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Shape Preserving Rational Cubic Fractal Interpolation Function

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Abstract

A new type of $\mathcal{C}^1$ Fractal Interpolation Function (FIF) is developed using the Iterated Function System (IFS) which contains the rational spline. The numerator of this rational spline contains cubic polynomial and the denominator of the rational spline contains quadratic polynomial. We find uniform error bound between the original function which belongs to the class $\mathcal{C}^2$ and the FIF. We described suitable conditions on scaling factors and shape parameters such that it preserves the shape properties which inherited in the data.

Keywords: Iterated function system, Fractal interpolation function, Positivity, Monotonicity, Convexity.

1. Introduction

Suppose the data $\mathcal{D} = \{(x_i,\ y_i) \in I \times \mathbb{R} :\ i = 1, 2, \dots, N\}$ is given, where $x_1 < x_2 < \cdots < x_N$ and $I = [x_1, x_N]$. Interpolation is the process of constructing a continuous function $\Phi : I \to \mathbb{R}$ such that $\Phi(x_i) = y_i$ for all $i = 1, 2, \dots, N$. The classical interpolants (polynomial, spline etc.) are infinitely differentiable or piecewise infinitely differentiable. In many situations, data comes from numerical experiments are highly irregular. So classical interpolation methods becomes unsuitable to interpolate these data. To interpolate irregular data, Barnsley [1] introduced a new interpolation method called Fractal Interpolation using special type of iterated function system. In order to approximate differentiable functions Barnsley and Harrington [2] introduced differentiable fractal interpolation functions. With the help of Barnsley and Harrington results, various classical spline methods are generalized for instance [3–5].

In many situations, it is required that interpolant should reflect the geometric characteristics of the data set. Constructing interpolant with sufficiently smooth and preserving geometric characteristics of the data is called shape preserving interpolation. To preserve shape properties of the data, various spline interpolants are developed, for instance [6–10]. The uniqueness of spline interpolation becomes unsuitable for shape modification problem. Späth [11] introduced rational function with shape parameters to preserve geometric characteristic attached to data set. Also, various researchers [12–19] have constructed shape preserving rational splines using shape parameters. Using fractal interpolation functions, Chand and coworkers [20–24] have initiated study on shape preserving.

In this paper, a new $\mathcal{C}^1$ fractal interpolation function using rational IFS which contains three families of shape parameters is constructed in such a way that it preserves shape properties of the data. The proposed scheme has many outstanding features.

- The proposed method is a best tool to approximate a function that is continuous and its derivatives are irregular (see Section 5).
- When all the scaling factors are zero, fractal interpolation function that obtained from proposed method, reduces into a classical rational cubic spline (see Remark 2 and Section 5).
- The proposed method is equally applicable for the data with derivatives or data without derivatives.

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- In the proposed method, extra knots are not needed to get the shape preserving interpolant.
- In the proposed scheme, shape preserving fractal interpolant is unique for the fixed scaling factors and the fixed shape parameters. By changing the scaling factors and the shape parameters, infinitely many shape preserving fractal interpolants can be obtained.
- This scheme is computationally economical because even though three families of shape parameters are involved, the data dependent conditions are prescribed on one family of the shape parameters. Remaining two families of the shape parameters can assume any positive values.

In Section 2, introduction about iterated function system and fractal interpolation functions are presented. In Section 3, construction of FIF and approximation property of this FIF are discussed. In Section 4, shape preserving aspects of this FIF are discussed. In Section 5, shape preserving aspects of FIF is checked with examples.

2. Fractal Interpolation Function

Let a set of data points $\{(x_i, y_i) : i = 1, 2, \dots, N\}$ be given such that $x_1 < x_2 < \cdots < x_N$. Let $J := \{1, 2, \dots, N-1\}$. Set $I_i = [x_i, x_{i+1}]$ and $I = [x_1, x_N]$. Let $L_i : I \to I_i,\ i \in J$, be contraction mappings such that

$$L_i(x_1) = x_i, \quad L_i(x_N) = x_{i+1}. \tag{1}$$

Let $K = I \times D$, where D is the suitable compact set containing all y_i's. Consider the mappings such that for all $i \in J$, $F_i : K \to D$ satisfying

$$\begin{aligned} F_i(x_1, y_1) = y_i, \quad F_i(x_N, y_N) = y_{i+1}, \\ |F_i(x, y) - F_i(x, y')| \le |s_i||y - y'|,\ x \in I;\ y,\ y' \in D, \end{aligned} \tag{2}$$

where $-1 < s_i < 1$. For each $i \in J$, define function $w_i : K \to K$ by $w_i(x, y) = (L_i(x), F_i(x, y))$ for all $(x, y) \in K$. The collection $\mathcal{J} = \{K; w_i : i \in J\}$ is called an Iterated Function System (IFS).

Proposition 1. *[25] The IFS $\{K; w_i : i \in J\}$ has a unique attractor G, and G is the graph of a continuous function $f : I \to \mathbb{R}$ which interpolates the data $\{(x_i, y_i) : i = 1, 2, \dots, N\}$, i.e., $f(x_i) = y_i$, $i = 1, 2, \dots, N$.*

The function f is called a Fractal Interpolation Function (FIF) corresponding to the IFS $\mathcal{J}$, and it can also be constructed based on the following.

Let $\mathcal{G} = \{g : I \to \mathbb{R}|\ g \text{ is continuous},\ g(x_1) = y_1 \text{ and } g(x_N) = y_N\}$. Then $\mathcal{G}$ is a complete metric space with respect to the uniform metric $\rho(g_1, g_2) = \max\{|g_1(x) - g_2(x)| : x \in I\}$. Define the Read-Bajraktarević operator T on $(\mathcal{G}, \rho)$ as

$$Tg(L_i(x)) = F_i(x, g(x)),\ x \in I,\ i \in J. \tag{3}$$

Using (1) and the first condition of (2), it is easy to verify that Tg is continuous on the interval $I_i,\ i \in J$, and all the interior points $x_2, x_3, \dots, x_{N-1}$. Also T is a contraction map on $(\mathcal{G}, \rho)$, i.e.,

$$\rho(Tg_1, Tg_2) \le |s|_\infty \rho(g_1, g_2),$$

where $|s|_\infty = \max\{|s_i| : i \in J\} < 1$. Therefore, by the Banach fixed point theorem, T has a unique fixed point f(say) on $\mathcal{G}$ such that $Tf(x) = f(x)$ for all $x \in I$. By (3), the FIF f satisfies the functional equation

$$f(L_i(x)) = F_i(x, f(x)),\ x \in I,\ i \in J.$$

The FIFs constructed so far by the following IFS $\{K; w_i : i \in J\}$

$$\left.\begin{aligned} L_i(x) = a_i x + b_i, \\ F_i(x, y) = s_i y + r_i(x), \end{aligned}\right\} \qquad i \in J, \tag{4}$$

where

$$a_i = \frac{x_{i+1} - x_i}{x_N - x_1},\ b_i = \frac{x_N x_i - x_1 x_{i+1}}{x_N - x_1},$$

$|s_i| < 1$ and $r_i : I \to \mathbb{R}$ is a continuous functions such that F_i satisfies (2). The number s_i is called the vertical scaling factor of the map w_i and $s = (s_1, s_2, \ldots, s_{N-1})$ is called the scale vector of the IFS. The following proposition ensures the existence of a differentiable FIF.

Proposition 2. *[2] Let $\{(x_i, y_i) : 1 = 1, 2, \ldots, N\}$ be a data set such that $x_1 < x_2 \cdots < x_N$. Let $L_i(x)$ be the affine functions satisfying $L_i(x_1) = x_i$, $L_i(x_N) = x_{i+1}$, $i \in J$ and $F_i(x, y) = s_i y + r_i(x)$, $i \in J$ satisfying (2). Suppose for some integer $n \geq 0$, $|s_i| < a_i^n$, $r_i \in \mathcal{C}^n(I)$, $i \in J$. Let*

$$F_{i,k}(x, y) = \frac{s_i y + r_i^{(k)}(x)}{a_i^k},\ y_{1,k} = \frac{r_1^{(k)}(x_1)}{a_1^k - s_1},\ y_{N,k} = \frac{r_{N-1}^{(k)}(x_N)}{a_{N-1}^k - s_{N-1}},$$

$k = 1, 2, \ldots, n$. If $F_{i-1,k}(x_N, y_{N,k}) = F_{i,k}(x_1, y_{1,k})$, $i = 2, 3, \ldots N-1$ and $k = 1, 2, \ldots, n$, then $\{(L_i(x), F_i(x, y)) : i = 1, 2, \ldots, N-1\}$ determines a FIF $f \in \mathcal{C}^n(I)$, and $f^{(k)}$ is the FIF determined by $\{(L_i(x),\ F_{i,k}(x, y)) : i = 1, 2, \ldots, N-1\}$, $k = 1, 2, \ldots, n$.

3. $\mathcal{C}^1$ rational FIF

In the present section, rational FIF with three families shape parameters is going to constructed based on Read-Bajraktarević operator [23, 26]. Let $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$ be a set of data points such that $x_1 < x_2 < \cdots < x_N$. Let d_i be the derivative value at the knot point x_i. Consider the IFS (4) with

$$r_i(x) = \frac{p_i(x)}{q_i(x)} \equiv \frac{P_i(\theta)}{Q_i(\theta)} = \frac{A_i(1-\theta)^3 + B_i\theta(1-\theta)^2 + C_i\theta^2(1-\theta) + D_i\theta^3}{u_i(1-\theta)^2 + \theta(1-\theta)(\gamma_i + u_i + v_i) + v_i\theta^2},$$

where $\theta = \frac{x - x_1}{x_N - x_1}$, $x \in [x_1, x_N]$. Here $A_i,\ B_i,\ C_i,\ D_i$ are constants such that it satisfies the required condition for the existence for $\mathcal{C}^1$ FIF. $u_i,\ v_i$ and γ_i are parameters such that $u_i > 0,\ v_i > 0$ and $\gamma_i \geq 0$. These parameters will ensure the positivity of denominator of $r_i(x)$.

Let $\mathcal{F}^* := \{\ \phi \in \mathcal{C}^1(I)|\ \phi(x_1) = y_1$ and $\phi(x_N) = y_N\}$. Then $(\mathcal{F}^*, \rho^*)$ is a complete metric space, where ρ^* is the metric induced by norm $\|g\| = \|g\|_\infty + \|g^{(1)}\|_\infty$ on $\mathcal{C}^1(I)$. Define Read-Bajraktarević operator T on $\mathcal{F}^*$ as

$$T\phi(L_i(x)) = s_i\phi(x) + r_i(x),\ x \in I,\ i \in J. \tag{5}$$

Let s_i such that $|s_i| < a_i$ for all $i \in J$. The fixed point Φ of $\mathcal{T}^*$ satisfies the functional equation

$$\Phi(L_i(x)) = s_i\Phi(x) + r_i(x),\ x \in I,\ i \in J. \tag{6}$$

Here, Φ is a FIF and derivative $\Phi^{(1)}$ is also a FIF which satisfy the following functional equation

$$\Phi'(L_i(x)) = \frac{s_i\Phi'(x) + r_i^{(1)}(x)}{a_i},\ x \in I,\ i \in J. \tag{7}$$

The constants $A_i,\ B_i,\ C_i$ and D_i are evaluated based on Hermite conditions $\Phi(x_i) = y_i,\ \Phi(x_{i+1}) = y_{i+1},\ \Phi^{(1)}(x_i) = d_i$ and $\Phi^{(1)}(x_{i+1}) = d_{i+1}$ for $i \in J$. These conditions are equivalent [20] to the conditions on $F_i(x, y)$ for generating a $\mathcal{C}^1$-FIF given in Proposition 2. So, $\Phi^{(1)}$ is the fixed point of the operator $T_* : \mathcal{F}_* \to \mathcal{F}_*$ defined by

$$(T_*\phi^*)(L_i(x)) = \frac{s_i\phi^*(x) + r_i^{(1)}(x)}{a_i},\ x \in I,\ i \in J,$$

where $\mathcal{F}_* := \{\phi^* \in \mathcal{C}(I)|\phi^*(x_1) = d_1$ and $\phi^*(x_N) = d_N\}$ is endowed with the uniform norm metric. Let $h_i = x_{i+1} - x_i$.

Substituting $x = x_1$ in (6) implies,

$$A_i = u_i[y_i - s_i y_1].$$

Substituting $x = x_N$ in (6) implies

$$D_i = v_i[y_{i+1} - s_i y_N].$$

The condition $\Phi'(x_i) = d_i$ in (7) gives

$$B_i = u_i h_i d_i + y_i(2u_i + v_i + \gamma_i) - s_i[u_i(x_N - x_1)d_1 + y_1(2u_i + v_i + \gamma_i)].$$

The condition $\Phi'(x_{i+1}) = d_{i+1}$ in (7) gives

$$C_i = -v_i h_i d_{i+1} + y_{i+1}(u_i + 2v_i + \gamma_i) + s_i[v_i(x_N - x_1)d_N - y_N(u_i + 2v_i + \gamma_i)].$$

Hence, the FIF with three families of shape parameters is

$$\Phi(L_i(x)) = s_i\Phi(x) + \frac{P_i(\theta)}{Q_i(\theta)}, \tag{8}$$

where

$$\begin{aligned} P_i(\theta) &= (u_i[y_i - s_i y_1])(1-\theta)^3 + (v_i[y_{i+1} - s_i y_N])\theta^3 \\ &\quad + (u_i h_i d_i + y_i(2u_i + v_i + \gamma_i) - s_i[u_i(x_N - x_1)d_1 + y_1(2u_i + v_i + \gamma_i)])\theta(1-\theta)^2 \\ &\quad + (-v_i h_i d_{i+1} + y_{i+1}(u_i + 2v_i + \gamma_i) + s_i[v_i(x_N - x_1)d_N - y_N(u_i + 2v_i + \gamma_i)])\theta^2(1-\theta), \end{aligned}$$

$Q_i(\theta) = u_i(1-\theta)^2 + \theta(1-\theta)(\gamma_i + u_i + v_i) + v_i\theta^2$, $\theta = \frac{x - x_1}{x_N - x_1}$, $x \in [x_1, x_N]$.

Arithmetic mean method

In the above constructed FIF, derivative values are needed. In most of the cases, derivative values are not given. In that situation derivative values are approximated using some approximation methods. In this paper Arithmetic Mean Method [12, 24] is used to find derivative values. Let $\Delta_i = \frac{y_{i+1} - y_i}{h_i}$, $i \in J$. At interior knots x_i, $i = 2, 3 \ldots, N-1$, set

$$d_i = \begin{cases} 0 & \text{if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0, \\ \frac{h_i\Delta_{i-1} + h_{i-1}\Delta_i}{h_{i-1} + h_i} & \text{otherwise, } i = 2, 3, \ldots, N-1. \end{cases}$$

At end knots x_1 and x_N, set

$$d_1 = \begin{cases} 0 & \text{if } \Delta_1 = 0 \text{ or } sgn(D_1^*) \neq sgn(\Delta_1), \\ D_1^* = \Delta_1 + \frac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2} & \text{otherwise,} \end{cases}$$

$$d_N = \begin{cases} 0 & \text{if } \Delta_{N-1} = 0 \text{ or } sgn(D_N^*) \neq sgn(\Delta_{N-1}), \\ D_N^* = \Delta_{N-1} + \frac{(\Delta_{N-1} - \Delta_{N-2})h_{N-1}}{h_{N-1} + h_{N-2}} & \text{otherwise.} \end{cases}$$

Remark 1. Rewrite the rational cubic FIF Φ given in (8) in the following form

$$\Phi(L_i(x)) = s_i\Phi(x) + \{[(1-\theta)y_i + \theta y_{i+1} + \frac{R_i(\theta)}{Q_i(\theta)}] - s_i[(1-\theta)y_1 + \theta y_N + \frac{T_i(\theta)}{Q_i(\theta)}]\},$$

where

$$R_i(\theta) = h_i\theta(1-\theta)[(\Delta_i - d_{i+1})v_i\theta + (d_i - \Delta_i)u_i(1-\theta)],$$

$$\begin{aligned} T_i(\theta) = \theta(1-\theta)[&\{(y_N - y_1) - (x_N - x_1)d_N\}v_i\theta + \\ &\{(x_N - x_1)d_1 - (y_N - y_1)\}u_i(1-\theta)]. \end{aligned}$$

From this expression, it follows that if $\gamma_i \to \infty$ then Φ converges to the following affine FIF

$$\Phi(L_i(x)) = s_i\Phi(x) + (y_i - s_i y_1)(1-\theta) + (y_{i+1} - s_i y_N)\theta.$$

Also, if $\gamma_i \to \infty$ and $s_i \to 0$ then Φ converges to the straight line segment in the interval $[x_i, x_{i+1}]$.

Remark 2. If $s_i = 0$ for all $i \in J$, then the rational cubic FIF given in (8) reduces to the classical rational interpolation C as

$$C(x) = \frac{U_i(\varphi)}{V_i(\varphi)}, \tag{9}$$

where

$$\begin{aligned} U_i(\varphi) &= u_i y_i (1-\varphi)^3 + (u_i h_i d_i + y_i(2u_i + v_i + \gamma_i)\varphi(1-\varphi)^2 \\ &\quad + (-v_i h_i d_{i+1} + y_{i+1}(u_i + 2v_i + \gamma_i)\varphi^2(1-\varphi) + v_i y_{i+1}\varphi^3, \\ V_i(\varphi) &= u_i(1-\varphi)^2 + \varphi(1-\varphi)(\gamma_i + u_i + v_i) + v_i\varphi^2,\ \varphi = \frac{x - x_i}{x_{i+1} - x_i},\ x \in [x_i, x_{i+1}]. \end{aligned}$$

This show that if $s_i \to 0$, then graph of our rational FIF on $[x_i, x_{i+1}]$ approaches the graph of the classical rational cubic interpolant given in [18].

Remark 3. If $s_i = \gamma_i = 0,\ u_i = v_i = 1$ on each subinterval $I_i = [x_i, x_{i+1}],\ i \in J$, then the rational FIF (8) reduces to the standard cubic Hermite spline
$\Phi(x) = (2\varphi^3 - 3\varphi^2 + 1)y_i + (\varphi^3 - 2\varphi^2 + \varphi)h_i d_i + (-2\varphi^3 + 3\varphi^2)y_{i+1} + (\varphi^3 - \varphi^2)h_i d_{i+1}$,
where $\varphi = \frac{x-x_i}{x_{i+1}-x_i},\ x \in [x_i, x_{i+1}]$.

3.1. Convergence Analysis of $\mathcal{C}^1$-Rational FIF

In this section, an upper bound of the uniform error between an original function $S \in \mathcal{C}^2[x_1, x_N]$ and the rational cubic FIF Φ is determined. The effectiveness of the FIF Φ in the approximation of a function S is derived with the help of classical rational cubic spline.

Theorem 3. *Let Φ as given in (8) and C as given in (9) respectively, be the rational cubic FIF and classical rational cubic interpolant with respect to the data $\{(x_i, y_i),\ i = 1, 2, \ldots, N\}$ generated from an original function $S \in \mathcal{C}^2[x_1, x_N]$. Let $d_i,\ i = 1, 2, \ldots, N$ denotes the derivative values at the knots. Then*

$$\|S - \Phi\|_\infty \leq \frac{|s|_\infty([M + h\bar{M}] + [M^* + |I|M_*])}{1 - |s|_\infty} + \|S^{(2)}\|_\infty h^2 c^*,$$

where $M := \max\{|y_i|,\ i = 1, 2, \ldots, N\}$, $\bar{M} := \max\{|d_i|,\ i = 1, 2, \ldots, N\}$, $M^* := \max\{|y_1|, |y_N|\}$, $M_* := \max\{|d_1|, |d_N|\}$, $|I| := x_N - x_1$, $h = \max\{h_i,\ i \in J\}$, $|s|_\infty = \max\{|s_i|,\ i \in J\}$,
$w(u_i, v_i, \gamma_i, \varphi) = \frac{\varphi^2(1-\varphi)^2 u_i^2 h_i^2 + \varphi^4(1-\varphi)^2 h_i^2(u_i+v_i+\gamma_i)^2}{2V_i(\varphi)[u_i+\varphi(u_i+v_i+\gamma_i)]} + \frac{\varphi^2(1-\varphi)^4 h_i^2(u_i+v_i+\gamma_i)^2 + \varphi^2(1-\varphi)^2 v_i^2 h_i^2}{2V_i(\varphi)[v_i+(u_i+v_i+\gamma_i)(1-\varphi)]}$,
$c_i^* := \max\{w(u_i,\ v_i\ \gamma_i, \varphi) : 0 \leq \varphi \leq 1\}$, $i \in J$ *and* $c^* = \max\{c_i^* : i \in J\}$.

Proof. From (5), the Read-Bajraktarević operator $T_s^* : \mathcal{F}^* \to \mathcal{F}^*$ such that

$$T_s^*\phi(x) = s_i\phi(L_i^{-1}(x)) + \frac{p_i(L_i^{-1}(x), s_i)}{q_i(L_i^{-1}(x))},\ x \in I_i,\ i \in J,$$

where $p_i(x) \equiv P_i(\theta)$ and $q_i(x) \equiv Q_i(\theta)$ are as in (8). It is evident that Φ is the fixed point of operator T_s^* with $s \neq \mathbf{0}$. Also, classical rational cubic spline C is the fixed point of T_s^* with $s = \mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^{N-1}$.

Let $s = (s_1, s_2, \ldots, s_{N-1})$ be a scale vector such that $|s_i| < a_i$ for all $i \in J$ and with at least one $s_i \neq 0$. For $s \neq \mathbf{0}$, T_λ^* is a contraction map with contraction factor $|s|_\infty$. Hence

$$\|T_\lambda^*\Phi - T_\lambda^* C\|_\infty \leq |s|_\infty \|\Phi - C\|_\infty. \tag{10}$$

Also

$$\begin{aligned} |T_\lambda^* C(x) - T_{\mathbf{0}}^* C(x)| &= \Big| s_i\ C \circ L_i^{-1}(x) + \frac{p_i(L_i^{-1}(x),\ s_i)}{q_i(L_i^{-1}(x))} - \frac{p_i(L_i^{-1}(x),\ 0)}{q_i(L_i^{-1}(x))} \Big|, \\ &\leq |s_i| \|C\|_\infty + \frac{|p_i(L_i^{-1}(x),\ s_i) - p_i(L_i^{-1}(x),\ 0)|}{q_i(L_i^{-1}(x))}. \end{aligned} \tag{11}$$

Using the mean value theorem of functions of several variables, there exist $\beta = (\beta_1, \beta_2, \dots, \beta_{N-1})$ such that $|\beta_i| < |s_i|$ and

$$p_i(L_i^{-1}(x),\ s_i) - p_i(L_i^{-1}(x),\ 0) = \frac{\partial p_i(L_i^{-1}(x),\ \beta_i)}{\partial s_i} s_i. \tag{12}$$

By (11) and (12),

$$|T_\lambda^* C(x) - T_{\mathbf{0}}^* C(x)| \le |s_i| \left(\|C\|_\infty + \left| \frac{\partial \left(\frac{p_i(L_i^{-1}(x), \beta_i)}{q_i(L_i^{-1}(x)} \right)}{\partial s_i} \right| \right). \tag{13}$$

Now let us concentrate to find the bound for the right hand side of the Equation (13). The classical rational cubic interpolant C can be written as

$$C(x) = w_1(u_i, v_i, \gamma_i, \varphi) y_i + w_2(u_i, v_i, \gamma_i, \varphi) y_{i+1} + w_3(u_i, v_i, \gamma_i, \varphi) d_i - w_4(u_i, v_i, \gamma_i, \varphi) d_{i+1}, \tag{14}$$

where
$w_1(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{u_i(1-\varphi)^3 + (2u_i + v_i + \gamma_i)\varphi(1-\varphi)^2}{V_i(\varphi)}$, $w_2(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{(u_i + 2v_i + \gamma_i)\varphi^2(1-\varphi) + v_i\varphi^3}{V_i(\varphi)}$,
$w_3(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{u_i h_i \varphi(1-\varphi)^2}{V_i(\varphi)}$, $w_4(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{v_i h_i \varphi^2(1-\varphi)}{V_i(\varphi)}$.
Now

$$w_1(u_i,\ v_i,\ \gamma_i,\ \varphi) + w_2(u_i,\ v_i,\ \gamma_i,\ \varphi) = 1.$$

Also

$$\begin{aligned} w_3(u_i,\ v_i,\ \gamma_i,\ \varphi) + w_4(u_i,\ v_i,\ \gamma_i,\ \varphi) &= \frac{u_i h_i \varphi(1-\varphi)^2 + v_i h_i \varphi^2(1-\varphi)}{u_i(1-\varphi)^2 + \varphi(1-\varphi)(\gamma_i + u_i + v_i) + v_i\varphi^2}, \\ &\le \frac{u_i h_i \varphi(1-\varphi)^2 + v_i h_i \varphi^2(1-\varphi)}{u_i(1-\varphi)^2 + v_i\varphi^2}. \end{aligned}$$

Since $u_i(1-\varphi)^2 + v_i\varphi^2 \ge \max\{u_i(1-\varphi)^2,\ v_i\varphi^2\}$,

$$\begin{aligned} w_3(u_i,\ v_i,\ \gamma_i,\ \varphi) + w_4(u_i,\ v_i,\ \gamma_i,\ \varphi) &\le \frac{u_i h_i \varphi(1-\varphi)^2}{u_i(1-\varphi)^2} + \frac{v_i h_i \varphi^2(1-\varphi)}{v_i\varphi^2}, \\ &= h_i. \end{aligned}$$

From (14),

$$\begin{aligned} |C(x)| &\le \max\{|y_i|, |y_{i+1}|\} + h_i \max\{|d_i|, |d_{i+1}|\}, \\ &= M_i + h_i \bar{M}_i, \end{aligned}$$

where $M_i = \max\{|y_i|, |y_{i+1}|\}$ and $\bar{M}_i = \max\{|d_i|, |d_{i+1}|\}$.
Thus

$$\|C\|_\infty \le M + h\bar{M}. \tag{15}$$

Now

$$\begin{aligned} \frac{\partial \left(\frac{p_i(L_i^{-1}(x), s_i)}{q_i(L_i^{-1}(x)} \right)}{\partial s_i} &= -w_1^*(u_i, v_i, \gamma_i, \varphi) y_1 - w_2^*(u_i, v_i, \gamma_i, \varphi) y_N - w_3^*(u_i, v_i, \gamma_i, \varphi) d_1 \\ &\quad + w_4^*(u_i, v_i, \gamma_i, \varphi) d_N, \end{aligned}$$

$w_1^*(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{u_i(1-\varphi)^3 + (2u_i + v_i + \gamma_i)\varphi(1-\varphi)^2}{V_i(\varphi)}$, $w_2^*(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{(u_i + 2v_i + \gamma_i)\varphi^2(1-\varphi) + v_i\varphi^3}{V_i(\varphi)}$,

$w_3^*(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{u_i(x_N-x_1)\varphi(1-\varphi)^2}{V_i(\varphi)}$, $w_4^*(u_i,\ v_i,\ \gamma_i,\ \varphi) = \frac{v_i(x_N-x_1)\varphi^2(1-\varphi)}{V_i(\varphi)}$.

$$\left|\frac{\partial\left(\frac{p_i(L_i^{-1}(x),s_i)}{q_i(L_i^{-1}(x)}\right)}{\partial s_i}\right| \leq |w_1^*(u_i,v_i,\gamma_i,\varphi)y_1|+|w_2^*(u_i,v_i,\gamma_i,\varphi)y_N|+|w_3^*(u_i,v_i,\gamma_i,\varphi)d_1| + |w_4^*(u_i,v_i,\gamma_i,\varphi)d_N|.$$

By using similar procedure for finding bound for $\|C\|_\infty$,

$$\left|\frac{\partial\left(\frac{p_i(L_i^{-1}(x),s_i)}{q_i(L_i^{-1}(x)}\right)}{\partial s_i}\right| \leq M^* + |I|M_*. \tag{16}$$

From (13),(15) and (16),

$$|T_s^*C(x) - T_{\mathbf{0}}^*C(x)| \leq |s|_\infty([M + h\bar{M}] + [M^* + |I|M_*]),$$

and hence

$$\|T_s^*C - T_{\mathbf{0}}^*C\|_\infty \leq |s|_\infty([M + h\bar{M}] + [M^* + |I|M_*]). \tag{17}$$

Using (10) and (17)

$$\|\Phi - C\|_\infty = \|T_s^*\Phi - T_{\mathbf{0}}^*C\|_\infty \leq \|T_s^*\Phi - T_s^*C\|_\infty + \|T_s^*C - T_{\mathbf{0}}^*C\|_\infty,$$

which implies

$$\|\Phi - C\|_\infty \leq \frac{|s|_\infty([M + h\bar{M}] + [M^* + |I|M_*])}{1 - |s|_\infty}. \tag{18}$$

Now, let us concentrate the uniform error bound between the original function S and it's classical rational cubic spline C. Assume that $S \in \mathcal{C}^2[x_1, x_N]$. For $x \in [x_i, x_{i+1}]$, let us consider the error function $E(S;x) = S(x) - C(x)$ as a linear functional which operates on S. Then using Peano Kernel Theorem [27]:

$$L[S] = E(S;x) = S(x) - C(x) = \int_{x_i}^{x_{i+1}} S^{(2)}(\tau)L_x[(x-\tau)_+]d\tau, \tag{19}$$

where

$$L_x[(x-\tau)_+] = \begin{cases} r(\tau,x) & \text{if } x_i < \tau < x, \\ s(\tau,x) & \text{if } x < \tau < x_{i+1}, \end{cases}$$

Here L_x is used to emphasize that the functional L is applied to the truncated power function $(x-\tau)_+^n$ considered as a function of x. Also

$$(x-\tau)_+^n := \begin{cases} (x-\tau)^n & \text{if } \tau < x, \\ 0 \quad \text{if } \tau > x, \end{cases}$$

$$r(\tau,x) = (x-\tau) - \frac{[(u_i + 2v_i + \gamma_i)(x_{i+1}-\tau) - v_ih_i]\varphi^2(1-\varphi) + v_i(x_{i+1}-\tau)\varphi^3}{u_i(1-\varphi)^2 + \varphi(1-\varphi)(\gamma_i + u_i + v_i) + v_i\varphi^2}, \tag{20}$$

$$s(\tau,x) = -\frac{[(u_i + 2v_i + \gamma_i)(x_{i+1}-\tau) - v_ih_i]\varphi^2(1-\varphi) + v_i(x_{i+1}-\tau)\varphi^3}{u_i(1-\varphi)^2 + \varphi(1-\varphi)(\gamma_i + u_i + v_i) + v_i\varphi^2}. \tag{21}$$

Therefore,

$$|S(x) - C(x)| \leq \|S^{(2)}\|_\infty \int_{x_i}^{x_{i+1}} |L_x[(x-\tau)_+]|d\tau. \tag{22}$$

The integral involved in (22) can be expressed as

$$\int_{x_i}^{x_{i+1}} |L_x[(x-\tau)_+]| d\tau = \int_{x_i}^{x} |r(\tau,x)| d\tau + \int_{x}^{x_{i+1}} |s(\tau,x)| d\tau. \tag{23}$$

In order to calculate the integral given in (23), the roots of $r(\tau,x)$ and $s(\tau,x)$ are calculated
Roots of $r(x,x)$, $s(x,x)$ in $[0,1]$ are $\varphi=0$ and $\varphi=1$,
Roots of $r(\tau,x)$ is $\tau^* = x - \frac{\varphi^2 h_i(u_i+v_i+\gamma_i)}{u_i+\varphi(u_i+v_i+\gamma_i)}$, and $\tau^* \in [x_i, x]$,
Roots of $s(\tau,x)$ is $\tau_* = x_{i+1} - \frac{v_i h_i(1-\varphi)}{v_i+(u_i+v_i+\gamma_i)(1-\varphi)}$, and $\tau_* \in [x, x_{i+1}]$.
The expression given in (20) and (21) can be simplifies as
$r(\tau,x) = \frac{[(1-\varphi)^2 u_i + \varphi(1-\varphi)^2(u_i+v_i+\gamma_i)](\tau^*-\tau)}{V_i(\varphi)}$,
$s(\tau,x) = \frac{[(u_i+v_i+\gamma_i)\varphi^2(1-\varphi)+\varphi^2 v_i](\tau-\tau_*)}{V_i(\varphi)}$,
respectively, where $V_i(\varphi)$ given in (9). Now

$$\begin{aligned}\int_{x_i}^{x} |r(\tau,x)| d\tau &= \int_{x_i}^{\tau^*} r(\tau,x)d\tau - \int_{\tau^*}^{x} r(\tau,x)d\tau,\\ &= \frac{\varphi^2(1-\varphi)^2 u_i^2 h_i^2}{2V_i(\varphi)[u_i+\varphi(u_i+v_i+\gamma_i)]} + \frac{\varphi^4(1-\varphi)^2 h_i^2 (u_i+v_i+\gamma_i)^2}{2V_i(\varphi)[u_i+\varphi(u_i+v_i+\gamma_i)]},\end{aligned}$$

$$\begin{aligned}\int_{x}^{x_{i+1}} |s(\tau,x)| d\tau &= -\int_{x}^{\tau_*} s(\tau,x)d\tau + \int_{\tau_*}^{x_{i+1}} s(\tau,x)d\tau,\\ &= \frac{\varphi^2(1-\varphi)^4 h_i^2(u_i+v_i+\gamma_i)^2}{2V_i(\varphi)[v_i+(u_i+v_i+\gamma_i)(1-\varphi)]} + \frac{\varphi^2(1-\varphi)^2 v_i^2 h_i^2}{2V_i(\varphi)[v_i+(u_i+v_i+\gamma_i)(1-\varphi)]}.\end{aligned}$$

By Equations (22) and (23),
$|S(x)-C(x)| \le c_i^* h_i^2 \|S^{(2)}\|_\infty, \quad$ where $c_i^* := \max\{w(u_i,v_i,\gamma_i,\varphi) : 0 \le \varphi \le 1\}$,
$w(u_i,v_i,\gamma_i,\varphi) = \frac{\varphi^2(1-\varphi)^2 u_i^2 h_i^2 + \varphi^4(1-\varphi)^2 h_i^2 (u_i+v_i+\gamma_i)^2}{2V_i(\varphi)[u_i+\varphi(u_i+v_i+\gamma_i)]} + \frac{\varphi^2(1-\varphi)^4 h_i^2(u_i+v_i+\gamma_i)^2 + \varphi^2(1-\varphi)^2 v_i^2 h_i^2}{2V_i(\varphi)[v_i+(u_i+v_i+\gamma_i)(1-\varphi)]}$.
Since the above inequality is true for $x \in [x_i, x_{i+1}]$, $i \in J$, the desired error bound is given by

$$\|S - C\|_\infty \le c^* h^2 \|S^{(2)}\|_\infty, \quad \text{where } c^* := \max\{c_i^* : i \in J\}. \tag{24}$$

Using (18) and (24) with the following inequality

$$\|S - \Phi\|_\infty \le \|S - C\|_\infty + \|C - \Phi\|_\infty,$$

gives our required bound for $\|S - \Phi\|$. □

Remark 4. In the above theorem, Assumed that $|s_i| < a_i = \frac{h_i}{x_N - x_1}$ for all $i \in J$. From this, when $h \to 0$ the FIF Φ (8) converges uniformly to the original function S. Also by assuming $|s_i| < a_i^2$, FIF Φ has same order of convergence as the classical rational interpolant C.

4. Shape Preserving Interpolation

In this section shape preserving aspects of $\mathcal{C}^1$ FIF (8) are investigated. There are different kinds of shape preserving aspects namely positivity, monotonicity, convexity etc. Choosing random scaling factors and shape parameters may not preserves these shapes. So sufficient conditions on scaling factors and shape parameters are derived to preserve these shape properties.

Proposition 4. *[28] The polynomial $\varrho = \varrho(t) = \zeta\upsilon^3 + \eta\upsilon^2 + \vartheta\upsilon + \kappa$ is nonnegative for all $\upsilon \ge 0$ if and only if $(\zeta,\eta,\vartheta,\kappa) \in T_1 \cup T_2$, where*
$T_1 = \{(\zeta,\eta,\vartheta,\kappa) : \zeta \ge 0,\ \eta \ge 0,\ \vartheta \ge 0 \text{ and } \kappa \ge 0\}$,
$T_2 = \{(\zeta,\eta,\vartheta,\kappa) : \zeta \ge 0,\ \kappa \ge 0,\ 4\zeta\vartheta^3 + 4\kappa\eta^3 + 27\zeta^2\kappa^3 + 27\zeta^2\kappa^2 - 18\zeta\eta\vartheta\kappa - \eta^2\vartheta^2 \ge 0\}$.

4.1. Positivity

Suppose $\{(x_i, y_i) : i = 1, 2, \dots, N\}$ be a data set such that $y_i > 0$. The following theorem gives sufficient conditions on scaling factors and shape parameters so that FIF (8) will preserve positivity.

Theorem 5. *Let $\{(x_i, y_i) : i \in J\}$ be a data set such that $y_i > 0$. Let Φ be the corresponding FIF defined in (8). Let d_i be the derivative at the knot x_i. Then sufficient conditions on scaling factors and shape parameters on each interval $I_i = [x_i, x_{i+1}]$, so that Φ preserve positivity are*

$$0 \le s_i < \min\Big\{a_i,\ \frac{y_i}{y_1},\ \frac{y_{i+1}}{y_N}\Big\},$$

$$u_i > 0,\ v_i > 0 \text{ and } \gamma_i > \max\Big\{0, \gamma^*_{1i}\ \gamma^*_{2i}\Big\},$$

where

$$\gamma^*_{1i} = \frac{-u_i h_i d_i - (2u_i + v_i)y_i + s_i[u_i d_1(x_N - x_1) + (2u_i + v_i)y_1]}{y_i - s_i y_1},$$
$$\gamma^*_{2i} = \frac{v_i h_i d_{i+1} - (u_i + 2v_i)y_{i+1} - s_i[v_i d_N(x_N - x_1) - (u_i + 2v_i)y_N]}{y_{i+1} - s_i y_N}.$$

Proof. Assume that $s_i \ge 0$ for all $i \in J$. Then $\Phi(L_i(x)) > 0$ if $\frac{P_i(\theta)}{Q_i(\theta)} > 0$. The parameters $u_i > 0,\ v_i > 0$ and $\gamma_i \ge 0$ for all $i \in J$, gives denominator $Q_i(\theta) > 0$. The positivity of the function $\frac{P_i(\theta)}{Q_i(\theta)}$ depends on the numerator $P_i(\theta)$. Now,

$$\begin{aligned} P_i(\theta) &= A_i(1-\theta)^3 + B_i\theta(1-\theta)^2 + C_i\theta^2(1-\theta) + D_i\theta^3, \\ &= t_{1i}\theta^3 + t_{2i}\theta^2 + t_{3i}\theta + t_{4i}, \end{aligned} \tag{25}$$

where
$t_{1i} = (u_i + v_i + \gamma_i)(y_i - s_i y_1) + (-u_i - v_i - \gamma_i)(y_{i+1} - s_i y_N) + u_i h_i(d_i - \frac{s_i}{a_i}d_1) + v_i h_i(d_{i+1} - \frac{s_i}{a_i}d_N)$,
$t_{2i} = (-u_i - 2v_i - 2\gamma_i)(y_i - s_i y_1) + (u_i + 2v_i + \gamma_i)(y_{i+1} - s_i y_N) - 2u_i h_i(d_i - \frac{s_i}{a_i}d_1) - v_i h_i(d_{i+1} - \frac{s_i}{a_i}d_N)$,
$t_{3i} = (-u_i + v_i + \gamma_i)(y_i - s_i y_1) + u_i h_i(d_i - \frac{s_i}{a_i}d_1)$,
$t_{4i} = u_i(y_i - s_i y_1)$.

By substituting $\theta = \frac{\upsilon}{\upsilon+1}$ in (25), $P_i(\theta) > 0$ for all $\theta \in [0,1]$ is equivalent to say
$\Theta_i(\upsilon) = t^*_{1i}\upsilon^3 + t^*_{2i}\upsilon^2 + t^*_{3i}\upsilon + t^*_{4i} > 0$ for all $\upsilon \ge 0$,
where
$t^*_{1i} = t_{1i} + t_{2i} + t_{3i} + t_{4i} = v_i(y_{i+1} - s_i y_N)$,
$t^*_{2i} = t_{2i} + 2t_{3i} + 3t_{4i} = (u_i + 2v_i + \gamma_i)(y_{i+1} - s_i y_N) - v_i h_i(d_{i+1} - \frac{s_i}{a_i}d_N)$,
$t^*_{3i} = t_{3i} + 3t_{4i} = (2u_i + v_i + \gamma_i)(y_i - s_i y_1) + u_i h_i(d_i - \frac{s_i}{a_i}d_1)$,
$t^*_{4i} = t_{4i} = u_i(y_i - s_i y_1)$.

From proposition 4, $\Theta_i(\upsilon) > 0$ for all $\upsilon \ge 0$ if and only if $(t^*_{1i},\ t^*_{2i},\ t^*_{3i},\ t^*_{4i}) \in T_1 \cup T_2$.
Conditions given in the region T_2 is omitted due to complication in calculation. Since conditions given in the region T_1 are computationally economical, the region T_1 is used to get positivity of Θ_i. Now,

$$t^*_{1i} > 0 \Longleftrightarrow v_i(y_{i+1} - s_i y_N) > 0 \Longleftrightarrow s_i < \frac{y_{i+1}}{y_N},$$
$$t^*_{4i} > 0 \Longleftrightarrow u_i(y_i - s_i y_1) > 0 \Longleftrightarrow s_i < \frac{y_i}{y_1},$$

$$
\begin{aligned}
t_{2i}^* > 0 &\Longleftrightarrow (u_i + 2v_i + \gamma_i)(y_{i+1} - s_i y_N) - v_i h_i (d_{i+1} - \frac{s_i}{a_i} d_N) > 0,\\
&\Longleftrightarrow \gamma_i > \frac{v_i h_i d_{i+1} - (u_i + 2v_i) y_{i+1} - s_i[v_i d_N (x_N - x_1) - (u_i + 2v_i) y_N]}{y_{i+1} - s_i y_N},\\
t_{3i}^* > 0 &\Longleftrightarrow (2u_i + v_i + \gamma_i)(y_i - s_i y_1) + u_i h_i (d_i - \frac{s_i}{a_i} d_1) > 0,\\
&\Longleftrightarrow \gamma_i > \frac{-u_i h_i d_i - (2u_i + v_i) y_i + s_i[u_i d_1 (x_N - x_1) + (2u_i + v_i) y_1]}{y_i - s_i y_1}.
\end{aligned}
$$

So $t_{1i}^*,\ t_{2i}^*,\ t_{3i}^*$and t_{4i}^* lies in the region T_1 if the scaling factors and shape parameters satisfy the conditions given in the statement of Theorem 5. □

Remark 5. When $s_i = 0$ for all $i \in J$, the sufficient conditions in the theorem 5 reduces to

$$
u_i > 0,\ v_i > 0,\ \text{and}\ \gamma_i > \max\Big\{0,\ \frac{-u_i h_i d_i - (2u_i + v_i) y_i}{y_i},\ \frac{v_i h_i d_{i+1} - (u_i + 2v_i) y_{i+1}}{y_{i+1}}\Big\}.
$$

This gives the sufficient conditions for the classical rational cubic spline (9).

4.2. Monotonicity

Let $\{(x_i,\ y_i) : i = 1, 2, \ldots, N\}$ be a monotonic data. Without loss of generality, assume that data is monotonically increasing, i.e., $y_1 \le y_2 \le \ldots \le y_N$. Then $\Delta_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \ge 0,\ i \in J$. The aim of this section is to find a suitable parameters such that the FIF (8) preserves monotonicity.

$$
\Phi^{(1)}(L_i(x)) = \frac{s_i \Phi^{(1)}(x)}{a_i} + \frac{\Psi_i(\theta)}{(Q_i(\theta))^2},
$$

where $\Psi_i(\theta) = \sum_{k=1}^{5} A_{k,i} \theta^{k-1} (1 - \theta)^{5-k}$,

$$
\begin{aligned}
A_{1,i} &= u_i^2 d_i^*,\\
A_{2,i} &= 2u_i \{(u_i + 2v_i + \gamma_i)\Delta_i^* - v_i d_{i+1}^*\},\\
A_{3,i} &= A_{2,i} + A_{4,i} - (A_{1,i} + A_{5,i}) + \gamma_i (u_i + v_i + \gamma_i)\Delta_i^* - \gamma_i (u_i d_i^* + v_i d_{i+1}^*),\\
A_{4,i} &= 2v_i \{(2u_i + v_i + \gamma_i)\Delta_i^* - u_i d_i^*\},\\
A_{5,i} &= v_i^2 d_{i+1}^*,
\end{aligned}
$$

$d_i^* = d_i - \frac{s_i d_1}{a_i},\ d_{i+1}^* = d_{i+1} - \frac{s_i d_N}{a_i}$ and $\Delta_i^* = \Delta_i - s_i \frac{y_N - y_1}{h_i}$.

Theorem 6. *Let $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$ be a monotonically increasing data, i.e., $y_1 \le y_2 \le \ldots \le y_N$. Let Φ be the corresponding rational cubic FIF defined in (8). Let d_i be the derivative at the knot point x_i. Let derivatives satisfy the necessary conditions for monotonicity, namely*

$$
sgn(d_i) = sgn(d_{i+1}) = sgn(\Delta_i) \quad \text{for } \Delta_i \ne 0.
$$

Then sufficient conditions on scaling factors and shape parameters so that Φ will preserve monotonic on the interval I are

$$
0 \le s_i < \left\{a_i,\ \frac{a_i d_i}{d_1},\ \frac{a_i d_{i+1}}{d_N},\ \frac{y_{i+1} - y_i}{y_N - y_1}\right\},
$$

$$
u_i > 0,\ v_i > 0 \text{ and } \gamma_i \ge \max\Big\{0, \frac{u_i d_i^* + v_i d_{i+1}^* - \Delta_i^*(u_i + v_i)}{\Delta_i^*}\Big\},
$$

for all $i \in J$.

Proof. Let $\{(x_i,\ y_i), i = 1, 2 \ldots, N\}$ be an monotonically increasing data. From single variable calculus, it is obvious that a function Φ is monotonic if $\Phi^{(1)}(x) \geq 0$, for $x \in [x_1, x_N]$. Assume that $s_i \geq 0,\ i \in J$. For each node $x_j,\ j = 1, 2, \ldots, N$

$$\Phi'(L_i(x_j)) = \frac{s_i \Phi'(x_j)}{a_i} + \frac{\Psi_i(\theta_j)}{(Q_i(\theta_j))^2},\ \theta_j = \frac{x_j - x_1}{x_N - x_1}. \tag{26}$$

From (26), $\Phi'(L_i(x_j)) \geq 0$ if $\frac{\Psi_i(\theta_j)}{(Q_i(\theta_j))^2} \geq 0$. It is obvious that $(Q_i(\theta_j))^2 \geq 0$ for all $\theta \in [0,\ 1]$. Then sufficient conditions for $\Psi_i(\theta_j) \geq 0$, $\theta \in [0, 1]$ are $A_{k,i} \geq 0$, $k = 1, 2, \ldots, 5$. Now

$$A_{1,i} \geq\ 0 \Leftrightarrow s_i \leq \frac{a_i d_i}{d_1}.$$

Similarly

$$A_{5,i} \geq\ 0 \Leftrightarrow s_i \leq \frac{a_i d_{i+1}}{d_N}.$$

Let $0 \leq s_i < \{\frac{a_i d_i}{d_1},\ \frac{a_i d_{i+1}}{d_N},\ \frac{y_{i+1} - y_i}{y_N - y_1}\}$.
Then $A_{2,i} \geq 0$ if

$$\gamma_i \geq \frac{d^*_{i+1} v_i - \Delta^*_i (u_i + 2v_i)}{\Delta^*_i}.$$

$A_{4,i} \geq 0$ if

$$\gamma_i \geq \frac{d^*_i u_i - \Delta^*_i (2u_i + v_i)}{\Delta^*_i}.$$

$A_{3,i}$ can be written as

$$\begin{aligned} A_{3,i} =& u_i\{(u_i + 2v_i + \gamma_i)\Delta^*_i - v_i d^*_{i+1}\} + v_i\{(2u_i + v_i + \gamma_i)\Delta^*_i - u_i d^*_i\} \\ &+ u_i\{(u_i + v_i + \gamma_i)\Delta^*_i - u_i d^*_i - v_i d^*_{i+1}\} + v_i\{(u_i + v_i + \gamma_i)\Delta^*_i - u_i d^*_i - v_i d^*_{i+1}\} \\ &+ \gamma_i\{(u_i + v_i + \gamma_i)\Delta^*_i - u_i d^*_i - v_i d^*_{i+1}\} + 2u_i v_i \Delta^*_i. \end{aligned}$$

Also

$$\frac{u_i d^*_i + v_i d^*_{i+1} - \Delta^*_i (u_i + v_i)}{\Delta^*_i} > \frac{d^*_{i+1} v_i - \Delta^*_i (u_i + 2v_i)}{\Delta^*_i},$$
$$\frac{u_i d^*_i + v_i d^*_{i+1} - \Delta^*_i (u_i + v_i)}{\Delta^*_i} > \frac{d^*_i u_i - \Delta^*_i (2u_i + v_i)}{\Delta^*_i}.$$

So $A_{3,i} \geq 0$ if

$$\gamma_i \geq \max\Big\{0, \frac{u_i d^*_i + v_i d^*_{i+1} - \Delta^*_i (u_i + v_i)}{\Delta^*_i}\Big\}.$$

From the above discussion, it is evident that $\Phi^{(1)}(L_i(x_j)) \geq 0$ for all $i \in J$, $j = 1, 2, \ldots, N$, if the scaling factors and shape parameters satisfies the sufficient conditions given in the statement of Theorem 6. $\{I; L_i(x) : i \in J\}$ is an IFS and $[x_1,\ x_N]$ is an its attractor. Since fractal interpolation function is defined as recursive structure, $\Phi^{(1)}(L_i(x_j)) \geq 0$ for all $i \in J$ and for every knot x_j imply that $\Phi^{(1)}(x) \geq 0$ for all $x \in [x_1,\ x_N]$. □

4.3. Convexity

A data $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$ is said to be convex if

$$\Delta_1 \leq \Delta_2 \leq \ldots \leq \Delta_{i-1} \leq \Delta_i \leq \ldots \leq \Delta_{N-1}.$$

Assume that data $\{(x_i, y_i) : i = 1, 2, ..., N\}$ is strictly convex, i.e., $\Delta_1 < \Delta_2 < ... < \Delta_{N-1}$. To avoid the possibility of straight line segments, assume that $d_1 < \Delta_1 < ... < d_i < \Delta_i < d_{i+1} < ... < \Delta_{N-1} < d_N$. Choosing arbitrary scaling factors and shape parameters may not give convex interpolant. The aim of this section is to find sufficient conditions on scaling factors and shape parameters so that rational FIF (8) preserves convexity.

Theorem 7. *Suppose* $\{(x_i, y_i) : i = 1, 2, ..., N\}$ *be a strictly convex data. Let the derivatives satisfy* $d_1 < \Delta_1 < ... < d_i < \Delta_i < d_{i+1} < ...\Delta_{N-1} < d_N$*. Let* Φ *be the corresponding rational cubic FIF defined in (8). Then the sufficient conditions for the scaling factors and shape parameters so that* Φ *preserve convexity on the interval* I *are*

$$0 \le s_i < \min\Big\{a_i^2,\ \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)},\ \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)}\Big\},$$

$$u_i > 0,\ v_i > 0 \text{ and } \gamma_i > \max\Big\{\frac{v_i(d^*_{i+1} - \Delta^*_i)}{(\Delta^*_i - d^*_i)},\ \frac{u_i(\Delta^*_i - d^*_i)}{(d^*_{i+1} - \Delta^*_i)}\Big\}.$$

Proof. From calculus, Φ is convex if $\Phi^{(2)}(x^+)$ or $\Phi^{(2)}(x^-)$ is exist and nonnegative for all $x \in (x_1, x_N)$ [21, 29]. So informally

$$\Phi^{(2)}(L_i(x)) = \frac{s_i\Phi^{(2)}(x)}{a_i^2} + \frac{\Psi^*_i(\theta)}{h_i(Q_i(\theta))^3}, \tag{27}$$

where $\Psi^*_i(\theta) = \sum_{k=1}^{6} B_{k,i}\theta^{k-1}(1-\theta)^{6-k}$,

$$\begin{aligned}
B_{1,i} &= 2u_i^3(\Delta^*_i - d^*_i) + 2u_i^2v_i(\Delta^*_i - d^*_i) + 2u_i^2[\gamma_i(\Delta^*_i - d^*_i) - v_i(d^*_{i+1} - \Delta^*_i)],\\
B_{2,i} &= 2B_{1,i} + 6u_i^2v_i(\Delta^*_i - d^*_i),\\
B_{3,i} &= B_{1,i} + 12u_i^2v_i(\Delta^*_i - d^*_i) + 6u_iv_i^2(d^*_{i+1} - \Delta^*_i),\\
B_{4,i} &= B_{6,i} + 12u_iv_i^2(d^*_{i+1} - \Delta^*_i) + 6u_i^2v_i(\Delta^*_i - d^*_i),\\
B_{5,i} &= 2B_{6,i} + 6u_iv_i^2(d^*_{i+1} - \Delta^*_i),\\
B_{6,i} &= 2v_i^3(d^*_{i+1} - \Delta^*_i) + 2v_i^2u_i(d^*_{i+1} - \Delta^*_i) + 2v_i^2[\gamma_i(d^*_{i+1} - \Delta^*_i) - u_i(\Delta^*_i - d^*_i)],
\end{aligned}$$

$d^*_i = d_i - \frac{s_id_1}{a_i}$, $d^*_{i+1} = d_{i+1} - \frac{s_id_N}{a_i}$, $\Delta^*_i = \Delta_i - s_i\frac{y_N - y_1}{h_i}$.

Now

$$\Phi^{(2)}(x_1^+) = \frac{B_{1,1}}{h_1u_1^3}\Big[1 - \frac{s_1}{a_1^2}\Big]^{-1}, \tag{28}$$

$$\Phi^{(2)}(x_N^-) = \frac{B_{6,N-1}}{h_{N-1}v_{N-1}^3}\Big[1 - \frac{s_{N-1}}{a_{N-1}^2}\Big]^{-1}, \tag{29}$$

$$\Phi^{(2)}(x_n^+) = \frac{s_n}{a_n^2}\Phi^{(2)}(x_1^+) + \frac{B_{1,n}}{u_n^3h_n}, \quad n = 2, 3, \ldots, N-1. \tag{30}$$

Let $0 \le s_i < a_i^2$, $i \in J$. From equations (28), (29) and (30) it is obvious that if $B_{1,i} \ge 0$ $(i \in J)$ and $B_{6,N-1} \ge 0$, then the right-handed second derivatives at the knots x_i, $i \in J$, and the left-handed second derivative at x_N are nonnegative. For a knot points x_n, $n \in J$

$$\Phi^{(2)}(L_i(x_n)^+) = \frac{s_i\Phi^{(2)}(x_n^+)}{a_i^2} + R_i(x_n),$$

where $R_i(x) = \frac{\Psi^*_i(\theta)}{h_i(Q_i(\theta))^3}$.

Assuming $B_{1,i} \ge 0$, $i \in J$, then $\Phi^{(2)}(L_i(x_n)^+) \ge 0$ if $R_i(x_n) \ge 0$. Note that $R_i(x_n) \ge 0$ if the coefficients $B_{j,i} \ge 0$ for $j = 1, 2, \ldots, 6$. Applying Three Chords Lemma [30], for a strict convex interpolant, the end point derivatives should necessarily satisfy $d_1 < \frac{y_N - y_1}{x_N - x_1} < d_N$. Because of this, we can get condition on scaling factor s_i such that the quantities, $\Delta^*_i - d^*_i > 0$ and $d^*_{i+1} - \Delta^*_i > 0$.

That is

$$\Big(\Delta_i - s_i\frac{y_N - y_1}{h_i}\Big) - \Big(d_i - \frac{s_id_1}{a_i}\Big) > 0 \text{ if } s_i < \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)},$$

$$\left(d_{i+1} - \frac{s_i d_N}{a_i}\right) - \left(\Delta_i - s_i \frac{y_N - y_1}{h_i}\right) > 0 \text{ if } s_i < \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)}.$$

Let

$$0 \leq s_i < \min\left\{a_i^2,\ \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)},\ \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)}\right\},\ i \in J. \tag{31}$$

The assumption on scaling factors given in (31) and if $\gamma_i \geq \frac{v_i(d^*_{i+1} - \Delta^*_i)}{(\Delta^*_i - d^*_i)}$ ensures $B_{1,i} \geq 0$, $B_{2,i} \geq 0$ and $B_{3,i} \geq 0$. Similarly by the assumption on scaling factors given in (31) and if $\gamma_i \geq \frac{u_i(\Delta^*_i - d^*_i)}{(d^*_{i+1} - \Delta^*_i)}$ ensures $B_{4,i} \geq 0$, $B_{5,i} \geq 0$ and $B_{6,i} \geq 0$.

Thus the condition on scaling factors and shape parameters given Theorem 7 statement ensures $B_{j,i} \geq 0$ for all $j = 1, 2, \ldots, 6$. This in turn ensures that nonnegative of $\Phi^{(2)}(L_i(x_n)^+)$ for $i, n \in J$, and $\Phi(x_N^-)$. The non negativity of $\Phi^{(2)}(x^+)$ or $\Phi^{(2)}(x^-)$ is follows from non negativity of $\Phi^{(2)}(L_i(x_n)^+)$ for $i, n \in J$, and $\Phi(x_N^-)$. Hence restrictions on scaling factors and shape parameters given in statement of the Theorem 7 ensure convexity of Φ. □

Remark 6. Suppose $\Delta_i - d_i = 0$ or $d_{i+1} - \Delta_i = 0$, then we take $s_i = 0$, $d_i = d_{i+1} = \Delta_i$. In this case, Φ become straight line $\Phi(L_i(x)) = y_i(1-\theta) + y_{i+1}\theta$ in the interval $[x_i, x_{i+1}]$.

5. Example

We consider the positive data $\{(2, 10),\ (3, 2),\ (7, 3),\ (8, 7),\ (9, 2),\ (13, 3),\ (14, 10)\}$. The derivatives are calculated based on arithmetic mean method and derivatives are $d_1 = -9.6500,\ d_2 = -6.3500,\ d_3 = 3.2500,\ d_4 = -0.5000\ d_5 = -3.9500\ d_6 = 5.6500,\ d_7 = 8.3500$. To preserve positivity, scaling factors are taken according to Theorem 5 and scaling factors ranges are $s_1 \in [0,\ 0.0833),\ s_2 \in [0,\ 0.2000),\ s_3 \in [0,\ 0.0833),\ s_4 \in [0,\ 0.0833),\ s_5 \in [0,\ 0.2000),\ s_6 \in [0,\ 0.0833)$.

Shape parameters and scaling factors are taken according to Theorem 5, using this Figure 1(a) is plotted, which is positive FIF. Figure 1(b) is plotted by perturbing scaling factor s_2 as 0.018 with respect to Figure 1(a). There is a significant change occurred in the interval $[x_2, x_3]$. There is a little change in the interval $[x_5, x_6]$ and changes in the other intervals are negligible. Figure 1(c) is plotted by perturbing shape parameter γ_2 as 300.6 with respect to Figure 1(a). In interval $[x_2, x_3]$ curve move little up and the changes in the other intervals are negligible. From this observation, scaling factors playing dominant role comparing to shape parameters.

Next by taking all the scaling factors $s_i = 0$ for all $i = 1, 2, \ldots, 6$, Figure 1(d) is plotted, which is classical rational cubic spline preserving positivity. Next Figure 1(e) is plotted by taking arbitrary scaling factors and shape parameters. By observing Figure 1(e), it is known that conditions given in Theorem 5 are sufficient not necessary. Figure 2 represents the derivative of the FIFs that are given in Figure 1. As mentioned in the introductory section, it is evident that proposed scheme is a best tool to approximate a continuous function that having irregular derivative. The parameters are used to plot Figures 1 and 2 are given in Table 1.

Next, we consider the monotonic increasing data $\{(7.99, 0), (8.09, 2.76429 \times 10^{-5}), (8.19, 4.37498 \times 10^{-2}), (8.70, 0.169183), (9.20, 0.469428), (10, 0.943740), (12,\ 0.998636), (15, 0.999919), (20, 0.999994)\}$. Derivatives are approximated using arithmetic mean method and derivatives are $d_1 = 0,\ d_2 = 0.2187,\ d_3$=0.4059, $d_4 = 0.4250,\ d_5 = 0.5976,\ d_6 = 0.4313,\ d_7 = 0.0166,\ d_8 = 0.0003\ d_9 = 0$. In order to preserve monotonicity, scaling factors are taken according to Theorem 6 and scaling factors ranges are $s_1 \in [0,\ 0.28 \times 10^{-4}),\ s_2 \in [0,\ 0.8326 \times 10^{-2}),\ s_3 \in [0,\ 0.4246 \times 10^{-1}),\ s_4 \in [0,\ 0.4163 \times 10^{-1}),\ s_5 \in [0,\ 0.6661 \times 10^{-1}),\ s_6 \in [0,\ 0.5489 \times 10^{-1}),\ s_7 \in [0,\ 0.1283 \times 10^{-2}),\ s_8 \in [0, 0.75 \times 10^{-4})$.

Taking arbitrary scaling factors and shape parameters (see Table(1)) Figure 3(a) is plotted, which is not preserving monotonicity of the data. So, scaling factors and shape parameters are selected based on Theorem 6, using this Figure 3(b) is plotted, which preserves monotonicity of the data. After perturbing scaling factor s_3, Figure 3(c) is plotted. Comparing Figure 3(c) with Figure 3(b), it is known that there is significant change occurred in the interval $[x_3, x_4]$, changes in the other intervals are negligible. Figure 3(d) is plotted after perturbing shape parameter γ_3 with respect to Figure 3(b). Comparing Figure 3(b) and Figure 3(d), it is observed that there is a slight change in the interval $[x_3, x_4]$, changes in the other intervals are negligible.

Taking all scaling factor $s_i = 0$ $i = 1, 2, \dots, 8$, classical rational cubic spline is plotted which is given in Figure 3(e). Derivative of the FIFs given in Figure 3 are constructed and plotted in Figure 4. Scaling factors and shape parameters are used to compute Figures 3 and 4 are given in Table 2.

Next consider the convex data $\{(-8, 4.5), (-7, 4), (2.2, 3.55), (7, 4), (10, 4.5), (12, 5)\}$. The derivatives are estimated using arithmetic mean method and derivatives are $d_1 = -0.5442$, $d_2 = -0.4558$, $d_3 = 0.0448$, $d_4 = 0.1386$, $d_5 = 0.2167$, $d_6 = 0.2833$. Scaling factors and shape parameters are calculated based on Theorem 7 and scaling factor ranges are $s_1 \in [0, 0.0025)$, $s_2 \in [0, 0.1669)$, $s_3 \in [0, 0.0206)$, $s_4 \in [0, 0.0074)$, $s_5 \in [0, 0.0059)$. Taking arbitrary scaling factors and shape parameters, Figure 5(a) is plotted. It is evident that Figure 5(a) is non convex fractal interpolation function. So scaling factors and shape parameters are taken according to Theorem 7, using this Figure 5(b) is plotted, which is convex.

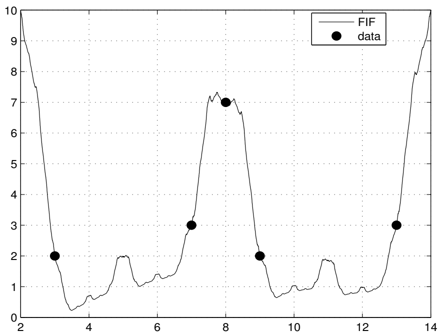
Figure 5(c) is plotted after perturbing scaling factor s_2 with respect to Figure 5(b). There is significant change occurred in the second interval, changes in the other intervals are negligible. Also, Figure 5(d) is plotted by perturbing shape parameter γ_2 with respect to Figure 5(b). By observing Figure 5(d) with Figure 5(b), it is known that there is small change is the second interval, curve little move up in that interval, changes in the other intervals are negligible. Figure 5(e) is plotted using scaling factors $s_i = 0$ for all $i = 1, 2, \dots, 5$. This is classical rational cubic spline. Derivative of the FIFs given in Figure 5 are constructed and plotted in Figure 6. Scaling factors and shape parameters are used to compute Figures 5 and 6 are given in Table 3.

Table 1: Parameters for positive interpolation with $u_i = 1.5$ and $v_i = 1.5$ for $i = 1, 2, \dots, 6$.

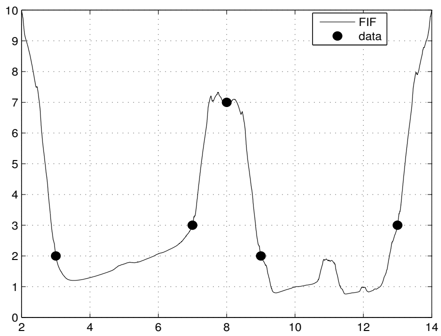
$\downarrow$ Parameter/Figure $\rightarrow$	1(a), 2(a)	1(b), 2(b)	1(c), 2(c)	1(d), 2(d)	1(e), 2(e)
s_1	0.0730	0.0730	0.0730	0	0.0900
s_2	0.1800	0.0180	0.1800	0	0.2100
s_3	0.0740	0.0740	0.0740	0	0.1200
s_4	0.0745	0.0745	0.0745	0	0.1400
s_5	0.1700	0.1700	0.1700	0	0.3000
s_6	0.0733	0.0733	0.0733	0	0.1000
γ_1	0.8000	0.8000	0.8000	0.8000	30.0000
γ_2	30.6000	30.6000	300.6000	15.0000	80.6000
γ_3	0.5000	0.5000	0.5000	0.5000	4.5000
γ_4	0.8000	0.8000	0.8000	0.8000	3.8000
γ_5	2.5000	2.5000	2.5000	7.5000	7.5000
γ_6	0.7000	0.7000	0.7000	0.7000	0.7200

Table 2: Parameters for monotone interpolation with $u_i = 1.5$ and $v_i = 1.5$ for $i = 1, 2, \dots, 8$.

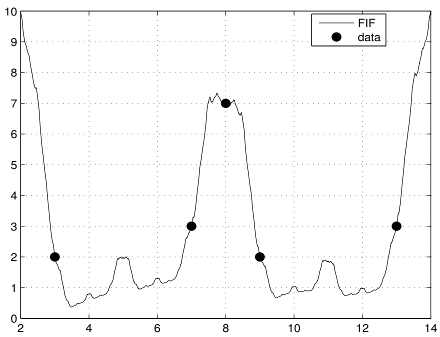
$\downarrow$ Parameter/Figure $\rightarrow$	3(a),4(a)	3(b),4(b)	3(c),4(c)	3(d),4(d)	3(e),4(e)
s_1	0.0010	0.21×10^{-4}	0.21×10^{-4}	0.21×10^{-4}	0
s_2	0.0004	0.81×10^{-2}	0.81×10^{-2}	0.81×10^{-2}	0
s_3	0.0500	0.422×10^{-1}	0.122×10^{-1}	0.422×10^{-1}	0
s_4	0.0007	0.413×10^{-1}	0.413×10^{-1}	0.413×10^{-1}	0
s_5	0.0043	0.662×10^{-1}	0.662×10^{-1}	0.662×10^{-1}	0
s_6	0.0060	0.543×10^{-1}	0.543×10^{-1}	0.543×10^{-1}	0
s_7	0.0340	0.12×10^{-2}	0.12×10^{-2}	0.12×10^{-2}	0
s_8	0.0650	0.7×10^{-4}	0.7×10^{-4}	0.7×10^{-4}	0
γ_1	1120	4936.4	4936.4	4936.4	1184.0
γ_2	23.1	0.8	0.8	0.8	0.8
γ_3	0.4	4.6	4.6	1600.6	2.1
γ_4	54.7	0.5	0.5	0.5	0.5
γ_5	43.5	0.5	0.5	0.5	0.5
γ_6	3.7	2250.7	2250.7	2250.7	21.5
γ_7	6.4	913.9	913.9	913.9	56.3
γ_8	9.7	406.3	406.3	406.3	24.3



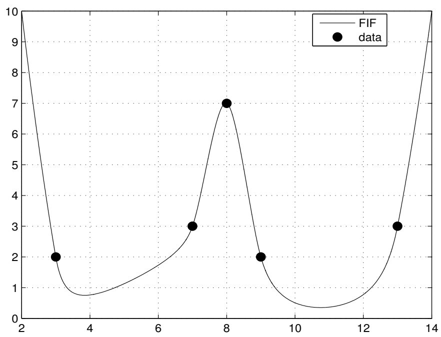
(a) Rational cubic FIF.



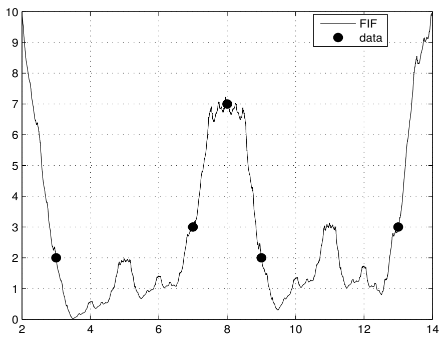
(b) Effect of s_2.



(c) Effect of γ_2.

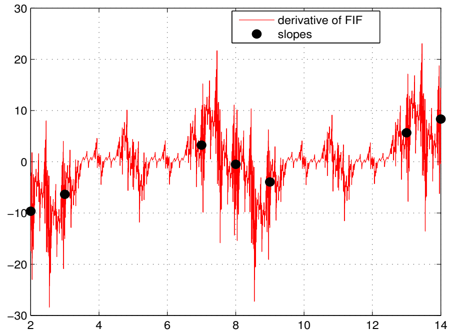


(d) Classical rational interpolant.

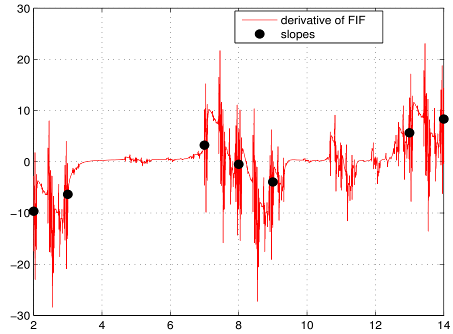


(e) FIF with arbitrary parameters.

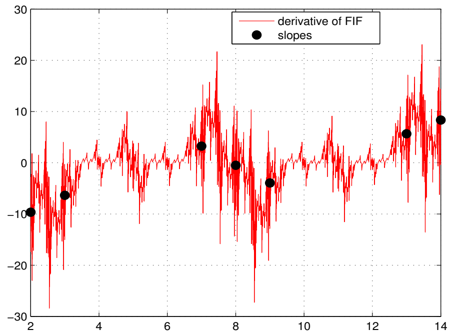
Figure 1: Positive preserving interpolation.



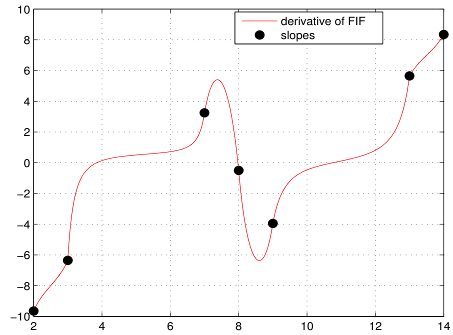
(a) Derivative of the FIF given in Figure 1(a).



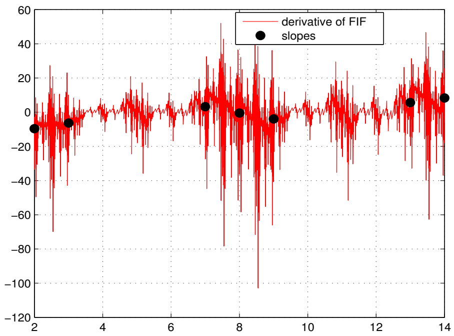
(b) Derivative of the FIF given in Figure 1(b).



(c) Derivative of the FIF given in Figure 1(c).

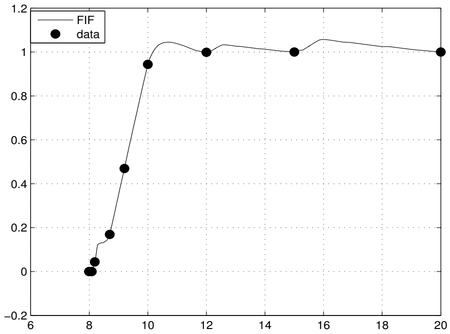


(d) Derivative of the FIF given in Figure 1(d).

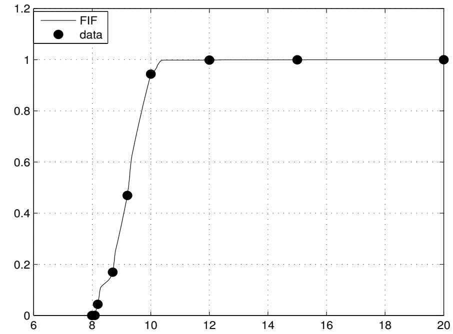


(e) Derivative of the FIF given in Figure 1(e).

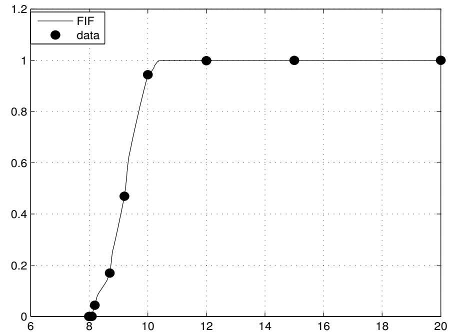
Figure 2: Derivatives of the FIFs given in Figure 1.



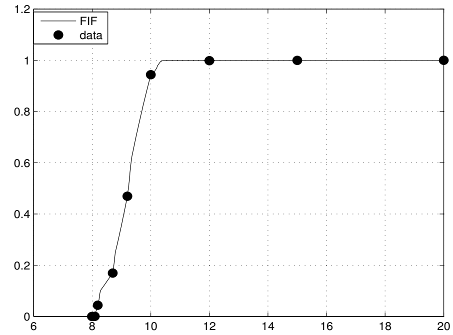
(a) Non monotonic FIF.



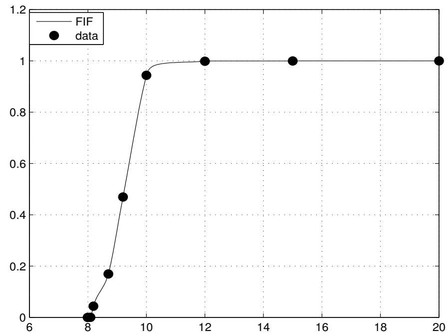
(b) Rational cubic FIF.



(c) Effect of s_3.

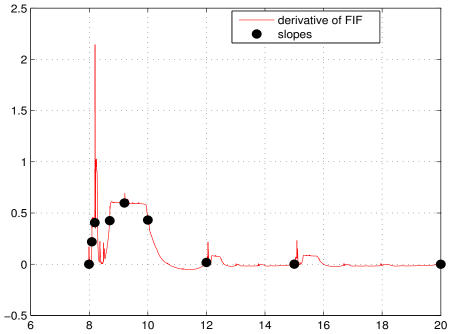


(d) Effect of γ_3.

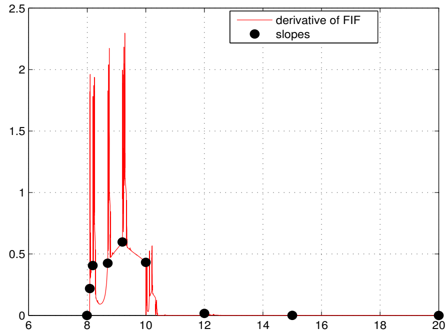


(e) Classical rational interpolant.

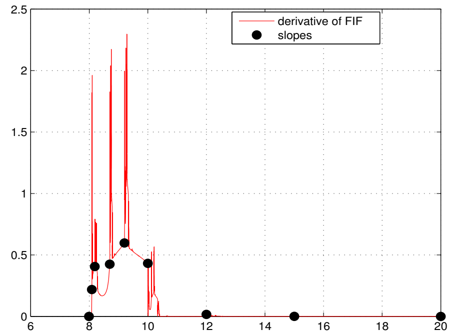
Figure 3: Monotonicity preserving interpolation.



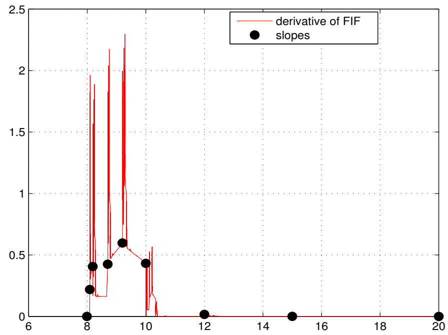
(a) Derivative of the FIF given in Figure 3(a).



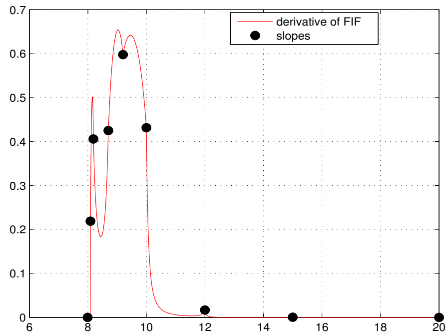
(b) Derivative of the FIF given in Figure 3(b).



(c) Derivative of the FIF given in Figure 3(c).

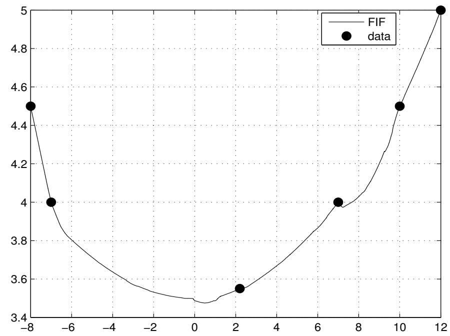


(d) Derivative of the FIF given in Figure 3(d).

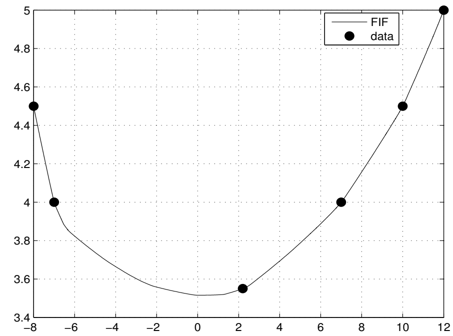


(e) Derivative of the FIF given in Figure 3(e).

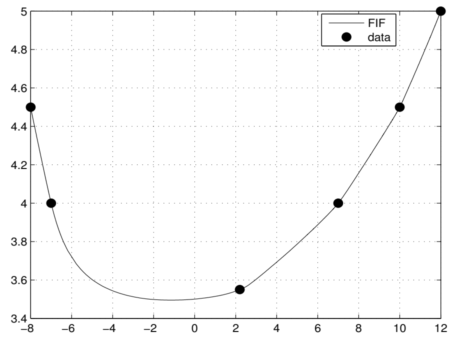
Figure 4: Derivatives of the FIFs given in Figure 3.



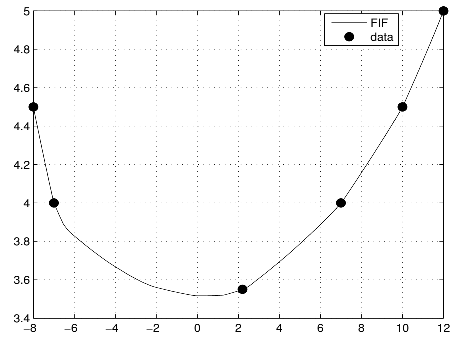
(a) Non convex FIF.



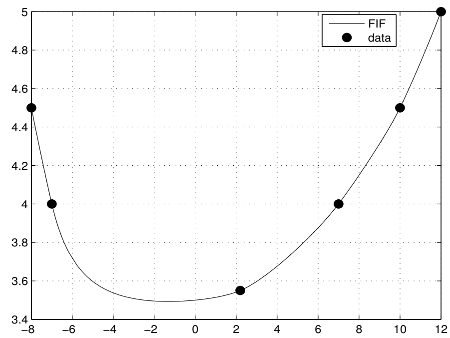
(b) Convex FIF.



(c) Effect of s_2.

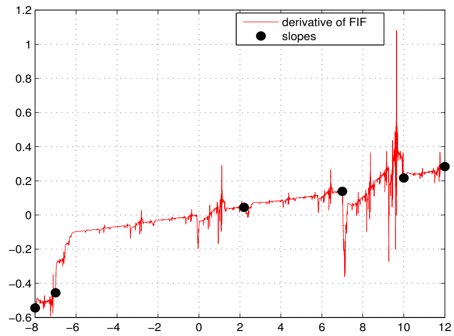


(d) Effect of γ_2.

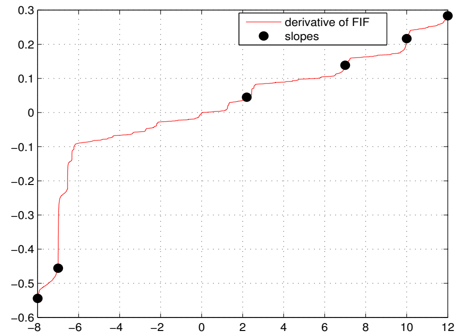


(e) Classical convex interpolant.

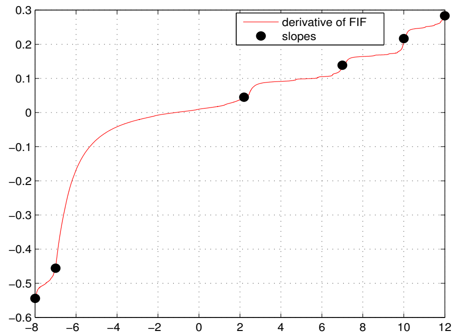
Figure 5: Convexity preserving interpolation.



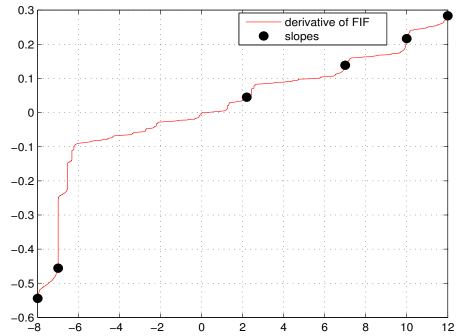
(a) Derivative of the FIF given in Figure 5(a).



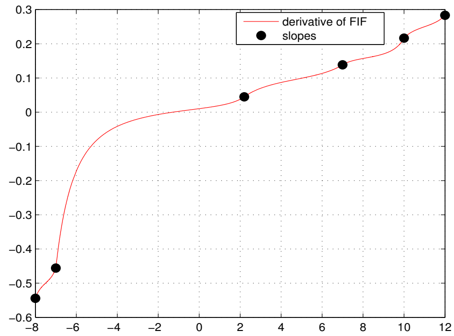
(b) Derivative of the FIF given in Figure 5(b).



(c) Derivative of the FIF given in Figure 5(c).



(d) Derivative of the FIF given in Figure 5(d).



(e) Derivative of the FIF given in Figure 5(e).

Figure 6: Derivatives of the FIFs given in Figure 5.

Table 3: Parameters for convex interpolation with $u_i = 1.5$ and $v_i = 1.5$ for $i = 1, 2, \ldots, 5$.

↓ Parameter/Figure →	5(a), 6(a)	5(b), 6(b)	5(c), 6(c)	5(d), 6(d)	5(e), 6(e)
s_1	0.0100	0.0020	0.0020	0.0020	0
s_2	0.1861	0.1661	0.0166	0.1661	0
s_3	0.0502	0.0202	0.0202	0.0202	0
s_4	0.1702	0.0070	0.0070	0.0070	0
s_5	0.0152	0.0052	0.0052	0.0052	0
γ_1	1.3694	2.3694	2.3694	2.3694	1.5000
γ_2	443.6127	643.6127	6.8637	90000.2163	6.5098
γ_3	28.5777	34.5777	34.5777	34.5777	1.6351
γ_4	31.4563	38.4293	38.4293	38.4293	2.6743
γ_5	5.9987	7.9948	7.9948	7.9948	1.5000

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6. References

[1] M. F. Barnsley, Fractal functions and interpolation, Constr. Approx. 2 (1) (1986) 303–329.

[2] M. F. Barnsley, A. N. Harrington, The calculus of fractal interpolation functions, J. Approx. Theory 57 (1) (1989) 14–34.

[3] A. K. B. Chand, G. P. Kapoor, Generalized cubic spline fractal interpolation functions, SIAM J. Numer. Anal. 44 (2) (2006) 655–676.

[4] A. K. B. Chand, M. A. Navascués, Natural bicubic spline fractal interpolation, Nonlinear Anal. 69 (11) (2008) 3679–3691.

[5] M. A. Navascués, M. V. Sebastián, Smooth fractal interpolation, J. Inequal. Appl., Article ID 78734 (2006) 1–20.

[6] C. D. Boor, B. Swartz, Piecewise monotone interpolation, J. Approx. Theory 21 (4) (1977) 411–416.

[7] F. N. Fritsch, R. E. Carlson, Monotone piecewise cubic interpolation, SIAM J. Numer. Anal. 17 (2) (1980) 238–246.

[8] E. Passow, J. A. Roulier, Monotone and convex spline interpolation, SIAM J. Numer. Anal. 14 (5) (1977) 904–909.

[9] L. I. Schumaker, On shape preserving quadratic spline interpolation, SIAM J. Numer. Anal. 20 (4) (1983) 854–864.

[10] K. W. Brodlie, S. Butt, Preserving convexity using piecewise cubic interpolation, Comput. Graph. 15 (1) (1991) 15–23.

[11] H. Späth, Spline algorithms for curves and surfaces, Utilitas Mathematica Pub., Winnipeg, 1974.

[12] R. Delbourgo, J. A. Gregory, $\mathcal{C}^2$ rational quadratic spline interpolation to monotonic data, IMA J. Numer. Anal. 3 (2) (1983) 141–152.

[13] R. Delbourgo, J. A. Gregory, Shape preserving piecewise rational interpolation, SIAM J. Sci. and Stat. Comput. 6 (4) (1985) 967–976.

[14] J. A. Gregory, Shape preserving spline interpolation, Comput.-Aided Des. 18 (1) (1986) 53–57.

[15] M. Sarfraz, M. Z. Hussain, Data visualization using rational spline interpolation, J. Comput. Appl. Math. 189 (1) (2006) 513–525.

[16] M. Z. Hussain, M. Hussain, Visualization of data preserving monotonicity, Appl. Math. Comput. 190 (2) (2007) 1353–1364.

[17] X. Han, Convexity-preserving piecewise rational quartic interpolation, SIAM J. Numer. Anal. 46 (2) (2008) 920–929.

[18] M. Abbas, A. A. Majid, J. M. Ali, Monotonicity-preserving $\mathcal{C}^2$ rational cubic spline for monotone data, Appl. Math. Comput. 219 (6) (2012) 2885–2895.

[19] M. Abbas, A. A. Majid, J. M. Ali, Local convexity-preserving rational cubic spline for convex data, The Scientific World Journal, Article ID 391568 (2014) 10 pages.

[20] A. K. B. Chand, P. Viswanathan, A constructive approach to cubic hermite fractal interpolation function and its constrained aspects, BIT 53 (4) (2013) 841–865.

[21] P. Viswanathan, A. K. B. Chand, R. P. Agarwal, Preserving convexity through rational cubic spline fractal interpolation function, J. Comput. Appl. Math. 263 (2014) 262–276.

[22] P. Viswanathan, A. K. B. Chand, A fractal procedure for monotonicity preserving interpolation, Appl. Math. Comput. 247 (2014) 190–204.

[23] A. K. B. Chand, P. Viswanathan, Shape preserving rational cubic spline fractal interpolation, arXiv preprint arXiv:1503.02458.

[24] A. K. B. Chand, N. Vijender, M. A. Navascués, Shape preservation of scientific data through rational fractal splines, Calcolo 51 (2) (2014) 329–362.

[25] M. F. Barnsley, Fractals everywhere, Academic press, Boston, 2014.

[26] P. Viswanathan, A. K. B. Chand, A $\mathcal{C}^1$-rational cubic fractal interpolation function: Convergence and associated parameter identification problem, Acta Appl. Math. 136 (1) (2015) 19–41.

[27] M. J. D. Powell, Approximation theory and methods, Cambridge university press, Cambridge, 1981.

[28] J. W. Schmidt, W. Hess, Positivity of cubic polynomials on intervals and positive spline interpolation, BIT 28 (2) (1988) 340–352.

[29] A. D. Miller, R. Vyborny, Some remarks on functions with one-sided derivatives, Amer. Math. Monthly (1986) 471–475.

[30] L. D. Berkovitz, Convexity and optimization in $\mathbb{R}^n$, Vol. 63, John Wiley and Sons, New York, 2003.