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Convergence of two-stage iterative methods using incomplete factorization☆

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Abstract

We first study the convergence of two-stage iterative methods using the incomplete factorization for solving a linear system whose coefficient matrix is an H-matrix, and then we study the convergence of two-stage iterative methods using the incomplete factorization for solving a linear system whose coefficient matrix is a symmetric positive definite matrix. Lastly, numerical experiments are provided to analyze theoretical results.

MSC: 65F10; 65F15

Keywords: Two-stage iterative method; Incomplete factorization; H-matrix; Symmetric positive definite matrix

1. Introduction

In this paper, we consider two-stage iterative methods for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse nonsingular matrix. A matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leqslant 0$ for $i \neq j$ and $A^{-1} \geqslant 0$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

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A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix. Let $\rho(A)$ denote the *spectral radius* of a square matrix A. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geqslant 0$ and $N \geqslant 0$, *weak regular* if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$, and *convergent* if $\rho(M^{-1}N) < 1$. When A is symmetric, a splitting $A=M-N$ is called *symmetric* if M is symmetric. A splitting $A=M-N$ is called an M-splitting if M is an M-matrix and $N \geqslant 0$, an H-splitting if $\langle M\rangle - |N|$ is an M-matrix, and an *H-compatible splitting* if $\langle A\rangle = \langle M\rangle - |N|$.

Let $A = M - N$ be a splitting of A and $M = F - G$ be a splitting of M. Then, the *stationary two-stage iterative method* for solving the linear system (1) is as follows.

Algorithm 1. Stationary two-stage iterative method

Given an initial vector x_0
For $k = 1, 2, \ldots,$ until convergence
  $y_0 = x_{k-1}$
  For $j = 1$ to p
    $Fy_j = Gy_{j-1} + Nx_{k-1} + b$
  $x_k = y_p$

If the number of inner iterations, p, in Algorithm 1 varies for each k (i.e., the p in Algorithm 1 is replaced by p_k), then we obtain the *nonstationary two-stage iterative method* (Algorithm 2). If the last line in Algorithm 1 is replaced with the following:

$$x_k = \omega y_p + (1-\omega)x_{k-1}, \quad \omega > 0,$$

then we obtain the *relaxed stationary two-stage iterative method* (Algorithm 3). Throughout the paper, it is assumed that $p \geqslant 1$ and $p_k \geqslant 1$ for all k. The convergence of two-stage iterative methods for solving (1) was studied by many authors [4,5,7,11]. Nichols [11] showed that if $\rho(M^{-1}N) < 1$ and $\rho(F^{-1}G) < 1$, then Algorithm 1 converges to the exact solution of the linear system (1) for large enough p. It was shown in [4,7] that Algorithms 1 and 2 converge to the exact solution of the linear system (1) for $A^{-1} \geqslant 0$ when the outer splitting $A = M - N$ is regular and the inner splitting $M = F - G$ is weak regular. It was also shown in [4] that Algorithms 1 and 2 converge to the exact solution of the linear system (1) for an H-matrix A when $A = M - N$ is an H-splitting and $M = F - G$ is an H-compatible splitting.

For a large sparse matrix A, a convenient way of obtaining a splitting of A is to use the incomplete factorization of A which was first introduced in [12] and studied in [9]. There is also a general algorithm for finding an incomplete factorization of a sparse matrix corresponding to a given zero pattern set. So, it is worth studying the convergence of two-stage iterative methods using the incomplete factorization as an inner splitting. This paper is organized as follows. In Section 2, we present some notation and preliminary results which we refer to later. In Section 3, we present convergence results of two-stage iterative methods using incomplete factorization for solving linear systems whose coefficient matrices are H-matrices or symmetric positive definite matrices. In Section 4, we present numerical results of the stationary two-stage iterative method using incomplete factorization for solving a linear system whose coefficient matrix is symmetric positive definite.

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geqslant 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geqslant y$ ($x > y$) means that $x - y \geqslant 0$ ($x - y > 0$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices. It follows that $|A| \geqslant 0$ for any matrix A and $|AB| \leqslant |A||B|$ for any two matrices A and B of compatible size.

It was shown in [4] that if $A = M - N$ is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) < 1$. Varga [13] showed that for any square matrices A and B, $|A| \leqslant B$ implies $\rho(A) \leqslant \rho(B)$. Note that M-matrices and strictly or irreducibly diagonally dominant matrices are contained in the class of all H-matrices. Actually, an $n \times n$ H-matrix $A = (a_{ij})$ can be equivalently characterized by being *generalized strictly diagonally dominant* [3], i.e.,

$$|a_{ii}|u_i > \sum_{j \neq i} |a_{ij}|u_j, \quad i = 1, 2, \ldots, n$$

for some vector $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} > 0$.

Lemma 2.1 (Lanzkron et al. [7]). *Given a nonsingular matrix A and a matrix H such that $I - H$ is nonsingular, there exists a unique pair of matrices B and C such that $A = B - C$ and $H = B^{-1}C$. Moreover, $B = A(I - H)^{-1}$.*

A general algorithm for building an incomplete factorization can be derived by performing Gaussian elimination and dropping some of elements in predetermined off-diagonal positions. Let S_n denote the set of all pairs of indices of off-diagonal matrix entries, that is,

$$S_n = \{(i, j) \mid i \neq j, \ 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant n\}.$$

Theorem 2.2 (Messaoudi [10]). *Let $A \in \mathbb{R}^{n \times n}$ be an H-matrix. Then, for every zero pattern set $Q \subset S_n$, there exist a unit lower triangular matrix $L = (l_{ij})$, an upper triangular matrix $U = (u_{ij})$, and a matrix $N = (n_{ij})$, with $l_{ij} = u_{ij} = 0$ if $(i, j) \in Q$ and $n_{ij} = 0$ if $(i, j) \notin Q$, such that $A = LU - N$. Moreover, the factors L and U are also H-matrices.*

In Theorem 2.2, $A = LU - N$ is called an incomplete LU (ILU) factorization of A corresponding to a zero pattern set $Q \subset S_n$. When A is an M-matrix, it was shown in [9] that the ILU factorization $A = LU - N$ in Theorem 2.2 is a regular splitting of A and the L and U are also M-matrices. The following theorem shows the relations between the ILU factorizations of an H-matrix A and $\langle A \rangle$.

Theorem 2.3 (Kim and Yun [6] and Messaoudi [10]). *Let $A \in \mathbb{R}^{n \times n}$ be an H-matrix. Let $A = LU - N$ and $\langle A \rangle = \tilde{L}\tilde{U} - N$ be the ILU factorizations of A and $\langle A \rangle$ corresponding to a zero pattern set $Q \subset S_n$, respectively. Then each of the following holds:*

$$\text{(a) } |L^{-1}| \leqslant \tilde{L}^{-1}, \quad \text{(b) } |U^{-1}| \leqslant \tilde{U}^{-1},$$

$$\text{(c) } |N| \leqslant \tilde{N}, \qquad \text{(d) } |(LU)^{-1}N| \leqslant (\tilde{L}\tilde{U})^{-1}\tilde{N}.$$

In Theorem 2.3, notice that LU is not an H-matrix and $\tilde{L}\tilde{U}$ is not an M-matrix even if L and U are H-matrices and $\tilde{L}$ and $\tilde{U}$ are M-matrices. For symmetric H-matrices, results similar to Theorems 2.2 and 2.3 are given in Theorems 2.4 and 2.5, respectively.

Theorem 2.4. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric H-matrix. Then, for every symmetric zero pattern set $Q \subset S_n$ (i.e., $(i,j) \in Q$ implies $(j,i) \in Q$), there exist an upper triangular matrix $U = (u_{ij})$, a diagonal matrix D whose kth diagonal element is u_{kk}^{-1}, and a symmetric matrix $N = (n_{ij})$, with $u_{ij} = 0$ if $(i,j) \in Q$ and $n_{ij} = 0$ if $(i,j) \notin Q$, such that $A = U^{\mathrm{T}}DU - N$. Moreover, U is an H-matrix.*

In Theorem 2.4, $A = U^{\mathrm{T}}DU - N$ is called an incomplete factorization of A corresponding to a symmetric zero pattern set Q. When A is a symmetric H-matrix with positive diagonal elements, the diagonal matrix D in the incomplete factorization $A = U^{\mathrm{T}}DU - N$ given in Theorem 2.4 has also positive diagonal elements and thus $U^{\mathrm{T}}DU$ is a symmetric positive definite matrix (see [8]). Note that the incomplete factorization $A = U^{\mathrm{T}}DU - N$ in Theorem 2.4 is a regular splitting of A when A is a symmetric M-matrix.

Theorem 2.5. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric H-matrix. Let $A = U^{\mathrm{T}}DU - N$ and $\langle A\rangle = U^{\mathrm{T}}\tilde{D}\tilde{U} - \tilde{N}$ be the incomplete factorizations of A and $\langle A\rangle$ corresponding to a symmetric zero pattern set $Q \subset S_n$, respectively. Then each of the following holds*:

(a) $|(U^{\mathrm{T}}D)^{-1}| \leqslant (\tilde{U}^{\mathrm{T}}\tilde{D})^{-1}$, (b) $|U^{-1}| \leqslant \tilde{U}^{-1}$,

(c) $|N| \leqslant \tilde{N}$, (d) $|(U^{\mathrm{T}}DU)^{-1}N| \leqslant (\tilde{U}^{\mathrm{T}}\tilde{D}\tilde{U})^{-1}\tilde{N}$.

3. Convergence of two-stage iterative methods

In this section, we study the convergence of two-stage iterative methods (i.e., Algorithms 1, 2 and 3) described in Section 1. Algorithm 1 can be written as

$$x_k = T_p x_{k-1} + K_p b, \quad k = 1, 2, \ldots, \tag{2}$$

where

$$\begin{aligned} T_p &= (F^{-1}G)^p + \sum_{j=0}^{p-1} (F^{-1}G)^j F^{-1}N \\ &= (F^{-1}G)^p + (I - (F^{-1}G)^p)M^{-1}N \\ &= I - (I - (F^{-1}G)^p)(I - M^{-1}N) \end{aligned}$$

and

$$K_p = \sum_{j=0}^{p-1} (F^{-1}G)^j F^{-1} = (I - (F^{-1}G)^p)M^{-1}.$$

The T_p is called an iteration matrix for Algorithm 1. Then, it is easy to show that $K_pA = I - T_p$. Hence, the exact solution ξ of the linear system (1) satisfies

$$\xi = T_p\xi + K_pb. \tag{3}$$

From (2) and (3), the error vector $e_k = x_k - \xi$ satisfies

$$e_k = T_pe_{k-1} = (T_p)^k e_0, \quad k = 1, 2, \ldots . \tag{4}$$

From (4), the sequence of vectors generated by iteration (2) converges to ξ for any initial vector x_0 if and only if $\rho(T_p) < 1$. If $\rho(F^{-1}G) < 1$, then $I - (F^{-1}G)^p$ is nonsingular for any integer $p \geqslant 1$. Thus, from Lemma 2.1, there exists a unique pair of matrices $B_p = M(I - (F^{-1}G)^p)^{-1}$ and C_p such that $M = B_p - C_p$ and $(F^{-1}G)^p = B_p^{-1}C_p$. It follows that

$$\begin{aligned} T_p &= (F^{-1}G)^p + (I - (F^{-1}G)^p)M^{-1}N \\ &= B_p^{-1}C_p + B_p^{-1}N \\ &= B_p^{-1}(C_p + N). \end{aligned}$$

Algorithm 2 can be written as

$$x_k = T_{p_k}x_{k-1} + K_{p_k}b, \quad k = 1, 2, \ldots, \tag{5}$$

where T_{p_k} and K_{p_k} are defined as in Algorithm 1 and the only difference is to use p_k instead of using p. The T_{p_k}'s are called iteration matrices for Algorithm 2. Then, it is easy to show that $K_{p_k}A = I - T_{p_k}$ for each k. Hence, the exact solution ξ of the linear system (1) satisfies

$$\xi = T_{p_k}\xi + K_{p_k}b, \quad k = 1, 2, \ldots . \tag{6}$$

From (5) and (6), the error vector $e_k = x_k - \xi$ satisfies

$$e_k = T_{p_k}e_{k-1} = T_{p_k}T_{p_{k-1}} \cdots T_{p_1}e_0, \quad k = 1, 2, \ldots . \tag{7}$$

From (7), the sequence of vectors generated by the iteration (5) converges to ξ for any initial vector x_0 if and only if

$$\lim_{k\to\infty} T_{p_k}T_{p_{k-1}} \cdots T_{p_1} = 0. \tag{8}$$

Algorithm 3 can be written as

$$x_k = T_{p,\omega}x_{k-1} + K_{p,\omega}b, \quad k = 1, 2, \ldots, \tag{9}$$

where

$$T_{p,\omega} = \omega T_p + (1 - \omega)I \quad \text{and} \quad K_{p,\omega} = \omega K_p.$$

The $T_{p,\omega}$ is called an iteration matrix for Algorithm 3. Then, $K_{p,\omega}A = I - T_{p,\omega}$. Hence, the exact solution ξ of the linear system (1) satisfies

$$\xi = T_{p,\omega}\xi + K_{p,\omega}b. \tag{10}$$

From (9) and (10), the error vector $e_k = x_k - \xi$ satisfies

$$e_k = T_{p,\omega} e_{k-1} = (T_{p,\omega})^k e_0, \quad k = 1, 2, \ldots \,. \tag{11}$$

From (11), the sequence of vectors generated by iteration (9) converges to ξ for any initial vector x_0 if and only if $\rho(T_{p,\omega}) < 1$.

Theorem 3.1 (Lanzkron et al. [7]). *Let $A^{-1} \geqslant 0$ and let $A = M - N = \hat{M} - \hat{N}$ be weak regular splittings such that $\hat{M}^{-1} \geqslant M^{-1}$. Let x and y be the nonnegative Frobenius eigenvectors of $M^{-1}N$ and $\hat{M}^{-1}\hat{N}$, respectively. If $\hat{N} y \geqslant 0$ or if $Nx \geqslant 0$ with $x > 0$, then*

$$\rho(\hat{M}^{-1}\hat{N}) \leqslant \rho(M^{-1}N).$$

Theorem 3.2. *Assume that $A^{-1} \geqslant 0$. Let $A = M - N$ be a regular splitting and let $M = F - G = \hat{F} - \hat{G}$ be a weak regular splittings. If $\hat{F}^{-1}\hat{G} \leqslant F^{-1}G$ and $\hat{G}\hat{F}^{-1} \geqslant 0$, then $\rho(\hat{T}_p) \leqslant \rho(T_p)$, where T_p and $\hat{T}_p$ are iteration matrices for Algorithm 1, that is,*

$$T_p = (F^{-1}G)^p + (I - (F^{-1}G)^p)M^{-1}N = B_p^{-1}(C_p + N),$$

$$\hat{T}_p = (\hat{F}^{-1}\hat{G})^p + (I - (\hat{F}^{-1}\hat{G})^p)M^{-1}N = \hat{B}_p^{-1}(\hat{C}_p + N),$$

$$B_p = M(I - (F^{-1}G)^p)^{-1}, \quad \hat{B}_p = M(I - (\hat{F}^{-1}\hat{G})^p)^{-1},$$

$$C_p = B_p - M, \quad \hat{C}_p = \hat{B}_p - M.$$

Proof. By simple calculation, it can be shown that

$$B_p^{-1} = (I - (F^{-1}G)^p)M^{-1} = \sum_{i=0}^{p-1} (F^{-1}G)^i F^{-1} \geqslant 0,$$

$$\hat{B}_p^{-1} = (I - (\hat{F}^{-1}\hat{G})^p)M^{-1} = \sum_{i=0}^{p-1} (\hat{F}^{-1}\hat{G})^i \hat{F}^{-1} \geqslant 0.$$

Since $B_p^{-1}C_p = (F^{-1}G)^p \geqslant 0$, $\hat{B}_p^{-1}\hat{C}_p = (\hat{F}^{-1}\hat{G})^p \geqslant 0$ and $N \geqslant 0$, $(F^{-1}G)^p \leqslant T_p$ and $(\hat{F}^{-1}\hat{G})^p \leqslant \hat{T}_p$. It follows that $\rho((F^{-1}G)^p) \leqslant \rho(T_p)$ and $\rho((\hat{F}^{-1}\hat{G})^p) \leqslant \rho(\hat{T}_p)$. By assumption, $\rho(\hat{F}^{-1}\hat{G}) \leqslant \rho(F^{-1}G)$ and thus $\rho((\hat{F}^{-1}\hat{G})^p) \leqslant \rho((F^{-1}G)^p)$. Consider first the case $\rho((\hat{F}^{-1}\hat{G})^p) = \rho(\hat{T}_p)$. Then, $\rho(\hat{T}_p) = \rho((\hat{F}^{-1}\hat{G})^p) \leqslant \rho((F^{-1}G)^p) \leqslant \rho(T_p)$. Thus, $\rho(\hat{T}_p) \leqslant \rho(T_p)$ is proved. Next, we consider the case $\rho((\hat{F}^{-1}\hat{G})^p) < \rho(\hat{T}_p)$. Since $\hat{T}_p \geqslant 0$, there exists an eigenvector $x \geqslant 0$ such that $\hat{T}_p x = \rho(\hat{T}_p)x$. Hence, one obtains

$$\rho(\hat{T}_p)\hat{B}_p x = (\hat{C}_p + N)x,$$

$$\rho(\hat{T}_p)M(I - (\hat{F}^{-1}\hat{G})^p)^{-1}x = M(I - (\hat{F}^{-1}\hat{G})^p)^{-1}(\hat{F}^{-1}\hat{G})^p x + Nx,$$

$$\rho(\hat{T}_p)(I - (\hat{F}^{-1}\hat{G})^p)^{-1}x = (I - (\hat{F}^{-1}\hat{G})^p)^{-1}(\hat{F}^{-1}\hat{G})^p x + M^{-1}Nx,$$

$$\rho(\hat{T}_p)(I-(\hat{F}^{-1}\hat{G})^p)^{-1}x = (\hat{F}^{-1}\hat{G})^p(I-(\hat{F}^{-1}\hat{G})^p)^{-1}x + M^{-1}Nx,$$

$$(I-(\hat{F}^{-1}\hat{G})^p)^{-1}x = (\rho(\hat{T}_p)I-(\hat{F}^{-1}\hat{G})^p)^{-1}M^{-1}Nx. \tag{12}$$

Using the last equality of Eq. (12) in the third equation of (13) and $M(\hat{F}^{-1}\hat{G})^p = (\hat{G}\hat{F}^{-1})^pM$, one obtains

$$\begin{aligned}(\hat{C}_p+N)x &= M(I-(\hat{F}^{-1}\hat{G})^p)^{-1}(\hat{F}^{-1}\hat{G})^px + Nx\\ &= M(\hat{F}^{-1}\hat{G})^p(I-(\hat{F}^{-1}\hat{G})^p)^{-1}x + Nx\\ &= M(\hat{F}^{-1}\hat{G})^p(\rho(\hat{T}_p)I-(\hat{F}^{-1}\hat{G})^p)^{-1}M^{-1}Nx + Nx\\ &= (\hat{G}\hat{F}^{-1})^pM(\rho(\hat{T}_p)I-(\hat{F}^{-1}\hat{G})^p)^{-1}M^{-1}Nx + Nx\\ &= (\hat{G}\hat{F}^{-1})^p(\rho(\hat{T}_p)I-M(\hat{F}^{-1}\hat{G})^pM^{-1})^{-1}Nx + Nx\\ &= (\hat{G}\hat{F}^{-1})^p(\rho(\hat{T}_p)I-(\hat{G}\hat{F}^{-1})^p)^{-1}Nx + Nx\end{aligned} \tag{13}$$

Since $\rho(\hat{G}\hat{F}^{-1}) = \rho(\hat{F}^{-1}\hat{G})$ and $\rho((\hat{F}^{-1}\hat{G})^p) < \rho(\hat{T}_p)$, $\rho((\hat{G}\hat{F}^{-1})^p) < \rho(\hat{T}_p)$ and hence

$$\rho\left(\frac{(\hat{G}\hat{F}^{-1})^p}{\rho(\hat{T}_p)}\right) < 1. \tag{14}$$

Using (14) and the hypothesis $\hat{G}\hat{F}^{-1} \geqslant 0$, we obtains

$$\begin{aligned}(\rho(\hat{T}_p)I-(\hat{G}\hat{F}^{-1})^p)^{-1} &= \frac{1}{\rho(\hat{T}_p)}\left(I-\frac{(\hat{G}\hat{F}^{-1})^p}{\rho(\hat{T}_p)}\right)^{-1}\\ &= \frac{1}{\rho(\hat{T}_p)}\sum_{j=0}^{\infty}\left(\frac{(\hat{G}\hat{F}^{-1})^p}{\rho(\hat{T}_p)}\right)^j \geqslant 0.\end{aligned} \tag{15}$$

From (13) and (15),

$$(\hat{C}_p+N)x \geqslant 0. \tag{16}$$

Since $0 \leqslant (\hat{F}^{-1}\hat{G})^p \leqslant (F^{-1}G)^p$,

$$B_p^{-1} = (I-(F^{-1}G)^p)M^{-1} \leqslant (I-(\hat{F}^{-1}\hat{G})^p)M^{-1} = \hat{B}_p^{-1}. \tag{17}$$

Notice that $A = B_p-(C_p+N) = \hat{B}_p-(\hat{C}_p+N)$ are weak regular splittings. Therefore, (16), (17) and Theorem 3.1 imply that $\rho(\hat{T}_p) \leqslant \rho(T_p)$. □

Theorem 3.2 presents a comparison result for iteration matrices of Algorithm 1. Now we give convergence results of two-stage iterative methods using the incomplete factorization for solving a linear system whose coefficient matrix is an H-matrix.

Theorem 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix and let $A = M - N$ be an H-splitting of A. Let $M = LU - G$ be the ILU factorization of M corresponding to a zero pattern set $Q \subset S_n$. Then, Algorithm 1 converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. Since $A = M - N$ is an H-splitting of A, $\langle M\rangle - |N|$ is an M-matrix and M is an H-matrix. Let $\langle M\rangle = \tilde{L}\tilde{U} - \tilde{G}$ be the ILU factorization of the M-matrix $\langle M\rangle$ corresponding to the zero pattern set Q. Then, $\langle M\rangle = \tilde{L}\tilde{U} - \tilde{G}$ is a regular splitting of $\langle M\rangle$. Using Theorem 2.3,

$$
\begin{aligned}
|T_p| &= \left| ((LU)^{-1}G)^p + \sum_{j=0}^{p-1} ((LU)^{-1}G)^j (LU)^{-1} N \right| \\
&\leqslant |(LU)^{-1}G|^p + \sum_{j=0}^{p-1} |(LU)^{-1}G|^j |(LU)^{-1}|\,|N| \\
&\leqslant ((\tilde{L}\tilde{U})^{-1}\tilde{G})^p + \sum_{j=0}^{p-1} ((\tilde{L}\tilde{U})^{-1}\tilde{G})^j (\tilde{L}\tilde{U})^{-1}|N|. \qquad (18)
\end{aligned}
$$

Let $\tilde{T}_p$ denote the matrix in the last line of Eq. (18). Then, $\tilde{T}_p$ is the iteration matrix of a stationary two-stage iterative method for the M-matrix $\langle M\rangle - |N|$ with the regular splittings $\langle M\rangle - |N|$ and $\langle M\rangle = \tilde{L}\tilde{U} - \tilde{G}$. Thus, $\rho(\tilde{T}_p) < 1$. From (18), $|T_p| \leqslant \tilde{T}_p$ and hence $\rho(T_p) \leqslant \rho(\tilde{T}_p)$. Therefore, $\rho(T_p) < 1$ is achieved. □

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix and let $A = M - N$ be an H-splitting of A. Let $M = LU - G$ be the ILU factorization of M corresponding to a zero pattern set $Q \subset S_n$. Then, Algorithm* 2 *converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. Let $\langle M\rangle = \tilde{L}\tilde{U} - \tilde{G}$ be the ILU factorization of $\langle M\rangle$ corresponding to the zero pattern set Q. Using (18), it is easy to show that $|T_{p_k}| < \tilde{T}_{p_k}$ for each k, where $\tilde{T}_{p_k}$ is defined as in the proof of Theorem 3.3. Hence, one obtains

$$
|T_{p_k} T_{p_{k-1}} \cdots T_{p_1}| \leqslant \tilde{T}_{p_k} \tilde{T}_{p_{k-1}} \cdots \tilde{T}_{p_1}. \qquad (19)
$$

Notice that $\tilde{T}_{p_k} \tilde{T}_{p_{k-1}} \cdots \tilde{T}_{p_1}$ is the matrix corresponding to k steps of a nonstationary two-stage iterative method for the M-matrix $\langle M\rangle - |N|$ with the regular splittings $\langle M\rangle - |N|$ and $\langle M\rangle = \tilde{L}\tilde{U} - \tilde{G}$. It follows that $\lim_{k\to\infty} \tilde{T}_{p_k} \tilde{T}_{p_{k-1}} \cdots \tilde{T}_{p_1} = 0$. Therefore, (19) implies that $\lim_{k\to\infty} T_{p_k} T_{p_{k-1}} \cdots T_{p_1} = 0$. □

Corollary 3.5. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix and let $A = M - N$ be an H -compatible splitting of A. Let $M = LU - G$ be the ILU factorization of M corresponding to a zero pattern set $Q \subset S_n$. Then, Algorithms* 1 *and* 2 *converge to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. Since A is an H-matrix and $A = M - N$ is an H-compatible splitting of A, $A = M - N$ is an H-splitting of A. Hence, Theorems 3.3 and 3.4 imply that Algorithms 1 and 2 converge to the exact solution of $Ax = b$ for any initial vector x_0. □

Corollary 3.6. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric H-matrix and let $A = M - N$ be a symmetric H-splitting of A. Let $M = U^{\mathrm{T}}DU - G$ be the incomplete factorization of M corresponding to a symmetric* *zero pattern set $Q \subset S_n$. Then, Algorithms* 1 *and* 2 *converge to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. This corollary can be proved in a similar way as was done for Theorems 3.3 and 3.4 by using Theorem 2.5. □

Theorem 3.7. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that $A^{-1} \geqslant 0$ and let $A = M - N$ be an M-splitting of A. Let $M = LU - G$ be the ILU factorization of M corresponding to a zero pattern set $Q \subset S_n$. Then, Algorithms* 1 *and* 2 *converge to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. It is clear that $A = M - N$ is a regular splitting of A. Since M is an M-matrix, $M = LU - G$ is also a regular splitting of M. Hence, the proof is complete. □

Theorem 3.8. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix and let $A = M - N$ be an H-splitting of A. Let $M = LU - G$ be the ILU factorization of M corresponding to a zero pattern set $Q \subset S_n$. If $0 < \omega < 2/(1 + \rho(T_p))$, then Algorithm* 3 *converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. We must claim that $\rho(T_{p,\omega}) < 1$, where $T_{p,\omega} = \omega T_p + (1 - \omega)I$. Let λ be an eigenvalue of T_p. Then, an eigenvalue of $T_{p,\omega}$ is $\lambda\omega + (1 - \omega)$. Note that

$$|\lambda\omega + (1 - \omega)| \leqslant \omega\rho(T_p) + |1 - \omega|. \tag{20}$$

From Theorem 3.3, $\rho(T_p) < 1$. Thus, for all $0 < \omega < 2/(1 + \rho(T_p))$,

$$\omega\rho(T_p) + |1 - \omega| < 1. \tag{21}$$

Since λ is an arbitrary eigenvalue of T_p, $\rho(T_{p,\omega}) < 1$ is obtained from (20) and (21). □

It was shown in [4] that Algorithms 1 and 2 converge to the exact solution of $Ax = b$ for any initial vector x_0 under the assumption that $A = M - N$ is an H-splitting and $M = F - G$ is an H-compatible splitting. The following example shows that the ILU factorization $M = LU - G$ used in Theorems 3.3 and 3.4 is not an H-compatible splitting. This means that Theorems 3.3 and 3.4 provide new convergence results for Algorithms 1 and 2 which are different from the convergence results given in [4].

Example 3.9. Consider a 3×3 matrix M of the form

$$M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Take a zero pattern set $Q = \{(2,3),(3,2)\} \subset S_3$. Then, M is an H-matrix since $\langle M \rangle^{-1} > 0$. The ILU factorization of M corresponding to Q is $M = LU - G$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 1 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

By simple calculation, one obtains

$$\langle LU \rangle - |G| = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Thus, $\langle M \rangle \neq \langle LU \rangle - |G|$, which shows that the ILU factorization of M is not an H-compatible splitting.

Next, we give convergence results of two-stage iterative methods using the incomplete factorization for solving a linear system whose coefficient matrix is symmetric positive definite. Let A be a symmetric positive definite matrix. Then, we can choose a nonnegative diagonal matrix Λ such that $\hat{A} = A + \Lambda$ is a generalized strictly diagonally dominant matrix (i.e., an H-matrix). For example, the easiest way of finding such a matrix $\hat{A}$ is to choose a nonnegative diagonal matrix Λ such that $\hat{A} = A + \Lambda$ is a strictly diagonally dominant matrix. From now on, the transformed matrix $\hat{A}=A+\Lambda$ is called a *diagonally dominated matrix* of A. Notice that $\hat{A}$ is a symmetric positive definite H-matrix.

Lemma 3.10. *Let A be a symmetric positive definite matrix and $\hat{A} = A + \Lambda$ be a diagonally dominated matrix of A. Let $\hat{A} = M - \hat{N}$ be a splitting with M being symmetric positive definite and let $N = M - A$. If the splitting $\hat{A} = M - \hat{N}$ is convergent, then the splitting $A = M - N$ is convergent.*

Proof. It is easy to see that $\hat{A}^{-1}A$ has positive eigenvalues and $\hat{A}^{-1}\Lambda$ has nonnegative eigenvalues. Since $\hat{A}^{-1}\Lambda = I - \hat{A}^{-1}A$, for every eigenvalue λ of $\hat{A}^{-1}A$

$$0 < \lambda \leqslant 1. \tag{22}$$

Since A, $\hat{A}$ and M are symmetric positive definite, from (22) and Lemma 2.1 in [1] one obtains

$$0 < \lambda_j(M^{-1}A) \leqslant \lambda_j(M^{-1}\hat{A})\lambda_{\max}(\hat{A}^{-1}A) \leqslant \lambda_j(M^{-1}\hat{A}), \tag{23}$$

where $\lambda_j(C)$ denotes the jth eigenvalue of C whose eigenvalues are assumed to be numbered in a nondecreasing order and $\lambda_{\max}(C)$ denotes the maximum eigenvalue of C. Since $M^{-1}N = I - M^{-1}A$ and $M^{-1}\hat{N} = I - M^{-1}\hat{A}$, from (23)

$$\lambda_j(M^{-1}\hat{N}) \leqslant \lambda_j(M^{-1}N) < 1. \tag{24}$$

By assumption, $|\lambda_j(M^{-1}\hat{N})| < 1$ for each j. Hence, from (24) $|\lambda_j(M^{-1}N)| < 1$ for each j. Therefore, $\rho(M^{-1}N) < 1$ is obtained. □

Lemma 3.10 implies that the problem of constructing a convergent splitting for a symmetric positive definite matrix can always be reduced to that of constructing a convergent splitting for a diagonally dominated matrix which is a symmetric positive definite H-matrix.

Lemma 3.11. *Let A be a symmetric positive definite matrix and $\hat{A}$ be a diagonally dominated matrix of A. Let $\hat{A} = M - \hat{N}$ be a symmetric H-splitting with M having positive diagonal elements. If we let $N = M - A$, then the splitting $A = M - N$ is convergent.*

Proof. Since $\hat{A} = M - \hat{N}$ is a symmetric H-splitting, M is a symmetric H-matrix and $\rho(M^{-1}\hat{N}) < 1$. Since M is a symmetric H-matrix with positive diagonal elements, M is symmetric positive definite. Hence, by Lemma 3.10 $\rho(M^{-1}N) < 1$. □

Lemma 3.12. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric H-matrix and $A = U^{\mathrm{T}}DU - N$ be the incomplete factorization of A corresponding to a symmetric zero pattern set $Q \subset S_n$. Then, the splitting $A = U^{\mathrm{T}}DU - N$ is convergent.*

Proof. Let $\langle A\rangle = \tilde{U}^{\mathrm{T}}\tilde{D}\tilde{U} - \tilde{N}$ be the incomplete factorization of $\langle A\rangle$ corresponding to the symmetric zero pattern set Q. For simplicity, let $M = U^{\mathrm{T}}DU$ and $\tilde{M} = \tilde{U}^{\mathrm{T}}\tilde{D}\tilde{U}$. Since $\langle A\rangle$ is an M-matrix, $\langle A\rangle = \tilde{M} - \tilde{N}$ is a regular splitting of $\langle A\rangle$ and thus $\rho(\tilde{M}^{-1}\tilde{N}) < 1$. From Theorem 2.5, $|M^{-1}N| \leqslant \tilde{M}^{-1}\tilde{N}$ and thus $\rho(M^{-1}N) < 1$. □

Theorem 3.13. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $\hat{A}$ be a diagonally dominated matrix of A. Let $\hat{A} = U^{\mathrm{T}}DU - \hat{N}$ be the incomplete factorization of $\hat{A}$ corresponding to a symmetric zero pattern set $Q \subset S_n$. If we let $N = U^{\mathrm{T}}DU - A$, then the splitting $A = U^{\mathrm{T}}DU - N$ is convergent.*

Proof. Since $\hat{A}$ is a symmetric positive definite H-matrix, $U^{\mathrm{T}}DU$ is symmetric positive definite. From Lemma 3.12, the splitting $\hat{A} = U^{\mathrm{T}}DU - \hat{N}$ is convergent. Hence, by Lemma 3.10 $\rho((U^{\mathrm{T}}DU)^{-1}N) < 1$. □

Theorem 3.14. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $\hat{A}$ be a diagonally dominated matrix of A. Let $\hat{A} = M - \hat{N}$ be a symmetric H-splitting with M having positive diagonal elements, and let $N = M - A$. Let $M = U^{\mathrm{T}}DU - G$ be the incomplete factorization of M corresponding to a symmetric zero pattern set $Q \subset S_n$. Then, Algorithm* 1 *converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. By assumption, it is clear that M is a symmetric positive definite H-matrix. Thus, $U^{\mathrm{T}}DU$ is symmetric positive definite. For simplicity, let $F = U^{\mathrm{T}}DU$. Let $\hat{T}_p$ denote the iteration matrix of a stationary two-stage iterative method for the matrix $\hat{A}$ with the splittings $\hat{A} = M - \hat{N}$ and $M = F - G$. Then, by Corollary 3.6 $\rho(\hat{T}_p) < 1$. Since $\rho(F^{-1}G) < 1$ from Lemma 3.12, there exist

$B_p = M(I - (F^{-1}G)^p)^{-1}$ and C_p such that

$$A = B_p - (C_p + N), \quad T_p = B_p^{-1}(C_p + N),$$

$$\hat{A} = B_p - (C_p + \hat{N}), \quad \hat{T}_p = B_p^{-1}(C_p + \hat{N}).$$

If we show that B_p is symmetric positive definite, then Lemma 3.10 implies $\rho(T_p) < 1$ since $\rho(\hat{T}_p) < 1$. Since $\rho(F^{-1}G) < 1$, B_p and B_p^{-1} can be written as

$$B_p = M\sum_{j=0}^{\infty}(F^{-1}G)^{pj} = M + \sum_{j=1}^{\infty} M(F^{-1}G)^{pj}, \tag{25}$$

$$B_p^{-1} = (I - (F^{-1}G)^p)\sum_{j=0}^{\infty}(F^{-1}G)^j F^{-1} = \sum_{j=0}^{p-1}(F^{-1}G)^j F^{-1}. \tag{26}$$

It is clear that B_1 is symmetric positive definite since $B_1 = F$. Let p be an even number. Then, for each $j = 1, 2, \ldots,$

$$M(F^{-1}G)^{pj} = (GF^{-1})^{pj/2} M(F^{-1}G)^{pj/2}. \tag{27}$$

From (27), $M(F^{-1}G)^{pj}$ is symmetric positive semidefinite for each $j = 1, 2, \ldots$. Thus, (25) implies that B_p is symmetric positive definite. On the other hand, from (26) one obtains

$$\begin{aligned} B_{p+1}^{-1} &= \sum_{j=0}^{p}(F^{-1}G)^j F^{-1} \\ &= B_p^{-1} + (F^{-1}G)^p F^{-1} \\ &= B_p^{-1} + (F^{-1}G)^{p/2} F^{-1}(GF^{-1})^{p/2}. \end{aligned} \tag{28}$$

Since B_p^{-1} and F^{-1} are symmetric positive definite, from (28) B_{p+1}^{-1} is also symmetric positive definite. Therefore, B_p is symmetric positive definite for any positive integer p. □

Theorem 3.15. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $\hat{A}$ be a diagonally dominated matrix of A. Let $\hat{A} = M - \hat{N}$ be a symmetric H-splitting with M having positive diagonal elements, and let $N = M - A$. Let $M = U^{\mathrm{T}}DU - G$ be the incomplete factorization of M corresponding to a symmetric zero pattern set $Q \subset S_n$. If $0 < \omega < 2/(1 + \rho(T_\rho))$, then Algorithm 3 converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. From Theorem 3.14, $\rho(T_p) < 1$. Hence, $\rho(T_{p,\omega}) < 1$ can be shown as in the proof of Theorem 3.8. □

Theorem 3.16 (Cao [2]). *Let A be a symmetric positive definite matrix. Let $A = M - N$ and $M = F - G$ be symmetric splittings with $\rho(M^{-1}N) < 1$ and $\rho(F^{-1}G) < 1$. Assume that Algorithm 1 converges to the exact solution of $Ax = b$ for any initial vector x_0. Then, Algorithm 2 converges too.*

Theorem 3.17. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $\hat{A}$ be a diagonally dominated matrix of A. Let $\hat{A} = M - \hat{N}$ be a symmetric H-splitting with M having positive diagonal elements, and let $N = M - A$. Let $M = U^{\mathrm{T}}DU - G$ be the incomplete factorization of M corresponding to a symmetric zero pattern set $Q \subset S_n$. Then, Algorithm 2 converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. Notice that $\rho(T_p) < 1$ from Theorem 3.14 and $\rho(M^{-1}N) < 1$ from Lemma 3.11. Since M is a symmetric H-matrix, $\rho(F^{-1}G) < 1$ from Lemma 3.12, where $F = U^{\mathrm{T}}DU$. Hence, the conclusion follows from Theorem 3.16. □

Theorem 3.18. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $\hat{A}$ be a diagonally dominated matrix of A. Let $\hat{A} = M - \hat{N}$ be a symmetric H-splitting with M having positive diagonal elements, and let $N = M - A$. Let $M = F - G$ be a symmetric H-compatible splitting with F having positive diagonal elements. Then, Algorithms 1 and 2 converge to the exact solution of $Ax = b$ for any initial vector x_0. For $0 < \omega < 2/(1 + \rho(T_\rho))$, Algorithm 3 also converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Proof. By assumptions, M and F are symmetric positive definite H-matrices, $\rho(F^{-1}G) < 1$ and $\rho(\hat{T}_p) < 1$. As in the proof of Theorem 3.14, it can be shown that B_p is symmetric positive definite. Hence, $\rho(T_p) < 1$, that is, Algorithm 1 converges for any initial vector x_0. Since $\rho(M^{-1}N) < 1$ from Lemma 3.11, Theorem 3.16 implies that Algorithm 2 converges for any initial vector x_0. Since $\rho(T_p) < 1$, convergence of Algorithm 3 for $0 < \omega < 2/(1 + \rho(T_p))$ follows by using the same arguments as was done for Theorem 3.8. □

Cao [2] also presented convergence results of two-stage iterative methods for solving a linear system whose coefficient matrix is symmetric positive definite. The main idea used in [2] is based on transforming a symmetric positive definite matrix A into a symmetric positive definite M-matrix $\hat{A}$, while the main idea used in this paper is based on transforming a symmetric positive definite matrix A into a symmetric positive definite H-matrix $\hat{A}$. Cao [2] showed the convergence results under the assumptions that $\hat{A} = M - \hat{N}$ is a symmetric regular splitting with M being symmetric positive definite and $M = F - G$ is a symmetric weak regular splitting with F being symmetric positive definite. Note that M-matrices are contained in the class of all H-matrices. In this respect, the convergence results presented in this paper can be viewed as an extension of those presented in [2]. The incomplete factorization of a large sparse matrix drops many fill-in elements according to a given zero pattern set, so one advantage of two-stage iterative methods using the incomplete factorization as an inner splitting is that the linear system required for each inner iteration can be solved cheaply compared to the standard two-stage iterative methods using a weak regular splitting as an inner splitting.

4. Numerical results

In this section, we present numerical results of the stationary two-stage iterative method (Algorithm 1) using incomplete factorization for solving a linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. For all test problems, the initial vector x_0 is set to zero and the right-hand side vector b is set to Ae, where $e=(1,1,\dots,1)^{\mathrm{T}}\in\mathbb{R}^n$. The stopping criterion for Algorithm 1 is $\|b-Ax_k\|_2/\|b\|_2<10^{-5}$, where $\|\cdot\|_2$ refers to L_2-norm. Let $B\in\mathbb{R}^{n\times n}$ be a symmetric positive definite matrix of the form

$$B=\begin{pmatrix} 16 & -8 & 0 & & & & & \\ -8 & 14 & -8 & 1 & & & & \\ 0 & -8 & 16 & -8 & 0 & & & \\ & 1 & -8 & 14 & -8 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 0 & -8 & 16 & -8 \\ & & & & 1 & -8 & 14 \end{pmatrix}, \tag{29}$$

where the order n of B is assumed to be even. Note that the matrix B is generated from the finite element formulation using piecewise quadratic basis functions for the one-dimensional boundary value problem

$$\begin{cases} -u''=g(x), \quad 0<x<1, \\ u(0)=u(1)=0. \end{cases}$$

Example 4.1. Let $m=n/8$, where n is assumed to be a multiple of 8. Let $A=B+uu^{\mathrm{T}}$, where B is defined by (29), $u=(u_1,u_2,\dots,u_n)^{\mathrm{T}}$ and

$$u_i=\begin{cases} 1 & \text{if } (i-1) \text{ or } i \text{ are multiples of } m, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that A is a symmetric positive definite matrix, but not an H-matrix. Clearly, A is irreducible since B is irreducible. If we choose a nonnegative diagonal matrix Λ such that $\hat{A}=A+\Lambda$ is an irreducibly diagonally dominant matrix, then $\hat{A}$ is a diagonally dominated matrix of A. Let

$$A=\begin{pmatrix} M_1 & C_1^{\mathrm{T}} \\ C_1 & M_2 \end{pmatrix} \quad \text{and} \quad \hat{A}=\begin{pmatrix} \hat{M}_1 & C_1^{\mathrm{T}} \\ C_1 & \hat{M}_2 \end{pmatrix},$$

where all submatrices of A and $\hat{A}$ are of order $n/2$. Let

$$\bar{M}=\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad M=\begin{pmatrix} \hat{M}_1 & 0 \\ 0 & \hat{M}_2 \end{pmatrix}, \quad \hat{N}=\begin{pmatrix} 0 & -C_1^{\mathrm{T}} \\ -C_1 & 0 \end{pmatrix}.$$

Then, $\hat{A}=M-\hat{N}$ is a symmetric H-splitting with M having positive diagonal elements. Let $N=M-A$ and let $M=U^{\mathrm{T}}DU-G$ be the incomplete factorization of M without fill-ins. Then, the iteration matrix of Algorithm 1 using splittings $A=M-N$ and $M=U^{\mathrm{T}}DU-G$ is

$$T_p=H^p+(1-H^p)M^{-1}N,$$

where $H=(U^{\mathrm{T}}DU)^{-1}G$. Numerical results for this problem are provided in Tables 1 and 2.

Table 1
The number of outer iterations of Algorithm 1

	Example 4.1				Example 4.2			
p	$n=64$	$n=128$	$n=256$	$n=512$	$n=64$	$n=128$	$n=256$	$n=512$
1	267	521	948	1135	346	679	567	413
2	266	525	970	1193	334	648	537	370
3	266	524	969	1194	334	649	539	373

Table 2
Spectral radius of the iteration matrix T_p of Algorithm 1

	Example 4.1				Example 4.2			
p	$n=64$	$n=128$	$n=256$	$n=512$	$n=64$	$n=128$	$n=256$	$n=512$
1	0.9913	0.9966	0.9988	0.9996	0.9946	0.9986	0.9997	0.9999
2	0.9916	0.9967	0.9989	0.9996	0.9951	0.9987	0.9997	0.9999
3	0.9916	0.9967	0.9989	0.9996	0.9950	0.9987	0.9997	0.9999

Example 4.2. Let $m=n/8$, where n is assumed to be a multiple of 8. Let $A=B+\sum_{i=1}^{m} S_i S_i^{\mathrm{T}}$, where B is defined by (29), $S_i=(s_{1i},s_{2i},\ldots,s_{ni})^{\mathrm{T}}$ and

$$s_{ji}=\begin{cases} 1 & \text{if } |j-i| \text{ is a multiple of } m,\\ 0 & \text{otherwise.}\end{cases}$$

Then, A is a symmetric positive definite matrix, but not an H-matrix. Numerical experiments for this problem are carried out exactly in the same way as was done for Example 4.1, and numerical results are provided in Tables 1 and 2.

In Examples 4.1 and 4.2, if we assume that $\bar{M}=\bar{U}^{\mathrm{T}}\bar{D}\bar{U}-\bar{G}$ is the incomplete factorization of $\bar{M}$ without fill-ins, then the iteration matrix of Algorithm 1 using splittings $A=\bar{M}-\hat{N}$ and $\bar{M}=\bar{U}^{\mathrm{T}}\bar{D}\bar{U}-\bar{G}$ is

$$\bar{T}_p=\bar{H}^p+(I-\bar{H}^p)\bar{M}^{-1}\hat{N},$$

where $\bar{H}=(\bar{U}^{\mathrm{T}}\bar{D}\bar{U})^{-1}\bar{G}$. The spectral radii of $\bar{T}_p$ are listed in Table 3 for several values of p and n.

According to Theorem 3.14, $\rho(T_p)<1$ for any value of p. Numerical experiments also show this theoretical result (see Tables 1 and 2). However, it is not true that $\rho(\bar{T}_p)<1$ (see Table 3). In other words, the stationary two-stage iterative method (Algorithm 1) derived from the matrix A is not convergent, while Algorithm 1 derived from the diagonally dominated matrix $\hat{A}$ of A is always convergent for any p.

Table 3
Spectral radius of the iteration matrix $\bar{T}_p$ of Algorithm 1

	Example 4.1				Example 4.2			
p	$n=64$	$n=128$	$n=256$	$n=512$	$n=64$	$n=128$	$n=256$	$n=512$
1	2.9196	3.2219	3.4188	3.5348	2.9080	2.9257	2.9312	2.9336
2	2.3759	3.3518	4.0567	4.4965	0.9993	0.9998	0.9999	0.99998
3	4.9022	7.1316	8.9332	10.1311	2.8301	2.8263	2.8203	2.8176

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