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Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems

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ABSTRACT

The purpose of this paper is to consider the convergence of a shrinking projection method for a finite family of quasi-ϕ-nonexpansive mappings and an equilibrium problem. Strong convergence theorems are established in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec–Klee property.

1. Introduction and preliminaries

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit (1.1) is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E has the Kadec–Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$ for more details on Kadec–Klee property; the readers is referred to [1–3] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec–Klee property.

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Let C be a nonempty closed and convex subset of a Banach space E and $T : C \to C$ a mapping. The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. A point $x \in C$ is a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

It is well known that if C is a nonempty bounded closed and convex subset of a uniformly convex Banach space E, then every nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [4] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{1.2}$$

Observe that, in a Hilbert space H, (1.2) is reduced to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y\in C} \phi(y, x)$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1,3–5]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.3}$$

Remark 1.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.3), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; see [1,3] for more details.

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [6] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive [4,7,8] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mappings was studied in [4,7,8]. The mapping T is said to be ϕ-nonexpansive if $\phi(Tx, Ty) \le \phi(x, y)$ for all $x, y \in C$. T is said to be quasi-ϕ-nonexpansive [9–11] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 1.2. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$.

In 2005, Matsushita and Takahashi [12] considered fixed point problems of a single relatively nonexpansive mapping in a Banach space. To be more precise, They proved the following theorem:

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = P_{H_n\cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.4}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. In this paper, we consider the following equilibrium problem. Find $p \in C$ such that

$$f(p, y) \ge 0, \quad \forall y \in C. \tag{1.5}$$

We use $EP(f)$ to denote the solution set of the equilibrium problem (1.5). That is,

$$EP(f) = \{p \in C : f(p, y) \ge 0, \ \forall y \in C\}.$$

Given a mapping $Q : C \to E^*$, let

$$f(x, y) = \langle Qx, y - x\rangle, \ \forall x, y \in C.$$

Then $p \in EP(f)$ if and only if

$$\langle Qp, y - p\rangle \geq 0, \quad \forall y \in C.$$

That is, p is a solution of the above variational inequality.

Numerous problems in physics, optimization and economics reduce to find a solution of (1.5); see [13–19]. For studying the equilibrium problem (1.5), let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0, \ \forall x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C$;
(A3)

$$\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y), \quad \forall\, x, y, z \in C;$$

(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semi-continuous.

Recently, some authors considered the problem of finding a common element in the set of fixed points of a relatively nonexpansive mapping which is a generalization of nonexpansive mappings in Hilbert spaces and in the set of solutions of the equilibrium problem (1.5) based on hybrid projection methods in the framework of real Banach spaces; see, for example [9,20–22] and the references therein.

In [20], Takahashi and Zembayshi obtained the following result on the equilibrium problem (1.5) and a relatively nonexpansive mapping.

Theorem TZ. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let T be a relatively nonexpansive mapping from C into itself such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases} \tag{1.6}$$

for every $n \geq 0$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)}x$, where $\Pi_{F(T)\cap EP(f)}$ is the generalized projection of E onto $F(T) \cap EP(f)$.

Recently, Qin, Cho and Kang [9] further improved Theorem TZ by considering a pair of quasi-ϕ-nonexpansive mappings based on shrinking projection methods which was considered by Takahashi, Takeuchi and Kubota [23] in Hilbert spaces. To be more precise, they proved the following results.

Theorem QCK. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $T, S : C \to C$ be two closed quasi-ϕ-nonexpansive mappings such that $\mathcal{F} = F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.7}$$

where J is the duality mapping on E, $\{r_n\}$ is a positive sequence and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ *satisfying the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$;
(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$.

Note that Theorems MT, TZ and QCK are all valid in uniformly convex and uniformly smooth Banach spaces. The following question naturally arises in connection with the above results on the framework of spaces.

Question 1.3. Can one weaken the restriction on the framework of spaces such that hybrid projection methods are still valid for the equilibrium problem (1.5)?

On the other hand, common fixed point problems recently have been studied by many authors; see, for example, [1,15,21,24–28]. Finding an optimal point in the intersection of the fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see [29–31]. The problem of finding an optimal point that minimizes a given cost function over common fixed point set of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see, e.g., [32–34].

In this paper, motivated by Theorems MT, TZ and QCK, we re-considered the problem of finding a common element in the common fixed point set of a family of quasi-ϕ-nonexpansive mappings and in the solution set of the equilibrium problem (1.5). Strong convergence theorems of common elements are established in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec–Klee property. Note that every uniformly convex Banach space enjoys the Kadec–Klee property. Our main convergence theorem gives an affirmative answer to Question 1.3. The results presented this paper mainly improve the corresponding results announced in Qin, Cho and Kang [9] and Takahashi and Zembayshi [20].

In order to establish our main results, we need the following lemmas.

Lemma 1.4 (*[4]*). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \forall y \in C.$$

Lemma 1.5 (*[4]*). *Let E be a reflexive, strictly convex and smooth Banach space, C a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 1.6. *Let E be a strictly convex and smooth Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Proof. Letting $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$, we prove that $p \in F(T)$. From the definition of T, we have $\phi(p_n, Tp) \leq \phi(p_n, p)$, which implies that $\phi(p_n, Tp) \to 0$ as $n \to \infty$. Note that

$$\phi(p_n, Tp) = \|p_n\|^2 - 2\langle p_n, J(TP)\rangle + \|Tp\|^2.$$

Letting $n \to \infty$ in the above equality, we see that $\phi(p, Tp) = 0$. This shows that $p = Tp$.

Next, we show that $F(T)$ is convex. To this end, for arbitrary $p_1, p_2 \in F(T)$, $t \in (0, 1)$, putting $p_3 = tp_1 + (1 - t)p_2$, we prove that $Tp_3 = p_3$. Indeed, from the definition of ϕ, we see that

$$\begin{aligned}
\phi(p_3, Tp_3) &= \|p_3\|^2 - 2\langle p_3, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 - 2\langle tp_1 + (1-t)p_2, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 - 2t\langle p_1, J(Tp_3)\rangle - 2(1-t)\langle p_2, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&\leq \|p_3\|^2 + t\phi(p_1, p_3) + (1-t)\phi(p_2, p_3) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&= \|p_3\|^2 - 2\langle tp_1 + (1-t)p_2, Jp_3\rangle - \|p_3\|^2 \\
&= \|p_3\|^2 - 2\langle p_3, Jp_3\rangle - \|p_3\|^2 \\
&= 0.
\end{aligned}$$

This implies that $p_3 \in F(T)$. This completes the proof. □

Lemma 1.7. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *Let $r > 0$ and $x \in E$. Then*

(a) *([13]). There exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall\, y \in C.$$

(b) *([9,20]). Define a mapping $T_r : E \to C$ by*

$$S_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle,\ \forall y \in C \right\}.$$

Then the following conclusions hold:

(1) S_r *is single-valued;*
(2) S_r *is a firmly nonexpansive-type mapping, i.e., for all* $x, y \in E$,
$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \le \langle S_r x - S_r y, Jx - Jy\rangle$$
(3) $F(S_r) = EP(f)$;
(4) S_r *is quasi-ϕ-nonexpansive;*
(5) $EP(f)$ *is closed and convex;*
(6)
$$\phi(q, S_r x) + \phi(S_r x, x) \le \phi(q, x), \quad \forall q \in F(S_r).$$

Lemma 1.8 *([35]). Let* $p > 1$ *and* $s > 0$ *be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_s(0) = \{x \in E : \|x\| \le s\}$ *and* $\lambda \in [0, 1]$, *where* $w_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

The following lemma can be obtained from Lemma 1.8 immediately.

Lemma 1.9. *Let E be a uniformly convex Banach space,* $s > 0$ *a positive number and* $B_s(0)$ *a closed ball of E. There exists a continuous, strictly increasing and convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\left\|\sum_{i=1}^{N}(\alpha_i x_i)\right\|^2 \le \sum_{i=1}^{N}(\alpha_i\|x_i\|^2) - \alpha_1\alpha_2 g(\|x_1 - x_2\|) \tag{1.8}$$

for all $x_1, x_2, \ldots, x_N \in B_s(0) = \{x \in E : \|x\| \le s\}$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_N \in [0, 1]$ *such that* $\sum_{i=1}^{N}\alpha_i = 1$.

2. Main results

Theorem 2.1. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec–Klee property, C a nonempty closed and convex subset of E and f bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4). *Let* $T_i : C \to C$ *be a closed and quasi-ϕ-nonexpansive mapping for each* $i \in \{1, 2, \ldots, N\}$. *Assume that* $\mathcal{F} = \cap_{i=1}^{N} F(T) \cap EF(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}x_0, \\ y_n = J^{-1}\left(\alpha_{n,0}Jx_n + \displaystyle\sum_{i=1}^{N}\alpha_{n,i}JT_i x_n\right), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \tag{Υ}$$

where $\{\alpha_{n,0}\}, \{\alpha_{n,1}\}, \ldots, \{\alpha_{n,N}\}$ *are real sequences in* $(0, 1)$, $\{r_n\}$ *is a real sequence in* $[a, \infty)$, *where a is some positive real number and J is the duality mapping on E. Assume that the control sequences satisfy* $\sum_{j=0}^{N}\alpha_{n,j} = 1$ *and* $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$, $\forall i \in \{1, 2, \ldots, N\}$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}}x_0$, *where* $\Pi_{\mathcal{F}}$ *is the generalized projection from E onto* $\mathcal{F}$.

Proof. First, we show that C_n is closed and convex for each $n \ge 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some integer h. For $z \in C_h$, we see that $\phi(z, u_h) \le \phi(z, x_h)$ is equivalent to

$$2\langle z, Jx_k - Ju_k\rangle \le \|x_k\|^2 - \|u_k\|^2.$$

It is easy to see that C_{h+1} is closed and convex. This proves that C_n is closed and convex for each $n \ge 1$. This in turn shows that $\Pi_{C_{n+1}}x_0$ is well defined. Putting $u_n = S_{r_n}y_n$, from Lemma 1.7 we see that S_{r_n} is quasi-ϕ-nonexpansive. Now, we are in a position to prove that $\mathcal{F} \subset C_n$ for each $n \ge 1$. Indeed, $\mathcal{F} \subset C_1 = C$ is obvious. Suppose that $\mathcal{F} \subset C_h$ for some h. Then, for $\forall w \in \mathcal{F} \subset C_h$, we have

$$\begin{aligned} \phi(w, u_h) &= \phi(w, S_{r_h}y_h) \\ &\le \phi(w, y_h) \\ &= \phi\left(w, J^{-1}\left(\alpha_{h,0}Jx_h + \sum_{i=1}^{N}\alpha_{h,i}JT_i x_h\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \|w\|^2 - 2\langle w, \alpha_{h,0} Jx_n + \sum_{i=1}^{N} \alpha_{h,i} JT_i x_n\rangle + \left\|\alpha_{h,0} Jx_n + \sum_{i=1}^{N} \alpha_{h,i} JT_i x_h\right\|^2 \\
&\le \|w\|^2 - 2\alpha_{h,0}\langle w, Jx_h\rangle - 2\sum_{i=1}^{N} \alpha_{h,i}\langle w, JT_i x_h\rangle + \alpha_{h,0}\|x_h\|^2 + \sum_{i=1}^{N}\alpha_{h,i}\|T_i x_h\|^2 \\
&= \alpha_{h,0}\phi(w, x_h) + \sum_{i=1}^{N}\alpha_{h,i}\phi(w, T_i x_h) \\
&\le \alpha_{h,0}\phi(w, x_h) + \sum_{i=1}^{N}\alpha_{h,i}\phi(w, x_h) \\
&= \phi(w, x_h),
\end{aligned}
\tag{2.1}
$$

which shows that $w \in C_{h+1}$. This implies that $\mathcal{F} \subset C_n$ for each $n \ge 1$. On the other hand, from Lemma 1.5 we see that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(w, x_0) - \phi(w, x_n) \le \phi(w, x_0),$$

for each $w \in \mathcal{F} \subset C_n$ and for each $n \ge 1$. This shows that the sequence $\phi(x_n, x_0)$ is bounded. From (1.3), we see that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup p$. Note that C_n is closed and convex for each $n \ge 1$. It is easy to see that $p \in C_n$ for each $n \ge 1$. Note that

$$\phi(x_n, x_0) \le \phi(p, x_0).$$

It follows that

$$\phi(p, x_0) \le \liminf_{n\to\infty} \phi(x_n, x_0) \le \limsup_{n\to\infty} \phi(x_n, x_0) \le \phi(p, x_0).$$

This implies that

$$\lim_{n\to\infty} \phi(x_n, x_0) = \phi(p, x_0).$$

Hence, we have $\|x_n\| \to \|p\|$ as $n \to \infty$. In view of the Kadec–Klee property of E, we obtain that

$$\lim_{n\to\infty} x_n = p. \tag{2.2}$$

Next, we show that $p \in F(T)$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. It follows that

$$
\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\
&\le \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).
\end{aligned}
$$

Letting $n \to \infty$, we obtain that $\phi(x_{n+1}, x_n) \to 0$. In view of $x_{n+1} \in C_{n+1}$, we obtain that

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n).$$

It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

From (1.3), we see that

$$\|u_n\| \to \|p\| \quad \text{as } n \to \infty. \tag{2.3}$$

It follows that

$$\|Ju_n\| \to \|Jp\| \quad \text{as } n \to \infty. \tag{2.4}$$

This implies that $\{Ju_n\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Ju_n \rightharpoonup x^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an $x \in E$ such that $Jx = x^*$. It follows that

$$
\begin{aligned}
\phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.
\end{aligned}
$$

Taking $\liminf_{n\to\infty}$ on both sides of equality above yields that

$$
\begin{aligned}
0 &\ge \|p\|^2 - 2\langle p, x^*\rangle + \|x^*\|^2 \\
&= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2
\end{aligned}
$$

$$\begin{aligned}&= \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2\\ &= \phi(p, x).\end{aligned}$$

That is, $p = x$, which in turn implies that $x^* = Jp$. It follows that $Ju_n \rightharpoonup Jp \in E^*$. Since (2.4) and E^* enjoys the Kadec–Klee property, we obtain that

$$Ju_n - Jp \to 0 \quad \text{as } n \to \infty.$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $u_n \rightharpoonup p$. Since (2.3) and E enjoys the Kadec–Klee property, we obtain that

$$\lim_{n\to\infty} u_n = p. \tag{2.5}$$

Note that

$$\|x_n - u_n\| \le \|x_n - p\| + \|p - u_n\|.$$

It follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.6}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{2.7}$$

Let $s = \sup_{n\ge 0}\{\|x_n\|, \|T_1x_n\|, \|T_2x_n\|, \dots, \|T_Nx_n\|\}$. Since E is uniformly smooth, we know that E^* is uniformly convex. In view of Lemma 1.9, we see that

$$\begin{aligned}\phi(w, u_n) &= \phi(w, S_{r_n}y_n)\\
&\le \phi(w, y_n)\\
&= \phi\left(w, J^{-1}\left(\alpha_{n,0}Jx_n + \sum_{i=1}^{N}\alpha_{n,i}JT_ix_n\right)\right)\\
&= \|w\|^2 - 2\langle w, \alpha_{n,0}Jx_n + \sum_{i=1}^{N}\alpha_{n,i}JT_ix_n\rangle + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^{N}\alpha_{n,i}JT_ix_n\right\|^2\\
&\le \|w\|^2 - 2\alpha_{n,0}\langle w, Jx_n\rangle - 2\sum_{i=1}^{N}\alpha_{n,i}\langle w, JT_ix_n\rangle + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{N}\alpha_{n,i}\|T_ix_n\|^2 - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|)\\
&= \alpha_{n,0}\phi(w, x_n) + \sum_{i=1}^{N}\alpha_{n,i}\phi(w, T_ix_n) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|)\\
&\le \alpha_{n,0}\phi(w, x_n) + \sum_{i=1}^{N}\alpha_{n,i}\phi(w, x_n) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|)\\
&\le \phi(w, x_n) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|).\end{aligned} \tag{2.8}$$

It follows that

$$\alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|) \le \phi(w, x_n) - \phi(w, u_n). \tag{2.9}$$

On the other hand, we have

$$\begin{aligned}\phi(w, x_n) - \phi(w, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n\rangle\\
&\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|w\|\|Jx_n - Ju_n\|.\end{aligned}$$

It follows from (2.6) and (2.7) that

$$\phi(w, x_n) - \phi(w, u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.10}$$

In view of (2.10) and the assumption $\liminf_{n\to\infty} \alpha_{n,0}(1 - \alpha_{n,1}) > 0$, we see that

$$g(\|Jx_n - JT_1x_n\|) \to 0 \quad \text{as } n \to \infty. \tag{2.11}$$

It follows from the property of g that

$$\|Jx_n - JT_1x_n\| \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

Since $x_n \to p$ as $n \to \infty$ and $J : E \to E^*$ is demi-continuous, we obtain that $Jx_n \rightharpoonup Jp \in E^*$. Note that

$$|\|Jx_n\| - \|Jp\|| = |\|x_n\| - \|p\|| \le \|x_n - p\|.$$

This implies that

$$\lim_{n\to\infty} \|Jx_n\| = \|Jp\|. \tag{2.13}$$

Since E^* enjoys the Kadec–Klee property, we see that

$$\lim_{n\to\infty} \|Jx_n - Jp\| = 0. \tag{2.14}$$

Note that

$$\|JT_1x_n - Jp\| \le \|JT_1x_n - Jx_n\| + \|Jx_n - Jp\|.$$

From (2.12) and (2.14), we arrive at

$$\lim_{n\to\infty} \|JT_1x_n - Jp\| = 0. \tag{2.15}$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $T_1x_n \rightharpoonup p$. On the other hand, we have

$$|\|T_1x_n\| - \|p\|| = |\|JT_1x_n\| - \|Jp\|| \le \|JT_1x_n - Jp\|.$$

In view of (2.15), we obtain that $\|T_1x_n\| \to \|p\|$ as $n \to \infty$. Since E enjoys the Kadec–Klee property, we obtain that

$$\lim_{n\to\infty} \|T_1x_n - p\| = 0. \tag{2.16}$$

It follows from the closedness of T_1 that $T_1p = p$. By repeating (2.8)–(2.16), we can obtain that $p \in \cap_{i=1}^N F(T_i)$.

Next, we show that $p \in EF(f)$. From (2.1), we arrive at

$$\phi(w, y_n) \le \phi(w, x_n). \tag{2.17}$$

In view of $u_n = S_{r_n}y_n$ and Lemma 1.7, we arrive at

$$\begin{aligned}\phi(u_n, y_n) &= \phi(S_{r_n}y_n, y_n)\\ &\le \phi(w, y_n) - \phi(w, S_{r_n}y_n)\\ &\le \phi(w, x_n) - \phi(w, S_{r_n}y_n)\\ &= \phi(w, x_n) - \phi(w, u_n).\end{aligned} \tag{2.18}$$

It follows from (2.10) that $\phi(u_n, y_n) \to 0$ as $n \to \infty$. From (1.3), we see that $\|u_n\| - \|y_n\| \to 0$ as $n \to \infty$. In view of $u_n \to p$ as $n \to \infty$, we arrive at

$$\|y_n\| - \|p\| \to 0 \quad \text{as } n \to \infty. \tag{2.19}$$

It follows that

$$\|Jy_n\| - \|Jp\| \to 0 \quad \text{as } n \to \infty. \tag{2.20}$$

Since E^* is reflexive, we may assume that $Jy_n \rightharpoonup f^* \in E^*$. In view of $J(E) = E^*$, we see that there exists $f \in E$ such that $Jf = f^*$. It follows that

$$\begin{aligned}\phi(u_n, y_n) &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|y_n\|^2\\ &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|Jy_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of equality above yields that

$$\begin{aligned}0 &\ge \|p\|^2 - 2\langle p, f^*\rangle + \|f^*\|^2\\ &= \|p\|^2 - 2\langle p, Jf\rangle + \|Jf\|^2\\ &= \|p\|^2 - 2\langle p, Jf\rangle + \|f\|^2\\ &= \phi(p, f).\end{aligned}$$

That is, $p = f$, which in turn implies that $f^* = Jp$. It follows that $Jy_n \rightharpoonup Jp \in E^*$. Since (2.20) and E^* enjoys the Kadec–Klee property, we obtain that

$$Jy_n - Jp \to 0 \quad \text{as } n \to \infty.$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $y_n \rightharpoonup p$. Since (2.19) and E enjoys the Kadec–Klee property, we obtain that

$$y_n \to p \quad \text{as } n \to \infty. \tag{2.21}$$

Note that

$$\|u_n - y_n\| \le \|u_n - p\| + \|p - y_n\|.$$

It follows from (2.6) and (2.21) that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{2.22}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the assumption $r_n \ge a$, we see that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{2.23}$$

In view of $u_n = S_{r_n} y_n$, we see that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall\, y \in C.$$

It follows from the condition (A2) that

$$\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} \ge \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge -f(u_n, y) \ge f(y, u_n), \quad \forall\, y \in C.$$

By taking the limit as $n \to \infty$ in the above inequality, from the condition (A4) and (2.23) we obtain that

$$f(y, p) \le 0, \quad \forall\, y \in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. It follows that $y_t \in C$, which yields that $f(y_t, p) \le 0$. It follows from the conditions (A1) and (A4) that

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, p) \le tf(y_t, y).$$

That is,

$$f(y_t, y) \ge 0.$$

Letting $t \downarrow 0$, from the condition (A3), we obtain that $f(p, y) \ge 0$, $\forall y \in C$. This implies that $p \in EP(f)$. This shows that $p \in \mathcal{F} = \cap_{i=1}^{N} F(T_i) \cap EP(f)$.

Finally, we prove that $p = \Pi_{\mathcal{F}} x_0$. From $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \ge 0, \quad \forall z \in C_n.$$

Since $\mathcal{F} \subset C_n$ for each $n \ge 1$, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0, \quad \forall\, w \in \mathcal{F}. \tag{2.24}$$

Letting $n \to \infty$ in (2.24), we see that

$$\langle p - w, Jx_0 - Jp\rangle \ge 0, \quad \forall\, w \in \mathcal{F}.$$

In view of Lemma 1.4, we can obtain that $p = \Pi_{\mathcal{F}} x_0$. This completes the proof. □

If $T_i = T$ for each $i \in \{1, 2, \ldots, N\}$, then Theorem 2.1 is reduced to the following results.

Corollary 2.2. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec–Klee property, C a nonempty closed and convex subset of E and f bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *Let T be a closed and quasi-ϕ-nonexpansive mapping. Assume that $\mathcal{F} = F(T) \cap EF(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0, 1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\{r_n\}$ *is a real sequence in* $[a, \infty)$, *where a is some positive real number and J is the duality mapping on E. Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_0$, *where* $\Pi_{\mathcal{F}}$ *is the generalized projection from E onto* $\mathcal{F}$.

Remark 2.3. Corollary 2.2 improves Theorem TZ in the following aspects.

(a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex space to a uniformly smooth and strictly convex Banach space which also enjoys the Kadec–Klee property (note that every uniformly convex Banach space enjoys the Kadec–Klee property).
(b) For the mappings, we extend the mapping from a relatively nonexpansive mapping to a quasi-ϕ-nonexpansive mapping (we remove the restriction $\widetilde{F}(T) = F(T)$, where $\widetilde{F}(T)$ denotes the asymptotic fixed point set).
(c) For the algorithm, we remove the set "W_n" in Theorem TZ.

For a special case that $N = 2$, we can obtain the following results on a pair of quasi-ϕ-nonexpansive mappings immediately from Theorem 2.1.

Corollary 2.4. *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec–Klee property, C a nonempty closed and convex subset of E and f bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4). *Let S and T be two closed and quasi-*ϕ*-nonexpansive mappings. Assume that* $\mathcal{F} = F(T) \cap F(S) \cap EF(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\big(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n\big), \\ u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \ge 0, \quad \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequences in* $(0, 1)$, $\{r_n\}$ *is a real sequence in* $[a, \infty)$, *where a is some positive real number and J is the duality mapping on E. Assume that the control sequences satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_0$, *where* $\Pi_{\mathcal{F}}$ *is the generalized projection from E onto* $\mathcal{F}$.

Remark 2.5. For the framework of spaces, Corollary 2.4 mainly improves Theorem QCK from a uniformly smooth and uniformly convex space to a uniformly smooth and strictly convex Banach space which also enjoys the Kadec–Klee property (note that every uniformly convex Banach space enjoys the Kadec–Klee property).

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