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Dynamics and local convergence of a family of derivative-free iterative processes

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Abstract

In this paper we establish the construction of a family of free derivative of point to point iterative processes, with quadratic convergence, from two known data in each previous iteration. Besides, we study the accessibility of this family by means of the basins of attraction and the convergence balls. We provide a local convergence analysis for the family of iterative processes free of derivatives, when the operator F is not necessarily Fréchet differentiable. The sufficient convergence conditions are weaker and more flexible than in earlier studies. An application is provided involving mixed Hammerstein nonlinear integral equation with application in real world problems. Finally, we show also a dynamical study and the convergence regions of some members of the family.

Keywords: Iterative processes, Steffensen method, local convergence, iterative processes derivative-free, divided differences.
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1 Introduction

One of the most studied problems in numerical analysis is the solution of nonlinear equations

$$F(x) = 0. \tag{1}$$

To give sufficient generality to the problem of approximating a solution of a nonlinear equation by iterative processes, throughout the paper we consider that $F : \Omega \subseteq X \to X$ is a nonlinear operator defined on a nonempty open convex subset Ω of a Banach space X, so that many scientific and engineering problems can be written as a nonlinear equation (see [9, 14]) in Banach space; for example, nonlinear integral equations, initial value problems, matrix equations, nonlinear PVF, etc..

It is well-known that Newton's method,

$$x_0 \in \Omega, \qquad x_n = x_{n-1} - [F'(x_{n-1})]^{-1} F(x_{n-1}), \quad n \in \mathbb{N}, \tag{2}$$

is the one of the most used iterative processes to approximate a solution x^* of $F(x) = 0$. The quadratic convergence and the low operational cost of Newton's method ensure that it has a good computational efficiency. But this method has a serious shortcoming: the derivative $F'(x)$ has to be evaluated at each iteration. This makes it inapplicable to equations with non-differentiable operators and in situations when evaluation of the derivative is too costly. In these cases, it is common to approximate derivatives by divided differences, so that iterative processes that use divided differences instead of derivatives are obtained. Remember that the operator $[u, v; F] \in \mathcal{L}(X, X)$, $u, v \in \Omega$ with $x \neq y$, is a first order divided difference [20, 21], which is a bounded linear operator such that

$$[u, v; F] : \Omega \subset X \longrightarrow X \quad \text{and} \quad [u, v; F](u - v) = F(u) - F(v). \tag{3}$$

Our first goal in this work consists on the construction, based on Newton's method (2), of a family of derivative-free iterative processes with the form:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [d_1(x_n), d_2(x_n); F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{4}$$

where $d_1(x_n)$ y $d_2(x_n)$ are known data at the point x_n. In that way, from this family of iterative processes, we want to establish a general theory of local convergence to that derivative-free point to point iterative processes that can be applied when operator F is non-differentiable. Occasionally, the study of the local convergence of derivative-free iterative processes shows a small contradiction. There are many known results of local convergence (see [7],[13],[15],[23],[25], and references therein given) which usually include the condition of the existence of the operator $[F'(x^*)]^{-1}$, forcing the operator F to be differentiable. However, in this paper, we obtain a type of result for the local convergence from requiring a weaker type of assumptions to obtain a local convergence result when operator F is non-differentiable.

Our second goal is to ensure that the order of convergence of the family of iterative processes included in (4) is quadratic as Newton's method (2). This fact allows us to study the conditions that data functions $d_1(t)$ and $d_2(t)$ must satisfy.

Notice that the methods using divided differences in their algorithm have a drawback, the accessibility of these methods to the solution of the equation (1) is poor, so that the domains of starting points are reduced. This is our third objective, to study the accessibiity of the iterative processes included in (4). In this work, we study the dynamical planes of the family of iterative processes considered and, on the other hand, we will study the accessibility in a theoretical way from the convergence balls associated to them.

The rest of the paper is structured as follows. Section 2 contains the construction of the family of iterative processes. A dynamical study for the iterative processes is given in Section 3. The local convergence results are given in Section 4. In the Section 5, another iterative processes with central divided differences are constructed. A numerical experiment is included in the Section 6. To finish, the conclusions are given in Section 7.

Moreover, we denote $\overline{B(x,\varrho)} = \{y \in X; \|y-x\| \leq \varrho\}$ and $B(x,\varrho) = \{y \in X; \|y-x\| < \varrho\}$, respectively for the closed and open balls with center in x and of radius $\varrho > 0$

2 Construction of the family

In this section, we consider the real-valued function $f : \mathbb{R} \to \mathbb{R}$ and we want to approximate a solution of the equations $f(t) = 0$. It is a know fact, that in order to approximate that solution the most used method is the Newton's one, defined as follows:

$$\begin{cases} t_0 \text{ given}, \\ t_{n+1} = N_f(t_n) = t_n - \dfrac{f(t_n)}{f'(t_n)}, \quad n \geq 0. \end{cases}$$

Firstly, we will use it by means of using divided differences which will allow us to extend our result to Banach spaces. Next, the convergence rate of the new family of iterative processes obtained should be quadratic. Finally, the family of iterative processes obtained should be a point-to-point family of iterative processes. So, we consider to construct an iterative process as

$$\begin{cases} t_0 \text{ given}, \\ t_{n+1} = G_f(t_n) = t_n - \frac{f(t_n)}{g(t_n)}, \quad n \geq 0, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$. As an example of this situation, Kung and Traub in [16] presented the Steffensen method as follows:

$$\begin{cases} t_0 \text{ given}, \\ t_{n+1} = S_f(t_n) = t_n + \dfrac{f(t_n)^2}{f(t_n) - f(t_n + f(t_n))}, \quad n \geq 0. \end{cases} \tag{5}$$

That is, they use the divided difference $g(t) = \dfrac{f(t+f(t)) - f(t)}{f(t)}$. Following this idea, we consider a value of t in which we will get two different data functions $d_1(t)$ and $d_2(t)$ that will allow us to define the divided difference and construct an iterative process as:

$$\begin{cases} t_0 \text{ given}, \\ t_{n+1} = G_f(t_n) = t_n - \dfrac{d_1(t) - d_2(t)}{f(d_1(t)) - f(d_2(t))} f(t_n), \quad n \geq 0, \end{cases} \tag{6}$$

In order to achieve the quadratic convergence, we wil replace $f'(t)$ by the divided differnece $g(t) = \dfrac{f(d_1(t)) - f(d_2(t))}{d_1(t) - d_2(t)}$ using the data functions $d_1(t)$ and $d_2(t)$. As the family of iterative processes constructed should be point-to-point, it is a well-known fact that to achieve the quadratic order, we need that $G_f(t^*) = t^*$ and $G'_f(t^*) = 0$, where t^* is a simple solution of the equation $f(t) = 0$ (see Shröder [24]).

Following Steffensen's method (5), we impose that $d_1(t^*) = d_2(t^*) = t^*$. Then, taking into account that

$$\lim_{t->t^*} \frac{d_1(t) - d_2(t)}{f(d_1(t)) - f(d_2(t))} = \frac{1}{f'(t^*)}$$

if the data functions d_1 and d_2 are derivable functions with $d'_1(t^*) \neq d'_2(t^*)$. It follows that $G_f(t^*) = t^*$,

On the other hand, in the previous conditions if there exist d''_1 and d''_2, we have

$$\lim_{t->t^*} \frac{(d_1(t^*) - d_2(t^*))f(t)}{f(d_1(t)) - f(d_2(t))} = 0,$$

and

$$\lim_{t->t^*} \frac{(d_1(t^*) - d_2(t^*))f'(t)}{f(d_1(t)) - f(d_2(t))} = 1,$$

Then, we obtain that $G_f(t^*) = t^*$ and so we get the quadratic convergence of the family of iterative processes (6).

Now, we observe that the most known iterative processes free-derivatives are included in the family of iterative processes constructed (6)..

(I) Steffensen method

If we consider $d_1(t) = t$ and $d_2(t) = t + f(t)$, we obtain the Steffensen method (5).

(II) Backward-Steffensen method

If we consider $d_1(t) = t - f(t)$ and $d_2(t) = t$, we obtain the Backward-Steffensen method:

$$\begin{cases} t_0 \text{ given}, \\ t_{n+1} = BS_f(t_n) = t_n + \dfrac{f(t_n)^2}{f(t_n - f(t_n)) - f(t_n)}, \quad n \geq 0, \end{cases} \tag{7}$$

notice that, in this case, used a backward divided difference .

(III) Central-Steffensen method
If we consider $d_1(t) = t - f(t)$ and $d_2(t) = t + f(t)$, we obtain the Central-Steffensen method:

$$\begin{cases} t_0 \text{ given}, \\ t_{n+1} = CS_f(t_n) = t_n + \dfrac{2f(t_n)^2}{f(t_n - f(t_n)) - f(t_n + f(t_n))}, \quad n \geq 0, \end{cases} \tag{8}$$

notice that, in this case, used a central divided difference .

(IV) Generalized Steffensen-type method
If we consider $d_1(t) = t - af(t)$ and $d_2(t) = t + bf(t)$, we obtain the Generalized Steffensen-type method:

$$\begin{cases} t_0 \text{ given}, \\ t_{n+1} = GS_f(t_n) = t_n + \dfrac{(a+b)f(t_n)^2}{f(t_n - af(t_n)) - f(t_n + bf(t_n))}, \quad n \geq 0. \end{cases} \tag{9}$$

3 A dynamical study

In this section we will compare the dynamical planes associated with the four special cases mentioned in Section 2. As all of these iterative methods are, derivative-free we will use the non-differentiable equation

$$f(z) \equiv z - \ \sigma z^3 - \mu|z| = 0 \tag{10}$$

where $\sigma = \mu = 1/40$

This equation has 3 different roots:

$$z_1 \approx -1.56155\ldots,$$
$$z_2 = 0$$

and

$$z_3 \approx 2.56155\ldots.$$

As in previous papers (see [2, 17, 18]) and books (see [3, 4, 5, 19]) we will define:

The basin of attraction of a root z^* is defined as the set of points whose iterations converge to the root z^*.

In order to draw the basins of attraction, a point is painted in red if the iteration of the method starting in the point converges to the root z_1, in blue if it converges to z_2, in yellow it it converges to z_3 and in black those points for which there is no convergence to any of the roots. We choose a tolerance of 10^{-3} and a limit of 200 iterations.

In Figures 1-6 the basins of attraction associated to the roots of the polynomial f are shown. As it can be seen the best methods in terms of convergence are the Generalized

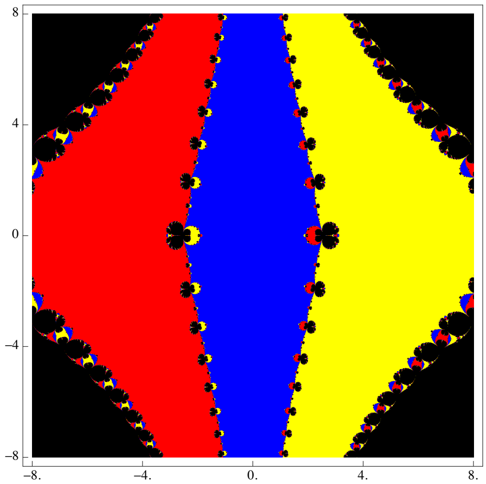

Figure 1: The Steffensen method

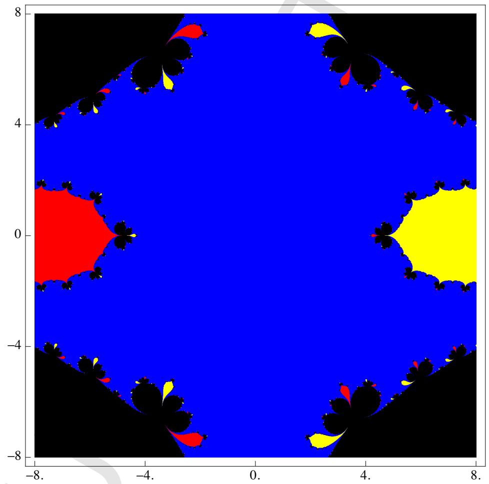

Figure 2: The Backward-Steffensen method

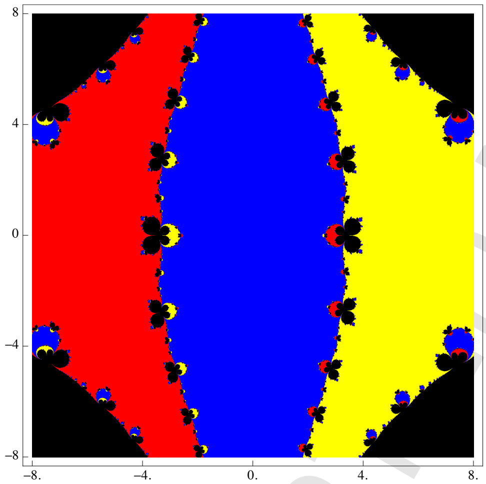

Figure 3: The Center Steffensen method

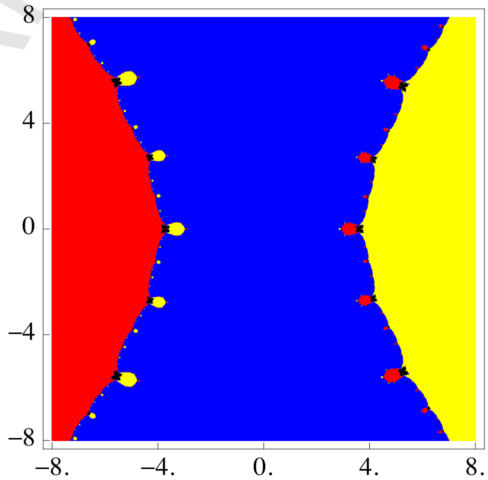

Figure 4: The Generalized Steffensen-type method with $a = b = 1/10$

Steffensen method with lower values of a and b and also values of both parameters should be close. In Figure 5 this fact can be seen as almost all points of the region converge to any of the three roots.

4 Local convergence

Taking into account the previous study, now, we consider the family of iterative processes in Banach spaces. To do this, we consider $F : \Omega \subseteq X \to X$ a continuous nonlinear operator, Ω is a non-empty open convex domain in the Banach space X and $d_i : X \to X$

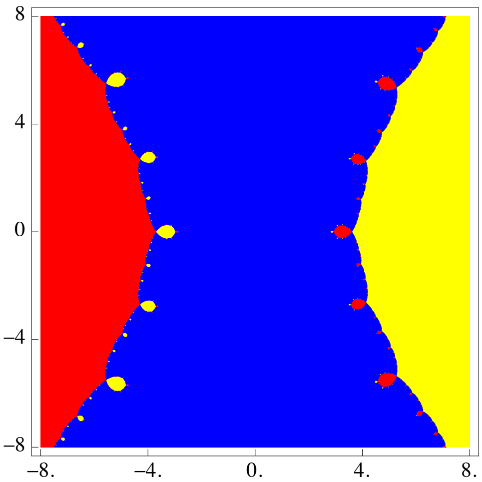

Figure 5: The Generalized Steffensen method with $a = b = 1/1000$

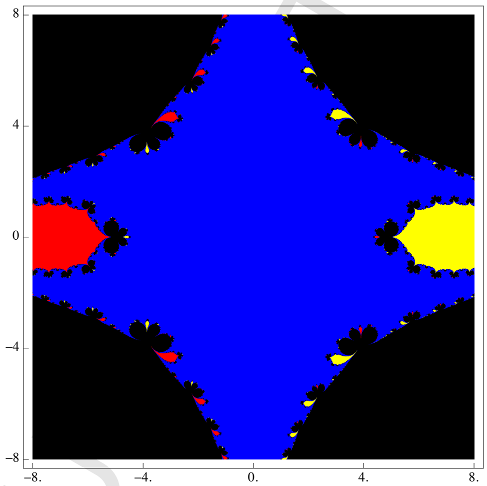

Figure 6: The Generalized Steffensen method with $a = 2$ and $b = 1/2$.

a derivable real function, for $i = 1, 2$, with $d_1(x^*) = d_2(x^*) = x^*$ and $d_1'(x^*) \neq d_2'(x^*)$, being x^* a solution of (1). Besides, we suppose that there exists a divided difference of order one $[z, w; F] \in \mathcal{L}(X, X)$ for each pair of distinct points $z, w \in \Omega$ and we assume that $d_1(x) = d_2(x)$ if and only if $x = x^*$, since that we must characterize the different data to define $[d_1(x), d_2(x); F]$. So, we will study the local convergence of iterative processes given by the following algorithm:

$$\left\{ \begin{array}{l} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [d_1(x_n), d_2(x_n); F]^{-1} F(x_n), \quad n \geq 0, \end{array} \right. \tag{11}$$

.

To analyze the local convergence of iterative processes that do not use derivatives in their algorithms, the condition usually required for the operator divided difference is known as Lipschitz continuous condition, which is given by

$$\|[x, u; F] - [y, v; F]\| \leq K(\|x - y\| + \|u - v\|); \quad x, y, u, v \in \Omega. \tag{12}$$

Another condition, under which the local convergence is also usually studied is when the operator divided difference is Hölder continuous in Ω. That is:

$$\|[x, u; F] - [y, v; F]\| \leq K(\|x - y\|^p + \|u - v\|^p); \quad x, y, u, v \in \Omega,\ p \in [0, 1], \tag{13}$$

which generalizes the Lipschitz continuous condition. Note that both conditions involve the operator F to be differentiable [10, 11]. To generalize the above conditions and even to consider situations in which operator F is non-differentiable, we will consider the condition

$$\|[x, u; F] - [y, v; F]\| \leq \omega(\|x - y\|, \|u - v\|); \quad x, y, u, v \in \Omega, \tag{14}$$

where $\omega : \mathbf{R}_* \times \mathbf{R}_* \to \mathbf{R}_*$ is a continuous nondecreasing function in its two arguments, with $\mathbf{R}_* = \mathbf{R}_+ \cup \{0\}$.

Obviously, we obtain (12) if $\omega(z) = Kz$ and (13) if $\omega(z) = Kz^p$. Moreover, as it is known ([10, 11]), if $\omega(0,0) = 0$ then F is a differentiable operator. Therefore, taking into account condition (14), we consider the case in which the operator F is non-differentiable. For example, situations where $\omega(0,0) \neq 0$, as we can see subsequently.

On the one hand, we assume the following conditions for the operator divided difference:

$(C1)$ There exist $x^* \in \Omega$ with $F(x^*) = 0$, $\delta > 0$ and $\tilde{x} \in \Omega$, with $\|\tilde{x} - x^*\| = \delta$, such that $[x^*, \tilde{x}; F]^{-1} \in \mathcal{L}(X, X)$, and suppose for $x \in \Omega$, $\|[x^*, \tilde{x}; F]^{-1}\| \leq \beta$.

$(C2)$ $\|[x, y; F] - [u, v; F])\| \leq \omega(\|x - u\|, \|y - v\|)$ holds for each pair of different points $(x, y), (u, v) \in \Omega \times \Omega$, where $\omega : \mathbb{R}_* \times \mathbb{R}_* \longrightarrow \mathbb{R}_*$ is a continuous non-decreasing function in its two variables.

$(C2')$ $\|[x, y; F] - [x^*, \tilde{x}; F])\| \leq \widetilde{\omega}(\|x - x^*\|, \|y - \tilde{x}\|)$ holds for each pair of different points $(x, y) \in \Omega \times \Omega$ divided difference of order one, where $\widetilde{\omega} : \mathbb{R}_* \times \mathbb{R}_* \longrightarrow \mathbb{R}_*$ is a continuous non-decreasing function in its two variables.

Notice that $(C2')$ is not an additional to $(C2)$ condition, since in practice the computation of function ω requires the computation of function $\widetilde{\omega}$ as a special case. Moreover, we clearly also have that $\widetilde{\omega}(s, t) \leq \omega(s, t)$ for each $s, t \in \mathbb{R}_*$ and the function $\dfrac{\omega}{\widetilde{\omega}}$ can be arbitrarily large ([1], [20]).

On the other hand, we assume the following condition for the data functions:

$(C3)$ $\|d_i(x) - d_i(x^*)\| \leq \omega_i(\|x - x^*\|)$ holds for each $x \in \Omega$, where $\omega_i : \mathbb{R}_* \longrightarrow \mathbb{R}_*$ is a continuous non-decreasing function for $i = 1, 2$.

The local study of the convergence is based on providing the so-called ball of convergence of iterative process, that shows the accessibility to x^* from the initial approximation x_0 belonging to the ball of convergence. We denote the ball of convergence as $B(x^*, R)$ and consider $x_0 \in B(x^*, R)$ with $x_0 \neq x^*$.

In first place, we must to prove the existence of the operator $[d_1(x_0), d_2(x_0); F]^{-1}$. As $d_1(x_0) \neq d_2(x_0)$ and $\|d_i(x_0) - x^*\| = \|d_i(x_0) - d_i(x^*)\| \leqslant \omega_i(\|x_0 - x^*\| < \omega_i(R)$ for $i = 1, 2$, if we assume that $B(x^*, \widetilde{R}) \subseteq \Omega$, with $\widetilde{R} = \max\{R, \omega_1(R), \omega_2(R)\}$, then $[d_1(x_0), d_2(x_0); F]$ is well defined. Therefore, we obtain

$$\|I - [x^*, \tilde{x}; F]^{-1}[d_1(x_0), d_2(x_0); F]\| \leq \|[x^*, \tilde{x}; F]^{-1}\|\|[x^*, \tilde{x}; F] - [d_1(x_0), d_2(x_0); F]\|$$

$$\leq \beta\, \widetilde{\omega}(\|x^* - d_1(x_0)\|, \|\tilde{x} - d_2(x_0))\| \leq \beta\, \widetilde{\omega}(\|d_1(x^*) - d_1(x_0)\|, \|\tilde{x} - x^*\| + \|d_2(x^*) - d_2(x_0)\|)$$

$$< \beta\, \widetilde{\omega}(\omega_1(R), \delta + \omega_2(R))$$

and, if we assume that $\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R)) < 1$, by the Banach Lemma we obtain that $[d_1(x_0), d_2(x_0); F]^{-1}$ exists with

$$\|[d_1(x_0), d_2(x_0); F]^{-1}\| \leq \frac{\beta}{1-\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R))}.$$

In second place, starting from one initial approximation x_0 of a solution x^* of the equation $F(x) = 0$, a sequence $\{x_n\}$ of approximations is constructed such that the sequence $\{\|x_n - x^*\|\}$ is decreasing and a better approximation to the solution x^* is then obtained at every step. Obviously, the interest focuses on $\lim_n x_n = x^*$. Therefore, we must to prove that $\|x_1 - x^*\| < \|x_0 - x^*\|$. So, we can write by method (11) that

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - [d_1(x_0), d_2(x_0); F]^{-1}F(x_0) + [d_1(x_0), d_2(x_0); F]^{-1}F(x^*) \\ &= [d_1(x_0), d_2(x_0); F]^{-1}([d_1(x_0), d_2(x_0); F] - [x_0, x^*; F])(x_0 - x^*), \end{aligned} \tag{15}$$

and

$$\begin{aligned} \|x_1 - x^*\| &\leq \|[d_1(x_0), d_2(x_0); F]^{-1}\|\|[d_1(x_0), d_2(x_0); F] - [x_0, x^*; F])\|\|x_0 - x^*\| \\ &\leq \frac{\beta\,\omega(\|d_1(x_0) - x_0\|, \|d_2(x_0) - x^*\|)}{1-\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R))}\|x_0 - x^*\| \\ &\leq \frac{\beta\,\omega(\|d_1(x_0) - x^*\| + \|x^* - x_0\|, \|d_2(x_0) - x^*\|)}{1-\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R))}\|x_0 - x^*\| \\ &< \frac{\beta\,\omega(\omega_1(R) + R, \omega_2(R))}{1-\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R))}\|x_0 - x^*\|. \end{aligned} \tag{16}$$

Then, if we assume that $\dfrac{\beta\omega(\omega_1(R) + R, \omega_2(R))}{1-\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R))} = 1$, we obtain that $\|x_1 - x^*\| < \|x_0 - x^*\|$ and besides $x_1 \in B(x^*, R)$.

Bearing in mind the first step we just study, we must assume the following conditions:

$(C4)$ The equation

$$\beta\,\left(\omega(\omega_1(t) + t, \omega_2(t)) + \widetilde{\omega}(\omega_1(t), \delta+\omega_2(t))\right) - 1 = 0 \tag{17}$$

has at least one positive real root, the smallest positive root is denoted by R.

$(C5)$ $B(x^*, \widetilde{R}) \subseteq \Omega$ and $\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R)) \neq 1$.

Notice that, from (17), we have

$$1-\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R)) = \beta\,\omega(\omega_1(R) + R, \omega_2(R) \geqslant 0$$

and, from $(C5)$, we obtain that $\beta\,\widetilde{\omega}(\omega_1(R), \delta+\omega_2(R)) < 1$.

To generalize the study carried out for the first step, we present an auxiliary perturbation result on the inverse of divided difference of order one for the operator F.

Lemma 1 *Suppose that conditions $(C1)$–$(C5)$ hold. If $x \in B(x^*, R)$, with $x \neq x^*$, then $[d_1(x), d_2(x); F]^{-1}$ exists and*

$$\|[d_1(x), d_2(x); F]^{-1}\| \leq \frac{\beta}{1 - \beta\,\widetilde{\omega}(\omega_1(R), \delta + \omega_2(R))} \tag{18}$$

Proof. We can write $\|d_i(x) - x^*\| \leq \|d_i(x) - d_i(x^*)\| \leq \omega_i(\|x - x^*\|) \leq \omega_i(R)$, for $i = 1, 2$. Therefore, from $(C5)$, we obtain that $d_1(x), d_2(x) \in B(x_0, \widetilde{R}) \subseteq \Omega$. Obviously, $d_1(x) \neq d_2(x)$ and then $[d_1(x), d_2(x); F]$ is well defined for each $x \in B(x^*, R)$, with $x \neq x^*$.

Then, using $(C2')$, $(C4)$ and $(C5)$, we obtain in turn that

$$\|I - [x^*, \tilde{x}; F]^{-1}[d_1(x), d_2(x); F]\| \leq \|[x^*, \tilde{x}; F]^{-1}\|\|[x^*, \tilde{x}; F] - [d_1(x), d_2(x); F]\|$$
$$\leq \beta\,\widetilde{\omega}(\|x^* - d_1(x)\|, \|\tilde{x} - d_2(x))\| \leq \beta\,\widetilde{\omega}(\|d_1(x^*) - d_1(x)\|, \|\tilde{x} - x^*\| + \|d_2(x^*) - d_2(x)\|)$$
$$< \beta\,\widetilde{\omega}(\omega_1(R), \delta + \omega_2(R)) < 1.$$

Then, by the Banach Lemma on invertible operators, the operator $[d_1(x), d_2(x); F]^{-1}$ exist so that (18) is satified. □

Taking into account the preceding notation, we shall show the main local convergence result for method (11) based on conditions $(C1)$–$(C5)$.

Theorem 2 *Suppose that conditions $(C1)$–$(C5)$ are satisfied. Then, sequence $\{x_n\}$ generated for $x_0 \in B(x^*, R)$ with $x_0 \neq x^*$, by the method (11) is well defined, remains in $B(x^*, R)$ for each $n \in \mathbb{N}$ and converges to x^*, a solution of the equation $F(x) = 0$.*

Proof. Previously, we have proved for, $x_0 \in B(x^*, R)$ with $x_0 \neq x^*$, that $[d_1(x_0), d_2(x_0); F]^{-1}$ exists, $\|x_1 - x^*\| < \|x_0 - x^*\|$ and besides $x_1 \in B(x^*, R)$.

Now, we suppose that $[d_1(x_{k-1}), d_2(x_{k-1}); F]^{-1}$ exists, $\|x_k - x^*\| < \|x_{k-1} - x^*\|$ and besides $x_k \in B(x^*, R)$. Let us assume that $x_k \neq x^*$, in other case the proof is completed. Then, by Lemma 1, we obtain that $[d_1(x_k), d_2(x_k); F]^{-1}$ exists and

$$\|[d_1(x_k), d_2(x_k); F]^{-1}\| \leq \frac{\beta}{1 - \beta\,\widetilde{\omega}(\omega_1(R), \delta + \omega_2(R))}.$$

So, x_{k+1} is well defined. Besides, from (16) and $(C4)$, we have $\|x_{k+1} - x^*\| < \|x_k - x^*\|$ and $x_{k+1} \in B(x^*, R)$.

So, we get by a mathematical induction procedure that $\|x_{n+1} - x^*\| < \|x_n - x^*\| < R$, which shows that $x_n \in B(x^*, R)$, for $n \in \mathbb{N}$, and $\lim\limits_{n \to +\infty} x_n = x^*$. □

Concerning the uniqueness of the solution x^*, we have the following result.

Theorem 3 *Under the conditions $(C1)$–$(C5)$ further suppose that there exists $R_u \geq R$ such that*

$$\beta\widetilde{\omega}(0, \delta + R_u) < 1. \tag{19}$$

Then, the limit point x^* is the only solution of equation $F(x) = 0$ in $\overline{B(x^*, R_u)} \cap \Omega$.

Proof. Let $y^* \in \overline{B(x^*, R_u)} \cap \Omega$ be such that $F(y^*) = 0$. Define $Q = [x^*, y^*; F]$. Then, using $(C2')$ and (19), we get in turn that

$$\|[x^*, \tilde{x}; F]^{-1}([x^*, y^*; F] - [x^*, \tilde{x}; F])\| \leq \beta\, \widetilde{\omega}(\|x^* - x^*\|, \|y^* - \tilde{x}\|) \leq \beta\, \widetilde{\omega}(0, \delta + R_u) < 1.$$

Hence, $Q^{-1} \in \mathcal{L}(X, X)$.

In view of the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. $\square$

4.1 Particular iterative processes in Banach spaces

In this section, we consider different particular cases of the family of iterative processes given by (11) in Banach spaces. For this study, notice that we must consider particular situations for the data functions d_1 and d_2. So, we will need to calculate the values of ω_1 and ω_2 such that the condition $(C3)$ is verified. Besides, when the functions $d_1(t)$ and $d_2(t)$ are known, in each case, the bounds obtained in (16) can be lower if we simplify the equation (17). Notice that, as the data known in each step are "x_n" and "$F(x_n)$" the idea to define a iterative process point to point is consider a combination of these values. In the particular cases that we study, we can consider some different conditions as for example that F is a Lipschitz-centered operator in the solution x^*, i. e., $\|F(x)-F(x^*)\| \leq k\|x-x^*\|$ or, equivalently in this case, that the operator "F" verifies that $\|F(x)\| \leq k\|x - x^*\|$. However, forward we will consider $\|[x, x^*; F]\| \leq \alpha$, for each $x \in \Omega$.

4.1.1 Steffensen method

In this first case, we consider

$$d_1(x) = x, \quad d_2(x) = x + F(x),$$

and therefore, we obtain

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n, x_n + F(x_n); F]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{20}$$

Then, in the condition $(C3)$, we have

$$\omega_1(t) = t, \quad \omega_2(t) = (1 + \alpha)t,$$

so, in $(C4)$, the equation (17) is reduced to

$$\beta\ (\omega(2t, (1 + \alpha)t) + \widetilde{\omega}(t, \delta + (1 + \alpha)t)) - 1 = 0 \tag{21}$$

and, in $(C5)$, we will consider $\widetilde{R} = (1 + \alpha)R$ and $\beta\, \widetilde{\omega}(R, \delta + (1 + \alpha)R) \neq 1$.

With the initial conditions $(C1)$–$(C3)$, from the previous conditions $(C4)$ and $(C5)$, the result of local convergence given in the Theorem 4, for Steffensen's method, is obtained.

4.1.2 Backward-Steffensen method

In second place, we consider

$$d_1(x) = x - F(x), \quad d_2(x) = x,$$

and, from (11), we obtain

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n - \ F(x_n), x_n; F]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{22}$$

For this method, we have

$$\omega_1 t) = (1+\alpha)t, \quad \omega_2(t) = t.$$

So, the previous conditions $(C4)$ and $(C5)$, can be expressed in the following form:

$(C4)$ The equation

$$\beta \ (\omega((2+\alpha)t, t) + \widetilde{\omega}((1+\alpha)t, \delta + t)) - 1 = 0 \tag{23}$$

has at least one positive real root, the smallest positive root is denoted by R.

$(C5)$ $B(x^*, \widetilde{R}) \subseteq \Omega$, with $\widetilde{R} = (1+\alpha)R$ and $\beta\,\widetilde{\omega}((1+\alpha)R, \delta + R) \neq 1$.

If the initial conditions $(C1)$–$(C3)$ are satisfied; in this situation, from the previous conditions $(C4)$ and $(C5)$, we can ensure the local convergence of the Backward-Steffensen's method, to a solution of the equation $F(x) = 0$, from Theorem 4.

4.1.3 Central-Steffensen method

In third place, we consider

$$d_1(x) = x - F(x), \quad d_2(x) = x + F(x),$$

and, if we apply these data functions to equation (11), we obtain

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n - F(x_n), x_n + F(x_n); F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{24}$$

For this iterative process, we have

$$\omega_1(t) = \omega_2(t) = (1+\alpha)t.$$

Then, in $(C4)$ the equation (17) is reduced to

$$\beta \ (\omega(((2+\alpha)t, (1+\alpha)t) + \widetilde{\omega}((1+\alpha)t, \delta + (1+\alpha)t)) - 1 = 0 \tag{25}$$

and, in $(C5)$, we have $\widetilde{R} = (1+\alpha)R$ and $\beta\,\widetilde{\omega}(R, \delta + (1+\alpha)R) \neq 1$.

If the initial conditions $(C1)$–$(C3)$ are satisfied and the specific conditions $(C4)$–$(C5)$ given for the Central-Steffensen method are satisfied, then, the sequence $\{x_n\}$ generated for $x_0 \in B(x^*, R)$, with $x_0 \neq x^*$, by the method (24) is well defined, remains in $B(x^*, R)$ for each $n \in \mathbb{N}$ and converges to x^*, a solution of the equation $F(x) = 0$.

4.1.4 Generalized Steffensen-type method

In the last place, we consider the data functions given by

$$d_1(x) = x - a\,F(x), \quad d_2(x) = x + b\,F(x),$$

for $a, b \in \mathbb{R}_+$. Then, from (11), we obtain

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n - a\,F(x_n), x_n + b\,F(x_n); F]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{26}$$

.

The condition $(C4)$ for this method tuns out to be

$$\omega_1(t) = (1 + a\,\alpha)t, \quad \omega_2(t) = (1 + b\,\alpha)t.$$

The equation (17) in $(C4)$ is now

$$\beta\ (\omega((2 + a\,\alpha)t, (1 + b\,\alpha)t) + \widetilde{\omega}((1 + a\,\alpha)t, \delta + (1 + b\,\alpha)t)) - 1 = 0, \tag{27}$$

and condition $(C5)$: $B(x^*, \widetilde{R}) \subseteq \Omega$, with $\widetilde{R} = \max\{(1 + a\,\alpha)R, (1 + b\,\alpha)R\}$, and $\beta\,\widetilde{\omega}((1 + a\,\alpha)R, \delta + (1 + b\,\alpha)R) \neq 1$.

If these previous conditions $(C4)$ and $(C5)$ and the initial conditions $(C1)$–$(C3)$ are satisfied, we get a result of local convergence for the Generalized Steffensen-type method, analogous to the above Theorem 4.

To finish the study of the particular cases for the family of iterative processes given by (11), we observe that the uniqueness Theorem 3 is valid for each of the methods described in this section, taking into account that the radius R will be obtained using (21), (23), (25) or (27).

4.2 On the accessibility from the ball of convergence

Finally, we study the theoretical accessibility of the methods developed in the previous section, we show as the balls of convergence are in these cases. As all of this iterative methods are, derivative-free methods, we will use the non-differentiable equation

$$f(z) \equiv z - \ \sigma z^3 - \mu|z| = 0 \tag{28}$$

where $\sigma = \mu = 1/40$ and we use the divided diference given by $[x, y; f] = \frac{f(x) - f(y)}{x - y}$, and can define

$$\|[x, y; f] - [u, v; f]\| \leq 2|\mu| + 3|\sigma|\|z\|(\|x - u\| + \|y - v\|), \text{ with } z \in \Omega$$

This equation (28) has 3 different roots:

$$z_1 \approx -6.4031242374328485, \quad z_2 = 0, \quad z_3 \approx 6.244997998398398$$

The basin of attraction of a root z^* is defined as the set of points whose iterations converge to the root z^*. In order to see the accessibility, we draw basins of attraction, a point is painted in red if the iteration of the method starting in the point converges to the root z_1, in blue if it converges to z_2, in yellow if it converges to z_3 and in black those points for which there is no convergence to any of the roots.

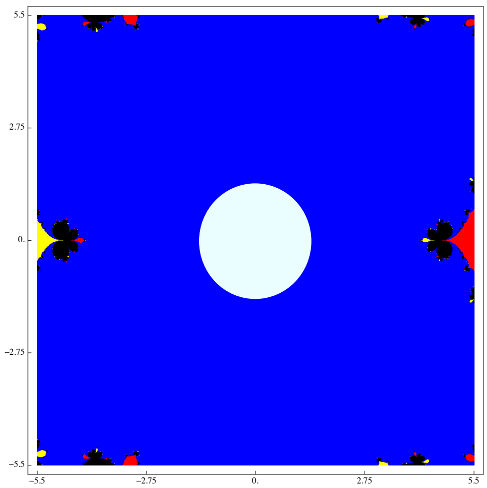

Figure 7: Back Steffensen method ball of convergence

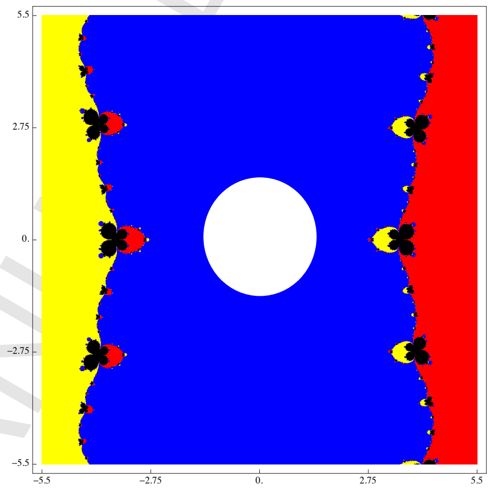

Figure 8: Center Steffensen method ball of convergence

Now, we consider $z^* = 0$ as solution of $f(z) = 0$. If $z \in B(0, r)$, it follows

$$\|[x, y; f] - [z^*, \tilde{z}; f]\| \leq 2|\mu| + 2|\sigma|\, r(\|x - z^*\| + \|y - \tilde{z}\|).$$

Therefore, we consider the functions

$$\omega(t_1, t_2) = 2|\mu| + 3|\sigma|\, r(t_1 + t_2), \quad \widetilde{\omega}(t_1, t_2) = 2|\mu| + 2|\sigma|\, r(t_1 + t_2)$$

Notice that the condition $(C4)$ for (IV) method giben in (27):

$$\beta\ (\omega((2 + a\,\alpha)t, (1 + b\,\alpha)t) + \widetilde{\omega}((1 + a\,\alpha)t, \delta + (1 + b\,\alpha)t)) - 1 = 0$$

in this case for (28) is

$$\beta\ (4|\mu| + |\sigma|\, r(5\alpha(a + b) + 13)t + 2|\sigma|\, r\delta) - 1 = 0,$$

and gets

$$t = \frac{1 - \beta(4|\mu| + 2|\sigma|\, r\delta)}{\beta|\sigma|\, r\,(5\alpha(a + b) + 13)} = \frac{40 - \beta(4 + 2r\delta)}{\beta\, r\,(5\alpha(a + b) + 13)}$$

Remark. We observe that in the previous equation in order to compute the ball of convergence, it appears the expression $a + b$. This fact, shows us that the methods for with the sum of both parameters we obtain the same radius of the ball of convergence. For example, for Steffensen's method ($a+b=0+1$), Backward-Steffensen's method ($a+b=1+0$) and some Generalized Steffensen-type methods ($a+b=0.5+0.5=0.25+0.75=...$) we obtain the same value of the radius.

Now, we take $\tilde{z}=0.01$, then $\delta=0.01$, and $\beta=1.02564$.

For Steffensen's method, Backward-Steffensen's method and Generalized Steffensen-type methods (such as $a+b=1$), we obtain:

$$R=1.35671, \quad (1+\alpha)R=2.98477, \quad \widetilde{\omega}(R,\delta+(1+\alpha)R)=0.354038<1$$

On the other hand, if in Generalized Steffensen-type methods we consider $a+b=0.5$ and $a+b=0.3$ the ball of convergence have the radii $R_1=1.47839$ and $R_2=1.5371$ respectively.

So, in all previous situations, the hypotheses of the local convergence result are fulfilled and the sequence $\{z_n\}$ is well defined and converges to $z^*=0$.

Using the radius of convergence associated to the methods we draw the ball in white. In Figures 9-10 the convergence balls associated to different methods are shown.

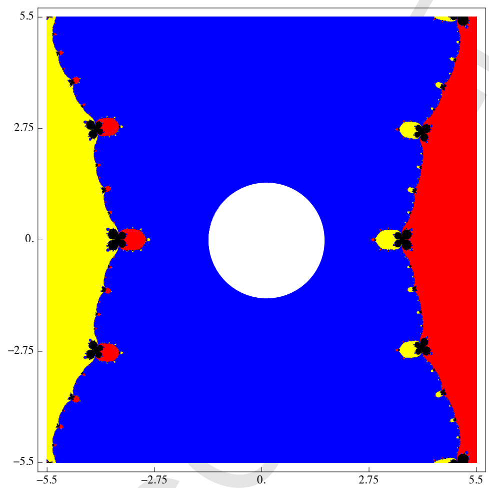

Figure 9: The Generalized Steffensen method with $a=b=1/2$

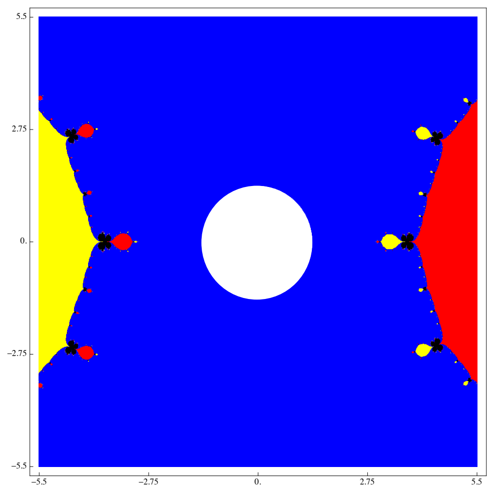

Figure 10: The Generalized Steffensen method with $a=b=15/100$

5 Another iterative processes with central divided differences

As we can seen, the iterative processes free-derivative defined by means central divided differences have a special role in the resolution of equations. In general, the central divided differences approximate better the derivative than other type of divided differences. In this situation, the approximation is better when the data are near of the point where we want approximate the derivative. For apply this idea, we consider the iterative processes given by the following algorithm:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n - \lambda_n F(x_n), x_n + \theta_n F(x_n); F]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{29}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ are two convergent sequences of positive real numbers. We denote $\lim_{n\to+\infty} \lambda_n = \lambda$ and $\lim_{n\to+\infty} \theta_n = \theta$. In general, as you can see in the Section 6, the best situations will appear when $\{\lambda_n\}, \{\theta_n\} \subseteq (0,1)$ and even when $\lambda = \theta = 0$.

Taking into account that, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_n - \lambda_n F(x_n) - x^*\| &\leq \|x_n - x^*\| + \lambda_n \|F(x_n) - F(x^*)\| \\ &\leq \|x_n - x^*\| + \Lambda \|[x_n, x^*; F](x_n - x^*)\| \|x_n - x^*\| \\ &\leq (1 + \Lambda\alpha) \|x_n - x^*\| \end{aligned} \tag{30}$$

and, analogously, we obtain

$$\|x_n - \theta_n F(x_n) - x^*\| \leq (1 + \Theta\,\alpha) \|x_n - x^*\|,$$

where $\Lambda = \max\{\lambda_n, n \in \mathbb{N}\}$ and $\Theta = \max\{\theta_n, n \in \mathbb{N}\}$. Then, we have that $\omega_1(t) = (1 + \Lambda\alpha)t$ and $\omega_2(t) = (1 + \Theta\alpha)t$.

On the other hand, notice that the equation (17) in $(C4)$ is now

$$\beta\ (\omega((2 + \Lambda\alpha)t, (1 + \Theta\,\alpha)t) + \widetilde{\omega}((1 + \Lambda\,\alpha)t, \delta + (1 + \Theta\,\alpha)t)) - 1 = 0, \tag{31}$$

and condition $(C5)$: $B(x^*, \widetilde{R}) \subseteq \Omega$, with $\widetilde{R} = \max\{(1 + \Lambda\,\alpha)R, (1 + \Theta\,\alpha)R\}$, and $\beta\,\widetilde{\omega}((1 + a\,\alpha)R, \delta + (1 + b\,\alpha)R) \neq 1$.

So, we obtain the following result for iterative methods given in (29).

Theorem 4 *Suppose that conditions $(C1)$–$(C3)$ are satisfied and we assume the following items:*

(i) The equation

$$\beta\ (\omega((2 + \Lambda\alpha)t, (1 + \Theta\,\alpha)t) + \widetilde{\omega}((1 + \Lambda\,\alpha)t, \delta + (1 + \Theta\,\alpha)t)) - 1 = 0$$

has at least one positive real root, the smallest positive root is denoted by R.

(ii) $B(x^*, \widetilde{R}) \subseteq \Omega$, *with* $\widetilde{R} = max\{(1+\Lambda\,\alpha)R, (1+\Theta\,\alpha)R\}$, *and* $\beta\,\widetilde{\omega}((1+\Lambda\,\alpha)R, \delta+(1+\Theta\,\alpha)R) \neq 1$.

Then, sequence $\{x_n\}$ *generated for* $x_0 \in B(x^*, R)$, *with* $x_0 \neq x^*$, *by the method (29) is well defined, remains in* $B(x^*, R)$ *for each* $n \in \mathbb{N}$ *and converges to* x^*, *a solution of the equation* $F(x) = 0$.

In order to draw the basins of attraction for iterative methods (29), a point is painted in red if the iteration of the method starting in the point converges to the root z_1, in blue if it converges to z_2, in yellow it it converges to z_3 and in black those points for which there is no convergence to any of the roots. We choose a tolerance of 10^{-3} and a limit of 200 iterations.

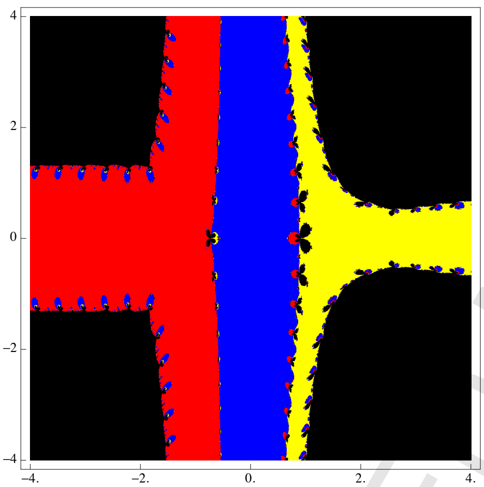

Figure 11: The method (29) with $\lambda_n = \theta_n = \frac{1}{n}$.

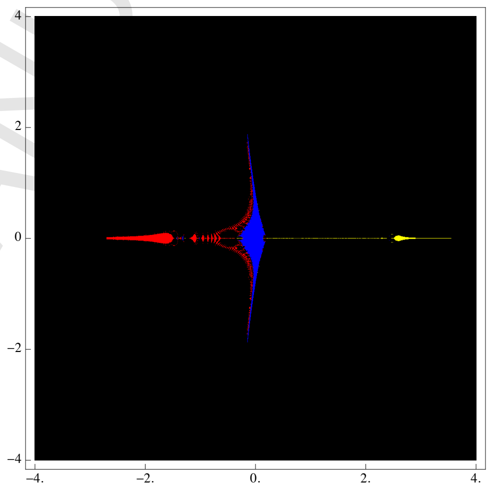

Figure 12: The method (29) with $\lambda_n = \frac{n}{n+1}$ and $\theta_n = \frac{5n}{n+1}$.

In Figures 11-14 the basins of attraction associated to the roots of the polynomial f, given in (28), are shown. As it can be seen, again, the best methods in terms of convergence are the methods in which λ_n and θ_n are close and moreover, as $n \to \infty$, both sequences tend to 0. Method even is better if the order of convergence to 0 is greater.

6 Numerical experiment

We consider as in [11], nonlinear integral equations of mixed Hammerstein type of type

$$x(s) = f(s) + \int_a^b G(s,t)\, H(t, x(t))\, dt, \quad s \in [a,b], \tag{32}$$

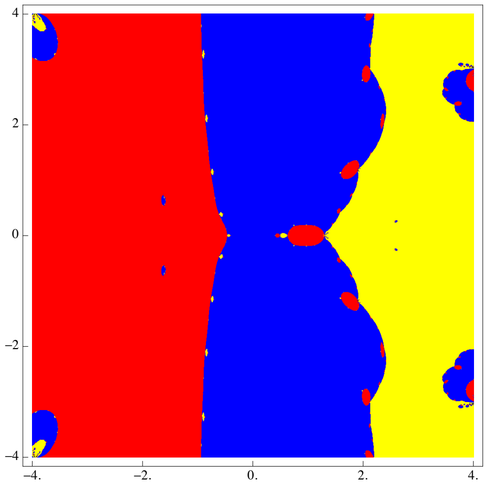

Figure 13: The method (29) with $\lambda_n = \theta_n = \frac{1}{n^4}$.

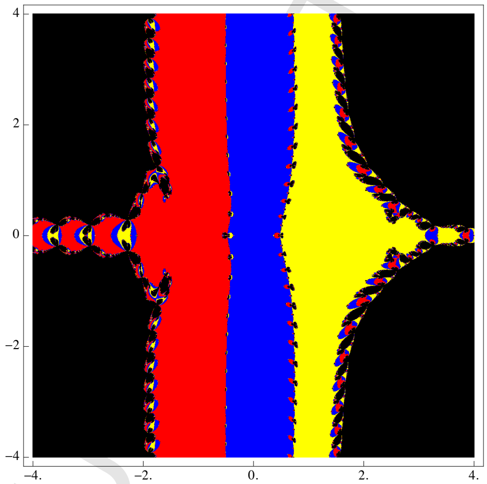

Figure 14: The method (29) with $\lambda_n = \frac{1}{n}$ and $\theta_n = \frac{1}{n^5}$.

where $-\infty < a < b < +\infty$, f, G y H are known functions and x is a solution to be determined. Integral equations of this type appear very often in several applications to real world problems. For example, in problems of dynamic models of chemical reactors [6], vehicular traffic theory, biology and queuing theory [8]. The Hammerstein integral equations also appear in the electro-magnetic fluid dynamics and can be reformulated as two-point boundary value problems with certain nonlinear boundary conditions and in multi-dimensional analogues which appear as reformulations of elliptic partial differentiable equations with nonlinear boundary conditions (see [22] and the references given there).

Solving equation (32) is equivalent to solve $\mathcal{F}(x) = 0$, where $\mathcal{F} : \Omega \subset \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ and

$$[\mathcal{F}(x)](s) = x(s) - f(s) - \int_a^b G(s,t)H(t,x(t))\,dt, \quad s \in [a,b].$$

Examples where the operator $\mathcal{F}$ is differentiable are found in [12]. Note that any operator F is differentiable if the divided difference of first order of F is Lipschitz or Hölder continuous in Ω, see [20].

If we consider (32) where G is the Green function in $[a,b] \times [a,b]$, we then use a discretization process to transform equation (32) into a finite dimensional problem by approximating the integral of (32) by a Gauss-Legendre quadrature formula with m nodes:

$$\int_a^b q(t)\,dt \simeq \sum_{i=1}^{m} w_i q(t_i),$$

where the nodes t_i, in $[a,b]$, and the weights w_i are determined.

If we denote the approximations of $x(t_i)$ and $f(t_i)$ by x_i and f_i, respectively, with $i = 1,2,\ldots,m$, then equation (32) is equivalent to the following system of nonlinear equations:

$$x_i = f_i + \sum_{j=1}^{m} a_{ij}\, H(t_j, x_j), \quad j = 1, 2, \ldots, m, \tag{33}$$

where

$$a_{ij} = w_j G(t_i, t_j) = \begin{cases} w_j \frac{(b-t_i)(t_j-a)}{b-a}, & j \le i, \\ w_j \frac{(b-t_j)(t_i-a)}{b-a}, & j > i. \end{cases}$$

Now, system (33) can be written as

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{f} - A\,\mathbf{z} = 0, \qquad F : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \tag{34}$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_m)^T, \qquad \mathbf{f} = (f_1, f_2, \ldots, f_m)^T, \qquad A = (a_{ij})_{i,j=1}^m,$$
$$\mathbf{z} = (H(t_1, x_1), H(t_2, x_2), \ldots, H(t_m, x_m))^T.$$

As in $\mathbb{R}^m$ we can consider divided difference of first order that do not need that the function F is differentiable (see [20]), we then use the divided difference of first order given by $[\mathbf{u}, \mathbf{v}; F] = ([\mathbf{u}, \mathbf{v}; F]_{ij})_{i,j=1}^m \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, where

$$[\mathbf{u}, \mathbf{v}; F]_{ij} = \frac{1}{u_j - v_j}\left(F_i(u_1, \ldots, u_j, v_{j+1}, \ldots, v_m) - F_i(u_1, \ldots, u_{j-1}, v_j, \ldots, v_m)\right),$$

$\mathbf{u} = (u_1, u_2, \ldots, u_m)^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_m)^T$.

We consider a nonlinear integral equation of Hammerstein-type defined in (32) with $a = 0$, $b = 1$, $G(s,t)$ is the Green function and $H(t, x(t)) = \lambda x(t)^3 + \mu |x(t)|$. Then, the system of nonlinear equations (34) is of the form

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{f} - A\left(\lambda \mathbf{v}_{\mathbf{x}} + \mu \mathbf{w}_{\mathbf{x}}\right) = 0, \qquad F : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \tag{35}$$

where

$$\mathbf{v}_{\mathbf{x}} = (x_1^3, x_2^3, \ldots, x_m^3)^T, \qquad \mathbf{w}_{\mathbf{x}} = (|x_1|, |x_2|, \ldots, |x_m|)^T,$$

$\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$, it is obvious that the function F defined in (35) is nonlinear and non-differentiable. In this case, we have

$$[\mathbf{x}, \mathbf{y}; F] = I - A \cdot \ \mathrm{Diag}\left(\lambda \begin{pmatrix} x_1^2 + x_1 y_1 + y_1^2 \\ x_2^2 + x_2 y_2 + y_2^2 \\ \cdots\cdots\cdots \\ x_m^2 + x_m y_m + y_m^2 \end{pmatrix} + \mu \begin{pmatrix} \frac{|x_1| - |y_1|}{x_1 - y_1} \\ \frac{|x_2| - |y_2|}{x_2 - y_2} \\ \cdots\cdots\cdots \\ \frac{|x_m| - |y_m|}{x_m - y_m} \end{pmatrix}\right)$$

Then, $[\mathbf{u}, \mathbf{v}; F] = I - (\lambda B + \mu C)$, where $B = (b_{ij})_{i,j=1}^m$ with $b_{ii} = a_{ii}(x_i^2 + x_i y_i + y_i^2)$ and $b_{ij} = 0$ if $i \neq j$, $C = (c_{ij})_{i,j=1}^m$ with $c_{ii} = a_{ii} \frac{|x_i| - |y_i|}{x_i - y_i}$ and $c_{ij} = 0$ if $i \neq j$.

Note that the divided difference of first order of the function F satisfies a condition of type

$$\|[x,y;F]-[u,v;F]\| \leq L+K(\|x-u\|+\|y-v\|); L,K \geq 0; x,y,u,v \in \Omega; x \neq y; u \neq v, \quad (36)$$

in $\mathbb{R}^m$, instead of a condition of Lipschitz or Hölder type. Then, we can solve equations where the operator F is non-differentiable, as for example equation (35).

Now, we consider $\mathbf{f} = \mathbf{0}$ in (35), $m = 8$ and $\Omega = B(0,\tau)$. Then, the system of nonlinear equations (35) is of the form

$$F(\mathbf{x}) = \mathbf{x} - A\,\mathbf{z}, \qquad z_j = \lambda x_j^3 + \mu|x_j|, \; j = 1,\ldots,m.$$

Obviously, in this case, $\mathbf{x}^* = \mathbf{0}$ is a solution of $F(\mathbf{x}) = \mathbf{0}$. In these conditions, we have

$$\|[\mathbf{x},\mathbf{y};F] - [\mathbf{u},\mathbf{v};F]\| \leq 3\tau|\lambda|\,\|A\|(\|\mathbf{x}-\mathbf{u}\| + \|\mathbf{y}-\mathbf{v}\|) + 2|\mu|\,\|A\|,$$

and then, it follows

$$\|[\mathbf{x},\mathbf{y};F] - [\mathbf{x}^*,\tilde{\mathbf{x}};F]\| \leq 2\tau|\lambda|\,\|A\|(\|\mathbf{x}-\mathbf{x}^*\| + \|\mathbf{y}-\tilde{\mathbf{x}}\|) + 2|\mu|\,\|A\|.$$

So, we obtain the following functions

$$\omega(s,t) = \|A\|(3\tau|\lambda|(s+t) + 2|\mu|) \text{ and } \quad \tilde{\omega}(\mathrm{s},\mathrm{t}) = \|\mathrm{A}\|(|\lambda|2\tau(\mathrm{s}+\mathrm{t}) + 2|\mu|),$$

Indeed, if we choose $\tau = 3.5$ and $\tilde{\mathbf{x}}$ with $\tilde{x}_j = 0.01$, $j = 1,\ldots,8$, and $\lambda = \mu = 1/9$, we obtain

$$\|A\| = 0.123632, \quad \delta = 0.01, \quad \alpha = 1.05495, \quad \beta = 1.0139.$$

where $\beta = \|[x^*,\tilde{x};F]^{-1}\|$.

For Steffensen and Backward-Steffensen methods, (20) and (22), has as unique solution for $R = 1.576497$ and it is verified

$$(1+\alpha)R = 3.239623 < \tau \quad \text{and} \quad \beta\widetilde{\omega}((1+a\,\alpha)R, \delta + (1+\alpha)R) = 0.498376 < 1.$$

Then, the hypotheses of Theorem 4 are fulfilled. The ball of convergence and the domain of uniqueness of solution are, respectively

$$\{\mathbf{x} \in \mathbb{R}^8 : \|\mathbf{x}\| \leqslant 1.576497\} \text{ and } \{\mathbf{x} \in \mathbb{R}^8 : \|\mathbf{x}\| \leqslant 5.81157\} \cap \Omega.$$

With the same initial conditions, we will show the behavior of the distinct methods. Then, we obtain the following results

The Central-Steffensen method, equation (24), The equation (25) has as unique solution $R = 1.10266$ and it is verified

$$(1+\alpha)R = 2.26591 < \tau \quad \text{and} \quad \beta\widetilde{\omega}((1+a\,\alpha)R, \delta + (1+\alpha)R) = 0.470658 < 1.$$

Then, the hypotheses of Theorem 4 are fulfilled.

In the case of Generalized Steffensen-type method, we consider, for example, $a = b = 0.25$. The equation (27) has as unique solution $R = 2.0079$ and

$$\widetilde{R} = \max\{R, (1+a\,\alpha)R, (1+b\,\alpha)R\} = 2.53749, \quad \beta\,\widetilde{\omega}((1+a\,\alpha)R, \delta+(1+b\,\alpha)R) = 0.523613 \neq 1$$

and therefore $B(x^*,\widetilde{R}) \subseteq \Omega$. Then, the hypotheses of Theorem 4 are fulfilled.

7 Conclusions

Many problems can be written in the form of equation (1) using Mathematical Modeling. In this study, in particular, we provide a local convergence analysis for iterative processes free of derivatives (4), when the operator F is not necessarily Féchet differentiable. The sufficient convergence conditions are weaker and more flexible than in earlier studies. An application is provided involving mixed Hammerstein nonlinear integral equation with application in real world problems. Finally, we show that the dynamical analysis of the generalized method can be as good as we want by menas of choosing appropriate values of the parameters a and b (small and close).

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