3. Concluding remarks

We investigated the IS-LM model with two time delays in the capital accumulation equation and concluded that when one time lag influences the qualitative behavior of the economic system, the other also changes the behavior. Therefore, for policy makers, they should seriously consider more time delays in the economic system and their impacts on the economic system.

From all the aforementioned economic model we could know that they are all fixed price disequilibrium dynamic models. For future research we could consider flexible price or Phillips curve in the dynamic economic system and the influence of time delay on the whole system.

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