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On the orthogonality of the derivative of the reciprocal sequence

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Abstract

Let $\{\Phi_n\}$ be a monic orthogonal polynomial sequence on the unit circle (MOPS). The study of the orthogonality properties of the derivative sequence $\{\Phi'_{n+1}/(n+1)\}$ is a classic problem of the orthogonal polynomials theory. In fact, it is well known that the derivative sequence is again a MOPS if and only if $\Phi_n(z)=z^n$.

A similar problem can be posed in terms of the reciprocal sequence of $\{\Phi_n\}$ as follows:

If $\Phi_{n+1}(0)\neq 0$, we can define the monic sequence $\{P_n\}$ by

$$P_n(z)=\frac{(\Phi^*_{n+1})'(z)}{(n+1)\overline{\Phi_{n+1}(0)}}\quad n\in\mathbb{N}=\{0,1,\ldots\},$$

where Φ^*_n denotes the reciprocal polynomial of Φ_n, and to study their orthogonality conditions.

In this paper we obtain a necessary and sufficient condition for the regularity of $\{P_n\}$ when the first reflection coefficient $\Phi_1(0)$ is a real number. Also, we give an explicit representation for $\{\Phi_n\}$ and $\{P_n\}$.

Moreover, we analyse some questions concerning to the associated functionals of them sequences and the positive definite and semiclassical character.

MSC: 42C05

Keywords: Orthogonal polynomials; Semiclassical orthogonal polynomials; Linear regular functionals; Measures on the unit circle

1. Introduction and motivation

Given a sequence of orthogonal polynomials, the problem of when the corresponding sequence of derivatives is again orthogonal is a classical problem in the theory of orthogonal polynomials.

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In this sense, this is a property characterizing the classical orthogonal polynomials on the real line [7].

On the unit circle, the results regarding this property are quite different [8].

On the other hand, when one is working with orthogonal polynomials on the unit circle, the behaviour of the reciprocal sequence is very important. The simple fact of entering in the recurrence relations, indicates the triviality of this comment.

Different authors have studied questions regarding the reciprocal sequence: relationship between zeros of Φ_n and Φ_n^*, orthogonality, etc.

Also, when we want to solve certain problems involving $\{\Phi_n'\}$ we get $\{(\Phi_n^*)'\}$ entering the picture. For example, in the study of the semiclassical orthogonal polynomials on the unit circle the sequence $\{(\Phi_n^*)'\}$ verifies a structure relation similar to the sequence $\{\Phi_n'\}$. This relation is crucial in obtaining the differential equation satisfied by the sequence of departure $\{\Phi_n\}$ [11].

In this paper a new question regarding the behaviour of the reciprocal sequence is studied: the orthogonality of the sequence of derivatives.

Specifically, we study the sequence defined by

$$P_n(z) = \frac{(\Phi_{n+1}^*)'(z)}{(n+1)\overline{\Phi_{n+1}(0)}}, \quad n \in \mathbb{N} = \{0, 1, \ldots\} \text{ and } \Phi_1(0) \in \mathbb{R}.$$

An equivalent form of posing this question, in terms of the Schur parameters [6] is: To study the orthogonality conditions of the sequence $\{P_n\}$ where

$$P_n(0) = \frac{\overline{\Phi_1(0)} + \Phi_1(0)\overline{\Phi_2(0)} + \cdots + \Phi_n(0)\overline{\Phi_{n+1}(0)}}{(n+1)\overline{\Phi_{n+1}(0)}}, \quad \Phi_1(0) \in \mathbb{R}.$$

The organization of this paper is the following. Section 2 is devoted to the preliminary definitions and results to be used later on. In Section 3 we analyse the regularity of the sequence $\{P_n\}$. Here, we conclude that the linear functional of moments $\tilde{u}$ of $\{P_n\}$ is related with the linear functional of moments u of $\{\Phi_n\}$ by

$$\tilde{u} = \left(\frac{az^2 + bz + \bar{a}}{z}\right) u. \qquad (*)$$

Therefore, in order to solve the posed problem we will use the known results regarding this kind $(*)$ of functionals [9]. The main conclusion of this section is that the equation $az^2 + bz + \bar{a} = 0$ must have a double root. Using this fact, we find the general term of the sequences $\{\Phi_n(0)\}$ and $\{P_n(0)\}$.

Section 4 is the main section of the paper. Here we give explicit formulas for the solutions $\{\Phi_n\}$ and $\{P_n\}$. Finally, in Section 5 we study some relevant properties of the solutions, such as their positive definite and semiclassical character.

2. Preliminary results

Let $\Lambda = span\{z^k, k \in \mathbb{Z}\}$ be the space of the Laurent polynomials with complex coefficients and let $u : \Lambda \to \mathbb{C}$ be a linear functional.

Definition 1. Denoting by $u_n = u(z^n)$, for $n \in \mathbb{Z}$, we say that [1,10]:

- u is hermitian if $\forall n \geqslant 0, u_{-n} = \overline{u_n}$.
- u is regular (positive definite) if the principal submatrices of the moment matrix are nonsingular (positive), i.e.,

$$\forall n \geqslant 0, \quad \Delta_n = \det\left(u(z^{i-j})\right)_{i=0\ldots n; j=0\ldots n} \neq 0 \quad (>0).$$

In any case we denote $\forall n \geqslant 0,\ e_n = \Delta_n / \Delta_{n-1}$ with $\Delta_{-1} = 1$.

It is well known that if u is positive definite a finite and positive Borel measure μ on the unit circle exists such that:

$$\forall P \in \Lambda, \quad u(P(z)) = \frac{1}{2\pi} \int_0^{2\pi} P(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\mu.$$

Definition 2. Let $\{\Phi_n(z)\}_0^{+\infty}$ be a complex polynomial sequence with $\deg \Phi_n(z) = n$. We say that $\{\Phi_n(z)\}_0^{+\infty}$ is a sequence of orthogonal polynomials (OPS) with respect to u if:

$$\forall n, m \geqslant 0, \quad u\left(\Phi_n(z)\overline{\Phi_m}\left(\frac{1}{z}\right)\right) = e_n \delta_{nm} \quad \text{with } e_n \neq 0.$$

In what follows we denote by $\{\Phi_n\}$ the monic orthogonal polynomials sequence (MOPS) relative to u. It is well known that $\{\Phi_n\}$ satisfies the following recurrence relations:

$$\forall n \geqslant 1,\ \Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z), \tag{2.1}$$

$$\forall n \geqslant 1,\ \Phi_n^*(z) = \Phi_{n-1}^*(z) + \overline{\Phi_n(0)} z\Phi_{n-1}(z), \tag{2.2}$$

$$\forall n \geqslant 1,\ \Phi_n(z) = (1 - |\Phi_n(0)|^2) z\Phi_{n-1}(z) + \Phi_n(0)\Phi_n^*(z), \tag{2.3}$$

$$\forall n \geqslant 1,\ \Phi_n^*(z) = (1 - |\Phi_n(0)|^2)\Phi_{n-1}^*(z) + \overline{\Phi_n(0)}\Phi_n(z), \tag{2.4}$$

where $\Phi_n^*(z) = z^n \overline{\Phi_n}(\frac{1}{z})$ is the reciprocal of $\Phi_n(z)$.

Also, we will use the relation:

$$\forall n \geqslant 1, \quad 1 - |\Phi_n(0)|^2 = \frac{e_n}{e_{n-1}}, \quad e_0 = 1. \tag{2.5}$$

Also, it is well known that the regular (positive definite) case is equivalent to $|\Phi_n(0)| \neq 1$ (<1).

Definition 3. We define the nth reproducing kernel for the linear functional u as a polynomial in two variables $K_n(z, y)$ given by $K_n(z, y) = \sum_{k=0}^{n} \Phi_k(z)\overline{\Phi_k(y)}/e_k$.

In what follows we denote by $\{K_n(z, y)\}$ the sequence of n-kernels relative to u and by $\{K_n^*(z, y)\}$ the corresponding reciprocal sequence.

It is well known that $K_n(z,y)$ satisfies:

$$\forall n \geqslant 0,\ K_n(z,y) = \frac{\Phi^*_{n+1}(z)\overline{\Phi^*_{n+1}(y)} - \Phi_{n+1}(z)\overline{\Phi_{n+1}(y)}}{e_{n+1}(1-z\bar{y})}. \tag{2.6}$$

$$\forall n \geqslant 0,\ K_n(z,y) = \frac{\Phi^*_{n}(z)\overline{\Phi^*_{n}(y)} - z\bar{y}\Phi_{n}(z)\overline{\Phi_{n}(y)}}{e_{n}(1-z\bar{y})}. \tag{2.7}$$

Moreover, we will use the following relation:

$$K^*_n(z,y) = y^n K_n\left(z, \frac{1}{\bar{y}}\right), \quad y \neq 0.$$

Definition 4. Let u be a regular and hermitian functional. For all $n \in \mathbb{N}$, $n \geqslant 1$, we define:

$$f_n(x) = \frac{z\Phi_{2n-1}(z) + \Phi^*_{2n-1}(z)}{2^n z^n}, \tag{2.8}$$

$$g_n(x) = \frac{z\Phi_{2n-1}(z) - \Phi^*_{2n-1}(z)}{i2^n z^n}, \tag{2.9}$$

where $x = (z+z^{-1})/2$. (See [2].)

The set $\{1\} \cup \{f_n, g_n\}_{n \geqslant 1}$ is a basis for Λ. The functions f_n and g_n can be written as follows:

$$f_n(x) = \frac{(1-\overline{\Phi_{2n}(0)})\Phi_{2n}(z) + (1-\Phi_{2n}(0))\Phi^*_{2n}(z)}{2^n z^n(1-|\Phi_{2n}(0)|^2)}, \tag{2.10}$$

$$g_n(x) = \frac{(1+\overline{\Phi_{2n}(0)})\Phi_{2n}(z) - (1+\Phi_{2n}(0))\Phi^*_{2n}(z)}{i2^n z^n(1-|\Phi_{2n}(0)|^2)}. \tag{2.11}$$

Definition 5. Let u be a regular and hermitian functional, and let a, b and c be any complex numbers. We define the linear functional $L : \Lambda \to \mathbb{C}$ as follows:

$$\forall P \in \Lambda, \quad L(P(z)) = u\left(\frac{az^2+bz+c}{z}P(z)\right).$$

We write $L = \left(\frac{az^2+bz+c}{z}\right)u$.

The regularity and some relevant properties about this functional has been studied in [9]. Here we recall some results.

Theorem 1. (i) *L is a hermitian functional if and only if $a = \bar{c}$ and $b = \bar{b}$.*

(ii) *Assume* (i) *and $a \neq 0$.*

If a_1, a_2 are the roots of the equation $az^2 + bz + \bar{a} = 0$, then:

L is regular if and only if $\forall n \in \mathbb{N}$, $K^*_n(a_1, a_2) \neq 0$.

In this case, denoting by $\{\Psi_n\}$ *the MPOS relative to* L, *we have*

$$\forall n \geqslant 1,\ (z-a_i)\Psi_{n-1}(z) = \Phi_n(z) - \frac{\Phi_n(a_i)}{K^*_{n-1}(a_1,a_2)} K^*_{n-1}(z,a_j);\ i,j=1,2.$$

(iii) *Let* L *be regular and let* $\varepsilon_n = L(\Psi_n(z)\overline{\Psi_n}(\frac{1}{z}))$ *then*:

$$\forall n \geqslant 1, \quad \varepsilon_{n-1} = -ae_n \frac{K^*_n(a_1,a_2)}{K^*_{n-1}(a_1,a_2)}.$$

Next we recall some facts concerning semiclassical functionals. For more details see [11].

Definition 6. Given a linear, regular and hermitian functional u, we say that u is semiclassical if there exist polynomials $\mathbf{A}(z) \neq 0$ and $\mathbf{B}(z)$ such that the following functional equation holds:

$$D(\mathbf{A}(z)u) = \mathbf{B}(z)u.$$

If $\deg \mathbf{A}(z) = p'$ and $\max\{p'-1, \deg[(p'-1)\mathbf{A}(z) + \mathrm{i}\mathbf{B}(z)]\} = q$ we say that u belongs to the class (p',q).

Given $\mathbf{A}(z) \in \Lambda$ the operator $\mathbf{A}(z)u$ is defined by

$$\forall P \in \Lambda,\ (\mathbf{A}(z)u)(P(z)) = u(\mathbf{A}(z)P(z))$$

and the derivative operator D is defined by

$$\forall P \in \Lambda,\ Du(P(z)) = -\mathrm{i}u(zP'(z)).$$

3. The regularity

Throughout Section 3 we assume that $\{\Phi_n(z)\}_{n=0}^{+\infty}$ is a MOPS such that $\Phi_n(0) \neq 0$, and $\{P_n(z)\}_{n=0}^{+\infty}$ is the sequence defined by

$$\forall n \geqslant 0, \quad P_n(z) = \frac{(\Phi^*_{n+1})'(z)}{(n+1)\overline{\Phi_{n+1}(0)}} \quad \text{where } \Phi^*_n(z) = z^n \overline{\Phi_n}\left(\frac{1}{z}\right). \tag{3.1}$$

Lemma 1. *For all* $n \in \mathbb{N}$, $n \geqslant 1$, *the following assertions hold*:

$$z\Phi'_n(z) = n(\Phi_n(z) - \Phi_n(0)P^*_{n-1}(z)). \tag{3.2}$$

$$(n+1)\overline{\Phi_{n+1}(0)}(P_n(z) - \Phi_n(z)) = n(\overline{\Phi_n(0)}P_{n-1}(z) - \overline{\Phi_{n+1}(0)}\Phi_n(0)P^*_{n-1}(z)). \tag{3.3}$$

$$(n+1)\overline{\Phi_{n+1}(0)}P_n(0) = n\overline{\Phi_n(0)}P_{n-1}(0) + \overline{\Phi_{n+1}(0)}\Phi_n(0). \tag{3.4}$$

Proof. We know that $z(\Phi^*_n)'(z) = n\Phi^*_n(z) - (\Phi'_n)^*(z)$ [11]. Using (3.1) and applying the $*n$-operator in the resulting expression we find (3.2).

- Taking derivatives in (2.2) written for $n+1$ and substituting (3.2) we have (3.3).
- Taking $z=0$ in (3.3) we get to (3.4). □

In what follows we will assume that $\{P_n(z)\}$ is a MOPS and $\tilde{u}$ the corresponding associated functional.

Proposition 1. *Denoting by* $E_n = \tilde{u}(P_n(z)\overline{P_n}(\frac{1}{z})) = \tilde{u}(P_n(z)z^{-n})$ *then*:

$$\forall n \in \mathbb{N}, \quad n \geqslant 1, \quad E_n = \frac{(n+2)\Phi_{n+2}(0)\Phi_1(0)e_{n+1}}{2(n+1)\Phi_{n+1}(0)\Phi_2(0)e_1}. \tag{3.5}$$

Proof. Denoting by $A_n = -n\overline{\Phi_n(0)}/(n+1)\overline{\Phi_{n+1}(0)}$ and by $B_n = n\Phi_n(0)/(n+1)$; from (3.3):

$$\forall n \geqslant 1, \quad \Phi_n(z) = P_n(z) + A_n P_{n-1}(z) + B_n P^*_{n-1}(z). \tag{3.6}$$

Using (2.1) we have

$$\forall n \geqslant 0,\ z\Phi_n(z) + \Phi_{n+1}(0)\Phi^*_n(z) = (z + A_{n+1})P_n(z) + (P_{n+1}(0) + B_{n+1})P^*_n(z).$$

Substituting (3.6) and $(3.6)^{*n}$:

$$\begin{aligned} &A_{n+1}P_n(z) + (P_{n+1}(0) + B_{n+1} - \Phi_{n+1}(0))P^*_n(z) \\ &\quad = z(A_n + \Phi_{n+1}(0)\overline{B_n})P_{n-1}(z) + z(B_n + \overline{A_n}\Phi_{n+1}(0))P^*_{n-1}(z). \end{aligned}$$

From (3.4) the coefficient of $P^*_n(z)$ is $-A_{n+1}P_n(0)$. Substituting this value in the previous equation and using (2.3) we get to

$$(A_n + \overline{B_n}\Phi_{n+1}(0))P_{n-1}(z) + (B_n + \Phi_{n+1}(0)\overline{A_n})P^*_{n-1}(z) = A_{n+1}(1 - |P_n(0)|^2)P_{n-1}(z).$$

Since that the coefficient of $P^*_{n-1}(z)$ is zero:

$$\forall n \geqslant 1, \quad 1 - |P_n(0)|^2 = \frac{n(n+2)\overline{\Phi_{n+2}(0)\Phi_n(0)}}{(n+1)^2\overline{\Phi_{n+1}(0)}^2}(1 - |\Phi_{n+1}(0)|^2). \tag{3.7}$$

Taking into account (2.5) and letting n take the successive values $n = 1, 2, \ldots$ we get to (3.5). $\square$

Corollary 1. *Denoting by* $\lambda = \Phi_2(0)\overline{\Phi_1(0)}/\overline{\Phi_2(0)}\Phi_1(0)$, *for all* $n \in \mathbb{N}$, *the following assertions hold*:

$$\frac{\overline{\Phi_{n+1}(0)}}{\Phi_{n+1}(0)} = \frac{1}{\lambda^{n-1}}\frac{\overline{\Phi_2(0)}}{\Phi_2(0)}. \tag{3.8}$$

$$\lambda\Phi_n(0)\overline{\Phi_{n+1}(0)} = \overline{\Phi_n(0)}\Phi_{n+1}(0). \tag{3.9}$$

Proof.
- Taking into account that (3.5) is a real number and letting n take the successive values $n=1,2,\ldots$ we get to (3.8).
- The relation (3.9) is immediate from the previous one. $\square$

Lemma 2. *For all $n \in \mathbb{N}$, $n \geqslant 1$, the following assertion holds*:

$$n\lambda\overline{\Phi_n(0)}P_{n-1}(0) = \eta + n\Phi_n(0)\overline{P_{n-1}(0)}, \tag{3.10}$$

where $\eta = \dfrac{\overline{\Phi_1(0)}^2\Phi_2(0) - \Phi_1(0)^2\overline{\Phi_2(0)}}{\Phi_1(0)\overline{\Phi_2(0)}} = \lambda\overline{\Phi_1(0)} - \Phi_1(0).$

Proof. We proceed by induction on n. When we apply the induction step we use (3.9) and (3.4). □

A crucial result in this section is the following proposition.

Proposition 2. *For all $n \in \mathbb{N}$, $n \geqslant 1$, the following assertion holds*:

$$\begin{aligned}(z^2 + C_n z + \lambda)P_{n-1}(z)\\ = \left(z - \frac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)}\right)\Phi_n(z) + \left(\frac{(n+1)\Phi_{n+1}(0)}{n} + \lambda P_{n-1}(0)\right)\Phi_n^*(z);\end{aligned} \tag{3.11}$$

where

$$C_n = \frac{\eta}{n} - \frac{e_n}{e_{n-1}}\left(\frac{(n+1)\Phi_{n+1}(0) + (n-1)\Phi_{n-1}(0)\lambda}{n\Phi_n(0)}\right) + 2\Phi_n(0)\overline{P_{n-1}(0)}.$$

Proof. From (3.6), (2.1) and (3.4):

$$\Phi_n(z) = \left(z - \frac{n\overline{\Phi_n(0)}}{(n+1)\overline{\Phi_{n+1}(0)}}\right)P_{n-1}(z) + \left(\Phi_n(0) + \frac{n\overline{\Phi_n(0)}P_{n-1}(0)}{(n+1)\overline{\Phi_{n+1}(0)}}\right)P_{n-1}^*(z).$$

Applying the $*n$-operator:

$$\Phi_n^*(z) = \left(\overline{\Phi_n(0)} + \frac{n\Phi_n(0)\overline{P_{n-1}(0)}}{(n+1)\Phi_{n+1}(0)}\right)zP_{n-1}(z) + \left(1 - z\frac{n\Phi_n(0)}{(n+1)\Phi_{n+1}(0)}\right)P_{n-1}^*(z).$$

Below we solve the system formed by the two previous equations and whose unknowns are $P_{n-1}(z)$ and $P_{n-1}^*(z)$.

Denoting by $D_n(z)$ the determinant of the matrix of the coefficients. We have:

$$\begin{aligned}D_n(z) = \frac{-z^2 n\Phi_n(0)}{(n+1)\Phi_{n+1}(0)} + z\left[\frac{n^2|\Phi_n(0)|^2}{(n+1)^2|\Phi_{n+1}(0)|^2} + 1 - \left|\Phi_n(0) + \frac{n\overline{\Phi_n(0)}P_{n-1}(0)}{(n+1)\overline{\Phi_{n+1}(0)}}\right|^2\right]\\ - \frac{n\overline{\Phi_n(0)}}{(n+1)\overline{\Phi_{n+1}(0)}}.\end{aligned}$$

It is easy to see that $D_n(z)$ has the property $D_n(z) = D_n^*(z)$, that is to say, it is self-reciprocal.

Taking into consideration the fact that $D_n(z)$ is a self-reciprocal polynomial of degree two, then it must be of the form:

$$D_n(z) = -\frac{n\Phi_n(0)}{(n+1)\Phi_{n+1}(0)}(z-a_n)\left(z-\frac{1}{\overline{a_n}}\right).$$

From (3.9): $a_n/\overline{a_n} = \lambda$.

Denoting by $C_n = -(a_n + 1/\overline{a_n})$, we have:

$$C_n = -\left[\frac{n^2|\Phi_n(0)|^2}{(n+1)^2|\Phi_{n+1}(0)|^2} + 1 - \left|\Phi_n(0) + \frac{n\overline{\Phi_n(0)}P_{n-1}(0)}{(n+1)\overline{\Phi_{n+1}(0)}}\right|^2\right]\frac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)}.$$

For $n = 1$ we obtain

$$C_1 = -2e_1\frac{\Phi_2(0)}{\Phi_1(0)} + \lambda\overline{\Phi_1(0)} + \Phi_1(0). \tag{3.12}$$

For $n \geqslant 2$ the value of C_n in the statement is obtained after a simple calculation in conjunction with (3.7), (3.9) and (3.10).

Since that,

$$D_n(z) = -\frac{n\Phi_n(0)}{(n+1)\Phi_{n+1}(0)}(z^2 + C_n z + \lambda),$$

solving the system, we get to (3.11). □

Proposition 3. *For all* $n \in \mathbb{N}$, $n \geqslant 1$, *it is verified* $C_n = C_1$.

Proof. Writing (3.11) for $n+1$ and using (2.1) and (2.2):

$$\begin{aligned}\forall n \geqslant 0,\ &(z^2 + C_{n+1}z + \lambda)P_n(z)\\ &= \left[z - \frac{(n+2)\Phi_{n+2}(0)}{(n+1)\Phi_{n+1}(0)}(1-|\Phi_{n+1}(0)|^2)\right.\\ &\quad \left.+\lambda P_n(0)\overline{\Phi_{n+1}(0)}\right] z\Phi_n(z) + (z\Phi_{n+1}(0) + \lambda P_n(0))\Phi_n^*(z).\end{aligned} \tag{3.13}$$

On the other hand,

$$(z^2 + C_{n+1}z + \lambda)P_n(z) = (z^2 + C_n z + \lambda)P_n(z) + (C_{n+1} - C_n)zP_n(z).$$

Substituting in the previous (3.11) and $(3.11)^{*(n+1)}$ and equating to (3.13):

$$\begin{aligned}&(C_{n+1} - C_n)P_n(z)\\ &= \left[\frac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)} - \frac{(n+2)\Phi_{n+2}(0)}{(n+1)\Phi_{n+1}(0)}\frac{e_{n+1}}{e_n} - \lambda\frac{P_n(0)\overline{\Phi_{n+1}(0)}}{n} - \overline{P_{n-1}(0)}P_n(0)\right]\Phi_n(z)\\ &\quad + \left[-\frac{\Phi_{n+1}(0)}{n} - \lambda P_{n-1}(0) + \frac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)}P_n(0)\right]\Phi_n^*(z).\end{aligned}$$

From (3.9) and (3.4) the coefficient of $\Phi_n^*(z)$ is zero. Identifying coefficients of degree n:

$$C_{n+1} - C_n = \left[\frac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)} - \frac{(n+2)\Phi_{n+2}(0)}{(n+1)\Phi_{n+1}(0)}\frac{e_{n+1}}{e_n} - \lambda\frac{P_n(0)\overline{\Phi_{n+1}(0)}}{n} - \overline{P_{n-1}(0)}P_n(0)\right].$$

Eliminating $(n+2)\Phi_{n+2}(0)e_{n+1}/e_n$ with (3.7):

$$C_{n+1} - C_n = P_n(0)\left(\overline{P_n(0)}\frac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)} - \lambda\frac{\overline{\Phi_{n+1}(0)}}{n} - \overline{P_{n-1}(0)}\right).$$

Again applying (3.4) we get to $C_{n+1} - C_n = 0$. $\square$

Remark 1. Eq. (3.11) suggests that $\{P_n\}$ can be orthogonal with respect to a functional of the kind $((az^2+bz+c)/z)u$ (see Definition 5). This fact is proved in the following Proposition.

On the other hand, the development of the more general case $\Phi_1(0) \in \mathbb{C}$ requires the distinction of several situations. Here, we focus our attention on the case $\Phi_1(0) \in \mathbb{R}$.

Proposition 4. *The functional $\tilde{u}$ verifies*:

$$\tilde{u} = -\frac{\Phi_1(0)}{2\Phi_2(0)e_1}\left(\frac{z^2+C_1z+1}{z}\right)u. \tag{3.14}$$

Furthermore, if $\Phi_1(0)$ is a real number, then: $\lambda = 1$, $\eta = 0$ *and* $\forall n \in \mathbb{N}$, $n \geqslant 1$, *the Schur parameters* $\Phi_n(0)$ *and* $P_n(0)$ *are real numbers.*

Proof. Let $\tilde{u}$ be the functional given in the statement and $P_{n-1}(z)$ as in (3.11). Then,

$$\tilde{u}(P_{n-1}(z)z^{-k}) = \begin{cases} \dfrac{\Phi_1(0)}{2\Phi_2(0)e_1}\dfrac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)}e_n = E_{n-1} & \text{if } k = n-1, \\ 0 & \text{if } 0 \leqslant k \leqslant n-2. \end{cases}$$

Therefore $\{P_{n-1}\}_{n\geqslant 1}$ is orthogonal with respect to $\tilde{u}$.

The last relations in the statement are consequences of Theorem 1(i) in conjunction with the hypothesis $\Phi_1(0) \in \mathbb{R}$. $\square$

Remark 2. Notice that as $\lambda = 1$ then the roots of $z^2 + C_1z + 1 = 0$ must be real numbers.

The previous results can be summed up in the following corollary.

Corollary 2. *For all $n \in \mathbb{N}$, $n \geqslant 1$, the following assertion holds*:

$$(z^2+C_1z+1)P_{n-1}(z) = \left(z - \frac{(n+1)\Phi_{n+1}(0)}{n\Phi_n(0)}\right)\Phi_n(z) + \left(\frac{(n+1)\Phi_{n+1}(0)}{n} + P_{n-1}(0)\right)\Phi_n^*(z); \quad C_1 \in \mathbb{R}. \tag{3.15}$$

Furthermore:

$$C_1 = -\frac{e_n}{e_{n-1}}\left(\frac{(n+1)\Phi_{n+1}(0)+(n-1)\Phi_{n-1}(0)}{n\Phi_n(0)}\right) + 2\Phi_n(0)P_{n-1}(0). \tag{3.16}$$

Corollary 3. *If $\tilde{u}$ is regular and α and $1/\alpha$ are the roots of the equation $z^2 + C_1 z + 1 = 0$ then,*

$$\forall n \geqslant 0, \quad \frac{(n+1)\Phi_{n+1}(0)}{\Phi_1(0)} = \begin{cases} \frac{\alpha^2\Phi_n^2(\alpha)-(\Phi_n^*(\alpha))^2}{\alpha^n(\alpha^2-1)e_n} & \text{if } \alpha^2 \neq 1, \\ \frac{K_n(\alpha,\alpha)}{\alpha^n} & \text{if } \alpha^2 = 1, \end{cases} \tag{3.17}$$

$$\forall n \geqslant 1, \quad (n+1)P_n(0) = \frac{\alpha^n K_n(\alpha, 1/\alpha)}{K_n(\alpha,\alpha)}. \tag{3.18}$$

Proof. Putting $z=\alpha$ and $z = 1/\alpha$ in (3.15) and solving the resulting system [5,9]. □

Corollary 4. *Let*

$$F_n(x) = \frac{zP_{2n-1}(z) + P^*_{2n-1}(z)}{2^n z^n}, \tag{3.19}$$

$$G_n(x) = \frac{zP_{2n-1}(z) - P^*_{2n-1}(z)}{i2^n z^n} \tag{3.20}$$

with $x = (z + z^{-1})/2$.

Then, there exist real numbers $\tilde{a}_n$ and $\tilde{b}_n$ such that:

$$\left(x + \frac{C_1}{2}\right) F_n(x) = f_{n+1}(x) + \tilde{a}_n f_n(x), \tag{3.21}$$

$$\left(x + \frac{C_1}{2}\right) G_n(x) = g_{n+1}(x) + \tilde{b}_n g_n(x), \tag{3.22}$$

where $f_n(x)$ and $g_n(x)$ are defined by (2.8) and (2.9).

Proof. See [5]. □

Now we show that the equation $z^2 + C_1 z + 1 = 0$ must have a double root, i.e., $C_1 = \pm 2$.

Proposition 5. *If $\tilde{u}$ is regular then*:

$$\tilde{u} = -\frac{\Phi_1(0)}{2\Phi_2(0)e_1}\left(\frac{z^2 \pm 2z + 1}{z}\right) u. \tag{3.23}$$

Proof. Assume that α and $1/\alpha$ with $\alpha \neq 1/\alpha$ are the roots of $z^2 + C_1 z + 1 = 0$.

From (2.10) and (2.11):

$$f_n\left(-\frac{C_1}{2}\right)=\frac{\Phi_{2n}(-C_1/2)+\Phi^*_{2n}(-C_1/2)}{(1+\Phi_{2n}(0))(-C_1)^n},$$

$$g_n\left(-\frac{C_1}{2}\right)=\frac{\Phi_{2n}(-C_1/2)-\Phi^*_{2n}(-C_1/2)}{\mathrm{i}(1-\Phi_{2n}(0))(-C_1)^n}.$$

From (2.8) to (2.9) in conjunction with (2.1) and (2.2):

$$f_{n+1}\left(-\frac{C_1}{2}\right)=\frac{-(C_1/2)(-(C_1/2)+\Phi_{2n+1}(0))\Phi_{2n}(-C_1/2)+(1-(C_1/2)\Phi_{2n+1}(0))\Phi^*_{2n}(-C_1/2)}{(-C_1)^{n+1}},$$

$$g_{n+1}\left(-\frac{C_1}{2}\right)=\frac{C_1/2((C_1/2)+\Phi_{2n+1}(0))\Phi_{2n}(-C_1/2)-(1+(C_1/2)\Phi_{2n+1}(0))\Phi^*_{2n}(-C_1/2)}{\mathrm{i}(-C_1)^{n+1}}.$$

We distinguish the following cases:
(i) There exists $n\in\mathbb{N}$, $n\geqslant 1$, such that $f_n(-C_1/2)=0$. $(\Phi_{2n}(-C_1/2)=-\Phi^*_{2n}(-C_1/2))$
From (3.21) $f_{n+1}(-C_1/2)=0$.
Since that $f_{n+1}(-C_1/2)=-(1-C_1^2/4)\frac{\Phi_{2n}(-C_1/2)}{(-C_1)^{n+1}}=0$ and $C_1\neq\pm 2$ then $\Phi_{2n}(-C_1/2)=0$. Therefore $\Phi^*_{2n}(-C_1/2)=0$ in contradiction with the fact that $\Phi_n(z)$ and $\Phi^*_n(z)$ have not common roots.
(ii) There exists $n\in\mathbb{N}$, $n\geqslant 1$, such that $g_n(-C_1/2)=0$.
We proceed as in (i) and again we get to a contradiction.
(iii) For all $n\in\mathbb{N}$, $n\geqslant 1$, $f_n(-C_1/2)\neq 0$ and $g_n(-C_1/2)\neq 0$.
From (3.17), (2.7) and (2.6):

$$\forall n\geqslant 1,\ n\Phi_n(0)=\frac{\Phi_n^2(\alpha)-(\Phi^*_n(\alpha))^2}{\alpha^{n-1}(\alpha^2-1)e_n}\Phi_1(0).$$

Since that $\Phi_n(0)\neq 0$ then $\Phi_n(\alpha)\neq\pm\Phi^*_n(\alpha)$ and $\alpha\Phi_n(\alpha)\neq\pm\Phi^*_n(\alpha)$.
From Theorem 1(ii):

$$\forall n\geqslant 1,\ (z-\alpha)P_{n-1}(z)=\Phi_n(z)-\frac{\Phi_1(0)\Phi_n(\alpha)}{n\Phi_n(0)}K^*_{n-1}\left(z,\frac{1}{\alpha}\right) \tag{3.24}$$

and

$$\forall n\geqslant 1,\ \left(z-\frac{1}{\alpha}\right)P_{n-1}(z)=\Phi_n(z)-\frac{\Phi_1(0)\Phi_n(1/\alpha)}{n\Phi_n(0)}K^*_{n-1}(z,\alpha). \tag{3.25}$$

Applying the $*n$-operator in (3.24):

$$\left(z-\frac{1}{\alpha}\right)P^*_{n-1}(z)=-\frac{1}{\alpha}\Phi^*_n(z)+\frac{\Phi_1(0)\Phi^*_n(1/\alpha)}{n\Phi_n(0)}zK^*_{n-1}(z,\alpha)$$

and eliminating $K^*_{n-1}(z,\alpha)$ with (3.25) we get to

$$e_nK_n(z,\alpha)=\Phi_n(\alpha)zP_{n-1}(z)+\Phi^*_n(\alpha)P^*_{n-1}(z).$$

Let $\tilde{A}_n = \frac{e_{2n}}{\Phi_{2n}(\alpha)+\Phi_{2n}^*(\alpha)}$, then,

$$F_n(x) = \tilde{A}_n \left(\frac{K_{2n}(z,\alpha) + K_{2n}^*(z,\alpha)}{2^n z^n} \right).$$

From (3.21):

$$(z^2 + C_1 z + 1)\tilde{A}_n \left(\frac{K_{2n}(z,\alpha) + K_{2n}^*(z,\alpha)}{2^{2n+1} z^{2n+1}} \right) = f_{n+1}(x) - \frac{f_{n+1}(-C_1/2)}{f_n(-C_1/2)} f_n(x),$$

wherefrom:

$$\begin{aligned}
&(z^2 + C_1 z + 1)\tilde{A}_n(1 + \Phi_{2n}(0)) f_n\left(-\frac{C_1}{2}\right)(K_{2n}(z,\alpha) + K_{2n}^*(z,\alpha)) \\
&\quad = z\Phi_{2n}(z)\left[z f_n\left(-\frac{C_1}{2}\right)(1 + \Phi_{2n}(0)) \right. \\
&\qquad \left. + \left(\Phi_{2n+1}(0) f_n\left(-\frac{C_1}{2}\right)(1 + \Phi_{2n}(0)) - 2 f_{n+1}\left(-\frac{C_1}{2}\right)\right)\right] \\
&\qquad + \Phi_{2n}^*(z)\left[z\left(\Phi_{2n+1}(0) f_n\left(-\frac{C_1}{2}\right)(1 + \Phi_{2n}(0)) \right.\right. \\
&\qquad \left.\left. -2 f_{n+1}\left(-\frac{C_1}{2}\right)\right) + f_n\left(-\frac{C_1}{2}\right)(1 + \Phi_{2n}(0))\right].
\end{aligned}$$

Developing the left-hand member with (2.7) we get to

$$H_{2n}\Phi_{2n}(z) = -H_{2n}\Phi_{2n}^*(z),$$

where

$$\begin{aligned}
H_{2n} = {} & f_n\left(-\frac{C_1}{2}\right)(1 + \Phi_{2n}(0)) \frac{\alpha^2 \Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha)}{\alpha(\Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha))} \\
& + \Phi_{2n+1}(0) f_n\left(-\frac{C_1}{2}\right)(1 + \Phi_{2n}(0)) - 2 f_{n+1}\left(-\frac{C_1}{2}\right).
\end{aligned}$$

Taking in consideration that $\Phi_n(z)$ and $\Phi_n^*(z)$ have not common roots, we have

$$2 f_{n+1}\left(-\frac{C_1}{2}\right) = \left[\Phi_{2n+1}(0) + \frac{\alpha^2 \Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha)}{\alpha(\Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha))} \right](1 + \Phi_{2n}(0)) f_n\left(-\frac{C_1}{2}\right).$$

Substituting $f_{n+1}(-\frac{C_1}{2})$ and $f_n(-\frac{C_1}{2})$:

$$\Phi_{2n}\left(-\frac{C_1}{2}\right) = -2\,\frac{\alpha^2 \Phi_{2n}(\alpha) - \Phi_{2n}^*(\alpha)}{(\alpha^2 + 1)(\Phi_{2n}(\alpha) - \Phi_{2n}^*(\alpha))}\,\Phi_{2n}^*\left(-\frac{C_1}{2}\right). \tag{3.26}$$

Let $\tilde{B}_n = \frac{e_{2n}}{\Phi_{2n}(\alpha)-\Phi_{2n}^*(\alpha)}$, then:

$$G_n(x) = \tilde{B}_n \left(\frac{K_{2n}(z,\alpha) - K_{2n}^*(z,\alpha)}{i2^n z^n} \right).$$

Proceeding as before we find:

$$2g_{n+1}\left(-\frac{C_1}{2}\right) = \left[-\Phi_{2n+1}(0) + \frac{\alpha^2\Phi_{2n}(\alpha) - \Phi_{2n}^*(\alpha)}{\alpha(\Phi_{2n}(\alpha) - \Phi_{2n}^*(\alpha))} \right] (1 - \Phi_{2n}(0)) g_n \left(-\frac{C_1}{2}\right),$$

wherefrom we obtain

$$\Phi_{2n}\left(-\frac{C_1}{2}\right) = 2\, \frac{\alpha^2\Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha)}{(\alpha^2+1)(\Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha))}\, \Phi_{2n}^*\left(-\frac{C_1}{2}\right). \tag{3.27}$$

From (3.26) and (3.27):

If $\Phi_{2n}^*(-C_1/2) = 0$ then $\Phi_{2n}(-C_1/2) = 0$ and this case is impossible. Therefore:

$$\frac{\alpha^2\Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha)}{(\Phi_{2n}(\alpha) + \Phi_{2n}^*(\alpha))} = -\frac{\alpha^2\Phi_{2n}(\alpha) - \Phi_{2n}^*(\alpha)}{(\Phi_{2n}(\alpha) - \Phi_{2n}^*(\alpha))}$$

and this implies $\alpha^2\Phi_{2n}^2(\alpha) = (\Phi_{2n}^*(\alpha))^2$ in contradiction with the hypothesis.

Consequently $\alpha = 1/\alpha$, i.e., $C_1 = \pm 2$. □

Corollary 5. *The sequences* $\{\Phi_n(0)\}_{n=0}^{+\infty}$ *and* $\{P_n(0)\}_{n=0}^{+\infty}$ *are given by*

$$\Phi_{n+1}(0) = \frac{K_n(\alpha,\alpha)}{(n+1)\alpha^n}\Phi_1(0), \tag{3.28}$$

$$P_n(0) = \frac{\alpha^n}{(n+1)}, \tag{3.29}$$

where $\alpha = \pm 1$.

Remark 3. The OPS whose Schur parameters are of the form $|P_n(0)| = 1/(n+1)$ have been studied in [3].

4. The solutions

Theorem 2. *Let* $\{\Phi_n\}$ *be a monic sequence of polynomials such that, for all* $n \in \mathbb{N}$, $\deg \Phi_n = n$, $\Phi_n(0) \neq 0$ *and* $\Phi_1(0) \in \mathbb{R}$, $\Phi_1(0) \neq 0$, $|\Phi_1(0)| \neq 1$.

Let $\{P_n\}$ *be a monic sequence of polynomials defined by* $P_n(z) = (\Phi_{n+1}^*)'(z)/(n+1)\overline{\Phi_{n+1}(0)}$.

If $\{\Phi_n\}$ *and* $\{P_n\}$ *are sequences of orthogonal polynomials then*:

(a)

$$\forall n \geqslant 1,\ \Phi_n(z) = z^n + \frac{\Phi_1(0)}{1-(n-1)\Phi_1(0)}\left(\frac{z^n-1}{z-1}\right),\ \Phi_1(0) \neq \frac{1}{n}. \tag{4.1}$$

In this case:

$$P_n(z) = z^n + \frac{n}{n+1} z^{n-1} + \frac{n-1}{n+1} z^{n-2} + \cdots + \frac{2}{n+1} z + \frac{1}{n+1}. \tag{4.2}$$

(b)

$$\forall n \geqslant 1, \quad \Phi_n(z) = z^n + \frac{\Phi_1(0)}{1+(n-1)\Phi_1(0)} \left(\frac{z^n + (-1)^{n-1}}{z+1} \right), \quad \Phi_1(0) \neq \frac{-1}{n}. \tag{4.3}$$

In this case:

$$P_n(z) = z^n - \frac{n}{n+1} z^{n-1} + \frac{n-1}{n+1} z^{n-2} + \cdots + \frac{2(-1)^{n+1}}{n+1} z + \frac{(-1)^n}{n+1}. \tag{4.4}$$

Proof. From (3.16) we have:

(a) If $C_1 = -2$ then $\Phi_2(0) = \Phi_1(0)/(1 - \Phi_1(0))$. From (3.28) we find $P_n(0) = 1/(n+1)$. Using (3.4): $\Phi_{n+1}(0) = \Phi_n(0)/(1 - \Phi_n(0))$, wherefrom:

$$\forall n \geqslant 1, \quad \Phi_n(0) = \frac{\Phi_1(0)}{1-(n-1)\Phi_1(0)}.$$

In [5] we can see that the sequence whose Schur parameters are $P_n(0) = 1/(n+1)$ is (4.2). Taking into account the definition of $P_n(z)$ we get to (4.1).

(b) If $C_1 = 2$, we proceed as in (a) and we find:

$$P_n(0) = \frac{(-1)^n}{n+1} \quad \text{and} \quad \Phi_n(0) = \frac{(-1)^{n+1}\Phi_1(0)}{1+(n-1)\Phi_1(0)}.$$

In order to compute $P_n(z)$ we use the following result [5]:

The MOPS $\{M_n(z)\}$ with sequence of Schur parameters $\{e^{in\varphi}P_n(0)\}$ verifies:

$$M_n(z) = e^{in\varphi} P_n(e^{-i\varphi} z).$$

Using (4.2) and the previous result with $\varphi = \pi$ the proof is complete. □

5. The linear associated functionals, the measures and the semiclassical character

In this section we denote by L_0 the functional associated with the normalized Lebesgue measure μ and by δ_a the Dirac distribution at a point a.

The following theorem is the reciprocal of Theorem 2.

Theorem 3. (a) *The sequence given by* (4.1) ((4.3)) *is orthogonal with respect to the functional*:

$$u = (1 \pm \Phi_1(0))L_0 \mp \Phi_1(0)\delta_{\pm 1}.$$

(b) *The sequences given by* (4.2) ((4.4)) *is orthogonal with respect to the functionals*:

$$\tilde{u} = \mp \frac{1}{2(1 \pm \Phi_1(0))} \frac{(z \mp 1)^2}{z} u.$$

Proof. Denoting by $u_n = u(z^n)$ and by $\tilde{u}_n = \tilde{u}(z^n)$, for $n \in \mathbb{N}$, we have the following cases:

(a) From the definition of u in the statement we have

$$u_n = \begin{cases} 1 & \text{if } n = 0, \\ -\Phi_1(0)((-1)^n \Phi_1(0)) & \text{if } n \neq 0. \end{cases}$$

Wherefrom:

$$u(\Phi_n(z)z^{-k}) = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant n-1, \\ -\frac{n(u_1-1)(u_1+1/n)}{1+(n-1)u_1} \neq 0 \ (-\frac{n(u_1+1)(u_1-1/n)}{1-(n-1)u_1} \neq 0) & \text{if } k = n. \end{cases}$$

(b) Proceeding as in (a):

$$\tilde{u}_n = \begin{cases} 1 & \text{if } n = 0, \\ -\frac{1}{2}(\frac{1}{2}) & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 2. \end{cases}$$

$$\tilde{u}(P_n(z)z^{-k}) = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant n-1, \\ \frac{n+2}{2(n+1)} \neq 0, (\frac{n+2}{2(n+1)} \neq 0) & \text{if } k = n. \end{cases} \qquad \square$$

Remark 4. (i) It is well known that the Jacobi measure on the unit circle is given by Szegö [10]:

$$\mu'(\theta) = \left(\sin\left(\frac{\theta}{2}\right)\right)^{2\alpha} \left(\cos\left(\frac{\theta}{2}\right)\right)^{2\beta}, \quad -\pi \leqslant 0 \leqslant \pi \quad \alpha, \beta > -\frac{1}{2}$$

and the corresponding sequence of Schur parameters is

$$\forall n \geqslant 0, \quad J_n(0) = \frac{\alpha + (-1)^n \beta}{n + \alpha + \beta}.$$

(ii) Given a probability measure μ on the unit circle a new probability measure on the unit circle can be defined as follows:

$$v = \frac{1}{1+M}(\mu + M\delta_{z_1}), \quad z_1 \in \mathbb{T}, \quad M > 0.$$

The sequences of orthogonal polynomials associated with v have been study in [4].

Theorem 4. (a) *If* $\Phi_1(0) \in (-1, 0)$ ($\Phi_1(0) \in (0, 1)$), *the sequence given by* (4.1) ((4.3)) *is orthogonal with respect to the measure*:

$$v = (1 \pm \Phi_1(0)) \left(\mu \mp \frac{\Phi_1(0)}{1 \pm \Phi_1(0)} \delta_{\pm 1}\right).$$

(b) *The sequence given by* (4.2) ((4.4)) *is a sequence of Jacobi kind and the corresponding measure is*

$$\sigma'(\theta) = \sin^2\left(\frac{\theta}{2}\right). \quad \left(\cos^2\left(\frac{\theta}{2}\right).\right)$$

Proof. In order to obtain (a) we impose the condition $|\Phi_n(0)| < 1$ [10] in (a) of Theorem 3. The case (b) is obvious from (i) of the previous remark. $\square$

Theorem 5. (a) *The sequence given by* (4.1) ((4.3)) *is semiclassical of class* (2,2) *and verifies*:

$$D((z \mp 1)^2 u) = 2\mathrm{i}z(z \mp 1)u.$$

(b) *The sequence given by* (4.2) ((4.4)) *is semiclassical of class* (1,1) *and verifies*:

$$D((z \mp 1)\tilde{u}) = \mathrm{i}(2z \pm 1)\tilde{u}.$$

Proof. If in Definition 6 we take $P(z) = z^n$ we obtain that the functionals in the statement verify the following difference equations:

(a)

$$(n+2)u_{n+2} \mp 2(n+1)u_{n+1} + nu_n = 0.$$

(b)

$$(n+2)\tilde{u}_{n+1} \mp (n-1)\tilde{u}_n = 0. \qquad \square$$

Now, using the results given in Proof 12 the statement follows.

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