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Convolution integral equations with Gegenbauer function kernel

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Abstract

In this paper we develop a concise and transparent approach for solving Mellin convolution equations where the convolutor is the product of an algebraic function and a Gegenbauer function. Our method is primarily based on

(1) the use of fractional integral/differential operators;

(2) a formula for Gegenbauer functions which is a fractional extension of the Rodrigues formula for Gegenbauer polynomials (see Theorem 3);

(3) an intertwining relation concerning fractional integral/differential operators (see Theorem 1), which in the integer case reads $(\mathrm{d}/\mathrm{d}x)^{2n+1} = (x^{-1}\,\mathrm{d}/\mathrm{d}x)^n x^{2n+1}(x^{-1}\,\mathrm{d}/\mathrm{d}x)^{n+1}$.

Thus we cover most of the known results on this type of integral equations and obtain considerable extensions. As a special illustration we present the Gegenbauer transform pair associated to the Radon transformation.

Key words: Gegenbauer functions; Gegenbauer transformations; Rodrigues formula; Fractional calculus; Integral equations

1. Weyl fractional calculus

First, for convenience we introduce the operators D and M_α, $\alpha \in \mathbb{R}$, by $Df = f'$ and $M_\alpha f = x^\alpha f$. On the space

$$S_{\to}(\mathbb{R}^+) = \{f \in C^\infty(\mathbb{R}^+)\colon M_k D^l f \in L_\infty([1,\infty)),\ k, l \in \mathbb{N}_0\},$$

the Erdélyi–Kober operators $W_{\mu,p}$, $\mu \in \mathbb{R}$, $p > 0$, are defined by (cf. [6])

$$(W_{\mu,p} f)(x) = \frac{p^{1-\mu}}{\Gamma(\mu)} \int_x^\infty f(t)(t^p - x^p)^{\mu-1} t^{p-1}\,\mathrm{d}t, \quad f \in S_{\to}(\mathbb{R}^+),\ x > 0,\ \mu > 0,$$

$$W_{0,p} = I, \qquad W_{-\mu,p} = W_{\alpha,p}(-M_{1-p}D)^n, \quad \mu = n - \alpha,\ 0 \leqslant \alpha < 1,\ n \in \mathbb{N}.$$

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The operators $W_{\mu,p}$, $\mu \in \mathbb{R}$, constitute a group on $S_{-}(\mathbb{R}^+)$. For $p = 1$, the operator $W_{\mu,1} = W_\mu$ is the classical Weyl operator; for $p = 2$, we use the notation $W_{\mu,2} = \mathbb{W}_\mu$.

The intertwining relation, stated in Theorem 1, is fundamental for this paper.

Theorem 1. *For* $\mu \in \mathbb{R}$,

$$W_{2\mu+1} = \mathbb{W}_{\mu+1} M_{-2\mu-1} \mathbb{W}_\mu.$$

Proof. Without loss of generality we may assume that $\mu > -1$, because, having proved the relation for $\mu > -1$, the relation for $\mu < 0$ then follows by replacing μ by $-\mu - 1$ and by inverting the operators on both sides of the equality. For $f \in S_{-}(\mathbb{R}^+)$ and $x > 0$,

$$\begin{aligned}
&(\mathbb{W}_{\mu+1} M_{-2\mu-1} \mathbb{W}_\mu f)(x) \\
&\quad = (\mathbb{W}_{\mu+1} M_{-2\mu-1} \mathbb{W}_{\mu+1} (\mathbb{W}_{-1} f))(x) \\
&\quad = \frac{2^{-2\mu}}{\Gamma^2(\mu+1)} \int_x^\infty \left(r^{-2\mu-1} \int_r^\infty (W_{-1} f)(t)(t^2 - r^2)^\mu \, dt \right) (r^2 - x^2)^\mu r \, dr \\
&\quad = \frac{2^{-2\mu}}{\Gamma^2(\mu+1)} \int_x^\infty (W_{-1} f)(t) \left(\int_x^t (r^2 - x^2)^\mu (t^2 - r^2)^\mu r^{-2\mu} \, dr \right) dt.
\end{aligned}$$

Substituting $\xi = (r^2 - x^2)/(t^2 - x^2)$ and using formulas [7, 2.1.3(10), 2.8(6)], the inner integral equals

$$\int_x^t (r^2 - x^2)^\mu (t^2 - r^2)^\mu r^{-2\mu} \, dr = 2^{2\mu} B(\mu+1, \mu+1)(t - x)^{2\mu+1}.$$

Therefore, we obtain

$$(\mathbb{W}_{\mu+1} M_{-2\mu-1} \mathbb{W}_\mu f)(x) = (W_{2\mu+2} W_{-1} f)(x) = (W_{2\mu+1} f)(x),$$

and the stated identity for $\mu > -1$ has been settled. □

Corollary 2. *For* $\lambda, \mu \in \mathbb{R}$,

(i) $W_\mu \mathbb{W}_\lambda = W_{\mu+2\lambda-1} \mathbb{W}_{-\lambda+1} M_{2\lambda-1}$;

(ii) $\mathbb{W}_\lambda W_\mu = M_{2\lambda+1} \mathbb{W}_{-\lambda-1} W_{\mu+2\lambda+1}$.

2. Rodrigues-type formulas for Gegenbauer functions

In Section 3 we study the operator products $W_\mu \mathbb{W}_\lambda$ and $\mathbb{W}_\lambda W_\mu$. For $\lambda + \mu > 0$, these operator products are integral operators with so-called Gegenbauer functions in their kernels. To prove this, we establish formulas for Gegenbauer functions which are extensions of the Rodrigues formula for the Gegenbauer polynomials.

We start with the introduction of some special functions. For $\mu > -\frac{1}{2}$ with $\mu \neq 0$ and $n \in \mathbb{N}_0$, the Gegenbauer polynomial C_n^μ may be defined by the Rodrigues formula, cf. [7, 3.15.1(10)],

$$C_n^\mu(x) = \frac{(-2)^n \Gamma(\mu+n)\Gamma(2\mu+n)}{n!\,\Gamma(\mu)\Gamma(2\mu+2n)}(1-x^2)^{1/2-\mu}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left[(1-x^2)^{n+\mu-1/2}\right].$$

The Chebyshev polynomial T_n and the Legendre polynomial P_n are special Gegenbauer polynomials:

$$C_n^0(x) = \lim_{\mu\to 0}\frac{1}{\mu}C_n^\mu(x) = \frac{2}{n}T_n(x) = \frac{2}{n}\cos(n \arccos x),$$

$$C_n^{1/2}(x) = P_n(x) = (2^n\, n!)^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^2-1)^n.$$

For $\mu, \nu \in \mathbb{R}$ the Legendre functions $\mathscr{P}_\nu^\mu$ and $\mathbb{P}_\nu^\mu$ and the Gegenbauer function $\boldsymbol{C}_\nu^\mu$ can be defined through hypergeometric functions, cf. [7, 3.2(16), 3.4(1), 3.15.1(4)],

$$\mathscr{P}_\nu^\mu(x) = \frac{2^{-\nu}}{\Gamma(1-\mu)}\frac{(x+1)^{\nu+\mu/2}}{(x-1)^{\mu/2}}\,{}_2F_1\left(-\nu, -\nu-\mu; 1-\mu; \frac{x-1}{x+1}\right), \quad x>1,$$

$$\mathbb{P}_\nu^\mu(x) = \frac{2^{-\nu}}{\Gamma(1-\mu)}\frac{(1+x)^{\nu+\mu/2}}{(1-x)^{\mu/2}}\,{}_2F_1\left(-\nu, -\nu-\mu; 1-\mu; \frac{x-1}{x+1}\right), \quad 0<x<1,$$

$$\boldsymbol{C}_\nu^\mu(x) = \frac{2^{-\nu-\mu+1/2}}{\Gamma(\mu+\frac{1}{2})}(x+1)^\nu\,{}_2F_1\left(-\nu-\mu+\tfrac{1}{2}, -\nu; \mu+\tfrac{1}{2}; \frac{x-1}{x+1}\right), \quad x>0.$$

For $\nu = n \in \mathbb{N}_0$ and $\mu > -\frac{1}{2}$, the function $\boldsymbol{C}_n^\mu$ equals C_n^μ up to a multiplicative constant,

$$\boldsymbol{C}_n^\mu(x) = 2^{1/2-\mu}\frac{\Gamma(2\mu)\,n!}{\Gamma(n+2\mu)\Gamma(\mu+\frac{1}{2})}C_n^\mu(x), \quad \mu \neq 0, \tag{1}$$

$$\boldsymbol{C}_n^0(x) = \sqrt{\frac{2}{\pi}}\,T_n(x), \qquad \boldsymbol{C}_n^{1/2}(x) = P_n(x). \tag{2}$$

Furthermore, we observe that for $\mu > -\frac{1}{2}$,

$$\boldsymbol{C}_0^\mu(x) = \frac{2^{1/2-\mu}}{\Gamma(\mu+\frac{1}{2})}. \tag{3}$$

For later convenience we introduce the notations $\mathscr{C}_\nu^\mu$ and $\mathscr{G}_\nu^\mu$ denoting the restrictions of $\boldsymbol{C}_\nu^\mu$ to $(1, \infty)$ and $(0, 1)$, respectively, so that

$$\mathscr{C}_\nu^\mu(x) = \boldsymbol{C}_\nu^\mu(x) = (x^2-1)^{1/4-\mu/2}\mathscr{P}_{\nu+\mu-1/2}^{1/2-\mu}(x), \quad x>1, \tag{4}$$

$$\mathscr{G}_\nu^\mu(x) = \boldsymbol{C}_\nu^\mu(x) = (1-x^2)^{1/4-\mu/2}\mathbb{P}_{\nu+\mu-1/2}^{1/2-\mu}(x), \quad 0<x<1. \tag{5}$$

Since $\mathscr{P}_\nu^\mu = \mathscr{P}_{-\nu-1}^\mu$ and $\mathbb{P}_\nu^\mu = \mathbb{P}_{-\nu-1}^\mu$, see [7, 3.3.1(1), 3.4(7)], we have

$$\mathscr{C}_\nu^\mu = \mathscr{C}_{-\nu-2\mu}^\mu, \qquad \mathscr{G}_\nu^\mu = \mathscr{G}_{-\nu-2\mu}^\mu. \tag{6}$$

The following integral relations for Gegenbauer functions, which can be deduced from [8, 13.1 (52), (54)], generalize the Rodrigues formula for Gegenbauer polynomials.

Theorem 3. (i) *For* $\nu \in \mathbb{R}$, $\mu > 0$, $\lambda > -\frac{1}{2}$ *and* $x > 1$,

$$\frac{1}{\Gamma(\mu)} \int_1^x (t^2-1)^{\lambda-1/2} \mathscr{C}_\nu^\lambda(t)(x-t)^{\mu-1}\,dt = (x^2-1)^{\lambda+\mu-1/2} \mathscr{C}_{\nu-\mu}^{\lambda+\mu}(x).$$

(ii) *For* $\nu \in \mathbb{R}$, $\mu > 0$, $\lambda > -\frac{1}{2}$ *and* $0 < x < 1$,

$$\frac{1}{\Gamma(\mu)} \int_x^1 (1-t^2)^{\lambda-1/2} \mathscr{G}_\nu^\lambda(t)(t-x)^{\mu-1}\,dt = (1-x^2)^{\lambda+\mu-1/2} \mathscr{G}_{\nu-\mu}^{\lambda+\mu}(x).$$

Take $\mu = \nu = n \in \mathbb{N}$ in Theorem 3 and differentiate n times; then the integral relations pass into the Rodrigues formula for Gegenbauer polynomials. Further we mention the following corollary.

Corollary 4. (i) *For* $\nu \in \mathbb{R}$, $n \in \mathbb{N}_0$, $\lambda > n - \frac{1}{2}$ *and* $x > 1$,

$$\mathscr{C}_{\nu+n}^{\lambda-n}(x) = (x^2-1)^{n-\lambda+1/2}\left(\frac{d}{dx}\right)^n \left[(x^2-1)^{\lambda-1/2} \mathscr{C}_\nu^\lambda(x)\right].$$

(ii) *For* $\nu \in \mathbb{R}$, $n \in \mathbb{N}_0$, $\lambda > n - \frac{1}{2}$ *and* $0 < x < 1$,

$$\mathscr{G}_{\nu+n}^{\lambda-n}(x) = (1-x^2)^{n-\lambda+1/2}\left(-\frac{d}{dx}\right)^n \left[(1-x^2)^{\lambda-1/2} \mathscr{G}_\nu^\lambda(x)\right].$$

Proof. For $n = 0$, the formulas are obvious. For $n \in \mathbb{N}$, the stated formulas follow by setting $\mu = n$, $\lambda := \lambda - n$, $\nu := \nu + n$ in Theorem 3, and by differentiating the integral relations thus obtained n times. □

3. The Gegenbauer transformations

For $\lambda + \mu > 0$ the operator products $W_\mu \mathbb{W}_\lambda$ and $\mathbb{W}_\lambda W_\mu$ are integral operators with Gegenbauer functions in their kernels. This result has motivated us to introduce so-called Gegenbauer transformations. We establish several relations between Gegenbauer transformations which will be used in the next section to solve certain integral equations.

Theorem 5. *Let* $\lambda, \mu \in \mathbb{R}$ *with* $\lambda + \mu > 0$. *For* $f \in S_{-1}(\mathbb{R}^+)$,

(i) $$(W_\mu \mathbb{W}_\lambda f)(x) = \int_x^\infty (t^2-x^2)^{\lambda+\mu-1} \mathscr{G}_{-\mu}^{\lambda+\mu-1/2}\left(\frac{x}{t}\right) f(t) t^{1-\mu}\,dt, \quad x > 0;$$

(ii) $$(\mathbb{W}_\lambda W_\mu f)(x) = x^{1-\mu} \int_x^\infty (t^2-x^2)^{\lambda+\mu-1} \mathscr{C}_{1-\mu}^{\lambda+\mu-1/2}\left(\frac{t}{x}\right) f(t)\,dt, \quad x > 0.$$

Proof. (i) First suppose $\lambda, \mu > 0$, $\lambda + \mu > 0$. Starting from the definitions of W_μ and $\mathbb{W}_\lambda$ and by (3), we obtain for $f \in S_{\rightarrow}(\mathbb{R}^+)$ and $x > 0$,

$$(W_\mu \mathbb{W}_\lambda f)(x) = \frac{2^{1-\lambda}}{\Gamma(\lambda)\Gamma(\mu)} \int_x^\infty \left(\int_r^\infty f(t)(t^2 - r^2)^{\lambda-1} t \, dt \right) (r-x)^{\mu-1} \, dr$$

$$= \frac{1}{\Gamma(\mu)} \int_x^\infty \left(\int_x^t (t^2 - r^2)^{\lambda-1} \mathcal{G}_0^{\lambda-1/2}\left(\frac{r}{t}\right)(r-x)^{\mu-1} \, dr \right) f(t) t \, dt.$$

By Theorem 3(ii) the inner integral equals

$$\frac{1}{\Gamma(\mu)} \int_x^t (t^2 - r^2)^{\lambda-1} \mathcal{G}_0^{\lambda-1/2}\left(\frac{r}{t}\right)(r-x)^{\mu-1} \, dr = (t^2 - x^2)^{\lambda+\mu-1} \mathcal{G}_{-\mu}^{\lambda+\mu-1/2}\left(\frac{x}{t}\right) t^{-\mu}.$$

Hence for $\lambda, \mu > 0$, $\lambda + \mu > 0$, the stated integral relation is established. Next suppose $\lambda > 0$, $\mu \leqslant 0$, $\lambda + \mu > 0$. Choose $n \in \mathbb{N}$ such that $\mu + n > 0$. Using the previous result and Corollary 4(ii), we derive for $f \in S_{\rightarrow}(\mathbb{R}^+)$ and $x > 0$,

$$(W_\mu \mathbb{W}_\lambda f)(x) = (W_{-n} W_{\mu+n} \mathbb{W}_\lambda f)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \int_x^\infty (t^2 - x^2)^{\lambda+\mu+n-1} \mathcal{G}_{-\mu-n}^{\lambda+\mu+n-1/2}\left(\frac{x}{t}\right) f(t) t^{1-\mu-n} \, dt$$

$$= \int_x^\infty (t^2 - x^2)^{\lambda+\mu-1} \mathcal{G}_{-\mu}^{\lambda+\mu-1/2}\left(\frac{x}{t}\right) f(t) t^{1-\mu} \, dt.$$

Thus we have proved the wanted integral relation for $\lambda, \mu \in \mathbb{R}$, with $\lambda > 0$, $\lambda + \mu > 0$. Now for $\lambda \leqslant 0$, $\mu > 0$, $\lambda + \mu > 0$, the wanted integral relation follows from the identities in Corollary 2(i) and (6), viz.,

$$W_\mu \mathbb{W}_\lambda = W_{2\lambda+\mu-1} \mathbb{W}_{-\lambda+1} M_{2\lambda-1}, \qquad \mathcal{G}_{-2\lambda-\mu+1}^{\lambda+\mu-1/2} = \mathcal{G}_{-\mu}^{\lambda+\mu-1/2}.$$

(ii) This relation can be proved with the same techniques. □

Now we come to the following definition of the Gegenbauer transformations $\mathcal{W}\mathcal{G}_{\nu,\lambda}$ and $\mathcal{W}\mathcal{C}_{\nu,\lambda}$.

Definition 6. For $\nu, \lambda \in \mathbb{R}$,

(i) $\mathcal{W}\mathcal{G}_{\nu,\lambda} = W_{-\nu} \mathbb{W}_{\nu+\lambda+1/2} M_{-\nu}$;

(ii) $\mathcal{W}\mathcal{C}_{\nu,\lambda} = M_{1-\nu} \mathbb{W}_{\nu+\lambda-1/2} W_{1-\nu}$.

For $\lambda > -\frac{1}{2}$ a Gegenbauer transformation is an integral operator by Theorem 5.

Theorem 7. *Let $\nu, \lambda \in \mathbb{R}$ with $\lambda > -\frac{1}{2}$. For $f \in S_{\leftarrow}(\mathbb{R}^+)$,*

(i) $$(\mathcal{W}\mathcal{G}_{\nu,\lambda} f)(x) = \int_x^\infty (t^2 - x^2)^{\lambda-1/2} \mathcal{G}_\nu^\lambda\left(\frac{x}{t}\right) f(t) t \, dt, \quad x > 0;$$

(ii) $$(\mathcal{W}\mathcal{C}_{\nu,\lambda} f)(x) = x \int_x^\infty (t^2 - x^2)^{\lambda-1/2} \mathcal{C}_\nu^\lambda\left(\frac{t}{x}\right) f(t) \, dt, \quad x > 0.$$

For $\lambda < -\frac{1}{2}$, a Gegenbauer transformation is the product of a (fractional) differential operator and an integral operator.

Theorem 8. *For* $\nu, \lambda, \rho \in \mathbb{R}$,
(i) $\mathscr{W}\mathscr{G}_{\nu,\lambda} = \mathscr{W}\mathscr{G}_{\nu,\lambda+\rho}(M_\nu \mathbb{W}_{-\rho} M_{-\nu})$;
(ii) $\mathscr{W}\mathscr{C}_{\nu,\lambda} = (M_{1-\nu}\mathbb{W}_{-\rho}M_{\nu-1})\mathscr{W}\mathscr{C}_{\nu,\lambda+\rho}$.

Corresponding to (6), viz., $\mathscr{C}_\nu^\lambda = \mathscr{C}^\lambda_{-\nu-2\lambda}$, $\mathscr{G}_\nu^\lambda = \mathscr{G}^\lambda_{-\nu-2\lambda}$, we have the following counterparts for Gegenbauer transformations.

Theorem 9. *For* $\nu, \lambda \in \mathbb{R}$,
(i) $\mathscr{W}\mathscr{G}_{\nu,\lambda} = \mathscr{W}\mathscr{G}_{-\nu-2\lambda,\lambda}$;
(ii) $\mathscr{W}\mathscr{C}_{\nu,\lambda} = \mathscr{W}\mathscr{C}_{-\nu-2\lambda,\lambda}$.

Proof. (i) By means of Definition 6 and Corollary 2(i) we deduce

$$\mathscr{W}\mathscr{G}_{\nu,\lambda} = W_{-\nu}\mathbb{W}_{\nu+\lambda+1/2}M_{-\nu} = W_{\nu+2\lambda}\mathbb{W}_{-\nu-\lambda+1/2}M_{\nu+2\lambda} = \mathscr{W}\mathscr{G}_{-\nu-2\lambda,\lambda}.$$

The proof of (ii) runs along the same lines. □

It is a straightforward consequence of Definition 6 that the inverse of a Gegenbauer transformation is another Gegenbauer transformation.

Theorem 10. *For* $\nu, \lambda \in \mathbb{R}$,
(i) $\mathscr{W}\mathscr{G}_{\nu,\lambda}^{-1} = \mathscr{W}\mathscr{C}_{1-\nu,-1-\lambda}$;
(ii) $\mathscr{W}\mathscr{C}_{\nu,\lambda}^{-1} = \mathscr{W}\mathscr{G}_{1-\nu,-1-\lambda}$.

It is remarkable that either $\mathscr{W}\mathscr{G}_{\nu,\lambda}$, $\mathscr{W}\mathscr{C}_{\nu,\lambda}$ or $\mathscr{W}\mathscr{G}_{\nu,\lambda}^{-1}$, $\mathscr{W}\mathscr{C}_{\nu,\lambda}^{-1}$ are integral operators for $\lambda \neq -\frac{1}{2}$.

Theorem 11. *For* $\nu, \lambda \in \mathbb{R}$,
(i) $\mathscr{W}\mathscr{G}_{\nu,\lambda}^{-1} = M_{-2\lambda-1}\mathscr{W}\mathscr{C}_{\nu,\lambda}W_{-2\lambda-1} = W_{-2\lambda-1}\mathscr{W}\mathscr{C}_{\nu,\lambda}M_{-2\lambda-1}$;
(ii) $\mathscr{W}\mathscr{C}_{\nu,\lambda}^{-1} = M_{-2\lambda-1}\mathscr{W}\mathscr{G}_{\nu,\lambda}W_{-2\lambda-1} = W_{-2\lambda-1}\mathscr{W}\mathscr{G}_{\nu,\lambda}M_{-2\lambda-1}$.

Proof. By Corollary 2(ii),

$$\begin{aligned}\mathscr{W}\mathscr{G}_{\nu,\lambda}^{-1} &= M_\nu\big(\mathbb{W}_{-\nu-\lambda-1/2}W_\nu\big) = M_{-2\lambda-1}\big(M_{1-\nu}\mathbb{W}_{\nu+\lambda-1/2}W_{1-\nu}\big)W_{-2\lambda-1}\\ &= M_{-2\lambda-1}\mathscr{W}\mathscr{C}_{\nu,\lambda}W_{-2\lambda-1}.\end{aligned}$$

Next it is sufficient to show that for $\lambda > -\frac{1}{2}$,

$$W_{2\lambda+1}M_{-2\lambda-1}\mathscr{W}\mathscr{G}_{\nu,\lambda} = \mathscr{W}\mathscr{G}_{\nu,\lambda}M_{-2\lambda-1}W_{2\lambda+1}, \tag{a}$$

$$W_{2\lambda+1}M_{-2\lambda-1}\mathscr{W}\mathscr{C}_{\nu,\lambda} = \mathscr{W}\mathscr{C}_{\nu,\lambda}M_{-2\lambda-1}W_{2\lambda+1}, \tag{b}$$

because having proved these relations for $\lambda > -\frac{1}{2}$, these relations for $\lambda < -\frac{1}{2}$ then follow by

replacing ν by $1-\nu$ and λ by $-1-\lambda$ and by inverting the operators on both sides of the equality. For $\lambda > -\frac{1}{2}$, $f \in S_{\to}(\mathbb{R}^+)$ and $x > 0$,

$$(W_{2\lambda+1}M_{-2\lambda-1}\mathscr{W}\mathscr{G}_{\nu,\lambda}f)(x)$$
$$= \frac{1}{\Gamma(2\lambda+1)}\int_x^\infty t^{-2\lambda-1}(t-x)^{2\lambda}\left(\int_t^\infty (s^2-t^2)^{\lambda-1/2}\mathscr{G}_\nu^\lambda\left(\frac{t}{s}\right)f(s)s\,\mathrm{d}s\right)\mathrm{d}t.$$

Now the substitution $\tau = xs/t$, $\sigma = s$ yields the validity of (a).

The proof of (b) is the same. □

4. Integral equations involving Gegenbauer functions

Based on the theory developed in Section 3 we can solve the integral equations

$$f(x) = \int_x^b (t^2-x^2)^{\lambda-1/2}\mathscr{G}_\nu^\lambda\left(\frac{x}{t}\right)t^{\mu+1}g(t)\,\mathrm{d}t, \quad 0<x<b, \tag{7}$$

$$f(x) = x^{\mu+1}\int_x^b (t^2-x^2)^{\lambda-1/2}\mathscr{C}_\nu^\lambda\left(\frac{t}{x}\right)g(t)\,\mathrm{d}t, \quad 0<x<b, \tag{8}$$

where ν, $\mu \in \mathbb{R}$, $\lambda > -\frac{1}{2}$, $0 < b \leqslant \infty$ and $f \in S_{\to b}(\mathbb{R}^+) = \{h \in S_{\to}(\mathbb{R}^+)\colon h(x) = 0 \text{ for } x > b\}$. In this section we discuss the integral equation (8) extensively. The integral equation (7) can be dealt with in the same manner. By Theorem 7(ii), the integral equation (8) can be written shortly as

$$M_\mu \mathscr{W}\mathscr{C}_{\nu,\lambda}g = f.$$

Hence, the solution g is given by

$$g = \mathscr{W}\mathscr{C}_{\nu,\lambda}^{-1}M_{-\mu}f.$$

Now the various expressions for $\mathscr{W}\mathscr{C}_{\nu,\lambda}^{-1}$, established in Section 3, provide integral/differential formulas for the solution g. From Theorems 10(ii), 8(i) and 9(i) we find for $\rho > \lambda + \frac{1}{2}$,

$$g(t) = (\mathscr{W}\mathscr{G}_{\nu+2\lambda-2\rho+1,-1-\lambda+\rho}M_{1-\nu}\mathbb{W}_{-\rho}M_{\nu-\mu-1}f)(t)$$
$$= \int_t^b (y^2-t^2)^{\rho-\lambda-3/2}\mathscr{G}_{\nu+2\lambda-2\rho+1}^{\rho-\lambda-1}\left(\frac{t}{y}\right)y^{2-\nu}\mathbb{W}_{-\rho}\{y^{\nu-\mu-1}f(y)\}\,\mathrm{d}y, \quad 0<t<b. \tag{9}$$

From Theorems 11(ii) and 7(i) we obtain the following alternative expressions for g:

$$g(t) = (M_{-2\lambda-1}\mathscr{W}\mathscr{G}_{\nu,\lambda}W_{-2\lambda-1}M_{-\mu}f)(t)$$
$$= t^{-2\lambda-1}\int_t^b (y^2-t^2)^{\lambda-1/2}\mathscr{G}_\nu^\lambda\left(\frac{t}{y}\right)yW_{-2\lambda-1}\{y^{-\mu}f(y)\}\,\mathrm{d}y, \quad 0<t<b, \tag{10}$$

and

$$g(t) = (W_{-2\lambda-1}\mathscr{W}\mathscr{G}_{\nu,\lambda}M_{-2\lambda-\mu-1}f)(t)$$
$$= W_{-2\lambda-1}\left\{\int_t^b (y^2-t^2)^{\lambda-1/2}\mathscr{G}_\nu^\lambda\left(\frac{t}{y}\right)y^{-2\lambda-\mu}f(y)\,\mathrm{d}y\right\}, \quad 0<t<b. \tag{11}$$

For specific values of the parameters λ, ν, the Gegenbauer function kernel in the integral equation (8) reduces to a Gegenbauer polynomial or to one of its special cases, a Legendre polynomial or a Chebyshev polynomial. Convolution integral equations with such polynomial kernels have been treated in [1–3,9,10]. In these papers the solution of the integral equation seems to have been found by clever guessing, supported in [2] by a formal application of the Mellin transformation. The proposed solution is then verified by substitution into the original integral equation. This verification requires the (lengthy) computation of a special convolution integral.

The solutions of [1–3,9,10] can be found systematically as special cases of our solution (9) with specific values assigned to the parameters λ, μ, ν, ρ, b. In addition, we have found alternative expressions for the solution of the integral equations considered, obtained by specialisation of (10) and (11).

Let us illustrate this by means of Buschman's paper [2], where he considered an integral equation involving the Gegenbauer polynomial $C_n^{k/2}$ with k, $n \in \mathbb{N}_0$, $0 \leqslant k < n$. For convenience we rewrite $C_n^{k/2}$ in terms of our notation $\boldsymbol{C}_n^{k/2}$, cf. (1). Then Buschman's integral equation takes the form

$$\int_x^1 (t^2 - x^2)^{k/2-1/2} \boldsymbol{C}_n^{k/2}\left(\frac{t}{x}\right) g(t)\, \mathrm{d}t = f(x), \quad 0 < x < 1, \tag{12}$$

with the solution

$$g(t) = \int_t^1 (y^2 - t^2)^{k/2-1/2} \boldsymbol{C}_{n-k-1}^{k/2}\left(\frac{t}{y}\right) y^{2-n} \left(-\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right)^{k+1} (y^n f(y))\, \mathrm{d}y, \quad 0 < t < 1, \tag{13}$$

provided f satisfies certain conditions.

Now assume $f \in S_{-1}(\mathbb{R}^+)$. Then the Gegenbauer transform pair (12), (13) is precisely the special case of (8), (9) with $\lambda = \frac{1}{2}k$, $\mu = -1$, $\nu = n$, $\rho = k+1$, $b = 1$. The corresponding specialisation of (10), (11) yields the following alternative expressions for g:

$$g(t) = t^{-k-1} \int_t^1 (y^2 - t^2)^{k/2-1/2} \boldsymbol{C}_n^{k/2}\left(\frac{t}{y}\right) y \left(-\frac{\mathrm{d}}{\mathrm{d}y}\right)^{k+1} (yf(y))\, \mathrm{d}y, \quad 0 < t < 1,$$

$$g(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k+1} \left\{ \int_t^1 (y^2 - t^2)^{k/2-1/2} \boldsymbol{C}_n^{k/2}\left(\frac{t}{y}\right) y^{1-k} f(y)\, \mathrm{d}y \right\}, \quad 0 < t < 1.$$

Sneddon [12] treated the general Mellin convolution equation which he solved by Mellin transform techniques. He recovered the solutions of [1–3,9,10] as special cases. However, his procedure requires a thorough knowledge of Mellin transforms.

For a detailed review of the literature on convolution integral equations with special function kernels, we refer to [13].

More recently, Deans [4,5] utilized the Radon transformation to determine a Gegenbauer transform pair with the Gegenbauer polynomial $C_n^{k/2}$, k, $n \in \mathbb{N}_0$, as its kernel. For convenience we rewrite $C_n^{k/2}$ in terms of our notation $\boldsymbol{C}_n^{k/2}$, cf. (1). Then Deans' Gegenbauer transform pair

takes the form

$$f(x) = \int_x^\infty (t^2 - x^2)^{(k-1)/2} C_n^{k/2}\left(\frac{x}{t}\right) tg(t)\, dt, \quad x > 0, \tag{14}$$

$$g(t) = t^{-k} \int_t^\infty (y^2 - t^2)^{(k-1)/2} C_n^{k/2}\left(\frac{y}{t}\right)\left(-\frac{d}{dy}\right)^{k+1} f(y)\, dy, \quad t > 0, \tag{15}$$

where k, $n \in \mathbb{N}_0$, provided f satisfies certain conditions.

Now assume $f \in S_{\to}(\mathbb{R}^+)$. Then the Gegenbauer transform pair (14), (15) follows from Theorems 7(i) and 11(i) with $\lambda = \frac{1}{2}k$, $\nu = n$, $b = \infty$. Moreover, Theorem 11(i) with $\lambda = \frac{1}{2}k$, $\nu = n$, yields the following alternative expression for g:

$$g(t) = \left(-\frac{d}{dt}\right)^{k+1} \left\{ t \int_t^\infty (y^2 - t^2)^{(k-1)/2} C_n^{k/2}\left(\frac{y}{t}\right) y^{-k-1} f(y)\, dy \right\}, \quad t > 0.$$

Gegenbauer transformations arise in the study of the Radon transformation $\mathscr{R}$ on $\mathbb{R}^q$ in the following way. Let ψ be a function on $\mathbb{R}^q$ of the form

$$\psi(r\omega) = f(r) Y_m(\omega),$$

where $r \geqslant 0$ and $\omega \in S^{q-1}$ (the unit sphere in $\mathbb{R}^q$) and where Y_m is a homogeneous harmonic polynomial of degree m. Then for $p \geqslant 0$ and $\omega \in S^{q-1}$,

$$\begin{aligned}(\mathscr{R}\psi)(p, \omega) &= (2\pi)^{(q-1)/2} (\mathscr{W}\mathscr{G}_{m,q/2-1} f)(p) Y_m(\omega) \\ &= (2\pi)^{(q-1)/2} \int_p^\infty (r^2 - p^2)^{(q-3)/2} C_m^{q/2-1}\left(\frac{p}{r}\right) rf(r)\, dr\, Y_m(\omega).\end{aligned}$$

See, e.g., [11, Lemma 5.2].

Appendix

Introduce the operator T_{-1} by $(T_{-1}f)(x) = f(1/x)$. On the space

$$S_{\leftarrow}(\mathbb{R}^+) = T_{-1}(S_{\to}(\mathbb{R}^+)) = \{f \in C^\infty(\mathbb{R}^+) : f^{(k)}(0^+) = 0,\ k \in \mathbb{N}_0\}$$

define the operators

$$I_\mu = M_{\mu-1} T_{-1} W_\mu T_{-1} M_{\mu+1}, \qquad \mathbb{I}_\mu = M_{2\mu-2} T_{-1} \mathbb{W}_\mu T_{-1} M_{2\mu+2},$$

$$\mathscr{I}\mathscr{G}_{\nu,\lambda} = M_{2\lambda} T_{-1} \mathscr{W}\mathscr{G}_{\nu,\lambda} T_{-1} M_{2\lambda+2}, \qquad \mathscr{I}\mathscr{C}_{\nu,\lambda} = M_{2\lambda} T_{-1} \mathscr{W}\mathscr{C}_{\nu,\lambda} T_{-1} M_{2\lambda+2},$$

corresponding to the Riemann–Liouville fractional calculus. Then Theorem 7 yields integral representations for $\mathscr{I}\mathscr{G}_{\nu,\lambda}$, $\mathscr{I}\mathscr{C}_{\nu,\lambda}$ with $\lambda > -\frac{1}{2}$; the identities in Definition 6 and Theorems 8–11 remain valid if we replace W, $\mathbb{W}$, $\mathscr{W}\mathscr{G}$, $\mathscr{W}\mathscr{C}$ by I, $\mathbb{I}$, $\mathscr{I}\mathscr{C}$, $\mathscr{I}\mathscr{G}$, respectively. Hence the following integral equations can be solved similarly:

$$f(x) = \int_a^x (x^2 - t^2)^{\lambda-1/2} \mathscr{C}_\nu^\lambda\left(\frac{x}{t}\right) t^{\mu+1} g(t)\, dt, \quad x > a,$$

$$f(x) = x^{\mu+1} \int_a^x (x^2 - t^2)^{\lambda-1/2} \mathscr{G}_\nu^\lambda\left(\frac{t}{x}\right) g(t)\, dt, \quad x > a,$$

where ν, $\mu \in \mathbb{R}$, $\lambda > -\frac{1}{2}$, $a \geqslant 0$ and $f \in S_{a\leftarrow}(\mathbb{R}^+) = \{h \in C^\infty(\mathbb{R}^+) : h(x) = 0 \text{ for } 0 < x < a\}$.

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