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Finite volume methods for degenerate chemotaxis model

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ABSTRACT

A finite volume method for solving the degenerate chemotaxis model is presented, along with numerical examples. This model consists of a degenerate parabolic convection–diffusion PDE for the density of the cell-population coupled to a parabolic PDE for the chemoattractant concentration. It is shown that discrete solutions exist, and the scheme converges.

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1. Introduction

The most known and most used model for chemotaxis is the Keller–Segel model. Almost 30 years after its proposal, the Keller and Segel model (see [1,2]) remains the most popular model for chemical control of cell movement. Generally, chemotaxis is the property of certain living organisms (e.g., the cellular slime mold *Dictyostelium discoideum*, which is a species of soil-living amoebae) to be repelled or attracted to chemical signals. The celebrated model in chemotaxis was introduced in [2]. Here we investigate the variant of the Keller–Segel chemotaxis model with a nonlinear degenerate diffusion law for the cells. Namely, we consider the modified Keller–Segel system:

$$\begin{cases} \partial_t u - \operatorname{div}(a(u)\nabla u - \chi(u)\nabla v) = 0 \quad \text{in } Q_T, \\ \partial_t v - \mathrm{d}\Delta v = g(u, v) \quad \text{in } Q_T, \end{cases} \tag{1.1}$$

with the no-flux boundary conditions on $\Sigma_T := \partial\Omega \times (0, T)$,

$$a(u)\frac{\partial u}{\partial \eta} = 0, \qquad \frac{\partial v}{\partial \eta} = 0, \tag{1.2}$$

and initial conditions on Ω:

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x). \tag{1.3}$$

Herein, $Q_T := \Omega \times (0, T)$, $T > 0$ is a fixed time, and Ω is a bounded domain in $\mathbb{R}^l$, $l = 2$ or $l = 3$, with Lipschitz boundary $\partial\Omega$ and outer unit normal η.

In the model above, the density of the cell-population and the chemoattractant (or chemorepellent) concentration are represented by $u = u(x, t)$ and $v = v(x, t)$, respectively. Next, $a(u)$ is a density-dependent diffusion coefficient, and d is

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a constant. Furthermore, the function $\chi(u)$ is the chemoattractant sensitivity. The function $g(u, v)$ describes the rates of production and degradation of the chemoattractant; here, we assume it is the linear function

$$g(u, v) = \alpha u - \beta v, \qquad \alpha, \beta \geq 0. \tag{1.4}$$

This assumption has also been used in the related literature (see, e.g., [3]). Notice that for a mathematical analysis of the model, the signs of α, β are essential, whereas the linearity assumption on g can be relaxed.

In this paper, we assume that $\chi(0) = 0$ and there exists a maximal density of cells u_m such that $\chi(u_m) = 0$. The threshold condition has a clear biological interpretation: the cells stop to accumulate at a given point of Ω after their density attains certain threshold value u_m, therefore the chemotactical sensitivity $\chi(u)$ vanishes when u tends to u_m. This interpretation is sometimes called the volume-filling effect, or prevention of overcrowding (see [4,5]). Secondly, we assume that the density-dependent diffusion coefficient $a(u)$ degenerates for $u = 0$ and $u = u_m$. This means that the diffusion vanishes when u approaches values close to the threshold u_m (see [6]), and also in the absence of cell-population. This interpretation was proposed in [7].

Upon normalization of u_m (one makes the transformation

$$\tilde{u} = \frac{u}{u_m}, \qquad \tilde{v} = v, \qquad \tilde{\chi}(\tilde{u}) = \frac{1}{u_m}\chi(\tilde{u}u_m), \qquad \tilde{a}(\tilde{u}) = a(\tilde{u}u_m), \qquad \tilde{g}(\tilde{u}, \tilde{v}) = g(\tilde{u}u_m, \tilde{v})$$

and omits the tildes in the notation), we can assume that $u_m = 1$. Then a typical example of χ is

$$\chi(u) = u(1 - u), \quad u \in [0, 1]. \tag{1.5}$$

The positivity of χ means that the chemical attracts the cells; the repellent case is the one of a negative χ. We can also assume that $0 \leq u_0 \leq 1$.

To summarize, along with (1.4) the following main assumptions are made:

$$\chi : [0, 1] \mapsto \mathbb{R} \quad \text{is continuous and } \chi(0) = \chi(1) = 0; \tag{1.6}$$

$$a : [0, 1] \mapsto \mathbb{R}^+ \quad \text{is continuous, } a(0) = a(1) = 0 \text{ and } a(s) > 0 \quad \text{for } 0 < s < 1. \tag{1.7}$$

The degeneracy of the diffusion coefficient a is a major concern for the mathematical and numerical treatment of system (1.1)–(1.3). Assumption (1.6) permits to confine the unknown solution u within the interval $[0, 1]$.

To put this paper in the proper perspective, we mention that the Keller–Segel model investigated by many authors: Murray [8] for a general background and Horstmann [9] for a fairly complete survey on the Keller–Segel model. Nonlinear diffusion equations for biological populations (that degenerate at least for $u = 0$) were proposed in the 1970s in [10]; more recent papers include those in [11,12,7,13]. Furthermore, well-posedness results for these kinds of models include, for example, the existence of radial solutions exhibiting chemotactic collapse [14], the local-in-time existence, uniqueness and positivity of classical solutions, and results on their blow-up behavior [15]. Burger et al. [12] prove the global existence and uniqueness of the Cauchy problem in $\mathbb{R}^N$ for linear and nonlinear diffusion with prevention of overcrowding.

Recently, in [7] Bendahmane et al. have proved the existence of weak solutions for (1.1)–(1.3) and studied the regularity of solutions (see also [13] for more general degenerate diffusion). Notice that by using the duality approach, uniqueness of weak solutions to (1.1)–(1.3) is proved in [6]. We refer for e.g. to [7] for a discussion of the uniqueness results for Keller–Segel type models.

From a numerical point of view, we mention that Filbet [16] analyzed an FV method for a simpler version of the Keller–Segel model:

$$\begin{cases} \partial_t u - \operatorname{div}(\nabla u - \chi(u)\nabla v) = 0 \quad \text{in } Q_T, \\ \Delta v - v + u = 0 \quad \text{in } Q_T, \end{cases} \tag{1.8}$$

in which the equation for concentration v has been replaced by an elliptic equation and the equation of cells u is a non-degenerate parabolic equation with $\chi(u) \equiv u$. In [16], existence, uniqueness and convergence of solutions to the FV scheme are proved. Regarding convergence analysis and error estimates for a Galerkin scheme for (1.8) we are only aware of the papers [17,18].

The present treatment is based on similar techniques in [16], but we here analyze a degenerate Chemotaxis model, and also include numerical experiments.

The plan of the paper is as follows. In Section 2 we describe the finite volume scheme chosen to approximate problem (1.1)–(1.3) and state the convergence of the scheme, as the discretization parameters tend to zero. The proof of the convergence result is split into several steps. First, analysis of the scheme is carried out, and *a priori* estimates are given (Section 3). Roughly speaking, we derive the discrete variants of the properties known for the solutions of the "continuous" problem (1.1)–(1.3). Then, the existence of a discrete solution is deduced by a fixed-point argument (Section 4). In Section 5 we show compactness of the set of discrete solutions; in Section 6, we identify the limits of the discrete solutions as weak solutions of the modified Keller–Segel system (1.1)–(1.3). The last section presents numerical experiments obtained with our finite volume scheme, in the model case (1.5). We make experiments illustrating the qualitative properties of the model; make a comparison of degenerate versus non-degenerate diffusion; illustrate pattern formation from randomly perturbed data.

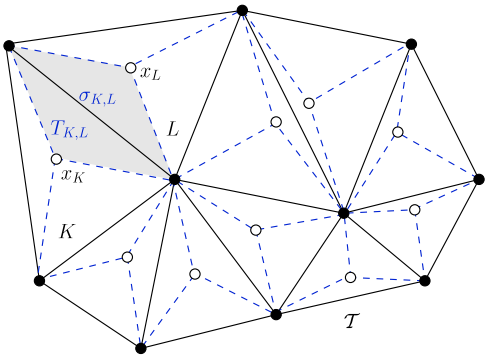


Fig. 1. Control volumes, centers and diamonds.

2. Finite volume approximation and main results

2.1. Weak solutions for modified Keller–Segel model

Before defining our finite volume scheme, let us recall the definition of a weak solution for the system (1.1)–(1.3).

Definition 2.1. Assume that u_0 is measurable, $0 \le u_0 \le 1$, and $v_0 \in L^\infty(\Omega)$. A weak solution of (1.1)–(1.3) is a pair (u, v) of functions on Q_T such that

$$0 \le u(t,x) \le 1, \qquad 0 \le v(t,x) \quad \text{a.e. in } Q_T, \tag{2.1}$$

$$\begin{aligned} &u \in L^\infty(Q_T), \qquad A(u) := \int_0^u a(r)\mathrm{d}r \in L^2(0,T;H^1(\Omega)) \\ &v \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)), \end{aligned} \tag{2.2}$$

and for all $\varphi, \psi \in \mathcal{D}([0,T) \times \overline{\Omega})$, (u, v) satisfy

$$-\int_\Omega u_0(x)\varphi(0,x)\mathrm{d}x - \iint_{Q_T} u\partial_t\varphi \mathrm{d}x\mathrm{d}t + \int_{Q_T} \nabla A(u)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t - \int_{Q_T} \chi(u)\nabla v\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t = 0, \tag{2.3}$$

$$-\int_\Omega v_0(x)\psi(0,x)\mathrm{d}x - \iint_{Q_T} v\partial_t\psi \mathrm{d}x\mathrm{d}t + \mathrm{d}\int_{Q_T} \nabla v\cdot\nabla\psi \mathrm{d}x\mathrm{d}t = \int_{Q_T} g(u,v)\psi \mathrm{d}x\mathrm{d}t. \tag{2.4}$$

2.2. Finite volume meshes and associated discrete functions

We assume that $\Omega \subset \mathbb{R}^l$, $l = 2$ (respectively, $l = 3$) is an open bounded polygonal (resp., polyhedral) connected domain with boundary $\partial\Omega$. Following [19], we consider a family $\mathcal{T}_h$ of admissible meshes of the domain Ω consisting of disjoint open and convex polygons (resp., polyhedra) called control volumes. The parameter h has the sense of an upper bound for the maximum diameter of the control volumes in $\mathcal{T}_h$. Whenever $\mathcal{T}_h$ is fixed, we will drop the subscript h in the notation.

A generic volume in $\mathcal{T}$ is denoted by K. Because we consider the zero-flux boundary condition, we do not need to distinguish between interior and exterior control volumes; only inner interfaces between volumes are needed in order to formulate the scheme.

For all $K \in \mathcal{T}$, denote by $|K|$ the l-dimensional Lebesgue measure of K. For all $K \in \mathcal{T}$, denote by $N(K)$ the set of the neighbors of K (i.e. the set of control volumes of $\mathcal{T}$ which have a common interface with K); a generic neighbor of K is often denoted by L. For all $L \in N(K)$, denote by $\sigma_{K,L}$, the interface between K and L; denote by $\eta_{K,L}$ the unit normal vector to $\sigma_{K,L}$ outward to K. We have $\eta_{L,K} = -\eta_{K,L}$. For an interface $\sigma_{K,L}$, denote by $|\sigma_{K,L}|$ its $(l-1)$-dimensional measure.

By saying that $\mathcal{T}$ is admissible, we mean that there exists a family $(x_K)_{K\in\mathcal{T}}$ such that the straight line $\overline{x_K x_L}$ is orthogonal to the interface $\sigma_{K,L}$. The point x_K is referred to as the center of K (notice that in general, x_K need not belong to K). In the case where $\mathcal{T}$ is a simplicial mesh of Ω (a triangulation, in dimension $l = 2$), one takes for x_K the center of the circumscribed ball of K. We also require that $\eta_{K,L}\cdot(x_L - x_K) > 0$ (in the case of simplicial meshes, this restriction amounts to the Delaunay condition, see e.g. [19]). The "diamond" constructed from the neighbor centers x_K, x_L and the interface $\sigma_{K,L}$ is denoted by $T_{K,L}$; e.g. in the case $x_K \in K$, $x_L \in L$, $T_{K,L}$ is the convex hull of x_K, x_L and $\sigma_{K,L}$ (see Fig. 1). We have $\Omega = \cup_{K\in\mathcal{T}}(\cup_{L\in N(K)} \overline{T}_{K,L})$. Next, we denote by $d_{K,L}$ the distance between x_K and x_L.

A discrete function W on the mesh $\mathcal{T}_h$ is a set $(W_K)_{K\in\mathcal{T}}$. Whenever convenient, we identify W with the piecewise constant function w_h on Ω such that $w|_K = W_K$. If w_h, v_h are discrete functions, the corresponding $L^2(\Omega)$ scalar product and norm can be computed as

$$(w_h, v_h)_{L^2(\Omega)} = \sum_{K\in\mathcal{T}} |K| W_K V_K, \qquad \|w_h\|^2_{L^2(\Omega)} = \sum_{K\in\mathcal{T}} |K||W_K|^2.$$

In addition, we can define the positive (but not definite) product and the corresponding "discrete H_0^1 seminorm" by

$$\langle w_h, v_h\rangle_{H_h(\Omega)} = l \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (W_L - W_K)(V_L - V_K),$$

$$|w_h|_{H_h(\Omega)} = (\langle w_h, w_h\rangle_{H_h(\Omega)})^{1/2}.$$

Finally, the discrete gradient $\nabla_h w_h$ of a constant per control volume function w_h is defined as the constant per diamond $T_{K,L}\mathbb{R}^l$-valued function with values

$$(\nabla_h w_h)|_{T_{K,L}} = \nabla_{K,L} w_h := l \frac{W_L - W_K}{d_{K,L}} \eta_{K,L}. \tag{2.5}$$

Notice that the l-dimensional measure $|T_{K,L}|$ of $T_{K,L}$ equals $\frac{1}{l}|\sigma_{K,L}| d_{K,L}$; therefore the seminorm $|w_h|_{H_h(\Omega)}$ coincides with the $L^2(\Omega)$ norm of $\nabla_h w_h$.

2.3. Finite volume scheme for modified Keller–Segel model

In order to discretize (1.1), we use the implicit order one discretization in time and a finite volume discretization in space. For a given mesh $\mathcal{T}$, at each time step n we consider discrete unknowns $(U_K^n, V_K^n)_{K\in\mathcal{T}}$ (for the time being, we drop the superscript n). As it is classical in the finite volume methods, we approximate the divergence operators in (1.1) by "integrating" them over each control volume K, using the Green–Gauss formula and then approximating the normal fluxes $\nabla A(u)\cdot\eta_{K,L}$, $\nabla v\cdot\eta_{K,L}$ across $\sigma_{K,L}\subset\partial K$.

The admissibility assumption on $\mathcal{T}_h$ allows us to approximate the normal fluxes $\nabla A(u)\cdot\eta_{K,L}$, $\nabla v\cdot\eta_{K,L}$ over the boundaries of the control volumes by means of the divided differences

$$\delta A(U)_{K,L} := \frac{|\sigma_{K,L}|}{d_{K,L}}(A(U_L) - A(U_K)), \qquad \delta V_{K,L} := \frac{|\sigma_{K,L}|}{d_{K,L}}(V_L - V_K)$$

of the values $A(U_L)$, $A(U_K)$ and V_L, V_K, respectively.

Next, we have to approximate $\chi(u)\nabla v\cdot\eta_{K,L}$ by means of the values U_K, U_L and the value $\delta V_{K,L} = \frac{|\sigma_{K,L}|}{d_{K,L}}(V_L - V_K)$ that are available in the neighborhood of the interface $\sigma_{K,L}$. To do this, we use a numerical flux function $G(U_K, U_L, \delta V_{K,L})$. Numerical convection flux functions G of arguments $(a, b, c)\in\mathbb{R}^3$ are required to satisfy the properties:

$$\begin{cases} \text{(a) } G(\cdot, b, c) \text{ is non-decreasing for all } b, c\in\mathbb{R}, \text{ and } G(a, \cdot, c) \text{ is non-increasing for all } a, c\in\mathbb{R};\\ \text{(b) } G(a, b, c) = -G(b, a, -c) \quad \text{for all } a, b, c\in\mathbb{R};\\ \text{(c) } G(a, a, c) = \chi(a)c \quad \text{for all } a, c\in\mathbb{R};\\ \text{(d) there exists } C>0 \text{ such that } \quad \forall a, b, c\in\mathbb{R} \quad |G(a, b, c)| \le C(|a| + |b|)|c|;\\ \text{(e) there exists a modulus of continuity } \omega:\mathbb{R}^+\to\mathbb{R}^+ \text{ such that}\\ \quad \forall a, b, a', b', c\in\mathbb{R} \quad |G(a, b, c) - G(a', b', c)| \le |c|\omega(|a - a'| + |b - b'|). \end{cases} \tag{2.6}$$

Remark. Note that the assumptions on the dependence of G on a, b are standard (see for e.g. [19]). Practical examples of numerical convective flux functions can be found in [19]. In our context, one possibility to construct the numerical flux G satisfying (2.6) is to split χ in the non-decreasing part $\chi_\uparrow$ and the non-increasing part $\chi_\downarrow$:

$$\chi_\uparrow(z) := \int_0^z (\chi'(s))^+ ds \qquad \chi_\downarrow(z) := -\int_0^z (\chi'(s))^- ds.$$

Herein, $s^+ = \max(s, 0)$ and $s^- = \max(-s, 0)$. Then we take

$$G(a, b; c) = c^+(\chi_\uparrow(a) + \chi_\downarrow(b)) - c^-(\chi_\uparrow(b) + \chi_\downarrow(a)). \tag{2.7}$$

Notice that in the case χ has a unique local (and global) maximum at the point $\bar u\in(0, 1)$, such as the flux (1.5), we have

$$\chi_\uparrow(z) = \chi(\min\{z, \bar u\}) \quad \text{and} \quad \chi_\downarrow(z) = \chi(\max\{z, \bar u\}) - \chi(\bar u).$$

We are now in a position to discretize problem (1.1)–(1.3). We denote by $\mathcal{D}$ an admissible discretization of Q_T, which consists of an admissible mesh of Ω and a time step $\Delta t > 0$. We give to the parameter h the sense of

$$\max\Big\{\Delta t, \max_{K\in\mathcal{T}_h} \operatorname{diam}(K), \max_{K\in\mathcal{T}_h} \max_{L\in N(K)} d_{K,L}\Big\}.$$

For h given, we fix a positive number $N = N(h)$ chosen as the smallest integer such that $(N+1)\Delta t \geq T$. We set $t^n = n\Delta t$ for $n \in [\![0..N]\!]$.

A finite volume scheme for the discretization of the problem (1.1)–(1.3) is given by the following set of equations: for all $K \in \mathcal{T}$,

$$U_K^0 = \frac{1}{|K|}\int_K u_0(x)\mathrm{d}x, \qquad V_K^0 = \frac{1}{|K|}\int_K v_0(x)\mathrm{d}x, \tag{2.8}$$

and for all $K \in \mathcal{T}$ and $n \in [\![0..N]\!]$,

$$\begin{aligned} &|K|\frac{U_K^{n+1}-U_K^n}{\Delta t} - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(A(U_L^{n+1}) - A(U_K^{n+1})) + \sum_{L\in N(K)} G(U_K^{n+1}, U_L^{n+1}; \delta V_{K,L}^{n+1}) = 0,\\ &|K|\frac{V_K^{n+1}-V_K^n}{\Delta t} - d\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(V_L^{n+1} - V_K^{n+1}) = |K| g(U_K^n, V_K^{n+1}), \end{aligned} \tag{2.9}$$

with the unknowns $U = (U_K^{n+1})_{K\in\mathcal{T}}$ and $V = (V_K^{n+1})_{K\in\mathcal{T}}, n \in [\![0..N]\!]$; recall that we have assigned $\delta V_{K,L}^{n+1} = \frac{|\sigma_{K,L}|}{d_{K,L}}(V_L^{n+1} - V_K^{n+1})$ and $A(s) = \int_0^s a(r)\mathrm{d}r$. Notice that the discrete zero-flux boundary conditions are implicitly contained in Eqs. (2.9). Indeed, we have for all $K \in \mathcal{T}_h$, $\cup_{L\in N(K)} \sigma_{K,L} = \partial K \setminus \partial\Omega$; the contribution of $\partial\Omega \cap \partial K$ to the approximation of $\int_{\partial K} \nabla v \cdot \eta$, $\int_{\partial K} \nabla A(u)\cdot\eta$ is zero, in compliance with (1.2).

Whenever convenient, we will assimilate a discrete solution of the scheme (2.9) with the couple of piecewise constant on Q functions (u_h, v_h) given by

$$\forall K \in \mathcal{T}_h\ \forall n \in [\![0..N]\!] \quad u_h|_{(t^n,t^{n+1}[\times K} = U_K^{n+1}, \qquad v_h|_{(t^n,t^{n+1}[\times K} = V_K^{n+1}.$$

Remark. In order to prove the existence of a solution to the scheme (2.9), we formally extend the functions χ and a by zero outside the segment $[0, 1]$. Later on, we will show that the discrete solution u_h keeps confined in the region $[0, 1]$ of physically meaningful values of u.

The main result of this paper is the following theorem.

Theorem 2.1. *Assume (1.4) and (1.6), (1.7). Assume that $v_0 \in L^\infty(\Omega)$, $v_0 \geq 0$, and that u_0 is measurable, $0 \leq u_0 \leq 1$ a.e. on Ω. Then there exists a solution (u_h, v_h) to the discrete system (2.9) with initial data (2.8). Further, any sequence $(h_m)_m$ decreasing to zero possesses a (not relabeled) subsequence such that (u_{h_m}, v_{h_m}) converge a.e. on Q_T to a solution (u, v) of the modified Keller–Segel system (1.1)–(1.3) in the sense of Definition 2.1.*

The proof is split in several lemmas and propositions gathered in Sections 3–6. The techniques are essentially those designed by Eymard, Gallouët and Herbin in [19] (see also [20]); yet we make an explicit use of the definition (2.5) of weakly convergent discrete gradient. We give full arguments of the proofs in order to bypass some unnecessary shape-regularity restrictions on the meshes used in [19].

Throughout this paper, C will represent a generic positive constant which may change from one expression to another; C is kept independent of the discretization parameter h but it may depend on Ω, T, $\|v_0\|_{L^\infty(\Omega)}$, on the nonlinearities χ and a, on α and β in (1.4), and on the constant in (2.6)(d).

3. A priori analysis of discrete solutions

3.1. Non-negativity of v_h, confinement of u_h

We have the following lemma.

Lemma 3.1. *Let $(U_K^{n+1}, V_K^{n+1})_{K\in\mathcal{T},\ n\in[\![0..N]\!]}$ be a solution of the finite volume scheme (2.8)–(2.9). Then for all $K \in \mathcal{T}$, for all $n \in [\![0..N]\!]$, $0 \leq U_K^{n+1} \leq 1$ and $0 \leq V_K^{n+1}$.*

Proof. Let us show by induction in n that for all $K \in \mathcal{T}$, $U_K^n \geq 0$. The claim is true for $n = 0$. Consider a volume K such that $U_K^n = \min\{U_L^{n+1}\}_{L\in\mathcal{T}}$. Arguing by contradiction, we assume that $U_K^{n+1} < 0$, while $U_K^n \geq 0$. Consider the first equation of (2.9) corresponding to the aforementioned volume K and multiply it by $-(U_K^{n+1})^-$. We find

$$\begin{aligned} &-|K|\frac{U_K^{n+1}-U_K^n}{\Delta t}(U_K^{n+1})^- + \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(A(U_L^{n+1}) - A(U_K^{n+1}))(U_K^{n+1})^-\\ &\quad - \sum_{L\in N(K)} G(U_K^{n+1}, U_L^{n+1}; \delta V_{K,L}^{n+1})(U_K^{n+1})^- = 0. \end{aligned} \tag{3.1}$$

Observe that $A(U_L^{n+1}) - A(U_K^{n+1}) \geq 0$ (recall that A is non-decreasing). This implies

$$\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (A(U_L^{n+1}) - A(U_K^{n+1}))(U_K^{n+1})^- \geq 0. \tag{3.2}$$

Using the assumptions on the numerical flux G (see (a) and (c) in (2.6)), using the extension of χ in Remark 2.3, we get

$$\begin{aligned} G(U_K^{n+1}, U_L^{n+1}; \delta V_{K,L}^{n+1})(U_K^{n+1})^- &\leq G(U_K^{n+1}, U_K^{n+1}; \delta V_{K,L}^{n+1})(U_K^{n+1})^- \\ &= \delta V_{K,L}^{n+1} \chi(U_K^{n+1})(U_K^{n+1})^- = 0. \end{aligned} \tag{3.3}$$

Using the identity $U_K^{n+1} = (U_K^{n+1})^+ - (U_K^{n+1})^-$ and the non-negativity of U_K^n, we deduce from (3.1)–(3.3) that $(U_K^{n+1})^- \leq 0$. According to the choice of K, $\min\{U_L^{n+1}\}_{L\in\mathcal{T}}$ is non-negative; this end the proof of our first claim.

The proof of non-negativity of V_K^n, $K \in \mathcal{T}$, $n \in [\![0..N]\!]$, follows the same lines; we use (1.4) and the non-negativity of U_K^n.

Finally, in order to prove (by induction) that $U_K^{n+1} \leq 1$, we take K such that U_K realizes $\max(U_L^{n+1})_{L\in\mathcal{T}}$.

Multiplying the first equation in (2.9) by $(U_K^{n+1} - 1)^+$, with the same arguments as in the above proof we find that $(U_K^{n+1} - 1)^+ \leq 0$. □

3.2. Discrete a priori estimates

Proposition 3.2. *Let $(U_K^{n+1}, V_K^{n+1})_{K\in\mathcal{T},\ n\in[\![0..N]\!]}$, be a solution of the finite volume scheme (2.8)–(2.9). Then there exists a constant M depending on $\|v_0\|_\infty$, α, β and T such that*

$$V_K^n \leq M. \tag{3.4}$$

Moreover, there exist a constant $C > 0$, depending on Ω, T, $\|v_0\|_\infty$, α, d and on the constant in (2.6) (d) such that

$$\frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |A(U_K^{n+1}) - A(U_L^{n+1})|^2 + \frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |V_K^{n+1} - V_L^{n+1}|^2 \leq C. \tag{3.5}$$

Proof. Let us prove (3.4). We construct the following constant in space discrete function $(H_K^{n+1})_{K\in\mathcal{T},n\in[\![0..N]\!]}$:

$$\forall n \in [\![0..N+1]\!] \forall K \in \mathcal{T} \quad H_K^n \equiv H^n := \|v_0\|_\infty + \alpha, n\Delta t. \tag{3.6}$$

The idea of (3.6) is that the discrete function $(H_K^{n+1})_{K\in\mathcal{T},\ n\in[\![0..N]\!]}$ is a super-solution of the second equation in (2.9). Indeed, thanks to (1.4) and Proposition 3.2,

$$\begin{cases} H^0 = \|v_0\|_{L^\infty(\Omega)} \geq v_K^0, \\ \dfrac{H^{n+1} - H^n}{\Delta t} = \alpha \geq \alpha U_K^{n+1} \geq g(U_K^n, H_K^n) \quad \text{for all } n \in [\![0..N]\!]. \end{cases} \tag{3.7}$$

Therefore we can prove by induction that $V_K^k \leq H^k$ for all $k \in [\![0..N+1]\!]$ and $K \in \mathcal{T}$. This claim is clear for $k = 0$ (recall that $H^0 = \|v_0\|_{L^\infty(\Omega)}$). Assume it holds true for $k = n$. Similarly to the proof of Proposition 3.2, take the volume K such that $V_K^{n+1} = \max(V_L^{n+1})_{L\in\mathcal{T}}$. Now, subtracting the scheme (2.9) for V_K^n from the scheme (3.7) for H^n, we get

$$|K|\frac{V_K^{n+1} - H^{n+1}}{\Delta t} + \beta|K|(V_K^{n+1} - H^{n+1}) - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (V_L^{n+1} - V_K^{n+1}) = |K|\frac{V_K^n - H^n}{\Delta t} + \alpha|K|(U_K^n - 1) \leq 0. \tag{3.8}$$

Multiplying (3.8) by $(V_K^{n+1} - H^{n+1})^+$, we deduce that $(V_K^{n+1} - H^{n+1})^+ \leq 0$. Finally, notice that $M := \sup_{N\in\mathbb{N}} \max_{0\leq n\leq N} H^{n+1} \leq \|v_0\|_\infty + (T+1)\alpha < +\infty$. This establishes (3.4).

Now let us prove (3.5). We multiply the first (respectively, the second) equation in (2.9) by $\Delta t A(u_K^{n+1})$ (resp., by $\Delta t v_K^{n+1}$) and sum up in $K \in \mathcal{T}$ and $n \in [\![0..N]\!]$. This yields

$$E_{1,1} + E_{1,2} + E_{1,3} = 0 \quad \text{and} \quad E_{2,1} + E_{2,2} = E_{2,3},$$

where

$$E_{1,1} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|(U_K^{n+1} - U_K^n) A(U_K^{n+1}),$$

$$E_{2,1} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|(V_K^{n+1} - V_K^n) V_K^{n+1},$$

$$E_{1,2} = -\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}(A(U_L^{n+1})-A(U_K^{n+1}))A(U_K^{n+1}),$$

$$E_{2,2} = -d\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}(V_L^{n+1}-V_K^{n+1})V_K^{n+1},$$

$$E_{1,3} = \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}G(U_K^{n+1},U_L^{n+1};\delta V_{K,L}^{n+1})A(U_K^{n+1}),$$

$$E_{2,3} = \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}|K|g(U_K^n,V_K^{n+1})V_K^{n+1}.$$

Let $\mathcal{B}(s)=\int_0^s A(r)\mathrm{d}r$; we have $A''(s)=a(s)\geq 0$, so that $\mathcal{B}$ is convex. From the convexity of $\mathcal{B}$ and of the function $s\mapsto\frac{1}{2}s^2$ we have the inequalities

$$\forall a,b\in\mathbb{R}\quad (a-b)A(a)\geq\mathcal{B}(a)-\mathcal{B}(b),\qquad (a-b)a\geq\frac{1}{2}(a^2-b^2).$$

We use these inequalities, then cancellations occur and we obtain

$$\begin{aligned}E_{1,1} &= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(U_K^{n+1}-U_K^n)A(U_K^{n+1})\\ &\geq \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(\mathcal{B}(U_K^{n+1})-\mathcal{B}(U_K^n))=\sum_{K\in\mathcal{T}}|K|\left(\mathcal{B}(U_K^{N+1})-\mathcal{B}(U_K^0)\right),\\ E_{2,1} &= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(V_K^{n+1}-V_K^n)V_K^{n+1}\\ &\geq \frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\left(|V_K^{n+1}|^2-|V_K^n|^2\right)=\frac{1}{2}\sum_{K\in\mathcal{T}}|K|\left(|V_K^{N+1}|^2-|V_K^0|^2\right).\end{aligned}$$

Further, in the terms $E_{1,2}, E_{1,3}$ and $E_{2,2}$, for every edge $\sigma_{K,L}$ the terms involving K and L appear twice. Thanks to the conservativity of the finite volume fluxes across $\sigma_{K,L}$, gathering by edges (see e.g. [19]) we find

$$E_{1,2} = \frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|A(U_K^{n+1})-A(U_L^{n+1})|^2$$

$$E_{2,2} = \frac{d}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|V_K^{n+1}-V_L^{n+1}|^2.$$

We also gather by edges in the term $E_{1,3}$. Recall that $\delta V_{K,L}:=\frac{|\sigma_{K,L}|}{d_{K,L}}(V_L-V_K)$; using in addition assumption (2.6)(d) together with the boundedness of U_K^{n+1},

$n\in[\![0..N]\!]$, $K\in\mathcal{T}$, by the weighted Young inequality we deduce

$$\begin{aligned}|E_{1,3}| &= \frac{1}{2}\Big|\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\left(\sum_{L\in N(K)}G(U_K^{n+1},U_L^{n+1};\delta V_{K,L}^{n+1})(A(U_K^{n+1})-A(U_L^{n+1}))\right)\Big|\\ &\leq C\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|V_K^{n+1}-V_L^{n+1}|^2+\frac{1}{4}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|A(U_K^{n+1})-A(U_L^{n+1})|^2,\end{aligned}$$

for some constant $C>0$. Next, using the form (1.4) of g and the L^∞ bound (3.4), we deduce

$$E_{2,3}\leq C':=\alpha MT|\Omega|.$$

Collecting the previous inequalities we readily deduce (3.5). This concludes the proof of Proposition 3.2. □

4. Existence of a discrete solution

The existence for the finite volume scheme is given in the following proposition.

Proposition 4.1. *Let $\mathcal{D}$ be an admissible discretization of Q_T as described in (2.2). Then problem (2.8)–(2.9) admits at least one solution $(U_K^{n+1}, V_K^{n+1})_{(K,n)\in\mathcal{T}\times[\![0..N]\!]}$.*

Proof. Denote $u_h^n := (U_K^n)_{K\in\mathcal{T}}$, $v_h^n := (V_K^n)_{K\in\mathcal{T}}$. We show the existence of u_h^n, v_h^n by induction on n. Note that the second equation in (2.9) is a standard time-implicit finite volume discretization of a uniformly parabolic equation, where the contribution of u in the right-hand side is discretized in the explicit way. Thus for a given u_h^n, v_h^n, we deduce the existence of the solution v_h^{n+1}. Now, we prove the existence of discrete solution u_h^{n+1}. Since $A(\cdot)$ is invertible, we can rewrite the scheme in terms of w_h^i with $u_h^i = A^{-1}(w_h^i)$, $i \in [\![0..N]\!]$. Assume that w_h^n and v_h^{n+1} exist. We choose the componentwise product $[\cdot,\cdot]$ as the scalar product on $\mathbb{R}^{\mathcal{T}}$. We define the mapping $\mathcal{M}$ that associates to the vector $\mathcal{W} = (W_K^{n+1})_{K\in\mathcal{T}}$ the expression

$$\mathcal{M}(\mathcal{W}) = \left(|K| \frac{A^{-1}(W_K^{n+1}) - A^{-1}(W_K^n)}{\Delta t} - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (W_L^{n+1} - W_K^{n+1}) + \sum_{L\in N(K)} G(A^{-1}(W_K^{n+1}), A^{-1}(W_L^{n+1}); \delta V_{K,L}^{n+1}) \right)_{K\in\mathcal{T}}$$

given by the first equation in (2.9). Now, using (3.4) and (3.5), and an application of Young's inequality to deduce

$$[\mathcal{M}(\mathcal{W}), \mathcal{W}] \geq C|U|^2 - C'|W| - C'' \geq 0 \quad \text{for } |\mathcal{W}| \text{ large enough},$$

for some constants C, C', $C'' > 0$. We deduce that

$$[\mathcal{M}(\mathcal{W}), \mathcal{W}] > 0 \quad \text{for } |\mathcal{W}| \text{ large enough}.$$

This implies (see for e.g. [21,22]): there exists $\mathcal{W}$ such that

$$\mathcal{M}(\mathcal{W}) = 0.$$

Thus w_n^{n+1} does exist. Then, we obtain the existence of at least one solution to the scheme (2.9). □

5. Compactness estimates on discrete solutions

In this section we derive estimates on differences of space and time translates of the function v_h which imply that the sequence v_h is relatively compact in $L^2(Q_T)$.

Lemma 5.1. *There exists positive a constant $C > 0$ depending on Ω, T, u_0 and v_0 such that*

$$\iint_{\Omega'\times(0,T)} |w_h(t, x+y) - w_h(t,x)|^2 \mathrm{d}x\mathrm{d}t \leq C|y|(|y| + 2h), \quad w_h = A(u_h),\ v_h, \tag{5.1}$$

for all $y \in \mathbb{R}^3$ with $\Omega' = \{x \in \Omega, [x, x+y] \subset \Omega\}$, and

$$\iint_{\Omega\times(0,T-\tau)} |w_h(t+\tau, x) - w_h(t,x)|^2 \mathrm{d}x\mathrm{d}t \leq C(\tau + \Delta t), \quad w_h = A(u_h),\ v_h, \tag{5.2}$$

for all $\tau \in (0, T)$.

Proof. The proof follows the guidelines of [20].

Proof of (5.1). First, to simplify the notation we write

$$\sum_{\sigma_{K,L}} \quad \text{instead of} \quad \sum_{\{(K,L)\in\mathcal{T}^2, K\neq L, m(\sigma_{K,L})\neq 0\}}.$$

Let $y \in \mathbb{R}^3$, $x \in \Omega'$, and $L \in N(K)$. We set

$$\beta_{\sigma_{K,L}} = \begin{cases} 1, & \text{if the line segment } [x, x+y] \text{ intersects } \sigma_{K,L},\ K \text{ and } L, \\ 0, & \text{otherwise.} \end{cases}$$

Next, the value $c_{\sigma_{K,L}}$ is defined by $c_{\sigma_{K,L}} = \frac{y}{|y|} \cdot \eta_{K,L}$ with $c_{\sigma_{K,L}} > 0$. We observe that (see [19] for more details)

$$\begin{aligned} &\int_{\Omega'} \beta_{\sigma_{K,L}}(x)\mathrm{d}x \leq m(\sigma_{K,L})|y|c_{\sigma_{K,L}}, \\ &\sum_{\sigma_{K,L}} \beta_{\sigma_{K,L}}(x) c_{\sigma_{K,L}} d_{K,L} \leq |y| + 2h. \end{aligned} \tag{5.3}$$

With (5.3) in hand, an application of the Cauchy–Schwarz inequality yields

$$\begin{aligned}
&\iint_{(0,T)\times\Omega'} |A(u_h)(t,x+y)-A(u_h)(t,x)|^2\mathrm{d}x\\
&\le \left(T\sum_{\sigma_{K,L}}\beta_{\sigma_{K,L}}(x)c_{\sigma_{K,L}}d_{K,L}\right)\sum_{n=0}^{N-1}\Delta t\sum_{\sigma_{K,L}}\frac{|A(U_L^{n+1})-A(U_K^{n+1})|^2}{c_{\sigma_{K,L}}d_{K,L}}\int_{\Omega'}\beta_{\sigma_{K,L}}(x)\mathrm{d}x\\
&\le T|y|(|y|+2h)\sum_{n=0}^{N-1}\Delta t\sum_{\sigma_{K,L}}\frac{|\sigma_{K,L}|}{d_{K,L}}|A(U_L^{n+1})-A(U_K^{n+1})|^2.
\end{aligned}$$

From the above estimate and from the bound (3.5) we deduce (5.1). □

Proof of (5.2). Let $\tau\in(0,T)$ and $t\in(0,T-\tau)$. We have

$$\mathcal{B}(t)=\int_\Omega |A(u_h)(t+\tau,x)-A(u_h)(t,x)|^2\mathrm{d}x.$$

Set $n_0(t)=[t/\Delta t]$ and $n_1(t)=[(t+\tau)/\Delta t]$, where

$$[x]=n \quad \text{for } x\in[n,n+1),\ n\in\mathbb{N}.$$

We get

$$\mathcal{B}(t)=\sum_{K\in\mathcal{T}}|K||A(U_K^{n_1(t)})-A(U_K^{n_0(t)})|^2,$$

which also implies

$$\mathcal{B}(t)\le C\sum_{K\in\mathcal{T}}\left((A(U_K^{n_1(t)})-A(U_K^{n_0(t)}))\times\sum_{t\le n\Delta t<t+\tau}|K|(U_K^{n+1}-U_K^n)\right).$$

Using the scheme (2.9), we obtain

$$\begin{aligned}
\mathcal{B}(t)\le C\sum_{t\le n\Delta t<t+\tau}\Delta t\sum_{K\in\mathcal{T}}\Big(A(U_K^{n_1(t)})-A(U_K^{n_0(t)})\Big)\Bigg(\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}A(U_L^{n+1})-A(U_K^{n+1})\\
+\sum_{L\in N(K)}G(U_K^{n+1},U_L^{n+1};\delta V_{K,L}^{n+1})\Bigg).
\end{aligned}\tag{5.4}$$

We observe that we can rewrite (5.4) as

$$\begin{aligned}
\mathcal{B}(t)\le\frac{C}{2}\sum_{t\le n\Delta t<t+\tau}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\Bigg[&\Big(A(U_K^{n+1})-A(U_L^{n+1})\Big)\Big(A(U_K^{n_1(t)})-A(U_L^{n_1(t)})\Big)\\
&+\Big(A(U_L^{n+1})-A(U_K^{n+1})\Big)\Big(A(U_L^{n_0(t)})-A(U_K^{n_0(t)})\Big)-G(U_K^{n+1},U_L^{n+1};\delta V_{K,L}^{n+1})\Big(A(U_K^{n_1(t)})-A(U_L^{n_1(t)})\Big)\\
&-G(U_K^{n+1},U_L^{n+1};\delta V_{K,L}^{n+1})\Big(A(U_L^{n_0(t)})-A(U_K^{n_0(t)})\Big)\Bigg].
\end{aligned}$$

We use the basic inequality "$ab\le\frac{1}{2}a^2+\frac{1}{2}b^2$" to deduce

$$\mathcal{B}(t)\le\frac{C}{2}\left(\mathcal{B}_1(t)+\frac{1}{2}\mathcal{B}_2(t)+\frac{1}{2}\mathcal{B}_3(t)+C'\mathcal{B}_4(t)+C'\mathcal{B}_5(t)\right),$$

for some constant $C'>0$, with

$$\begin{aligned}
\mathcal{B}_1(t)&=\sum_{t\le n\Delta t<t+\tau}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|A(U_K^{n+1})-A(U_L^{n+1})|^2,\\
\mathcal{B}_2(t)&=\sum_{t\le n\Delta t<t+\tau}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|A(U_K^{n_1(t)})-A(U_L^{n_1(t)})|^2,
\end{aligned}$$

$$\mathcal{B}_3(t) = \sum_{t\le n\Delta t<t+\tau} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |A(U_K^{n_0(t)}) - A(U_L^{n_0(t)})|^2,$$

$$\mathcal{B}_4(t) = \sum_{t\le n\Delta t<t+\tau} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left(|A(U_K^{n_1(t)}) - A(U_L^{n_1(t)})|^2 + |V_K^{n+1} - V_L^{n+1}|^2 \right),$$

$$\mathcal{B}_5(t) = \sum_{t\le n\Delta t<t+\tau} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \left(\frac{|\sigma_{K,L}|}{d_{K,L}} |A(U_K^{n_0(t)}) - A(U_L^{n_0(t)})|^2 + |V_K^{n+1} - V_L^{n+1}|^2 \right).$$

Now, we introduce the characteristic function β defined by $\beta(n, t_1, t_2) = 1$ if $t_1 < (n+1)\Delta t \le t_2$ and $\beta(n, t_1, t_2) = 0$ otherwise. Then we have for any sequence $(a^n)_{n\in\mathbb{N}}$ of non-negative numbers that

$$\int_0^{T-\tau} \sum_{t\le n\Delta t<t+\tau} a^n \mathrm{d}t \le \sum_{n=0}^{\left[\frac{T}{\Delta t}\right]} a^n \int_0^{T-\tau} \beta(n, t, t+\tau)\mathrm{d}t \le \tau \sum_{n=0}^{\left[\frac{T}{\Delta t}\right]} a^n \tag{5.5}$$

and for any $\xi \in [0, \tau]$

$$\int_0^{T-\tau} \sum_{t\le n\Delta t<t+\tau} a^{[(t+\xi)/\Delta t]} \le \tau \sum_{n=0}^{\left[\frac{T}{\Delta t}\right]} a^n. \tag{5.6}$$

From (5.5), we deduce

$$\begin{aligned}\int_0^{T-\tau} \mathcal{B}_1(t)\mathrm{d}t &\le \sum_{n=0}^{N-1} \Delta t \int_0^{T-\tau} \beta(n, t, t+\tau) \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |A(U_K^{n+1}) - A(U_L^{n+1})|^2 \mathrm{d}t \\ &\le \tau \sum_{n=0}^{N-1} \Delta t \sum_{(K,L)\in\mathcal{T}^2} \frac{|\sigma_{K,L}|}{d_{K,L}} |A(U_K^{n+1}) - A(U_L^{n+1})|^2.\end{aligned}$$

In view of (3.5), this implies that there exists a constant $C > 0$ such that

$$\int_0^{T-\tau} \mathcal{B}_1(t)\mathrm{d}t \le \tau C. \tag{5.7}$$

Next, we consider $\mathcal{B}_2(t)$ and $\mathcal{B}_3(t)$. We use (5.6) with $\xi = \tau$ for $\mathcal{B}_2(t)$ and (5.5) for $\mathcal{B}_3(t)$ to obtain

$$\int_0^{T-\tau} \mathcal{B}_2(t)\mathrm{d}t \le \tau \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |A(U_K^{n+1}) - A(U_L^{n+1})|^2$$

and

$$\int_0^{T-\tau} \mathcal{B}_3(t)\mathrm{d}t \le \tau \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |A(U_K^{n+1}) - A(U_L^{n+1})|^2.$$

We use (3.5) to deduce that

$$\int_0^{T-\tau} \mathcal{B}_2(t)\mathrm{d}t \le \tau C, \qquad \int_0^{T-\tau} \mathcal{B}_3(t)\mathrm{d}t \le \tau C, \tag{5.8}$$

for some constant $C > 0$. Reasoning along the same lines for (5.7) and (5.8) yield

$$\int_0^{T-\tau} \mathcal{B}_4(t)\mathrm{d}t \le \tau C, \qquad \int_0^{T-\tau} \mathcal{B}_5(t)\mathrm{d}t \le \tau C,$$

for some constant $C > 0$. This concludes the proof of the lemma. □

6. Convergence of the finite volume scheme

The translation estimates of Section 5 result in compactness of the set of discrete solutions. More precisely, we have the following lemma.

Lemma 6.1. *There exists a sequence $(h_m)_{m\in\mathbb{N}}$, $h_m \to 0$ as $m \to \infty$, and functions u, v on Q_T such that $0 \le u \le 1$, both $A(u)$ and v belong to $L^2(0,T;H^1(\Omega))$, and*

$$\begin{aligned} &\text{(i) } u_{h_m} \to u \quad \text{and} \quad v_{h_m} \to v \text{ a.e. in } Q_T \text{ and strongly in } L^p(Q_T) \text{ for all } p < +\infty, \\ &\text{(ii) } \nabla_{h_m} A(u_{h_m}) \to \nabla A(u) \quad \text{and} \quad \nabla_{h_m} v_{h_m} \to \nabla v \text{ weakly in } (L^2(Q_T))^l. \end{aligned} \tag{6.1}$$

Proof. Proof of (i). Observe that from Lemma 5.1 and Kolmogorov's compactness criterion (see, e.g., [23, Theorem IV.25]), we deduce that there exists a (not labeled) subsequence of u_h such that

$$A(u_h) \to \overline{A} \quad \text{strongly in } L^2(Q_T).$$

Because A is strictly monotone, there exists a unique u such that $A(u) = \overline{A}$. Thus,

$$A(u_h) \to A(u) \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.$$

As A^{-1} is well defined and continuous, applying the L^∞ bound on u_h and the dominated convergence theorem to $u_h = A^{-1}(A(u_h))$ we get

$$u_h \to u \quad \text{a.e. in } Q_T \text{ and strongly in } L^p(Q_T),\ p < +\infty.$$

Similarly, the translation estimates and the L^∞ bound (3.4) on v_h ensure that, up to extraction of a subsequence, $v_h \to v$ a.e. on Q and strongly in $L^p(Q_T)$ for $1 \le p < \infty$. □

Proof of (ii). The proof of the claim (ii) is similar to that of Lemma 4.4 in [24], therefore we omit the details. The idea is to use (3.5) to bound $\nabla_h w_h$ in $L^2(Q_T)$, for $w_h = A(u_h)$ and for $w_h = v_h$. Upon extraction of a further subsequence, we have $w_h \to w$ in $L^2(Q_T)$ and $\nabla_h w_h \to \chi$ in $(L^2(Q_T))^l$. Then one takes a smooth compactly supported vector-function ϕ on Q_T and shows that, from the discrete summation by parts and the consistency of the finite volume approximation of $\operatorname{div}\phi$,

$$\int_0^T \int_\Omega \nabla w \cdot \phi = -\int_0^T \int_\Omega w \operatorname{div}\phi.$$

This shows that $w \in L^2(0,T;H^1(\Omega))$ and $\chi = \nabla w$. □

Our final goal is to prove the following lemma.

Lemma 6.2. *Assume (1.4), (1.6) and (1.7). Let u_0, v_0 be as in Theorem 2.1. Then the limit functions u, v constructed in Lemma 6.1 constitute a weak solution of problem (1.1)–(1.3).*

Proof. Le $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$. Set $\varphi_K^n := \varphi(t^n, x_K)$ for all $K \in \mathcal{T}$ and $n \in [\![0..(N+1)]\!]$. Multiply the first equation in (2.9) by $\Delta t \varphi_K^{n+1}$, and sum up in $K \in \mathcal{T}$ and $n \in [\![0..N]\!]$. This yields

$$S_1^h + S_2^h + S_3^h = 0,$$

where

$$\begin{aligned} S_1^h &:= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K| (U_K^{n+1} - U_K^n) \varphi_K^{n+1}, \\ S_2^h &:= -\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (A(U_L^{n+1}) - A(U_K^{n+1})) \varphi_K^{n+1}, \\ S_3^h &:= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} G(U_K^{n+1}, U_L^{n+1}, \delta V_{K,L}^{n+1}) \varphi_K^{n+1}. \end{aligned}$$

Performing summation by parts in time and keeping in mind that $\varphi_K^{N+1} = 0$ for all $K \in \mathcal{T}$, we obtain

$$\begin{aligned} S_1^h &= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K| U_K^n (\varphi_K^{n+1} - \varphi_K^n) - \sum_{K\in\mathcal{T}} |K| U_K^0 \varphi_K^0 \\ &= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K u_h(t,x) \partial_t \varphi(t, x_K) \mathrm{d}x \mathrm{d}t - \sum_{K\in\mathcal{T}} \int_K u_0(x) \varphi(0, x_K) \mathrm{d}x. \end{aligned}$$

Then proceeding as in [25], it is clear from Lemma 6.1(i) that

$$\lim_{m\to\infty} S_1^{h_m} = -\int_0^T \int_\Omega u \partial_t \varphi - \int_\Omega u_0 \varphi(0,\cdot). \tag{6.2}$$

Now, let us show that

$$\lim_{m\to\infty} S_2^{h_m} = \int_0^T \int_\Omega \nabla A(u)\cdot\nabla\varphi. \tag{6.3}$$

Gathering by edges and using the definition (2.5) of ∇_h, we have

$$\begin{aligned} S_2^h &= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{1}{l}|\sigma_{K,L}|d_{K,L} l \frac{A(U_L^{n+1})-A(U_K^{n+1})}{d_{K,L}}\frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{d_{K,L}} \\ &= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} |T_{K,L}|\Big(\nabla_{K,L}A(u_h^{n+1})\cdot\eta_{K,L}\Big)\Big(\nabla\varphi(t^{n+1},\overline{x_{K,L}})\cdot\eta_{K,L}\Big), \end{aligned}$$

where $\overline{x_{K,L}}$ is some point on the segment with the endpoints x_K, x_L. Moreover, because the values of $\nabla_{K,L}$ are directed by $\eta_{K,L}$, we actually have

$$\Big(\nabla_{K,L}u_h^{n+1}\cdot\eta_{K,L}\Big)\Big(\nabla\varphi(t^{n+1},\overline{x_{K,L}})\cdot\eta_{K,L}\Big) \equiv \nabla_{K,L}u_h^{n+1}\cdot\nabla\varphi(t^{n+1},\overline{x_{K,L}}).$$

Since each term corresponding to $T_{K,L}$ appears twice in the above formula,

$$S_2^h = \int_0^T\int_\Omega \nabla_h A(u_h)\cdot(\nabla\varphi)_h,$$

where

$$(\nabla\varphi)_h|_{(t^n,t^{n+1}]\times T_{K,L}} := \nabla\varphi(t^{n+1},\overline{x_{K,L}}).$$

Observe that from the continuity of φ we get $(\nabla\varphi)_h \to \nabla\varphi$ in $L^\infty(Q_T)$. Hence (6.4) follows by Lemma 6.1(ii).

Finally, we show that

$$\lim_{m\to\infty} S_3^{h_m} = \int_0^T\int_\Omega \chi(u)\nabla v\cdot\varphi \mathrm{d}x\mathrm{d}t. \tag{6.4}$$

Gathering by edges (thanks to the consistency of the fluxes, see (2.6)(b)), we find

$$S_3^h = -\frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} G(U_K^{n+1},U_L^{n+1},\delta V_{K,L}^{n+1})\frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{d_{K,L}}.$$

For each couple of neighbors K, L, pick for $U_{K,L}^{n+1}$ the minimum of U_K^{n+1} and U_L^{n+1}. Set

$$S_3^{h,*} := -\frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \chi(U_{K,L}^{n+1})\delta V_{K,L}^{n+1}\frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{d_{K,L}}.$$

Introduce $\overline{u_h}$, $\underline{u_h}$ by

$$\overline{u_h}|_{(t^n,t^{n+1}]\times T_{K,L}} := \max\{U_K^{n+1},U_L^{n+1}\},\qquad \underline{u_h}|_{(t^n,t^{n+1}]\times T_{K,L}} := \min\{U_K^{n+1},U_L^{n+1}\}.$$

As previously, using the definitions of ∇_h and $(\nabla\varphi)_h$ we rewrite

$$S_3^{h,*} = -\int_0^T\int_\Omega \chi(\underline{u_h})\nabla_h v_h\cdot(\nabla\varphi)_h.$$

By the monotonicity of A and thanks to the estimate (3.5), we have

$$\begin{aligned} \int_0^T\int_\Omega |A(\overline{u_h})-A(\underline{u_h})|^2 &\le \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}||A(U_L^{n+1})-A(U_K^{n+1})|^2 \\ &\le Ch^2\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|A(U_L^{n+1})-A(U_K^{n+1})|^2 \le Ch^2. \end{aligned}$$

Because A^{-1} is continuous, up to extraction of another subsequence, we deduce

$$|\overline{u_{h_m}} - \underline{u_{h_m}}| \to 0 \quad \text{a.e. on } Q_T. \tag{6.5}$$

In addition, $\underline{u_{h_m}} \le u_{h_m} \le \overline{u_{h_m}}$; moreover, by Lemma 6.1(i), $u_{h_m} \to u$ a.e. on Q_T. Thus we see that $\chi(\underline{u_{h_m}}) \to \chi(u)$ a.e. on Q_T and in $L^p(Q_T)$, for $p < +\infty$. Using again Lemma 6.1(ii) and the strong convergence of $(\nabla\varphi)_h$ to $\nabla\varphi$, we infer that

$$\lim_{m\to\infty} S_3^{h_m,*} = -\int_0^T \int_\Omega \chi(u)\nabla v \cdot \nabla\varphi.$$

It remains to show that

$$\lim_{m\to\infty} |S_3^{h_m} - S_3^{h_m,*}| = 0. \tag{6.6}$$

By properties (2.6) we have

$$\begin{aligned} |G(U_K^{n+1}, U_L^{n+1}, \delta V_{K,L}^{n+1}) - \chi(U_{K,L}^{n+1})\delta V_{K,L}^{n+1}| &= |G(U_K^{n+1}, U_L^{n+1}, \delta V_{K,L}^{n+1}) - G(U_{K,L}^{n+1}, U_{K,L}^{n+1}, \delta V_{K,L}^{n+1})| \\ &\le |\delta V_{K,L}^{n+1}|\omega(2|U_L^{n+1} - U_K^{n+1}|). \end{aligned}$$

In view of the definitions of $\overline{u_h}$, $\underline{u_h}$ and $\nabla_h v_h$, $(\nabla\varphi)_h$, this yields

$$|S_3^h - S_3^{h,*}| \le \int_0^T \int_\Omega \omega(2|\overline{u_h} - \underline{u_h}|)|\nabla_h v_h \cdot (\nabla\varphi)_h|.$$

Using the Cauchy–Schwarz inequality, the uniform bound on $\nabla_h v_h$ stated in (3.5) and the convergence (6.5), we establish (6.6).

This concludes the proof of (2.3). Reasoning along the same lines as above, using in addition the uniform estimate (5.2) of the time translates of u_h in order to pass to the limit in the term $g(u_h(t-\Delta t, x), v_h(t,x))$ (this term comes from the right-hand side of the second equation of the scheme (2.9)), we conclude that also (2.4) holds. □

7. Numerical examples

In this section we show some numerical experiments with the proposed numerical scheme. The algorithm used to compute numerical solution of the system (2.9) is the following: at each time step, we first calculate V^{n+1} solution of the linear system given by the second equation of (2.9) and next we compute U^{n+1} as the solution of the nonlinear system defined by the first equation of (2.9). Then, a Newton algorithm is implemented to approach the solution of nonlinear system coupled with a bigradient method to solve linear systems arising from the Newton algorithm process.

Mass conservation. Note that the mass of u and v are conserved in the following sense: integrate Eqs. (1.1) in time and space, we deduce that:

$$\int_\Omega u(t,x)\mathrm{d}x = \int_\Omega u_0(x)\mathrm{d}x, \tag{7.1}$$

and

$$\int_\Omega v(t,x)\mathrm{d}x = \mathrm{e}^{-\beta t}\int_\Omega v_0(x)\mathrm{d}x + (1-\mathrm{e}^{-\beta t})\frac{\alpha}{\beta}\int_\Omega u_0(x)\mathrm{d}x. \tag{7.2}$$

Observe that these properties are satisfied for our finite volume scheme (2.9). In fact, summing Eqs. (2.9) over $K \in \mathcal{T}$, we get

$$\sum_{K\in\mathcal{T}} |K|\frac{U_K^{n+1} - U_K^n}{\Delta t} - \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(A(U_L^{n+1}) - A(U_K^{n+1})) + \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} G(U_K^{n+1}, U_L^{n+1}; \delta V_{K,L}^{n+1}) = 0, \tag{7.3}$$

$$\sum_{K\in\mathcal{T}} |K|\frac{V_K^{n+1} - V_K^n}{\Delta t} - d\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(V_L^{n+1} - V_K^{n+1}) = \sum_{K\in\mathcal{T}} |K| g(U_K^n, V_K^{n+1}). \tag{7.4}$$

Note that the second and the third terms in (7.3) and the second term in (7.4) vanish due to the main property of the finite volume scheme, namely the conservation of numerical fluxes. Then the above equations are reduced to

$$\begin{aligned} &\sum_{K\in\mathcal{T}} |K|U_K^{n+1} = \sum_{K\in\mathcal{T}} |K|U_K^n, \\ &\sum_{K\in\mathcal{T}} |K|V_K^{n+1} = \sum_{K\in\mathcal{T}} |K|V_K^n + \Delta t\alpha\sum_{K\in\mathcal{T}} U_K^n - \Delta t\beta\sum_{K\in\mathcal{T}} |K|V_K^{n+1}. \end{aligned}$$

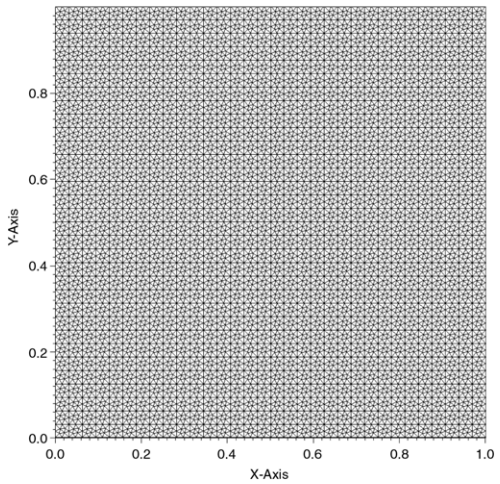


Fig. 2. Mesh with 14 336 triangles.

Therefore, by induction, we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}}|K|U_K^n=\sum_{K\in\mathcal{T}}|K|U_K^0,\\&\sum_{K\in\mathcal{T}}|K|V_K^n=(1+\beta\Delta t)^{-n}\sum_{K\in\mathcal{T}}|K|V_K^0+\frac{\alpha}{\beta}(1-(1+\beta\Delta t)^{-n})\sum_{K\in\mathcal{T}}U_K^0.\end{aligned}\tag{7.5}$$

The first equality indicates that the mass of cell is conserved along time, whereas the second shows a discrete form of the mass conservation (7.2) of the concentration. In the numerical tests, we illustrate the total mass $\sum_{K\in\mathcal{T}}|K|U_K^n$ and $\sum_{K\in\mathcal{T}}|K|V_K^n$ along time in order to show that the implemented algorithm for approximate solving of nonlinear system does not affect the quality of the solutions.

We will provide three tests made on a nonuniform admissible grid.

Test 1. Consider the following data: $L_x=1, L_y=1$ (the length and the width of the domain); the rectangular domain is covered by 14 336 triangles. Further, $\mathrm{d}t=0.002, \alpha=0.01, \beta=0.05, d=0.0001, A(u)=D\left(\frac{u^2}{2}-\frac{u^3}{3}\right)$, with $D=0.05$, $\chi(u)=u(1-u)^2$. Then, we have $a(u)=A'(u)=Du(1-u)$ and in the definition of the numerical fluxes G defined by (2.7), we consider

$$\chi_\uparrow(z)=\chi\left(\min\left\{z,\frac{1}{3}\right\}\right)\quad\text{and}\quad\chi_\downarrow(z)=\chi\left(\max\left\{z,\frac{1}{3}\right\}\right)-\chi\left(\frac{1}{3}\right).$$

The degenerate diffusion function A is cubic, so the Newton algorithm converges rapidly with four to six iterations.

In Fig. 2, we show the mesh with 14 336 triangles.

In Fig. 3, the initial conditions are defined by region. The initial cell density is defined as $u_0(x,y)=1$ in the square $(x,y)\in[0.45,0.55]\times[0.45,0.55]$ and 0 otherwise. The initial chemoattractant is defined as $v_0(x,y)=5$ in the union of four squares $(x,y)\in([0.2,0.3]\times[0.7,0.8])\cup([0.2,0.3]\times[0.2,0.3])\cup([0.7,0.8]\times[0.7,0.8])\cup([0.7,0.8]\times[0.2,0.3])$ and 0 otherwise.

In Fig. 4, we show the evolution of the cell density at time $t=1$, $t=3$ and $t=20$. At first stage of evolution we observe the effect of the diffusion of cell without interactions with the chemoattractant, next we see the rule of the chemo-attraction since the cells are locally present in the chemoattractant regions.

In Fig. 5, we observe the influence of the degeneracy of the diffusive term since this operator vanishes in the region where the cell density is maximal. The spread of the chemoattractant is limited due to the fact that the coefficient of diffusion ($d=0.0001$) is small compared to the coefficient of diffusion of cells ($D=0.01$).

In Fig. 6, we plot the mass evolution of solutions during the time simulation. We verify the mass conservation along time of the cell density, whereas the chemoattractant tends to disappear as time grows.

In Fig. 7, we show the evolution of the density of cell and the concentration of chemoattractant at fixed points in the domain with respect to time. The evolution at the point (0.5, 0.5) (the middle point in the domain) shows that the cell density decreases due to the diffusion phenomena, whereas the concentration of chemoattractant remains null along time. The evolution at the point (0.25, 0.75) (a point where initially the concentration of chemoattractant is maximal and the cells are not present) indicates the spread of the chemoattractant and the time for which the cells reach the attractive zone and the cell density increases to become asymptotically maximal.

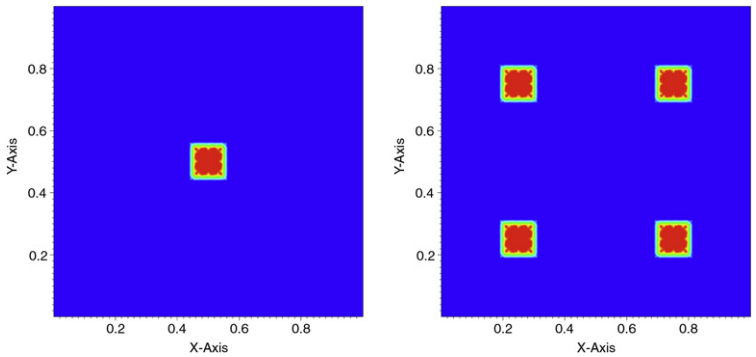


Fig. 3. Initial condition for the cell density $(u)u(0, x, y) = 1$ in the region where $(x, y) \in [0.45, 0.55] \times [0.45, 0.55]$ and 0 otherwise (left) and the chemoattractant (v) (right) $v(0, x, y) = 5$ in the region where $(x, y) \in ([0.2, 0.3] \times [0.7, 0.8]) \cup ([0.2, 0.3] \times [0.2, 0.3]) \cup ([0.7, 0.8] \times [0.7, 0.8]) \cup ([0.7, 0.8] \times [0.2, 0.3])$ and 0 otherwise.

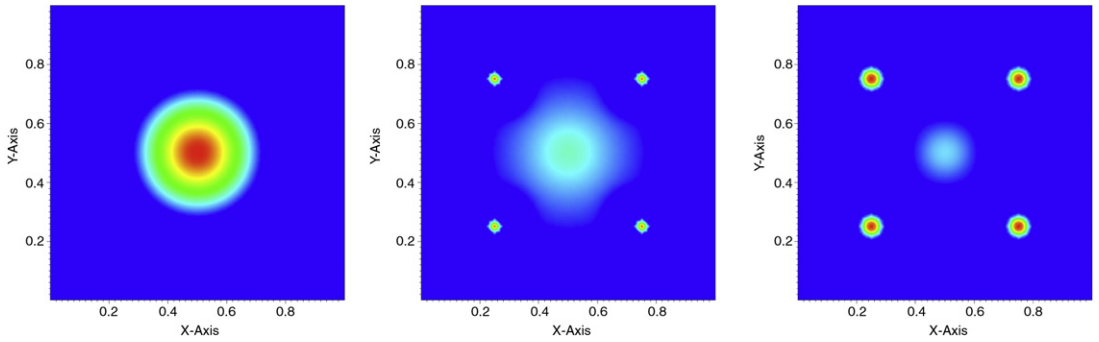


Fig. 4. Evolution of the cell density (u), at time $t = 1$ with $0 \leq u \leq 0.1626$ (left), at time $t = 3$ with $0 \leq u \leq 0.3947$ and at time $t = 20$ with $0 \leq u \leq 0.7420$ (right).

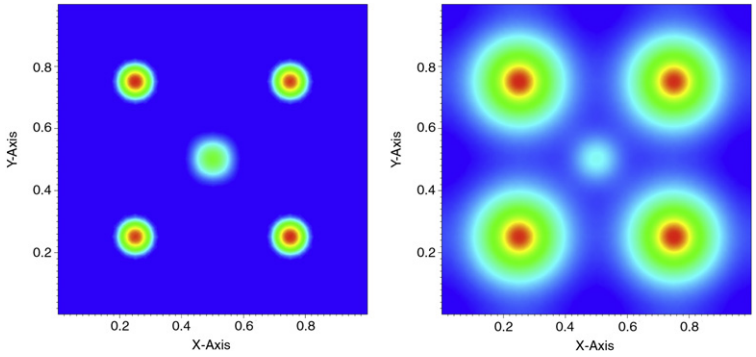


Fig. 5. The cell density (u), at time $t = 60$ with $0 \leq u \leq 0.3275$ (left) and the chemoattractant (v), at time 60 with $0 \leq v \leq 0.0565$ (right).

Test 2. This test is devoted to an illustration of the influence of the diffusive degeneracy of the cell density. The only difference versus Test 1 is considering $A(u) = Du$ and then $a(u) = D$. In Fig. 8, we show the evolution of the cell density at points (0.5, 0.5) and (0.25, 0.75). Note that, the degeneracy retards considerably the diffusion.

Test 3. In this test we consider a random initial distribution of cell density in the same domain as in Fig. 2. The initial condition of chemoattractant is considered to be 5 in eight squares as shown in Fig. 10. For this test we take the following data: $\alpha = 0.01$, $\beta = 0.05$, $A(u) = D\left(\frac{u^2}{2} - \frac{u^3}{3}\right)$, with $D = 0.01$, $\chi(u) = cu(1-u)^2$, with $c = 0.01$.

In Figs. 9 and 10, we show the evolution of the concentration of the cell density at times $t = 0.25$ and $t = 5$ respectively. We observe during the stage of evolution the effect of the chemo-attraction, since the cells are present in the chemoattractant regions. Finally, in Fig. 10, we illustrate the distribution of the chemoattractant at time $t = 5$.

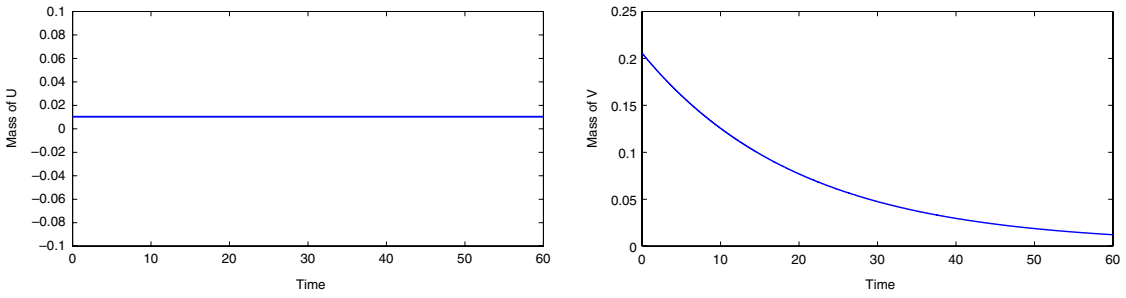


Fig. 6. Evolution of the total cell density $\sum_{K\in\mathcal{T}} |K|U_K^n$ (left), and the total chemoattractant $\left(\sum_{K\in\mathcal{T}} |K|V_K^n\right)$ (right).

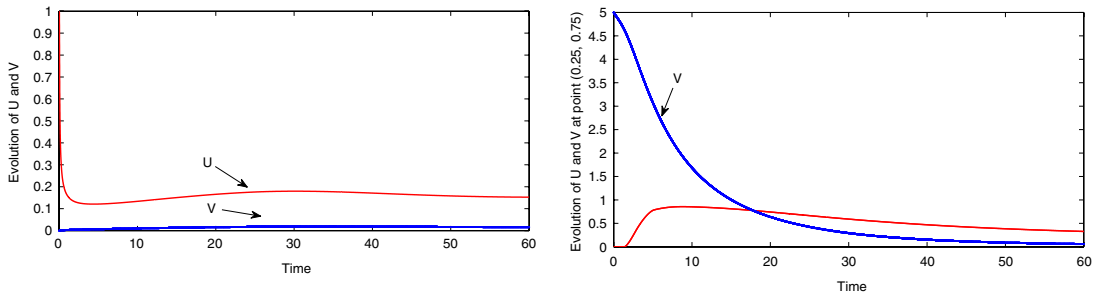


Fig. 7. Evolution of the cell density and the concentration of the chemoattractant at point (0.5, 0.5) (left), at point (0.25, 0.75) (right).

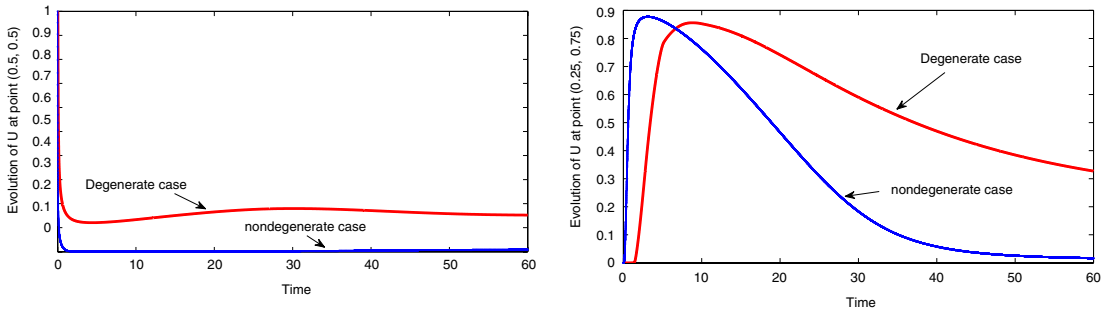


Fig. 8. Comparison between the degenerate case $a(u) = Du(1-u)$ and a non-degenerate case $a(u) = c$. Evolution of the cell density (0.5, 0.5) (left), at point (0.25, 0.75) (right).

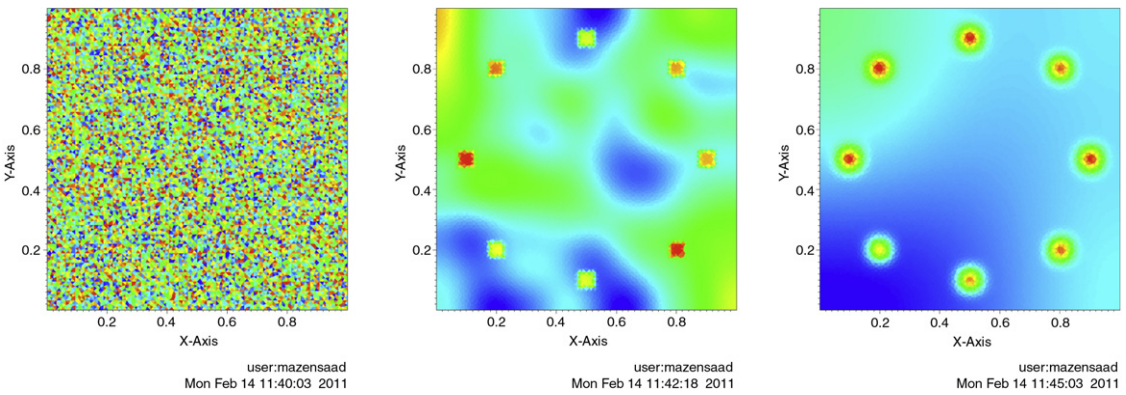


Fig. 9. Cell density, random initial condition with $0 \le u \le 1$ (left), solution at time $t = 0.25$ with $0 \le u \le 0.5570$ (center) and solution at time $t = 5$. With $0 \le u \le 0.5254$ (right).

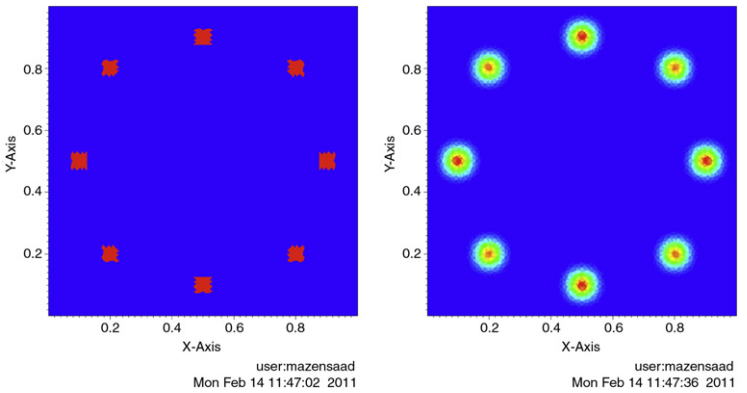


Fig. 10. Initial condition with $0 \leq v \leq 5$ (left) and distribution of chemoattractant at time $t = 5$. With $0 \leq v \leq 2.25$ (right).

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