Journal of Computational and Applied Mathematics 235 (2011) 4532–4536

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On a characterization of the uniform distribution by generalized order statistics

G. Arslan

Department of Statistics and Computer Science, Başkent University, 06810 Ankara, Turkey

ARTICLE INFO

Article history:
Received 28 January 2010
Received in revised form 18 February 2010

Keywords:
Characterization
Generalized order statistics
Uniform distribution

ABSTRACT

In this paper a new variant of the Choquet–Deny theorem is obtained and used to prove a characterization of the uniform distribution based on spacings of generalized order statistics. This result extends two recent characterizations of the uniform distribution.

1. Introduction

The characterization of the uniform distribution has been considered by many authors and is also an important problem in applications. For some recent examples, one may refer to the papers of Hamedani and Volkmer [1], Arslan et al. [2] and Ahsanullah [3], among many others. In this paper a characterization of the uniform distribution is obtained using generalized order statistics.

Let $\{X_i, i \geq 1\}$ be a set of independent and identically distributed (i.i.d.) random variables and denote the corresponding order statistics by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. The upper record values will be denoted by $X_{U(1)}, X_{U(2)}, \ldots$. For recent developments in the theory of order statistics, one may refer to [4]. For details on the theory of order statistics, see also [5–8].

Although several characterization results involving spacings of order statistics can be found in the literature we note that, as pointed out in [9], there are only a few results involving identical distributions of linear forms. Puri and Rubin [10], for example, investigated the relation $X_{1:1} \stackrel{d}{=} X_{2:2} - X_{1:2}$ and showed all possible distributions satisfying this relation. Later Gather [11] considered a more general relation; $X_{j-i:n-i} \stackrel{d}{=} X_{j:n} - X_{i:n}$. There are even fewer results based on spacings of record values. Arslan et al. [2] have, for example, considered a characterization of the uniform distribution based on the relation $X_{U(r)} - X_{U(r-1)} \stackrel{d}{=} X_{L(r)}$, where $X_{U(r)}$ and $X_{L(r)}$ are the rth upper and lower records, respectively. Note that the result obtained in [2] is an example of a mixed type (lower and upper records) of characterization. A related result is also given in [12, p. 175].

The main objective of this paper is to obtain a new variant of the Choquet–Deny theorem and to use it to prove some characterization results for the uniform distribution, based on spacings. This result may be considered as a further example of variants presented in [13]. Using this theorem two characterizations of the uniform distribution, one obtained using order statistics [1] and other obtained using record statistics [2], are combined in one theorem by using generalized order statistics.

After presenting some basic notations and a variant of the Choquet–Deny theorem in the next section, an application to a characterization of the uniform distribution is given in the third section.

E-mail address: guvenca@baskent.edu.tr.

0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.02.040

2. Some notations and a variant of the Choquet–Deny theorem

Generalized order statistics were introduced in [14]. Let F be an absolutely continuous distribution with density function f. The random variables $X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are called generalized order statistics based on F if their joint density function is given by

$$f^{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, \ldots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i}f(x_i)\right)(1-F(x_n))^{k-1}f(x_n),$$

where $F^{-1}(0) < x_1 \leq \cdots \leq x_n < F^{-1}(1)$, $n \in \mathbb{N}$, $n \geq 2$, $k > 0$, $\tilde{m} = (m1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, and $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k+n-r+M_r > 0$ for all $r \in \{1, \ldots, n-1\}$. Choosing the parameters appropriately, different models, such as ordinary statistics, record values, progressive-type II censored order statistics may be obtained. In this paper we only consider the special case where $m = m_i$, for all $i \in \{1, \ldots, n-1\}$. Note that if $m = 0$ and $k = 1$, then $X(r, n, m, k)$ reduces to the ordinary rth order statistics. On the other hand if $m = -1$ and $k = 1$, then $X(r, n, m, k)$ reduces to the rth upper record value.

In a similar way, the lower (dual) generalized order statistics can be defined. The random variables $X_l(1, n, m, k), X_l(2, n, m, k), \ldots, X_l(n, n, m, k)$ are called lower (dual) generalized order statistics based on F if their joint density function is given by

$$f_l^{X_l(1,n,m,k),\ldots,X_l(n,n,m,k)}(x_1, \ldots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(F(x_i))^{m}f(x_i)\right)(F(x_n))^{k-1}f(x_n),$$

where $F^{-1}(1) > x_1 \geq \cdots \geq x_n > F^{-1}(0)$, $n \in \mathbb{N}$, $n \geq 2$, $k > 0$, $m \geq -1$, and such that $\gamma_r = k + (n-r)(m+1) > 0$ for all $r \in \{1, \ldots, n-1\}$. For details on generalized order statistics, one may refer to [14], and for details and some applications of lower generalized order statistics one may refer to [15,3].

It has been already shown in some papers that many characterization results proved by several approaches can also be proved by using some version of the Choquet–Deny theorem. Some examples may be found in [13]. For information about some versions of the Choquet–Deny theorem one may refer to [16,17], among others.

The main result may be considered as a variant of the following theorem [13].

Theorem 1. *Let p be a positive integer and A be a non-empty subset of $[0, \infty)^p \setminus \{\mathbf{0}\}$ with the property that $\boldsymbol{x} \in A$ implies $[\mathbf{0}, \boldsymbol{x}] \setminus \{\mathbf{0}\} \subset A$, where $\mathbf{0} = (0, \ldots, 0)$. Also, let for each $\boldsymbol{x} \in A$, $B_{\boldsymbol{x}} = [\mathbf{0}, \boldsymbol{x}] \setminus \left(\{\boldsymbol{x}\} \bigcup \{\mathbf{0}\}\right)$ and $\{\mu_{\boldsymbol{x}} : \boldsymbol{x} \in A\}$ be a family of probability measures (on $\mathbb{R}^p$) such that for each $\boldsymbol{x}$, $\mu_{\boldsymbol{x}}$ is concentrated on $B_{\boldsymbol{x}}$. Then a continuous real-valued function H on A, such that $H(\boldsymbol{x})$ has a limit as $\|\boldsymbol{x}\|$ tends to $0+$, satisfies*

$$H(\boldsymbol{x}) = \int_{B_{\boldsymbol{x}}} H(\boldsymbol{x}-\boldsymbol{y})\mu_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y}), \quad \boldsymbol{x} \in A$$

if and only if it is identically equal to a constant.

Arslan [2] have used the following corolloray of Theorem 1 to prove some characterization results.

Corollary 1. *Let $A = (0, \beta)$ and for each $x \in A$, $B_x = (0, \beta - x)$. Also, let $\{\mu_x : x \in A\}$ be a family of probability measures such that for each $x \in A$, μ_x is concentrated on B_x. Then a continuous bounded nonnegative real-valued function h on A such that $h(x)$ has a limit as $\|x\|$ tends to $\beta-$, satisfies*

$$h(x) = \int_{B_x} h(x+y)\mu_x(\mathrm{d}y), \quad x \in A$$

if and only if it is identically equal to a constant.

The next theorem represents another variant of the Choquet–Deny theorem that is used to prove some recent characterization results, which will be given in the next section.

Theorem 2. *Let $A = (0, \beta)$ and for each $x \in A$, $B_x = (0, \beta - x)$. Also, let $\{\mu_x : x \in A\}$ be a family of (nondegenerate) probability measures such that for each $x \in A$, μ_x is concentrated on B_x. Then a continuous bounded nonnegative real-valued function H on A satisfies*

$$H(\beta - x) = \int_{B_x} H(x+y)\mu_x(\mathrm{d}y), \quad x \in A, \tag{2.1}$$

if and only if either $H \equiv 0$, or $H(x)$ is constant on A.

Proof. The "if" part of the theorem is trivial. To show the "only if" part suppose that H satisfies the integral equation (2.1) and assume that $H(x) > 0$ for some $x \in (0, \beta)$, and $H(x)$ is not identically equal to a constant on A. It will be shown that this leads to a contradiction.

Integral equation (2.1) can be rewritten as

$$H(x)=\int_{\beta-x}^{\beta}H(t)\nu_x(\mathrm{d}t), \tag{2.2}$$

where for any $\mu_{\beta-x}$-measurable set B, $\nu_x(B)=\mu_{\beta-x}(B)$.

Since (2.2) holds for every $x\in A$, $\exists x_0\in A$ and a Borel set $B_0\subseteq(\beta-x_0,\beta)$ such that $\nu_{x_0}(B_0)>0$ and

$$H(x_0)<H(y),\quad \forall y\in B_0. \tag{2.3}$$

Hence

$$H(x_0)\nu_{x_0}(B_0)<\int_{B_0}H(t)\nu_{x_0}(\mathrm{d}t). \tag{2.4}$$

Using the Kolmogorov consistency theorem, it follows that there exists an infinite sequence $\{X_n:n=1,2,\ldots\}$ of 0-1 valued exchangeable random variables such that for each $n\geq 1$

$$P\{X_1=1,\ldots,X_n=1\}=[H(x_0)]^{-1}\int_{B_0}\cdots\int_{B_0}H(y_1+y_2+\cdots+y_n)\,\nu_{x_0}(\mathrm{d}y_n)\cdots\nu_{x_0}(\mathrm{d}y_1). \tag{2.5}$$

Now, it can be shown that (see the proof of the theorem in [17])

$$P\{X_1=1,X_2=1,\ldots,X_{2^n}=1\}\geq(P\{X_1=1\})^{2^n},\quad n\geq 1. \tag{2.6}$$

Consequently,

$$\frac{P\{X_1=1,X_2=1,\ldots,X_{2^n}=1\}}{\left(\nu_{x_0}(B_0)\right)^{2^n}}\geq\left(\frac{P\{X_1=1\}}{\nu_{x_0}(B_0)}\right)^{2^n},\quad n\geq 1, \tag{2.7}$$

leading to a contradiction since the left-hand side of (2.7) is bounded relative to n while the right-hand side tends to ∞ as $n\to\infty$. Using (2.2), (2.5) and the fact that H is bounded there exists a positive constant C such that

$$P\{X_1=1,X_2=1,\ldots,X_{2^n}\}\leq\frac{C}{H(x_0)}\left[\nu_{x_0}(B_0)\right]^{2^n},$$

from which the boundedness of the left-hand side follows. The unboundedness of the right-hand side of (2.7) follows from (2.4) and (2.5). This contradiction proves the theorem. □

3. A characterization of the uniform distribution

In this section we consider a relation characterizing the uniform distribution based on spacings of generalized order statistics. This generalizes some previous characterization results and uses upper as well as lower generalized order statistics.

The uniform distribution with cumulative distribution function (CDF) F defined on (a,b), where $a<b$ will be denoted by $F\sim U(a,b)$.

Hamedani and Volkmer [1] have investigated the following relation

$$X_{s:n}-X_{r:n}\stackrel{d}{=}X_{s-r:n},$$

where $1\leq r<s\leq n$. They obtained several characterizations of the uniform distribution by considering special families of distributions such as subadditive and symmetric families. On the other hand, Arslan et al. [2] have shown that

$$X_{U(r)}-X_{U(r-1)}\stackrel{d}{=}X_{L(r)},$$

where $1<r$, is also a characteristic property of the uniform distribution, if the family of distributions is assumed to be symmetric.

In the next theorem it will be shown that, under suitable conditions, the relation

$$X(r,n,m,k)-X(r-1,n,m,k)\stackrel{d}{=}X_l(r,n,m,k),\quad r>1 \tag{3.1}$$

implies that the random variables X_i are from a Uniform distribution.

Theorem 3. *Let $X_1,X_2,\ldots,X_n$ be a sequence of i.i.d. nonnegative random variables with an absolutely continuous distribution function F and symmetric about $\beta/2$. Given the following statements:*

(1) $X(r,n,m,k)-X(r-1,n,m,k)\stackrel{d}{=}X_l(r,n,m,k)$ *is true for some* $\gamma_r=k+(n-r)(m+1)\geq 1$ *with* $r>1$.
(2) $\gamma_r=1$ *and* $F\sim U(0,\beta)$,

it follows that $(1)\Rightarrow(2)$.

Proof. The pdf of $X(r,n,m,k)-X(r-1,n,m,k)$ is given by

$$f_{r-1,r}(x)=\frac{c_{r-1}}{(r-2)!}\int_0^{\beta-x}[1-F(y)]^m g_m^{r-2}(F(y))\times[1-F(y+x)]^{\gamma_r-1}f(y)f(y+x)\,\mathrm{d}y,$$

where $\gamma_r=k+(n-r)(m+1)$. After some simplifications, this may be written as

$$f_{r-1,r}(x)=\frac{c_{r-1}}{\Gamma(r)}\int_0^{\beta-x}\left[\bar F(y+x)\right]^{\gamma_r-1}f(y+x)\,\mathrm{d}g_m^{r-1}(F(y)),$$

where $c_{r-1}=\prod_{j=1}^{r}\gamma_j$, and

$$g_m(x)=\begin{cases}\dfrac{1}{m+1}\left[1-(1-x)^{m+1}\right], & m\neq -1\\ -\ln(1-x), & m=-1.\end{cases}$$

The pdf of $X_l(r,n,m,k)$, on the other hand is given by

$$f_r^*(x)=\frac{c_{r-1}}{(r-1)!}F^{\gamma_r-1}(x)\left[g_m^*(F(x))\right]^{r-1}f(x), \tag{3.2}$$

where

$$g_m^*(x)=\begin{cases}\dfrac{1}{m+1}\left(1-x^{m+1}\right), & m\neq -1\\ -\ln(x), & m=-1.\end{cases}$$

Thus, relation (3.1) implies that

$$F^{\gamma_r-1}(x)\left[g_m^*(F(x))\right]^{r-1}f(x)=\int_0^{\beta-x}\left[\bar F(y+x)\right]^{\gamma_r-1}f(y+x)\,\mathrm{d}g_m^{r-1}(F(y))$$

and, since $g_m(F(y))=g_m^*(\bar F(y))$,

$$F^{\gamma_r-1}(x)\left[g_m^*(F(x))\right]^{r-1}f(x)=\int_0^{\beta-x}\left[\bar F(y+x)\right]^{\gamma_r-1}f(y+x)\,\mathrm{d}\left[g_m^*(\bar F(y))\right]^{r-1},$$

or

$$F^{\gamma_r-1}(x)f(x)=\int_0^{\beta-x}\left[\bar F(y+x)\right]^{\gamma_r-1}f(y+x)\,\frac{\mathrm{d}\left[g_m^*(\bar F(y))\right]^{r-1}}{\left[g_m^*(F(x))\right]^{r-1}},\quad x\in(0,\beta).$$

Since f is symmetric about $\beta/2$, this can be written as

$$\bar F^{\gamma_r-1}(\beta-x)f(\beta-x)=\int_0^{\beta-x}\left[\bar F(y+x)\right]^{\gamma_r-1}f(y+x)\,\frac{\mathrm{d}\left[g_m^*(\bar F(y))\right]^{r-1}}{\left[g_m^*(F(x))\right]^{r-1}},\quad x\in(0,\beta). \tag{3.3}$$

This functional integral equation can be solved by using Theorem 2. Let $H(x)=\bar F^{\gamma_r-1}(x)f(x)$, and define

$$\mu_x(B)=\int_{B\cap B_x}\frac{\mathrm{d}\left[g_m^*(\bar F(y))\right]^{r-1}}{\left[g_m^*(F(x))\right]^{r-1}},$$

where $B_x=(0,\beta-x)$. Then Eq. (3.3) can be written as

$$H(\beta-x)=\int_{B_x}H(x+y)\mu_x(\mathrm{d}y),\quad x\in A,$$

where $A=(0,\beta)$.

From Theorem 2 it follows that

$$H(x)=\bar F^{\gamma_r-1}(x)f(x)$$

is constant on A. The symmetry of F implies that $\gamma_r=1$ and $F\sim U(0,\beta)$. □

Note that $\gamma_r=1$ leads to two characterizations, which correspond to characterizations with order statistics ($k=1$, $m=0$, and $n=r$) and record values ($k=1$ and $m=-1$). The relations resulting from this special case are given below:

$$X_{n:n}-X_{n-1:n}\overset{d}{=}X_{1:n}, \tag{3.4}$$

and

$$X_{U(r)} - X_{U(r-1)} \stackrel{d}{=} X_{L(r)}. \tag{3.5}$$

Relation (3.4) corresponds to the special case for $r = n - 1$ in Theorem 5.2 given in [1], and (3.5) corresponds to the result given in [2], respectively.

Acknowledgements

The author thanks the referees for their comments and suggestions which helped to improve the presentation of this paper.

References

[1] G.G. Hamedani, H. Volkmer, Certain characterizations of the uniform distribution, Metrika 61 (2005) 117–125.
[2] G. Arslan, M. Ahsanullah, I.G. Bayramoglu, On characteristic properties of the uniform distribution, Sankhyā 67 (2005) 715–721.
[3] M. Ahsanullah, A Characterization of the uniform distribution by dual generalized order statistics, Communications in Statistics: Theory and Methods 33 (2004) 2921–2928.
[4] N. Balakrishnan, E. Castillo, J.M. Sarabia (Eds.), Advances in Distribution Theory, Order Statistics, and Inference, Birkhäuser, 2006.
[5] M. Ahsanullah, V. Nevzorov, Ordered Random Variables, Nova Science Publishers Inc., New York, 2002.
[6] B.C. Arnold, N. Balakrishnan, H.N. Nagaraja, A First Course in Order Statistics, John Wiley & Sons Inc., New York, 1992.
[7] N. Balakrishnan, C.R. Rao, Handbook of Statistics 16: Order Statistics: Theory & Methods, Elsevier Science, 1998.
[8] H.A. David, Order Statistics, John Wiley & Sons Inc., New York, 1981.
[9] U. Gather, G. Székely, Characterizations of distributions by forms of order statistics, Communications in Statistics 33 (2004) 2913–2919.
[10] S. Puri, H. Rubin, A characterization based on the absolute difference of two i.i.d. random variables, Annals of Mathematical Statistics 41 (1970) 2113–2122.
[11] U. Gather, On a characterization of the exponential distribution by properties of order statistics, Statistics & Probability Letters 7 (1989) 93–96.
[12] M. Ahsanullah, Record Values—Theory and Applications, University Press of America, Lanmham, MD, USA, 2004.
[13] E.B. Fosam, D.N. Shanbhag, Variants of the Choquet–Deny theorem with applications, Journal of Applied Probability 34 (1997) 101–106.
[14] U. Kamps, A Concept of Generalized Order Statistics, B.G. Teubner, Stuttgart, 1995.
[15] M. Burkschat, E. Cramer, U. Kamps, Dual generalized order statistics, Metron 61 (2003) 13–26.
[16] A.A. Alzaid, K.S. Lau, R. Rao, D.N. Shanbhag, Solution of Deny convolution equation restricted to a half line via a random walk approach, Journal of Multivariate Analysis 24 (1988) 309–329.
[17] C.R. Rao, D.N. Shanbhag, An elemantary proof for an extended version of the Choquet–Deny theorem, Journal of Multivariate Analysis 38 (1991) 141–148.