



Weak convergence theorems for a countable family of Lipschitzian mappings

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ABSTRACT

This paper is concerned with convergence of an approximating common fixed point sequence of countable Lipschitzian mappings in a uniformly convex Banach space. We also establish weak convergence theorems for finding a common element of the set of fixed points, the set of solutions of an equilibrium problem, and the set of solutions of a variational inequality. With an appropriate setting, we obtain and improve the corresponding results recently proved by Moudafi [A. Moudafi, Weak convergence theorems for nonexpansive mappings and equilibrium problems. *J. Nonlinear Convex Anal.* 9 (2008) 37–43], Tada–Takahashi [A. Tada and W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem. *J. Optim. Theory Appl.* 133 (2007) 359–370], and Plubtieng–Kumam [S. Plubtieng and P. Kumam, Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings. *J. Comput. Appl. Math.* (2008) doi:10.1016/j.cam.2008.05.045]. Some of our results are established with weaker assumptions.

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1. Introduction

Let C be a subset of a real Banach space E . A mapping $T : C \rightarrow E$ is said to be *Lipschitzian* if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be L -Lipschitzian. Clearly, if T is L_1 -Lipschitzian and $L_1 < L_2$, then T is L_2 -Lipschitzian. Throughout the paper, we assume that every Lipschitzian mapping is L -Lipschitzian with $L \geq 1$. If $L = 1$, then T is known as a nonexpansive mapping. We denote by $F(T)$ the set of fixed points of T . If C is a nonempty bounded closed convex subset of a uniformly convex Banach space and T is a nonexpansive self-mapping of C , then $F(T) \neq \emptyset$ (see [1]). There are many methods for approximating fixed points of a mapping. In 1953, Mann [2] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, Aoyama [3] extended iteration (1.1) to obtain weak convergence to a common fixed point of a countable family of nonexpansive mappings $\{T_n\}_{n=1}^\infty$ by the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

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where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, the present authors [4] proved that the iteration (1.2) converges weakly to a common fixed point of a countable family of Lipschitzian mappings in a real Hilbert space.

In this paper, we establish weak convergence theorem for finding common fixed points of a countable family of Lipschitzian mappings in a uniformly convex Banach space. We also establish weak convergence of iterative sequences for finding a common element of the set of fixed points, the set of solutions of an equilibrium problem, and the set of solutions of a variational inequality. With an appropriate setting, we obtain the corresponding results due to Moudafi [5], Tada–Takahashi [6], and Plubtieng–Kumam [7]. Some of our results are established with weaker assumptions.

2. Preliminaries

A real Banach space E is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \quad \text{imply} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

We need some facts and tools in a uniformly convex Banach space which are listed as lemmas below.

Lemma 2.1 ([8]). Let E be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq d$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq d$ and $\lim_{n \rightarrow \infty} \|\lambda_n x_n + (1 - \lambda_n)y_n\| = d$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 2.2 ([9], Theorem 2). Let E be a uniformly convex Banach space and $B_r := \{x \in E : \|x\| \leq r\}$, where $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|), \quad (2.1)$$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$.

We write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x .

Lemma 2.3 ([10], Lemma 1.1). Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E . Then there is a strictly increasing and continuous convex function $\gamma : [0, \infty) \rightarrow [0, \infty)$ with $\gamma(0) = 0$ such that, for every L -Lipschitzian mapping $T : C \rightarrow E$ and, the following inequality holds

$$\gamma \left(\frac{1}{L} \|tTx + (1 - t)Ty - T(tx + (1 - t)y)\| \right) \leq \|x - y\| - \frac{1}{L} \|Tx - Ty\|,$$

for all $x, y \in C$ and $t \in [0, 1]$.

We also need the following lemma (see [11], Lemma 1).

Lemma 2.4. Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \text{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Recall that a mapping $T : C \rightarrow E$ is *demi-closed at* y , if $x_n \rightharpoonup x$ and $Tx_n \rightarrow y$, then $Tx = y$.

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a real Banach space E , let $\{T_n\}$ be a family of mappings of C into E with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\omega_w\{z_n\}$ denotes the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C . $\{T_n\}$ is said to satisfy

(a) the *AKTT-condition (I)* [12] if for each bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

(b) the *AKTT-condition (II)* [3] if for each bounded subset B of C and each increasing sequence $\{n_i\}$ of \mathbb{N} , there exists a mapping $T : C \rightarrow E$ with $I - T$ is demi-closed at 0 and a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$\lim_{j \rightarrow \infty} \sup\{\|T_{n_{i_j}}z - Tz\| : z \in B\} = 0 \quad \text{and} \quad F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

(c) the *NST-condition* [13] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \quad \text{implies} \quad \omega_w\{z_n\} \subset \bigcap_{n=1}^{\infty} F(T_n).$$

Remark 2.5. If $\{T_n\}$ satisfies the NST-condition and $T_n z \rightarrow z \in C$, then $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.6 ([12], Lemma 3.2). Let C be a nonempty closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into E which satisfies the AKTT-condition (I), then the mapping $T : C \rightarrow E$ defined by

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in C \quad (2.2)$$

satisfies

$$\lim_{n \rightarrow \infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of C . In particular, if $I - T$ is demi-closed at 0 and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, then $\{T_n\}$ satisfies the AKTT-condition (II).

From now on, we will write $(\{T_n\}, T)$ satisfies AKTT-condition (I) if $\{T_n\}$ satisfies AKTT-condition (I) and T is defined by (2.2).

Lemma 2.7 ([14], Theorem 10.4). Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Then $I - T$ is demi-closed at 0.

Lemma 2.8. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition (I) and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{T_n\}$ satisfies the AKTT-condition (II).

Proof. It follows from the definition of T and Lemma 2.7 that T is nonexpansive and $I - T$ is demiclosed at 0, respectively. Applying Lemma 2.6, $\{T_n\}$ satisfies the AKTT-condition (II). \square

Lemma 2.9. Let C be a nonempty closed convex subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. If $\{T_n\}$ satisfies AKTT-condition (II), then $\{T_n\}$ satisfies the NST-condition.

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$ and $z_0 \in \omega_w\{z_n\}$. Then $z_{n_i} \rightharpoonup z_0$ for some subsequence $\{n_i\}$ of $\{n\}$. Since $\{T_n\}$ satisfies AKTT-condition (II), there exists a mapping $T : C \rightarrow E$ with $I - T$ is demi-closed at 0 and a subsequence $\{n_{ij}\}$ of $\{n_i\}$ such that

$$\lim_{j \rightarrow \infty} \sup\{\|T_{n_{ij}} z - Tz\| : z \in \{z_n\}\} = 0 \quad \text{and} \quad F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

Then $z_{n_{ij}} \rightharpoonup z_0$ and

$$\begin{aligned} \|z_{n_{ij}} - T_{n_{ij}} z_{n_{ij}}\| &\leq \|z_{n_{ij}} - T_{n_{ij}} z_{n_{ij}}\| + \|T_{n_{ij}} z_{n_{ij}} - Tz_{n_{ij}}\| \\ &\leq \|z_{n_{ij}} - T_{n_{ij}} z_{n_{ij}}\| + \sup\{\|T_{n_{ij}} z - Tz\| : z \in \{z_n\}\} \rightarrow 0, \end{aligned}$$

as $j \rightarrow \infty$. Since $I - T$ is demi-closed at 0, $z_0 \in F(T)$ and hence $\omega_w\{z_n\} \subset F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. This implies that $\{T_n\}$ satisfies the NST-condition. \square

Lemma 2.10. Let C and K be nonempty closed convex subsets of a real Banach space E . Let $\mathcal{S} := \{S_n\}$ be a family of L_n -Lipschitzian mappings of C into K with $L_n \rightarrow 1$ and $F(\mathcal{S}) := \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\mathcal{W} := \{W_n\}$ be a family of nonexpansive mappings of K into C with $F(\mathcal{W}) := \bigcap_{n=1}^{\infty} F(W_n) \neq \emptyset$ and

$$\|W_n x - u\|^2 \leq \|x - u\|^2 - a_n \|W_n x - x\|^2 \quad \text{for all } x \in K, u \in F(\mathcal{W}) \text{ and } n \in \mathbb{N},$$

where $\{a_n\}$ is a sequence in $[a, \infty) \subset (0, \infty)$. Let $\{T_n\}$ be a family of mappings defined by

$$T_n = S_n W_n \quad \text{for all } n \in \mathbb{N}.$$

If $\{S_n\}$ and $\{W_n\}$ satisfy NST-condition and $F(\mathcal{S}) \cap F(\mathcal{W}) \neq \emptyset$, then $\{T_n\}$ is a family of L_n -Lipschitzian mappings of K into itself satisfies NST-condition and $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{S}) \cap F(\mathcal{W})$.

Proof. It is easy to see that $\{T_n\}$ is a family of L_n -Lipschitzian mappings of K into itself. To show that $\{T_n\}$ satisfies NST-condition, let $\{z_n\}$ be a bounded sequence in K such that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0.$$

Let $u \in F(\mathcal{S}) \cap F(\mathcal{W})$ be given. Then

$$\begin{aligned} \|T_n z_n - u\|^2 &= \|S_n W_n z_n - u\|^2 \\ &\leq L_n^2 \|W_n z_n - u\|^2 \\ &\leq L_n^2 \|z_n - u\|^2 - a_n L_n^2 \|W_n z_n - z_n\|^2 \\ &\leq L_n^2 \|z_n - u\|^2 - a \|W_n z_n - z_n\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} a\|W_n z_n - z_n\|^2 &\leq (L_n^2 - 1)\|z_n - u\|^2 + (\|z_n - u\|^2 - \|T_n z_n - u\|^2) \\ &= (L_n^2 - 1)\|z_n - u\|^2 + (\|z_n - u\| - \|T_n z_n - u\|)(\|z_n - u\| + \|T_n z_n - u\|) \\ &\leq (L_n^2 - 1)\|z_n - u\|^2 + \|z_n - T_n z_n\|(\|z_n - u\| + \|T_n z_n - u\|). \end{aligned}$$

Since $\{z_n - u\}$ and $\{T_n z_n - u\}$ are bounded,

$$\lim_{n \rightarrow \infty} \|W_n z_n - z_n\| = 0.$$

So, we get

$$\begin{aligned} \|S_n W_n z_n - W_n z_n\| &\leq \|S_n W_n z_n - z_n\| + \|W_n z_n - z_n\| \\ &= \|T_n z_n - z_n\| + \|W_n z_n - z_n\| \rightarrow 0. \end{aligned}$$

Since $\{S_n\}$ and $\{W_n\}$ satisfy the NST-condition, we have $\omega_w\{z_n\} \subset F(\mathcal{S}) \cap F(\mathcal{W})$. It is easy to see that $F(\mathcal{S}) \cap F(\mathcal{W}) \subset \bigcap_{n=1}^{\infty} F(T_n)$. To see the reverse inclusion, let $z \in \bigcap_{n=1}^{\infty} F(T_n)$. Follow the first part of the proof above but now let $z_n \equiv z$. Then $z \in F(\mathcal{S}) \cap F(\mathcal{W})$. This implies that $\{T_n\}$ satisfies the NST-condition. This completes the proof. \square

Lemma 2.11. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. If $\{T_n\}$ satisfies the NST-condition, then $\bigcap_{n=1}^{\infty} F(T_n)$ is closed and convex.

Proof. It follows from the continuity of T_n that $F(T_n)$ is closed and so is $\bigcap_{n=1}^{\infty} F(T_n)$. Now we prove that $\bigcap_{n=1}^{\infty} F(T_n)$ is convex. To this end, let $x, y \in \bigcap_{n=1}^{\infty} F(T_n)$. Put $z = tx + (1-t)y$, where $t \in (0, 1)$. Let $\tilde{x}_n = t(T_n z - x)$ and $\tilde{y}_n = (1-t)(y - T_n z)$. Then

$$\begin{aligned} \|\tilde{x}_n\| &= t\|T_n z - x\| \leq tL_n\|z - x\| = t(1-t)L_n\|x - y\|, \\ \|\tilde{y}_n\| &= (1-t)\|y - T_n z\| \leq (1-t)L_n\|y - z\| = t(1-t)L_n\|x - y\|, \end{aligned}$$

and

$$\|(1-t)\tilde{x}_n + t\tilde{y}_n\| = \|(1-t)t(T_n z - x) + t(1-t)(y - T_n z)\| = t(1-t)\|x - y\|.$$

This together with $L_n \rightarrow 1$ gives

$$\limsup_{n \rightarrow \infty} \|\tilde{x}_n\| \leq t(1-t)\|x - y\|, \quad \limsup_{n \rightarrow \infty} \|\tilde{y}_n\| \leq t(1-t)\|x - y\|$$

and

$$\lim_{n \rightarrow \infty} \|(1-t)\tilde{x}_n + t\tilde{y}_n\| = t(1-t)\|x - y\|.$$

By Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \|T_n z - z\| = \lim_{n \rightarrow \infty} \|\tilde{x}_n - \tilde{y}_n\| = 0.$$

Then $z \in \bigcap_{n=1}^{\infty} F(T_n)$ and hence $\bigcap_{n=1}^{\infty} F(T_n)$ is convex. This completes the proof. \square

3. Main results

In this section, we prove several weak convergence theorems. We first prove a weak convergence theorem of the sequence $\{x_n\}$ which defined by (1.2) in a uniformly convex Banach space satisfying Opial's condition. Recall that E satisfies Opial's condition [15] if $x_n \rightharpoonup x$ and $x \neq y$ imply that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

It is well known [15] that all Hilbert spaces and ℓ^p spaces, $1 \leq p < \infty$, have this property, while all L^p spaces do not have this property unless $p = 2$.

Lemma 3.1. Let C be a nonempty closed convex subset of a Banach space E and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by (1.2). Then

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for each $p \in \bigcap_{n=1}^{\infty} F(T_n)$,
- (ii) there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)g(\|x_n - T_n x_n\|) < \infty$.

Proof. To see (i), let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Then

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|T_n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) L_n \|x_n - p\| \\ &= (1 + (1 - \alpha_n)(L_n - 1)) \|x_n - p\| \\ &\leq (1 + (L_n - 1)) \|x_n - p\|,\end{aligned}\quad (3.1)$$

for all $n \in \mathbb{N}$. From $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and Lemma 2.4, we have (i). Note that $\{x_n - p\}$ and $\{T_n x_n - p\}$ are bounded. We may assume that such sequences belong to B_r where $r > 0$. By Lemma 2.2, there exists $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous strictly increasing and convex function with $g(0) = 0$ such that (2.1) is satisfied. Then

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|T_n x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|) \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) L_n^2 \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|) \\ &\leq (1 + (1 - \alpha_n)(L_n^2 - 1)) \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|).\end{aligned}$$

That is,

$$\alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1 - \alpha_n)(L_n^2 - 1)M,$$

where $M = \sup\{\|x_n - p\|^2 : n \in \mathbb{N}\}$. Summing from 1 to m and tending to infinity for m , we have (ii). This completes the proof. \square

Theorem 3.2. Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $\{T_n\}$ satisfies the NST-condition, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\}$ is a sequence in $[a, b] \subset (0, 1)$, converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. It follows from Lemma 3.1(i) that $\{x_n\}$ is bounded. By Lemma 3.1(ii) and $\alpha_n \in [a, b] \subset (0, 1)$, we have

$$\sum_{n=1}^{\infty} g(\|x_n - T_n x_n\|) < \infty.$$

In particular, $\lim_{n \rightarrow \infty} g(\|x_n - T_n x_n\|) = 0$ and so

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

Since $\{T_n\}$ satisfies the NST-condition, we have $\omega_w \{x_n\} \subset \bigcap_{n=1}^{\infty} F(T_n)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ for some $z \in C$. Then $z \in \bigcap_{n=1}^{\infty} F(T_n)$. It follows from Opial's condition and $\lim_{n \rightarrow \infty} \|x_n - p\|$ for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$ that $x_n \rightharpoonup z \in \bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. \square

In the presence of the stronger condition than NST-condition, a variable control sequence $\{\alpha_n\}$ is taken into consideration.

Theorem 3.3. Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $(\{T_n\}, T)$ satisfies AKTT-condition (I) and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. By Lemma 3.1(ii) and $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, we have

$$\liminf_{n \rightarrow \infty} g(\|x_n - T_n x_n\|) = 0.$$

This implies that

$$\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

We next prove that the limit $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ actually exists. Since $\{x_n\}$ is bounded, it follows that

$$\sum_{n=1}^{\infty} \sup\{\|T_n z - T_{n+1} z\| : z \in \{x_n\}\} < \infty. \quad (3.2)$$

Notice that

$$\begin{aligned}\|x_{n+1} - T_{n+1} x_{n+1}\| &= \|\alpha_n (x_n - T_n x_n) + (T_n x_n - T_{n+1} x_{n+1})\| \\ &\leq \alpha_n \|x_n - T_n x_n\| + \|T_n x_n - T_{n+1} x_{n+1}\|\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|x_n - T_n x_n\| + \|T_n x_n - T_n x_{n+1}\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
&\leq \alpha_n \|x_n - T_n x_n\| + L_n \|x_n - x_{n+1}\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
&= (1 + (1 - \alpha_n)(L_n - 1)) \|x_n - T_n x_n\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
&\leq (1 + (1 - \alpha_n)(L_n - 1)) \|x_n - T_n x_n\| + \sup\{\|T_n z - T_{n+1} z\| : z \in \{x_n\}\}.
\end{aligned}$$

By Lemma 2.4 and (3.2), the limit $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ exists. Thus

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

By Lemmas 2.8 and 2.9, $\{T_n\}$ satisfies the NST-condition. As in the proof of Theorem 3.2, $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. \square

Secondly, we prove a weak convergence of the iteration (1.2) in a uniformly convex Banach space E whose dual E^* has the Kadec–Klee property. Most of the weak convergence theorems are proved in a uniformly convex Banach space and the presence of Opial's condition or the Fréchet differentiability of the norm (see e.g., [3, 11]). Before going on, let us recall some additional geometric properties. A Banach space E has

- (a) the Kadec–Klee property [14] if for every $\{x_n\}$ in E , $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ together imply $x_n \rightarrow x$.
- (b) a Fréchet differentiable norm if, for any $x \in S_E$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is approached uniformly as y varies over S_E , where S_E denotes the unit sphere of E .

It is known that if E is reflexive and has a Fréchet differentiable norm, then E^* has the Kadec–Klee property (see also [16, Lemma 1]). However, there exist uniformly convex Banach spaces which have neither Opial's condition nor a Fréchet differentiable norm but their duals do have the Kadec–Klee property (see also [17, 14]).

The following lemma is our main tool for proving the weak convergence theorem.

Lemma 3.4 ([17], Lemma 3.2). *Let E be a uniformly convex Banach space such that its dual E^* has the Kadec–Klee property. Suppose $\{x_n\}$ is a bounded sequence in E such that*

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$. Then $\omega_w\{x_n\}$ is a singleton.

Lemma 3.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and let $\{x_n\}$ be a sequence in C defined by (1.2). Then the limit*

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded, so we let $R > 0$ be such that $\{x_n\} \subset K := B_R \cap C$. Then K is a nonempty closed convex bounded subset of E . Let $p, q \in \bigcap_{n=1}^{\infty} F(T_n)$ and set

$$a_n(t) := \|tx_n + (1 - t)p - q\|.$$

Then $\lim_{n \rightarrow \infty} a_n(0) = \|p - q\|$, and from Lemma 3.1, $\lim_{n \rightarrow \infty} a_n(1) = \lim_n \|x_n - q\|$ exists. Now, we consider the case $t \in (0, 1)$. Hence, we can define $S_n : C \rightarrow C$ by

$$S_n x = \alpha_n x + (1 - \alpha_n) T_n x, \quad x \in C.$$

Then S_n is an L_n -Lipschitzian mapping. In fact, for all $x, y \in C$,

$$\begin{aligned}
\|S_n x - S_n y\| &\leq \alpha_n \|x - y\| + (1 - \alpha_n) \|T_n x - T_n y\| \\
&\leq \alpha_n \|x - y\| + (1 - \alpha_n) L_n \|x - y\| \\
&\leq \alpha_n L_n \|x - y\| + (1 - \alpha_n) L_n \|x - y\| = L_n \|x - y\|.
\end{aligned}$$

Moreover, $x_{n+1} = S_n x_n$ and $\bigcap_{n=1}^{\infty} F(T_n) \subset \bigcap_{n=1}^{\infty} F(S_n)$. Set

$$U_{n,m} := S_{n+m-1} S_{n+m-2} \cdots S_n, \quad n, m \in \mathbb{N}.$$

Then $U_{n,m} x_n = x_{n+m}$ and $\bigcap_{n=1}^{\infty} F(T_n) \subset \bigcap_{n,m=1}^{\infty} F(U_{n,m})$. Moreover,

$$\|U_{n,m} x - U_{n,m} y\| \leq L_{n,m} \|x - y\|, \quad x, y \in K,$$

where $L_{n,m} = \prod_{j=n}^{n+m-1} L_j$. We note that $\lim_{n,m \rightarrow \infty} L_{n,m} = 1$. Set

$$b_{n,m}(t) := \|tU_{n,m}x_n + (1-t)p - U_{n,m}(tx_n + (1-t)p)\|.$$

By Lemma 2.3, we have

$$\gamma\left(\frac{1}{L_{n,m}}b_{n,m}(t)\right) \leq \|x_n - p\| - \frac{1}{L_{n,m}}\|U_{n,m}x_n - U_{n,m}p\| = \|x_n - p\| - \frac{1}{L_{n,m}}\|x_{n+m} - p\|,$$

where $\gamma : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing continuous convex and $\gamma(0) = 0$. By using Lemma 3.1 and $\lim_{n,m \rightarrow \infty} L_{n,m} = 1$, we have $\lim_{n,m \rightarrow \infty} \gamma\left(\frac{1}{L_{n,m}}b_{n,m}(t)\right) = 0$, and hence $\lim_{n,m \rightarrow \infty} b_{n,m} = 0$. Observe that

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &= \|tU_{n,m}x_n + (1-t)p - q\| \\ &\leq b_{n,m}(t) + \|U_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,m}(t) + \|tx_n + (1-t)p - q\| = b_{n,m}(t) + a_n(t). \end{aligned}$$

Consequently,

$$\limsup_{n \rightarrow \infty} a_n(t) \leq \lim_{n,m \rightarrow \infty} b_{n,m}(t) + \liminf_{n \rightarrow \infty} a_n(t) = \liminf_{n \rightarrow \infty} a_n(t).$$

This implies that $\lim_{n \rightarrow \infty} a_n(t)$ exists for any $t \in (0, 1)$ and the proof is finished. \square

Theorem 3.6. Let C be a nonempty closed convex subset of a uniformly convex Banach space E such that its dual E^* has the Kadec–Klee property. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings from C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $\{T_n\}$ satisfies the NST-condition, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. As in the proof of Theorem 3.2, we have $\omega_w\{x_n\} \subset \bigcap_{n=1}^{\infty} F(T_n)$. Then, by Lemma 3.5, the limit

$$\lim_{n \rightarrow \infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$. Hence $x_n \rightharpoonup z \in \bigcap_{n=1}^{\infty} F(T_n)$ by Lemma 3.4. This completes the proof. \square

Similarly, using the same arguments as in Theorems 3.3 and 3.6 yields the following.

Theorem 3.7. Let C be a nonempty closed convex subset of a uniformly convex Banach space E such that its dual E^* has the Kadec–Klee property. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $(\{T_n\}, T)$ satisfies AKTT-condition (I) and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, converges weakly to a common fixed point of $\{T_n\}$.

Setting $L \equiv 1$, we have the following.

Theorem 3.8. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and satisfy one of the following control conditions:

- (i) $\{T_n\}$ satisfies the NST-condition and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$;
- (ii) $(\{T_n\}, T)$ satisfies AKTT-condition (I), $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

If, in addition, E satisfies Opial's condition or E^* has the Kadec–Klee property, then the sequence $\{x_n\}$ in C defined by (1.2), converges weakly to a common fixed point of $\{T_n\}$.

Remark 3.9. Theorem 3.8(i) extends and improves Lemma 3.2 of [3] in the following ways:

- (i) The geometric property imposed on a Banach space is weakened.
- (ii) The AKTT-condition (II) is weakened and replaced by the more general NST-condition (see Lemma 2.9).

4. Common solutions of a fixed point problem and an equilibrium problem

In this section, we present several related results which can be deduced by corresponding convergence theorems obtained in Section 3. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Then

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle \quad (4.1)$$

for all $x, y \in H$. It is also known that H satisfies Opial's condition and has the Kadec–Klee property.

Let C be a nonempty closed convex subset of H . Then, for any $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the *metric projection* of H onto C . We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{if and only if} \quad \langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C.$$

Lemma 4.1 ([4], Lemma 4). Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{x_n\}$ be a sequence in H and let $\{\delta_n\}$ be a sequence in $[0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and

$$\|x_{n+1} - y\| \leq (1 + \delta_n)\|x_n - y\| \quad \text{for all } y \in C \text{ and } n \in \mathbb{N}.$$

Then the sequence $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

Theorem 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and satisfy one of the following control conditions:

- (i) $\{T_n\}$ satisfies the NST-condition and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$;
- (ii) [4, Theorem 5] ($\{T_n\}, T$) satisfies AKTT-condition (I), $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then $\{x_n\}$ in C defined by (1.2), converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(T_n)}(x_n)$.

Proof. By Theorems 3.2, 3.3, 3.6 and 3.7, we have $x_n \rightharpoonup z$ for some $z \in \bigcap_{n=1}^{\infty} F(T_n)$. We prove that $\lim_{n \rightarrow \infty} z_n = z$, where $z_n = P_{\bigcap_{n=1}^{\infty} F(T_n)}(x_n)$. By (3.1) and Lemma 4.1, there is $z' \in \bigcap_{n=1}^{\infty} F(T_n)$ such that $z_n \rightarrow z'$. From $z_n = P_{\bigcap_{n=1}^{\infty} F(T_n)}(x_n)$ and $z \in \bigcap_{n=1}^{\infty} F(T_n)$, we have

$$\langle x_n - z_n, z_n - z \rangle \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

It follows from $z_n \rightarrow z'$ and $x_n \rightharpoonup z$ that

$$\langle z - z', z' - z \rangle \geq 0$$

and then $z' = z$. This completes the proof. \square

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction and let $A : C \rightarrow H$ be a nonlinear mapping. Then, we consider the following equilibrium problem:

$$\text{Find } z \in C \text{ such that } f(z, y) + \langle Az, y - z \rangle \geq 0 \quad \text{for all } y \in C. \quad (4.2)$$

The set of such z is denoted by $EP(f, A)$, i.e.,

$$EP(f, A) = \{z \in C : f(z, y) + \langle Az, y - z \rangle \geq 0 \text{ for all } y \in C\}.$$

In the case of $A \equiv 0$, $EP(f, A)$ is denoted by $EP(f)$. In the case of $f \equiv 0$, $EP(f, A)$ is denoted by $VI(C, A)$. The problem (4.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games (see [18–20]).

A mapping $A : C \rightarrow H$ is said to be

(1) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \text{for all } x, y \in C;$$

(2) α -*inverse-strongly-monotone*, where $\alpha > 0$, if

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \quad \text{for all } x, y \in C.$$

Note that every α -inverse-strongly-monotone mapping is monotone and $(1/\alpha)$ -Lipschitzian.

Recently Tada and Takahashi [6] considered iterative methods for finding an element of $EP(f) \cap F(S)$, where $S : C \rightarrow H$ is a nonexpansive mapping. On the other hand, Takahashi and Toyoda [21] introduced an iterative methods for finding an element of $VI(C, A) \cap F(S)$, where $A : C \rightarrow H$ is an α -inverse-strongly-monotone mapping. Very recently Moudafi [5] introduced an iterative methods for finding an element of $EP(f, A) \cap F(S)$ and then proved a weak convergence theorem.

Motivated by Tada–Takahashi [6] and Moudafi [5], we prove a weak convergence theorem for finding a common element of a common element of the common fixed point set for a countable family of mappings and the set of solutions of an equilibrium problem in a Hilbert space.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (see [18]):

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for any $x, y \in C$;

(A3) f is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [18, Corollary 1] and [22, Lemma 2.12], we have the following lemma.

Lemma 4.3. Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and let $r > 0$ and $x \in H$. Then there exists unique $x^* \in C$ such that

$$f(x^*, y) + \frac{1}{r} \langle y - x^*, x^* - x \rangle \geq 0 \quad \text{for all } y \in C.$$

Let T_r be a mapping of H into C defined by

$$T_r(x) = x^*$$

for all $x \in H$. Then, the following hold:

(i) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|(I - T_r)x - (I - T_r)y\|^2;$$

(ii) $F(T_r) = EP(f)$;

(iii) $EP(f)$ is closed and convex.

We need the following lemmas.

Lemma 4.4. Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and let $r > 0$. Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H with $EP(f, A) \neq \emptyset$, then

$$\|T_r(I - rA)x - u\|^2 \leq \|x - u\|^2 - \frac{2\alpha - r}{2\alpha} \|T_r(I - rA)x - x\|^2 \quad (4.3)$$

for all $x \in C$ and $u \in F(T_r(I - rA)) = EP(f, A)$. Furthermore, if $0 \leq r \leq 2\alpha$, then $T_r(I - rA)$ is a nonexpansive mapping of C into itself.

Proof. For $r > 0$, we note that $u \in EP(f, A)$ if and only if $u = T_r(I - rA)u$. That is, $F(T_r(I - rA)) = EP(f, A)$. Let $x \in C$ and $u \in EP(f, A)$. Since T_r is firmly nonexpansive, A is α -inverse-strongly-monotone and (4.1), we have

$$\begin{aligned} \|T_r(I - rA)x - u\|^2 &\leq \|(I - rA)x - (I - rA)u\|^2 - \|(I - T_r)(I - rA)x - (I - T_r)(I - rA)u\|^2 \\ &= \|(x - u) - r(Ax - Au)\|^2 - \|(x - T_r(I - rA)x) - r(Ax - Au)\|^2 \\ &= (\|x - u\|^2 - 2r\langle x - u, Ax - Au \rangle + r^2\|Ax - Au\|^2) \\ &\quad - (\|x - T_r(I - rA)x\|^2 - 2r\langle x - T_r(I - rA)x, Ax - Au \rangle + r^2\|Ax - Au\|^2) \\ &= \|x - u\|^2 - 2r\langle x - u, Ax - Au \rangle + 2r\langle x - T_r(I - rA)x, Ax - Au \rangle - \|x - T_r(I - rA)x\|^2 \\ &\leq \|x - u\|^2 - 2\alpha r\|Ax - Au\|^2 + 2r\|x - T_r(I - rA)x\|\|Ax - Au\| - \|x - T_r(I - rA)x\|^2 \\ &= \|x - u\|^2 - 2\alpha r \left(\|Ax - Au\| - \frac{1}{2\alpha} \|x - T_r(I - rA)x\| \right)^2 - \frac{2\alpha - r}{2\alpha} \|x - T_r(I - rA)x\|^2 \\ &\leq \|x - u\|^2 - \frac{2\alpha - r}{2\alpha} \|x - T_r(I - rA)x\|^2. \end{aligned}$$

Hence (4.3) holds. Finally, let $0 \leq r \leq 2\alpha$. Since T_r is nonexpansive and (4.1), we have

$$\begin{aligned} \|T_r(I - rA)x - T_r(I - rA)y\|^2 &\leq \|(I - rA)x - (I - rA)y\|^2 \\ &= \|x - y\|^2 - 2r\langle Ax - Ay, x - y \rangle + r^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r\alpha\|Ax - Ay\|^2 + r^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 - r(2\alpha - r)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 \end{aligned}$$

for all $x, y \in C$ and hence $T_r(I - rA)$ is nonexpansive. \square

Lemma 4.5. Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfies (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H with $EP(f, A) \neq \emptyset$. Let $\{W_n\}$ be a sequence of mappings of C into itself defined by

$$W_n = T_{r_n}(I - r_n A),$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$. Then $\{W_n\}$ satisfies NST-condition.

Proof. We note that $\bigcap_{n=1}^{\infty} F(W_n) = EP(f, A) \neq \emptyset$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - W_n z_n\| = 0$ and $z \in \omega_w\{z_n\}$. Then $z_{n_i} \rightarrow z$ for some subsequence $\{n_i\}$ of $\{n\}$. For each $n \in \mathbb{N}$, let $y_n = W_n z_n$. Then $y_{n_i} \rightarrow z$. Since $\liminf_{n \rightarrow \infty} r_n > 0$ and A is $(1/\alpha)$ -Lipschitzian, we have

$$\lim_{n \rightarrow \infty} \left\| \frac{z_n - y_n}{r_n} \right\| = \lim_{n \rightarrow \infty} \frac{1}{r_n} \|z_n - W_n z_n\| = 0 \quad (4.4)$$

and

$$\lim_{n \rightarrow \infty} \|Az_n - Ay_n\| \leq \lim_{n \rightarrow \infty} \frac{1}{\alpha} \|z_n - y_n\| = 0, \quad (4.5)$$

respectively. Notice that

$$f(y_n, y) + \frac{1}{r_n} \langle y - y_n, y_n - (z_n - r_n Az_n) \rangle \geq 0 \quad \text{for all } y \in C.$$

So, from (A2) and A is monotone,

$$\begin{aligned} f(y, y_n) &\leq \left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle + \langle y - y_n, Az_n \rangle \\ &= \left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle + \langle y - y_n, Az_n - Ay_n \rangle + \langle y - y_n, Ay_n - Ay \rangle + \langle y - y_n, Ay \rangle \\ &\leq \left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle + \langle y - y_n, Az_n - Ay_n \rangle + \langle y - y_n, Ay \rangle. \end{aligned}$$

In particular,

$$f(y, y_{n_i}) \leq \left\langle y - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{r_{n_i}} \right\rangle + \langle y - y_{n_i}, Az_{n_i} - Ay_{n_i} \rangle + \langle y - y_{n_i}, Ay \rangle.$$

This together with (4.4), (4.5), (A4) and $y_{n_i} \rightarrow z$ gives

$$f(y, z) \leq \langle y - z, Ay \rangle. \quad (4.6)$$

Put $z_t = ty + (1 - t)z$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (4.6) we have

$$f(z_t, z) \leq \langle z_t - z, Az_t \rangle. \quad (4.7)$$

It follows from (A1), (A4) and (4.7) that

$$\begin{aligned} 0 &= f(z_t, z_t) \\ &\leq tf(z_t, y) + (1 - t)f(z_t, z) \\ &\leq tf(z_t, y) + (1 - t)\langle z_t - z, Az_t \rangle \\ &= tf(z_t, y) + (1 - t)t\langle y - z, Az_t \rangle \end{aligned}$$

and hence

$$0 \leq f(z_t, y) + (1 - t)\langle y - z, Az_t \rangle$$

Letting $t \rightarrow 0$ and using (A3), we get

$$0 \leq f(z, y) + \langle y - z, Az \rangle \quad \text{for all } y \in C$$

and hence $z \in EP(f, A) = \bigcap_{n=1}^{\infty} F(W_n)$. This implies that $\{W_n\}$ satisfies NST-condition. This completes the proof. \square

Theorem 4.6. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of

L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ such that $\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} f(y_n, y) + \langle Ax_n, y - y_n \rangle + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0 & \text{for all } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n y_n, \end{cases} \quad (4.8)$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $\{S_n\}$ satisfies NST-condition, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)}(x_n)$.

Proof. For each $n \in \mathbb{N}$, we have $y_n = T_{r_n}(I - r_n A)x_n$. By Lemmas 2.10 and 4.5, $\{S_n T_{r_n}(I - r_n A)\}$ is a family of L_n -Lipschitzian mappings of C into itself satisfying NST-condition. Applying Theorem 4.2, $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)}(x_n)$. \square

From Theorem 4.6, letting $L_n \equiv 1$ gives

Corollary 4.7. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (4.8), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $\{S_n\}$ satisfies NST-condition, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)}(x_n)$.

From Corollary 4.7, letting $S_n \equiv S$ gives

Corollary 4.8 ([5], Theorem 3.1). Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let S be a nonexpansive mapping of C into itself such that $F(S) \cap EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} f(y_n, y) + \langle Ax_n, y - y_n \rangle + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0 & \text{for all } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. Then the sequence $\{x_n\}$ converges weakly to $z \in F(S) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap EP(f, A)}(x_n)$.

We obtain [6, Theorem 4.1] from Corollary 4.8 by letting $A \equiv 0$.

Corollary 4.9. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and

$$\begin{cases} f(y_n, y) + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0 & \text{for all } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap EP(f)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap EP(f)}(x_n)$.

Theorem 4.10. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n P_C(x_n - r_n A x_n), \quad (4.9)$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $\{S_n\}$ satisfies NST-condition, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)}(x_n)$.

Proof. Setting $f \equiv 0$ in Theorem 4.6, we have $EP(f, A) = VI(C, A)$ and $T_r = P_C$ for all $r > 0$. \square

We immediately obtain [7, Theorem 4] from our Theorem 4.10 by letting $L_n \equiv 1$ and applying Lemmas 2.8 and 2.9.

Corollary 4.11. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (4.9), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $(\{S_n\}, S)$ satisfies AKTT-condition and $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)}(x_n)$.

Remark 4.12 ([7, Theorem 4]). (See Corollary 4.11) is also a direct consequence of [3, Lemma 3.2] since $\{S_n P_C(I - r_n A)\}$ is a family of nonexpansive mappings satisfying AKTT-condition (II). In fact, by Lemma 2.10,

$$\bigcap_{n=1}^{\infty} F(S_n P_C(I - r_n A)) = \bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) = F(S) \cap VI(C, A) \neq \emptyset.$$

Let $\{n_i\}$ be a subsequence of $\{n\}$. Since $\{r_{n_i}\} \subset [c, d]$, there exists a subsequence $\{n_{ij}\}$ of $\{n_i\}$ such that $r_{n_{ij}} \rightarrow r \in [c, d]$. Then $SP_C(I - rA)$ is a nonexpansive mapping of C into itself and

$$F(SP_C(I - rA)) = F(S) \cap VI(C, A) = \bigcap_{n=1}^{\infty} F(S_n P_C(I - r_n A)).$$

Let B be a bounded subset of C . Since A is $(1/\alpha)$ -Lipschitzian and $P_C(I - rA)$ is nonexpansive of C into itself, $\{Az : z \in B\}$ and $\tilde{B} = \{P_C(I - rA)z : z \in B\} \subset C$ are bounded. By Lemma 2.6, we have

$$\lim_{n \rightarrow \infty} \sup\{\|Sy - S_n y\| : y \in \tilde{B}\} = 0.$$

Let $M = \sup\{\|Az\| : z \in B\}$. Then

$$\begin{aligned} & \|SP_C(I - rA)z - S_{n_{ij}} P_C(I - r_{n_{ij}} A)z\| \\ & \leq \|SP_C(I - rA)z - S_{n_{ij}} P_C(I - rA)z\| + \|S_{n_{ij}} P_C(I - rA)z - S_{n_{ij}} P_C(I - r_{n_{ij}} A)z\| \\ & \leq \sup\{\|Sy - S_{n_{ij}} y\| : y \in \tilde{B}\} + \|P_C(I - rA)z - P_C(I - r_{n_{ij}} A)z\| \\ & \leq \sup\{\|Sy - S_{n_{ij}} y\| : y \in \tilde{B}\} + \|(I - rA)z - (I - r_{n_{ij}} A)z\| \\ & \leq \sup\{\|Sy - S_{n_{ij}} y\| : y \in \tilde{B}\} + |r - r_{n_{ij}}| M \end{aligned}$$

for all $z \in B$ and $j \in \mathbb{N}$. So, we get

$$\lim_{j \rightarrow \infty} \sup\{\|SP_C(I - rA)z - S_{n_{ij}} P_C(I - r_{n_{ij}} A)z\| : z \in B\} = 0.$$

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