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Convergence of a FEM and two-grid algorithms for elliptic problems on disjoint domains

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ABSTRACT

In this paper, we analyze a FEM and two-grid FEM decoupling algorithms for elliptic problems on disjoint domains. First, we study the rate of convergence of the FEM and, in particular, we obtain a superconvergence result. Then with proposed algorithms, the solution of the multi-component domain problem (simple example — two disjoint rectangles) on a fine grid is reduced to the solution of the original problem on a much coarser grid together with solution of several problems (each on a single-component domain) on fine meshes. The advantage is the computational cost although the resulting solution still achieves asymptotically optimal accuracy. Numerical experiments demonstrate the efficiency of the algorithms.

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1. Introduction

In this paper, we consider coupling elliptic transmission problems on a multi-component domain with nonlocal interface conditions on parts of the boundary of the components. The study of such problems could be motivated physically by the occurrence of various nonstandard boundary and coupling conditions in modern physics, biology and engineering [1–3]. A typical physical example is the stationary problem for radiative–conductive heat transfer in a system on opaque bodies [1,2]. Mathematical results in this direction were first obtained in [3]. Later, many works were devoted to the solvability of such problems. A survey concerning the mathematical theory of heat transfer in conducting and radiating bodies can be found in [1].

Some one dimensional problems of this type was studied numerically in [4–7]. The two-grid method was proposed in [8,9], independently of each other, for a linearization of nonlinear elliptic problems. The two-grid finite element method was also used by Xu and many other scientists (see the reference in [10]) for discretizing nonsymmetric indefinite elliptic and parabolic equations. By employing two finite element spaces of different scales, one coarse and one fine space, the method was used for symmetrization of nonsymmetric problems, which reduces the solution of a nonsymmetric problem on a fine grid to the solution of a corresponding (but much smaller) nonsymmetric problem, discretized on the coarse grid and the solution of a symmetric positive definite problem on the fine grid. This method was also used for location and parallelization for solving a large class of partial differential equations [11,12]. There are many other authors who have used this method for many different applications; see [13–15] among others.

The two-grid approach in the present paper is an extension of the idea in [10], where it was used to decouple a Shrödinger system of differential equations. The system of partial differential equations is first discretized on the coarse grid, then a decoupled system is discretized on the fine mesh. As a result, the computational complexity of solving the Shrödinger system is comparable with solving two decoupled Poisson equations on the same grid.

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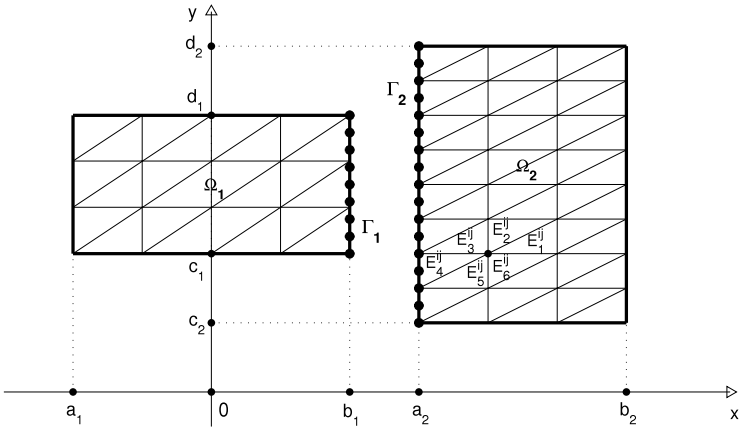


Fig. 1. Domains Ω_1 and Ω_2.

However, here we want to illustrate the two-grid idea in a new direction, *namely we use the two-grid discretization method to decouple the multi-component domain problem to several elliptic equations, each of them solved on its own domain*. For clarity, we use a simple model problem of linear elliptic equation on two disjoint rectangles.

In some industrial applications such as crystal growth, radiative heat transfer in materials that are conductive, gray and semitransparent, situations are relevant in which a transparent medium is enclosed by or several opaque, of diffusive gray bodies, for example *rectangles* in the 2D models in [2,16].

Also, our analysis should provide some insights on how a multiscale idea can be applied to multi-component domain problems when for example, some of the domains require very fine mesh. Finally, for nonlinear problems, iterative processes are often used to overcome the nonlinearities, see for example [17,4] and the resulting linear problems have the form (1)–(5).

In this paper we consider a system of two elliptic equations, each of them solved on its own rectangle. The two problems are coupled with nonlocal interface conditions on parts of the rectangles' boundary. First, the original system is approximated on the coarse mesh and then a decoupled system is discretized on the fine grid.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem, used to illustrate our method. In the next section, we discuss the convergence property of the FEM. Two-grid algorithms are proposed an analyzed in Section 4. In the last section we present results of numerical experiments, showing the effectiveness of our method.

In the next section, by C we denote a positive constant, independent of the boundary value solution and mesh sizes.

2. Two-rectangle model problem

As a model example we study the elliptic problem, defined on the disjoint rectangles $\Omega_n = (a_n, b_n) \times (c_n, d_n)$ with boundaries $\partial\Omega_n,\ n = 1, 2$, see Fig. 1,

$$-\frac{\partial}{\partial x}\left(p^n\frac{\partial u^n}{\partial x}\right) - \frac{\partial}{\partial y}\left(q^n\frac{\partial u^n}{\partial y}\right) + r^n u^n = f^n, \quad (x, y) \in \Omega_n, \tag{1}$$

$$u^n(x, c_n) = u^n(x, d_n) = 0, \quad a_n \le x \le b_n, \tag{2}$$

$$u^1(a_1, y) = 0, \quad c_1 \le y \le d_1, \qquad u^2(b_2, y) = 0, \quad c_2 \le y \le d_2, \tag{3}$$

$$p^1(b_1, y)\frac{\partial u^1}{\partial x}(b_1, y) + s^1(y)u^1(b_1, y) = \int_{c_2}^{d_2} \varphi^2(y, \eta)u^2(a_2, \eta)\mathrm{d}\eta, \quad c_1 \le y \le d_1, \tag{4}$$

$$-p^2(a_2, y)\frac{\partial u^2}{\partial x}(a_2, y) + s^2(y)u^2(a_2, y) = \int_{c_1}^{d_1} \varphi^1(y, \eta)u^1(b_1, \eta)\mathrm{d}\eta, \quad c_2 \le y \le d_2. \tag{5}$$

Throughout the paper we assume that the input datum satisfy the usual regularity and ellipticity conditions in Ω_n, $n = 1, 2$,

$$p^n, q^n, r^n \in L_\infty(\Omega_n), \tag{6}$$

$$0 < p_0^n \le p^n(x, y), \qquad 0 < q_0^n \le q^n(x, y), \tag{7}$$

$$s^n \in L_\infty(c_n, d_n), \qquad \varphi^n \in L_\infty((c_{3-n}, d_{3-n}) \times (c_n, d_n)). \tag{8}$$

In the real physical problems (see [1,2]) we often have $s^n, \varphi^n > 0,\ n = 1, 2$.

We introduce the product space $L = \{v = (v^1, v^2) \mid v^n \in L_2(\Omega_n),\ n = 1, 2\}$, endowed with the inner product and the associated norm

$$(u, v)_L = (u^1, v^1)_{L_2(\Omega_1)} + (u^2, v^2)_{L_2(\Omega_2)}, \qquad \|v\|_L = (v, v)_L^{1/2}, \quad \text{where}$$
$$(u^n, v^n)_{L_2(\Omega_n)} = \int_{\Omega_n} u^n v^n \mathrm{d}x\mathrm{d}y, \quad n = 1, 2.$$

We also define the spaces $H^m = \{v = (v^1, v^2) \mid v^n \in H^m(\Omega_n)\},\ m = 1, 2, \ldots$ endowed with the inner products and norms

$$(u, v)_{H^m} = (u^1, v^1)_{H^m(\Omega_1)} + (u^2, v^2)_{H^m(\Omega_2)}, \qquad \|v\|_{H^m} = (v, v)_{H^m}^{1/2} \quad (H^0 = L_2), \text{ where}$$
$$(u^n, v^n)_{H^m(\Omega_n)} = \sum_{j=0}^{m} \sum_{l=0}^{j} \left(\frac{\partial^j u^n}{\partial x^l \partial y^{j-l}}, \frac{\partial^j v^n}{\partial x^l \partial y^{j-l}} \right)_{L_2(\Omega_n)}, \quad n = 1, 2,\ m = 1, 2, \ldots.$$

In particular, we set $H_0^1 = \{v = (v^1, v^2) \in H^1 \mid v^n = 0 \text{ on } \Gamma_n, n = 1, 2\},\ m = 1, 2, \ldots$, where $\Gamma_1 = \partial\Omega_1 \setminus \{(b_1, y) \mid y \in (c_1, d_1)\}$ and $\Gamma_2 = \partial\Omega_2 \setminus \{(a_2, y) \mid y \in (c_2, d_2)\}$, see Fig. 1. Finally, with $u = (u^1, u^2)$ and $v = (v^1, v^2)$ we define the following bilinear form:

$$a(u, v) = \mathsf{a}^1(u^1, v^1) + \mathsf{a}^2(u^2, v^2) + \mathsf{b}^1(u^2, v^1) + \mathsf{b}^2(u^1, v^2), \tag{9}$$

where

$$\mathsf{a}^1(u^1, v^1) = \int_{\Omega_1} \left(p^1 \frac{\partial u^1}{\partial x} \frac{\partial v^1}{\partial x} + q^1 \frac{\partial u^1}{\partial y} \frac{\partial v^1}{\partial y} + r^1 u^1 v^1 \right) \mathrm{d}x\,\mathrm{d}y + \int_{c_1}^{d_1} s^1(y) u^1(b_1, y) v^1(b_1, y)\,\mathrm{d}y,$$
$$\mathsf{a}^2(u^2, v^2) = \int_{\Omega_2} \left(p^2 \frac{\partial u^2}{\partial x} \frac{\partial v^2}{\partial x} + q^2 \frac{\partial u^2}{\partial y} \frac{\partial v^2}{\partial y} + r^2 u^2 v^2 \right) \mathrm{d}x\,\mathrm{d}y + \int_{c_2}^{d_2} s^2(y) u^2(a_2, y) v^2(a_2, y)\,\mathrm{d}y,$$
$$\mathsf{b}^1(u^2, v^1) = -\int_{c_1}^{d_1} \int_{c_2}^{d_2} \varphi^2(y, \eta) u^2(a_2, \eta) v^1(b_1, y)\,\mathrm{d}\eta\mathrm{d}y,$$
$$\mathsf{b}^2(u^1, v^2) = -\int_{c_2}^{d_2} \int_{c_1}^{d_1} \varphi^1(y, \eta) u^1(b_1, \eta) v^2(a_2, y)\,\mathrm{d}\eta\mathrm{d}y$$

and the linear form

$$c(v) = \mathsf{c}^2(v^2) + \mathsf{c}^2(v^2), \quad \text{where } \mathsf{c}^n(v^n) = \int_{\Omega_n} f^n v^n \mathrm{d}x\mathrm{d}y,\ n = 1, 2.$$

Lemma 1. *Under the conditions (6) and (8) the bilinear form $a(u, v)$, defined by (9), is bounded on $H^1 \times H^1$. If besides it, the conditions (7) are fulfilled, this form satisfies the Gårding's inequality on H_0^1, i.e. there exist positive constants m and κ such that*

$$a(u, u) + \kappa \|u\|_L^2 \geq m \|u\|_{H^1}^2, \quad \forall u \in H_0^1.$$

Proof. Boundedness of a follows from (6), (8) and the trace theorem

$$\|u_i\|_{L_2(\partial\Omega_i)} \leq C \|u_i\|_{H^1(\Omega_i)}.$$

From (7) and Poincaré inequality we immediately obtain

$$\sum_{i=1}^{2} \int_{\Omega_i} \left[p_i \left(\frac{\partial u_i}{\partial x} \right)^2 + q_i \left(\frac{\partial u_i}{\partial y} \right)^2 \right] \mathrm{d}x\,\mathrm{d}y \geq c_0 \|u\|_{H^1}^2,$$

where c_0 is a computable constant depending on input data. The remainder terms of $a(u, u)$ can be estimated by $\varepsilon \|u\|_{H^1}^2 + \frac{C}{\varepsilon} \|u\|_L^2$ using well known estimate

$$\|u_i\|_{L_2(\partial\Omega_i)} \leq \varepsilon \| \, |\nabla u_i| \, \|_{L_2(\Omega_i)} + \frac{C}{\varepsilon} \|u_i\|_{L_2(\Omega_i)}, \quad \varepsilon > 0.$$

In such a way, result follows for sufficiently small ε. □

The weak form of the problem (1)–(5) is defined as follows: *find* $(u^1, u^2) \in H^1$ *such that*

$$a(u, v) = c(v), \quad \forall v \in H_0^1. \tag{10}$$

Using Theorem 17.11 from [18] we immediately obtain the next assertion.

Theorem 2. *Let the conditions (6)–(8) hold and* $f = (f_1, f_2) \in L$,

$$p^n, q^n(x, y) \in W_\infty^1(\Omega), \qquad s^n \in W_\infty^1(c_n, d_n), \qquad \varphi^n \in W_\infty^1((c_{3-n}, d_{3-n}) \times (c_n, d_n)),$$

for $n = 1, 2$. *Let* 0 *is not the eigenvalue of the corresponding spectral problem for (1)–(5). Then there exists the unique solution* $u \in H_0^1 \cap H^2$ *of the problem (1)–(5) and a priori estimate* $\|u\|_{H^2} \le C\|f\|_{L_2}$ *holds.*

3. Finite element method solution

The discretization of $\Omega_1 \cup \Omega_2$ is generated by the uniform mesh $\overline{\omega}^h = \overline{\omega}_1^h \cup \overline{\omega}_2^h$,

$$\overline{\omega}_n^h = \{(x_i, y_j) : x_i = a_n + (i-1)h_n, i = 1, \ldots, N_n, x_{N_n} = b_n,$$
$$y_j = c_n + (j-1)k_n, j = 1, \ldots, M_n, y_{M_n} = d_n\}, \quad n = 1, 2.$$

Now, we consider a standard piecewise linear finite element space $V_h = V_h^1 \times V_h^2 \subset H_0^1$, associated with $\overline{\omega}^h$. Namely, we use linear triangular elements, see Fig. . With E_s^{ij}, $s = 1, \ldots, 6$ we denote the sixth elements, associated with the grid node (x_i, y_j). Let $\{\Phi_h^n\}$ is a basis of V_h^n, $n = 1, 2$, $\Phi_h = (\Phi_h^1, \Phi_h^2)$. We seek the finite element solution $u_h = (u_h^1, u_h^2)$, $u_h^n \in V_h^n$, $n = 1, 2$, in the form

$$u_h^1(x, y) = \sum_{i=1}^{N_1} \sum_{j=1}^{M_1} U_{i,j}^1 \Phi_{h_{i,j}}^1(x, y), \qquad u_h^2(x, y) = \sum_{i=1}^{N_2} \sum_{j=1}^{M_2} U_{i,j}^2 \Phi_{h_{i,j}}^2(x, y). \tag{11}$$

Then the finite element approximation of problem (9) is defined as follows: *Find* $u_h \in V_h$ *such that*

$$a(u_h, v_h) = c(v_h), \quad \forall v_h \in V_h. \tag{12}$$

The error analysis of the above finite element discretization can be achieved by standard techniques, see e.g. [19].

Theorem 3. *Under the assumptions in Theorem 2, solution* u_h *of the finite element scheme (12) satisfies error estimate*

$$\|u - u_h\|_{H^s} \le Ch^{2-s}\|u\|_{H^2}, \quad s = 0, 1.$$

Proof. From (10) and (12) follows

$$a(u - u_h, u - u_h) = a(u - u_h, u - v_h), \quad \forall v_h \in V_h,$$

whereby, using Lemma 1

$$\|u - u_h\|_{H^1}^2 \le C_1\|u - u_h\|_{H^1} \inf_{v_h \in V_h} \|u - v_h\|_{H^1} + C_2\|u - u_h\|_L^2. \tag{13}$$

Further we have

$$\|u - u_h\|_L = \sup_{g \in L} \frac{(u - u_h, g)_L}{\|g\|_L}. \tag{14}$$

For given $g \in L$ we determine $w_g \in H_0^1$ such that:

$$a(v, w_g) = (v, g)_L, \quad \forall v \in H_0^1. \tag{15}$$

Hence, for all $w_h \in V_h$ we have

$$(u - u_h, g)_L = a(u - u_h, w_g) = a(u - u_h, w_g - w_h) \le C\|u - u_h\|_{H^1} \|w_g - w_h\|_{H^1},$$

whereby follows

$$\|u - u_h\|_L \le \sup_{g \in L} \left(\frac{C}{\|g\|_L} \|u - u_h\|_{H^1} \inf_{w_h \in V_h} \|w_g - w_h\|_{H^1} \right). \tag{16}$$

Using approximation properties of finite element spaces [19] and Theorem 2 we obtain

$$\inf_{w_h \in V_h} \|w_g - w_h\|_{H^1} \le Ch\|w_g\|_{H^2} \le Ch\|g\|_L. \tag{17}$$

From (16) and (17) follows

$$\|u - u_h\|_L \le Ch\|u - u_h\|_{H^1} \tag{18}$$

whereby using (13) for sufficiently small h we obtain

$$\|u - u_h\|_{H^1} \le C \inf_{v_h \in V_h} \|u - v_h\|_{H^1} \le Ch\|u\|_{H^2}. \tag{19}$$

Error estimate in H^0-norm immediately follows from (18) and (19). □

The finite element problem (12) reduces to a system of linear equations with unknowns $U^n_{i,j}$.

In practice we lump the mass matrix by nodal-point integration, that is we get nonzero contributions only for diagonal elements of the local (element) mass matrix. In fact mass-lumping removes the extra terms and simplify the computations. For right hand side of (12) we use the product approximation formula. With integrals of known function φ and unknown solution U we deal as follows: for $n = 1, 2$

$$\int_{c_n}^{d_n} \varphi^n(\cdot,\eta)U^n(\cdot,\eta)\mathrm{d}\eta \approx \frac{1}{2}\sum_{l=1}^{M_n-1}[U^n(\cdot,y_l)+U^n(\cdot,y_{l+1})]\int_{y_l}^{y_{l+1}}\varphi^n(\cdot,\eta)\mathrm{d}\eta. \tag{20}$$

The remaining integral can be computed exactly, depending on the function φ^n. We choose approximation preserving the second order of convergence rate.

In such a way, instead of (12) we obtain perturbed problem:

$$a_h(u_h, v_h) = c_h(v_h), \quad \forall v_h \in V_h \tag{21}$$

where the bilinear form a_h and linear form c_h are obtained from a and c using numerical integration. The resulting discrete approximation of (1)–(5) is as follows

$$AU = (A^1U, A^2U) = F = (F^1, F^2), \tag{22}$$

where

$$\begin{aligned}
(A^nU)_{i,j} &= -\frac{\hat P^n_{i,j}}{h_n^2}U^n_{i+1,j} - \frac{\check P^n_{i,j}}{h_n^2}U^n_{i-1,j} + \left(\frac{\hat P^n+\check P^n}{h_n^2}+\frac{\hat Q^n+\check Q^n}{k_n^2}+r^n\right)_{i,j}U^n_{i,j}\\
&\quad -\frac{\hat Q^n_{i,j}}{k_n^2}U^n_{i,j+1} - \frac{\check Q^n_{i,j}}{k_n^2}U^n_{i,j-1}, \quad n=1,2,\ i=2,\dots,N_n-1,\ j=2,\dots,M_n-1,\\
(A^1U)_{N_1,j} &= -\frac{2\check P^1_{N_1,j}}{h_1^2}U^1_{N_1-1,j} + \left(\frac{2\check P^1}{h_1^2}+\frac{\hat Q^1+\check Q^1}{k_1^2}+r^1+\frac{2s^1}{h_1}\right)_{N_1,j}U^1_{N_1,j}\\
&\quad -\frac{\hat Q^1_{N_1,j}}{k_1^2}U^1_{N_1,j+1} - \frac{\check Q^1_{N_1,j}}{k_1^2}U^1_{N_1,j-1} - \frac{2}{h_1}\sum_{l=2}^{M_2-1}\widetilde\varphi^2_{j,l}U^2_{1,l}, \quad j=2,\dots,M_1-1,\\
(A^2U)_{1,j} &= -\frac{2\hat P^2_{1,j}}{h_2^2}U^2_{2,j} + \left(\frac{2\check P^2}{h_2^2}+\frac{\hat Q^2+\check Q^2}{k_2^2}+r^2+\frac{2s^2}{h_2}\right)_{1,j}U^2_{1,j} - \frac{\hat Q^2_{1,j}}{k_2^2}U^2_{1,j+1}\\
&\quad -\frac{\check Q^2_{1,j}}{k_2^2}U^2_{1,j-1} - \frac{2}{h_2}\sum_{l=2}^{M_1-1}\widetilde\varphi^1_{j,l}U^1_{N_1,l}, \quad j=2,\dots,M_2-1,\\
U^n_{i,1} &= U^1_{1,j} = U^2_{N_2,j} = U^n_{i,M_n} = 0, \quad n=1,2,\ i=1,\dots,N_n,\ j=1,\dots,M_n,
\end{aligned}$$

$U^n_{i,j} = u^n_h(x_i,y_j)$, $F^n_{i,j} = f^n(x_i,y_j)$, and the coefficients $\hat P^n$, $\check P^n$, $\hat Q^n$, $\check Q^n$, r^n, s^n, $\widetilde\varphi^n$, f^n, $n = 1, 2$, can be computed exactly or numerically. For example,

$$\hat P^n_{i,j} = \left\{\begin{array}{lll}
\dfrac{1}{h_nk_n}\left(\displaystyle\int_{E_1^{ij}}+\int_{E_6^{ij}}\right)p^n\mathrm{d}x\mathrm{d}y, & \text{exact} & \dfrac{1}{h_nk_n}\left(\displaystyle\int_{E_5^{ij}}+\int_{E_6^{ij}}\right)q^n\mathrm{d}x\mathrm{d}y\\
p^n_{i+\frac12,j}, & \text{midpont approx.} & q^n_{i,j-\frac12}\\
\dfrac12(p^n_{i,j}+p^n_{i+1,j}), & \text{trapezoidal approx.} & \dfrac12(q^n_{i,j}+q^n_{i,j-1})
\end{array}\right\} = \check Q^n_{i,j},$$

$$\check P^n_{i,j} = \hat P^n_{i-1,j}, \qquad \hat Q^n_{i,j} = \check Q^n_{i,j+1}, \qquad r^n_{i,j} = r^n(x_i,y_j), \qquad s^n_j = s^n(y_j),$$

and $\widetilde\varphi^n_{j,l}$ is equal to:

$$\frac12\int_{y_{l-1}}^{y_{l+1}}\varphi^n(y_j,\xi)\mathrm{d}\xi \quad \text{or} \quad \frac{k_n}{2}\left(\varphi^n_{j,l+\frac12}+\varphi^n_{j,l-\frac12}\right) \quad \text{or} \quad \frac{k_n}{4}\left(\varphi^n_{j,l+1}+2\varphi^n_{j,l}+\varphi^n_{j,l-1}\right),$$

for exact computation, midpoint or trapezoidal approximation, respectively.

If the input data of (1)–(5) are sufficiently smooth, the finite element scheme with numerical integration (21) keeps the same order of convergence as the scheme with exact integration (12).

Theorem 4. *Let the assumptions of Theorem 3 hold and let* $f^n, p^n, q^n, r^n \in C^1(\bar{\Omega}_n)$, $s^n \in C^1[c_n, d_n]$, $\varphi^n \in C^1([c_{3-n}, d_{3-n}] \times [c_n, d_n])$, $n = 1, 2$. *Then the solution* u_h *of (21) satisfies error estimate*

$$\|u - u_h\|_{H^1} \le Ch.$$

Proof. From (10) and (21), analogously as in the proof of Theorem 3, one obtains

$$\begin{aligned} a(u - u_h, u - u_h) &= a(u - u_h, u - v_h) + [c(v_h - u_h) - c_h(v_h - u_h)] \\ &\quad - [a(u_h, v_h - u_h) - a_h(u_h, v_h - u_h)], \quad \forall v_h \in V_h. \end{aligned}$$

From this, using Lemma 1 we have

$$\|u - u_h\|_{H^1}^2 \le C_1\|u - u_h\|_{H^1} \inf_{v_h \in V_h} \|u - v_h\|_{H^1} + C_2\|u - u_h\|_L^2 + C_3(\alpha + \beta)\left(\|u - u_h\|_{H^1} + \inf_{v_h \in V_h} \|u - v_h\|_{H^1}\right), \quad (23)$$

where denoted

$$\alpha = \|c - c_h\|^* = \sup_{w_h \in V_h} \frac{c(w_h) - c_h(w_h)}{\|w_h\|_{H^1}}, \qquad \beta = \|a(u_h, \cdot) - a_h(u_h, \cdot)\|^*.$$

Further, from (23) follows

$$\|u - u_h\|_{H^1}^2 \le C\left(\inf_{v_h \in V_h} \|u - v_h\|_{H^1}^2 + \alpha^2 + \beta^2 + \|u - u_h\|_L^2\right). \quad (24)$$

Let $g \in L$ and let $w_g \in H_0^1 \cap H^2$ satisfies (15). Hence,

$$\begin{aligned} (u - u_h, g)_L &= a(u - u_h, w_g - w_h) + [c(w_h) - c_h(w_h)] - [a(u_h, w_h) - a_h(u_h, w_h)] \\ &\le C\|u - u_h\|_{H^1}\|w_g - w_h\|_{H^1} + (\alpha + \beta)\left(\|w_h - w_g\|_{H^1} + \|w_g\|_{H^1}\right). \end{aligned}$$

Taking infimum on $w_h \in V_h$ and using representation (14) we have

$$\|u - u_h\|_L \le C\big(h\|u - u_h\|_{H^1} + \alpha + \beta\big). \quad (25)$$

By direct estimation we obtain

$$\alpha \le Ch \sum_{n=1}^{2} \|f^n\|_{C^1(\bar{\Omega}_n)} \quad (26)$$

and

$$\beta \le Ch \sum_{n=1}^{2} \Big(\|p^n\|_{C^1(\bar{\Omega}_n)} + \|q^n\|_{C^1(\bar{\Omega}_n)} + \|r^n\|_{C^1(\bar{\Omega}_n)} + \|s^n\|_{C^1[c_n,d_n]} + \|\varphi^n\|_{C^1([c_{3-n},d_{3-n}]\times[c_n,d_n])}\Big). \quad (27)$$

Result follows from (24)–(27). □

Remark 1. If $f^n, p^n, q^n, r^n \in C^2(\bar{\Omega}_n)$, $\varphi^n \in C^2([c_{3-n}, d_{3-n}] \times [c_n, d_n])$, $s^n \in C^2[c_n, d_n]$, $n = 1, 2$, then $\alpha, \beta = O(h^2)$, whereby

$$\|u - u_h\|_{H^0} \le Ch^2.$$

Let us introduce the space L_h of discrete functions $U = (U^1, U^2)$, $U^n = \{U_{ij}^n \mid i = 1, \ldots, N_n,\ j = 1, \ldots, M_1\}$ satisfying $U_{i,1}^n = U_{1,j}^1 = U_{N_2,j}^2 = U_{i,M_n}^n = 0$. We define the following inner product

$$(U, W)_{L_h} = \sum_{i=2}^{N_1} \sum_{j=2}^{M_1-1} U_{ij}^1 W_{ij}^1 \hbar_{1,i} k_1 + \sum_{i=1}^{N_2-1} \sum_{j=2}^{M_2-1} U_{ij}^2 W_{ij}^2 \hbar_{2,i} k_2,$$

where denoted $\hbar_{1,N_1} = h_1/2$, $\hbar_{2,1} = h_2/2$ and $\hbar_{n,i} = h_n$ for $n = 1, 2$, $i = 2, \ldots, N_n - 1$. We also define discrete norms

$$\|U\|_{L_h} = (U, U)_{L_h}^{1/2}$$

and

$$\|U\|_{H_h^1}^2 = \|U\|_{L_h}^2 + \sum_{n=1}^{2} \left[\sum_{i=1}^{N_n-1} \sum_{j=2}^{M_n-1} (D_x^+ U_{ij}^n)^2 h_n k_n + \sum_{i=3-n}^{N_n+1-n} \sum_{j=1}^{M_n-1} (D_y^+ U_{ij}^n)^2 \hbar_{n,i} k_n\right],$$

where D_x^+ and D_y^+ are upward finite differences:

$$D_x^+ U_{ij}^n = \frac{U_{i+1,j}^n - U_{i,j}^n}{h_n} = D_x^- U_{i+1,j}^n, \qquad D_y^+ U_{ij}^n = \frac{U_{i,j+1}^n - U_{i,j}^n}{k_n} = D_y^- U_{i,j+1}^n.$$

The following analogue of Lemma 1 holds.

Lemma 5. *Let the functions p^n, q^n, r^n, s^n and φ^n, $n = 1, 2$ be continuous. Then the bilinear form $a_h(u, v)$ is bounded on $H_h^1 \times H_h^1$. If besides it, the conditions (7) are fulfilled, this form satisfies the Gårding's inequality on $L_h \cap H_h^1$, i.e. there exist positive constants m and κ such that*

$$a_h(U, U) + \kappa \|U\|_{L_h}^2 \geq m \|U\|_{H_h^1}^2, \quad \forall U \in L_h \cap H_h^1.$$

By direct computation (comp. [20]) one verifies the next assertion.

Lemma 6. *For $v_h \in V_h$*

$$\|v_h\|_{L_h} \asymp \|v_h\|_L \quad \text{and} \quad \|v_h\|_{H_h^1} \asymp \|v_h\|_{H^1}.$$

Now we are ready to prove the following superconvergence result for finite element scheme (21)–(22).

Theorem 7. *Let the assumptions of Theorem 3 hold and let $p^n, q^n \in C^3(\bar{\Omega}_n)$, $f^n, r^n \in C^2(\bar{\Omega}_n)$, $s^n \in C^2[c_n, d_n]$, $\varphi^n \in C^2([c_{3-n}, d_{3-n}] \times [c_n, d_n])$, $u^n \in C^4(\bar{\Omega}_n)$, $n = 1, 2$, where $u = (u^1, u^2)$ is the solution of (1)–(5). Then the solution u_h of (21)–(22) satisfies error estimate*

$$\|u - u_h\|_{H_h^1} \leq Ch^2.$$

Proof. Let $u = (u^1, u^2)$ be the solution of (1)–(5) and $u_h = (u_h^1, u_h^2)$—the solution of finite element scheme (21)–(22). Then the error $z = (z^1, z^2) = u - u_h$ satisfies equation

$$AZ = \Psi, \tag{28}$$

where $Z_{ij}^n = z^n(x_i, y_j)$, $\Psi^n = \tilde{\Psi}^n + \bar{\Psi}^n$, $n = 1, 2$, and

$$\tilde{\Psi}_{ij}^n = \left[\frac{\partial}{\partial x}\left(P^n \frac{\partial u^n}{\partial x}\right) - D_x^-(\hat{P}^n D_x^+ u^n) + \frac{\partial}{\partial y}\left(q^n \frac{\partial u^n}{\partial y}\right) - D_y^-(\hat{Q}^n D_y^+ u^n) \right]_{(x_i, y_j)},$$
$$n = 1, 2,\ i = 2, \ldots, N_n - 1,\ j = 2, \ldots, M_n - 1,$$
$$\tilde{\Psi}_{N_1,j}^1 = 0, \quad j = 2, \ldots, M_n - 1,$$
$$\tilde{\Psi}_{1,j}^2 = 0, \quad j = 2, \ldots, M_n - 1,$$
$$\bar{\Psi}_{ij}^n = 0, \quad n = 1, 2,\ i = 2, \ldots, N_n - 1,\ j = 2, \ldots, M_n - 1,$$
$$\bar{\Psi}_{N_1,j}^1 = \left[\frac{\partial}{\partial x}\left(p^1 \frac{\partial u^1}{\partial x}\right) + \frac{2}{h_1}\left(\check{P}^1 D_x^- u^1 - p^1 \frac{\partial u^1}{\partial x}\right) + \frac{\partial}{\partial y}\left(q^1 \frac{\partial u^1}{\partial y}\right) - D_y^-(\hat{Q}^1 D_y^+ u^1) \right]_{(x_{N_1}, y_j)}$$
$$+ \frac{2}{h_1} \int_{c_2}^{d_2} \varphi^2(y_j, \eta) u^2(a_2, \eta) \mathrm{d}\eta - \frac{2}{h_1} \sum_{l=2}^{M_2-2} \tilde{\varphi}_{jl}^2 u^2(a_2, y_j), \quad j = 2, \ldots, M_1 - 1,$$
$$\bar{\Psi}_{1,j}^2 = \left[\frac{\partial}{\partial x}\left(p^2 \frac{\partial u^2}{\partial x}\right) - \frac{2}{h_2}\left(\hat{P}^2 D_x^+ u^2 - p^2 \frac{\partial u^2}{\partial x}\right) + \frac{\partial}{\partial y}\left(q^2 \frac{\partial u^2}{\partial y}\right) - D_y^-(\hat{Q}^2 D_y^+ u^2) \right]_{(x_1, y_j)}$$
$$+ \frac{2}{h_2} \int_{c_1}^{d_1} \varphi^1(y_j, \eta) u^1(b_1, \eta) \mathrm{d}\eta - \frac{2}{h_2} \sum_{l=2}^{M_1-2} \tilde{\varphi}_{jl}^1 u^1(b_1, y_j), \quad j = 2, \ldots, M_2 - 1.$$

From (28) follows

$$a_h(Z, Z) = (AZ, Z)_{L_h} = (\Psi, Z)_{L_h} \leq \|\hat{\Psi}\|_{L_h}^2 + \|Z\|_{L_h}^2 + \varepsilon \left[\sum_{j=2}^{M_1-1} \left(Z_{N_1,j}^1\right)^2 k_1 + \sum_{j=2}^{M_2-1} \left(Z_{1,j}^2\right)^2 k_2 \right]$$
$$+ \frac{1}{16\varepsilon} \left[\sum_{j=2}^{M_1-1} \left(\bar{\Psi}_{N_1,j}^1\right)^2 h_1^2 k_1 + \sum_{j=2}^{M_2-1} \left(\bar{\Psi}_{1,j}^2\right)^2 h_2^2 k_2 \right],$$

hence, for sufficiently small ε, using discrete Poincaré inequality and Lemma 5:

$$\|Z\|^2_{H^1_h} \le C\left[\|\tilde{\Psi}\|^2_{L_h} + \sum_{j=2}^{M_1-1}\left(\bar{\Psi}^1_{N_1,j}\right)^2 h_1^2 k_1 + \sum_{j=2}^{M_2-1}\left(\bar{\Psi}^2_{1,j}\right)^2 h_2^2 k_2 + \|Z\|^2_{L_h}\right]. \tag{29}$$

Using Taylor expansion we easily obtain

$$\|\tilde{\Psi}\|_{L_h}, \left[\sum_{j=2}^{M_1-1}\left(\bar{\Psi}^1_{N_1,j}\right)^2 h_1^2 k_1 + \sum_{j=2}^{M_2-1}\left(\bar{\Psi}^2_{1,j}\right)^2 h_2^2 k_2\right]^{1/2} = O(h^2). \tag{30}$$

From Lemma 6 follows

$$\|Z\|_{L_h} = \|u - u_h\|_{L_h} \le C\|I_h u - u_h\|_L,$$

where I_h is interpolation operator [19] from H^1 into V_h. Using Remark 1 and interpolation properties of finite element space V_h (see [19]) from here we have

$$\|Z\|_{L_h} \le C\left(\|u - u_h\|_L + \|I_h u - u\|_L\right) = O(h^2). \tag{31}$$

The result follows from (29)–(31). □

4. Two-grid FEM decoupling algorithms

Let define a new mesh (*coarse mesh*) $\overline{\omega}^H = \overline{\omega}^H_1 \cup \overline{\omega}^H_2$, analogically to $\overline{\omega}^h$ with mesh steps: $H_n > h_n$ and $K_n > k_n$, $n = 1, 2$. The mesh $\overline{\omega}^h$ we will call *fine mesh*. As before, we consider a standard piecewise finite element space V_H (with basis Φ_H), associated with $\overline{\omega}^H$. Let $\mathbf{U}^n_i = [U^n(x_i, y_1), U^n(x_i, y_2), \ldots, U^n(x_i, y_{M_n-1}), U^n(x_i, M_n)]^T$, $n = 1, 2$, $i = 1, \ldots, N_n$.

Algorithm 1

step 1. Find $\mathbf{U}^2_1$ from (22) on the coarse mesh $\overline{\omega}^H$.

step 2. For all $\Phi_h \in V_h$ on the fine mesh $\overline{\omega}^h$:

(a) Find $\mathring{u}^1_h \in V^1_h$ such that $\mathsf{a}^1(\mathring{u}^1_h, \Phi^1_h) = \mathsf{c}^1(\Phi^1_h) - \mathsf{b}^1(u^2_H, \Phi^1_h)$.

(b) Find $\mathring{u}^2_h \in V^2_h$ such that $\mathsf{a}^2(\mathring{u}^2_h, \Phi^2_h) = \mathsf{c}^2(\Phi^2_h) - \mathsf{b}^2(\mathring{u}^1_h, \Phi^2_h)$.

Note that at step 2 the problems in Ω_1 and Ω_2 are computed consecutively and separately. Now, Algorithm 1 can be improved in a successive fashion, organizing an iteration process, such that at each step to account the last computed values of $\mathbf{U}^1_{N_1}$ and $\mathbf{U}^2_1$.

Algorithm 2

step 1. Find $\mathring{u}^n_h$, $n = 1, 2$ from Algorithm 1, $\mathring{u}^{n(1)}_h := \mathring{u}^n_h$

step 2. For $k = 2, 3, \ldots$ and all $\Phi_h \in V_h$ on the fine mesh $\overline{\omega}^h$:

(a) Find $\mathring{u}^{1(k)}_h \in V^1_h$: $\mathsf{a}^1(\mathring{u}^{1(k)}_h, \Phi^1_h) = \mathsf{c}^1(\Phi^1_h) - \mathsf{b}^1(\mathring{u}^{2(k-1)}_h, \Phi^1_h)$.

(b) Find $\mathring{u}^{2(k)}_h \in V^2_h$: $\mathsf{a}^2(\mathring{u}^{2(k)}_h, \Phi^2_h) = \mathsf{c}^2(\Phi^2_h) - \mathsf{b}^2(\mathring{u}^{1(k)}_h, \Phi^2_h)$.

The proofs of the next theorems are based on the ideas in [10].

Theorem 8. *Under the assumptions in Theorem 2, $\mathring{u}^n_h$ for $n = 1, 2$, computed with Algorithm 1 has the following error estimate*

$$\|u^n - \mathring{u}^n_h\|_{H^1_h} \le C(h_n^2 + k_n^2 + H_n^3 + K_n^3), \quad n = 1, 2.$$

The next theorem shows that $\mathring{u}^{n(k)}_h$, $k = 1, 2, \ldots$ can reach the optimal accuracy in H^1-norm, if the coarse mesh step size H_n (K_n) is taken to be $h_n^{2/(k+2)}$ ($k_n^{2/(k+2)}$). As the dimension V_h is *much smaller* than V_H, the efficiency of the algorithms is then evident.

Theorem 9. *Under the assumptions in Theorem 2, $\mathring{u}^{n(k)}_h$ for $n = 1, 2$, computed with Algorithm 2, has the following error estimate*

$$\|u^n_h - \mathring{u}^{n(k)}_h\|_{H^1_h} \le C(H_n^{k+2} + K_n^{k+2}), \quad n = 1, 2, \ k \ge 0.$$

Consequently,

$$\|u^n - \mathring{u}^{n(k)}_h\|_{H^1_h} \le C(h_n^2 + k_n^2 + H_n^{k+2} + K_n^{k+2}), \quad n = 1, 2.$$

Namely, $\mathring{u}^{n(k)}_h$, $k \ge 1$ has the same accuracy as $\mathring{u}^n_h$ in H^1-norm, if $H_n = h_n^{2/(k+2)}$ and $K_n = k_n^{2/(k+2)}$, $n = 1, 2$.

Table 1
Error and CR in different norms, standard one-grid procedure.

Mesh		Ω_1			Ω_2		
H	K	max	L^2	H^1	max	L^2	H^1
1/4	1/5	1.81	8.61e−1	4.55	2.57e−1	1.30e−1	5.93e−1
1/8	1/10	2.04e−1	7.34e−2	4.27e−1	1.32e−1	5.61e−2	6.13e−1
1/16	1/20	3.23e−2	8.08e−3	1.77e−1	3.13e−2	1.24e−2	1.54e−1
		(***2.6545***)	(***3.1825***)	(***1.2707***)	(***2.0802***)	(***2.1730***)	(***1.9941***)
1/32	1/40	7.33e−3	1.77e−3	5.05e−2	7.72e−3	3.00e−3	4.18e−2
		(***2.1414***)	(***2.1921***)	(***1.8078***)	(***2.0175***)	(***2.0562***)	(***1.8785***)
1/64	1/80	1.79e−3	4.25e−4	1.33e−2	1.92e−3	7.37e−4	1.08e−2
		(***2.0326***)	(***2.0566***)	(***1.9275***)	(***2.0063***)	(***2.0208***)	(***1.9481***)
1/128	1/160	4.45e−4	1.05e−4	3.46e−3	4.80e−4	1.83e−4	2.75e−3
		(***2.0085***)	(***2.0189***)	(***1.9417***)	(***2.0004***)	(***2.0086***)	(***1.9817***)

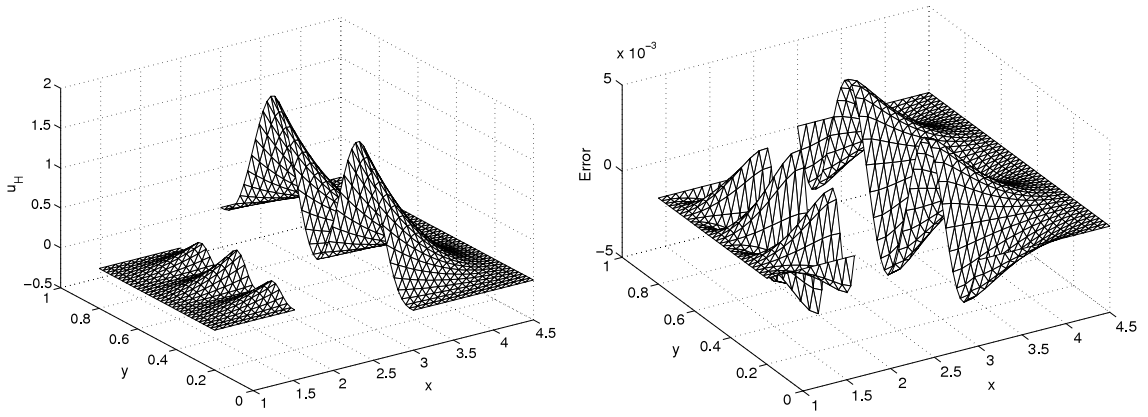


Fig. 2. Numerical solution (left) and error (right) for $H = 0.1$, $K = 0.02$, Algorithm 1, step 1.

5. Numerical examples

The test example is problem (1)–(5), with $a_1 = 1, b_1 = 2, c_1 = 0.2, d_1 = 0.8, a_2 = 3, b_2 = 4.5, c_2 = 0, d_2 = 1$ and zero Dirichlet boundary conditions. The coefficients are:

$$p^1(x,y) = e^{x+y}, \quad q^1(x,y) = \sin(x+y), \quad r^1(x,y) = x+y, \quad s^1(y) = 2b_1 y,$$
$$p^2(x,y) = x^2+y^2, \quad q^2(x,y) = x(1+y), \quad r^2(x,y) = x-y, \quad s^2(y) = 2(a_2+y),$$
$$\varphi^1(y,\eta) = (2+\eta^2)(2+y^2), \quad \varphi^2(y,\eta) = (1+\eta)(1+y).$$

In the right hand side of Eqs. (4), (5) we determine $f^n(x,y)$, $n = 1, 2$, in such a manner that $u = (u^1, u^2)$,

$$u^1(x,y) = 2[\cos(10\pi y - \pi) + 1](x-a_1)^2(y-c_1)(d_1-y),$$
$$u^2(x,y) = 2[\cos(4\pi y - \pi) + 1](x-b_2)^2(y-c_2)(d_2-y),$$

is the exact solution of the problem (1)–(5). In the examples, the coefficients $\hat{P}$, $\check{P}$, $\hat{Q}$ and $\check{Q}$ are computed using midpoint approximation, while for the right hand side we use trapezoidal rule approximation. Mesh parameters are $H_1 = H_2 = H, K_1 = K_2 = K$ for the coarse mesh and $h_1 = h_2 = \mathbf{h}$, $k_1 = k_2 = \mathbf{k}$ for the fine mesh. The results are given in H^1 ($\|\cdot\|_{H_h^1}$), L^2 and max discrete norms and the convergence rate (CR) is calculated using double mesh principle: $E^{\mathbf{h}} = \|u_h - u\|$, CR $= \log_2[E^{\mathbf{h}}/E^{\frac{\mathbf{h}}{2}}]$. All computations are performed by MATLAB 7.10. The generated linear systems of algebraic equations are solved, using MATLAB libraries function 'mldivide'.

Example 1 (*One-Grid Procedure*)**.** We compute the FEM solution (22) for the test problem on uniform mesh $\overline{\omega}^H$, i.e. we apply *only* step 1 of Algorithm 1. Errors and convergence rates in corresponding discrete norms are given in Table 1. In Fig. 2 numerical solution and error are plotted for $H = 0.1$, $K = 0.02$.

Example 2 (*Algorithm 1*)**.** We chose $\mathbf{h} = H^{3/2}$ and $\mathbf{k} = K^{3/2}$, in order to test the accuracy of the solution, computed with Algorithm 1. For the solution $u_H^2(a_2, y_j)$ at step 2a we use a linear interpolation. The results listed in Table 2, show that $\|\mathring{u}_h - u\|_{H_h^1} \approx O(H^3 + K^3) = O(H^3 + K^3 + \mathbf{h}^2 + \mathbf{k}^2)$, i.e. agree with the assertion of Theorem 8.

Table 2
Error and CR in different norms $\|\mathring{u}_h^n - u^n\|$, $\mathbf{h} = H^{\frac{3}{2}}$, $\mathbf{k} = K^{\frac{3}{2}}$, Algorithm 1.

Mesh		Ω_1			Ω_2		
H	K	max	L^2	H^1	max	L^2	H^1
1/4	1/5	8.19e−2	2.96e−2	6.17e−1	1.08e−1	4.40e−2	6.07e−1
1/8	1/10	1.75e−2	4.10e−3	9.76e−2	1.29e−2	4.88e−3	7.40e−2
		(***2.2233***)	(***2.8492***)	(***2.6598***)	(***3.0619***)	(***3.1716***)	(***3.0371***)
1/16	1/20	1.44e−3	3.43e−4	1.18e−2	1.54e−3	5.89e−4	8.94e−3
		(***3.6022***)	(***3.5803***)	(***3.0429***)	(***3.0712***)	(***3.0508***)	(***3.0489***)
1/32	1/40	2.18e−4	4.34e−5	1.49e−3	1.94e−4	7.31e−5	1.12e−3
		(***2.7309***)	(***2.9825***)	(***2.9885***)	(***2.9878***)	(***3.0093***)	(***3.0024***)

Table 3
Error and CR in different norms for $\mathbf{h} \leq H^{\frac{\mathbf{k}+2}{2}}$, $\mathbf{k} \leq K^{\frac{\mathbf{k}+2}{2}}$, Algorithm 2.

Mesh		Ω_1			Ω_2		
H	K	max	L^2	H^1	max	L^2	H^1
		$k = 2$, $\mathbf{h} \simeq H^2$, $\mathbf{k} \simeq K^2-$ fixed for all fine mesh iterations					
1/4	1/5	1.85e−2	4.91e−3	1.58e−1	1.93e−2	7.89e−3	1.18e−1
1/8	1/10	1.14e−3	2.71e−4	9.43e−3	1.23e−3	4.72e−4	7.18e−3
		(***4.0201***)	(***4.1785***)	(***4.0660***)	(***3.9767***)	(***4.0643***)	(***4.0429***)
1/16	1/20	6.96e−5	1.67e−5	5.84e−4	7.69e−5	2.92e−5	4.46e−4
		(***4.0320***)	(***4.0211***)	(***4.0148***)	(***3.9973***)	(***4.0118***)	(***4.0103***)
		$k = 3$, $\mathbf{h} \simeq H^{5/2}$, $\mathbf{k} \simeq K^{5/2}-$fixed for all fine mesh iterations					
1/4	1/5	3.56e−3	8.95e−4	3.08e−2	3.96e−3	1.53e−3	2.32e−2
1/8	1/10	1.84e−4	3.31e−5	9.69e−4	1.21e−4	4.68e−5	7.13e−4
		(***4.2688***)	(***4.7576***)	(***4.9920***)	(***5.0296***)	(***5.0283***)	(***5.0264***)
		$k = 4$, $\mathbf{h} \simeq H^3$, $\mathbf{k} \simeq K^3-$fixed for all fine mesh iterations					
1/4	1/5	7.14e−4	1.73e−4	6.03e−3	7.84e−4	3.02e−4	4.60e−3
		$k = 5$, $\mathbf{h} \simeq H^{7/2}$, $\mathbf{k} \simeq K^{7/2}-$fixed for all fine mesh iterations					
1/4	1/5	1.41e−4	3.43e−5	1.20e−3	1.57e−4	6.01e−5	9.17e−4

Table 4
Error and CR in different norms for $\mathbf{h} \leq H^{\frac{\mathbf{k}+2}{2}}$, $\mathbf{k} \leq K^{\frac{\mathbf{k}+2}{2}}$, Algorithm 2.

Mesh		Ω_1			Ω_2		
H	K	max	L^2	H^1	max	L^2	H^1
		$k = 2$					
1/4	1/5	1.73e−2	4.93e−3	1.58e−1	1.92e−2	7.89e−3	1.18e−1
1/8	1/10	1.21e−3	2.76e−4	9.45e−3	1.24e−3	4.71e−4	7.18e−3
		(***3.8376***)	(***4.1609***)	(***4.0638***)	(***3.9560***)	(***4.0653***)	(***4.0427***)
1/16	1/20	7.24e−5	1.68e−5	5.84e−4	7.72e−5	2.92e−5	4.46e−4
		(***4.0635***)	(***4.0369***)	(***4.0159***)	(***4.0051***)	(***4.0114***)	(***4.0104***)
		$k = 3$					
1/4	1/5	3.52e−3	8.96e−4	3.08e−2	3.97e−3	1.53e−3	2.32e−2
1/8	1/10	1.78e−4	3.20e−5	9.68e−4	1.22e−4	4.68e−5	7.13e−4
		(***4.3046***)	(***4.8100***)	(***4.9937***)	(***5.0255***)	(***5.0286***)	(***5.0264***)

Example 3 (*Algorithm 2*)**.** Here we will demonstrate the efficiency of Algorithm 2.

- *Two-grid computations*. In order to check the convergence rate, each $\mathring{u}_h^{n(k)}$, $n = 1, 2$ are computed with fixed fine mesh step size $\mathbf{h} = H^{(k+2)/2}$, $\mathbf{k} = K^{(k+2)/2}$ for all $1, \ldots, k$ iterations. The results, errors in different discrete norms and convergence rate, are shown in Table 3. We can see that $\|\mathring{u}_h^{n(k)} - u^n\|_{H_h^1} \approx O(\mathbf{h}^2 + H^{k+2})$, which agree with the statement of Theorem 9.
- *Multi-grid computations*. Algorithm 2 can be performed in more efficient manner. Now, at each iteration we use a different fine mesh step sizes, determined such that at k-th iteration $\mathbf{h} = H^{(k+2)/2}$ and $\mathbf{k} = K^{(k+2)/2}$, $k = 1, 2, \ldots$. For example, if we compute up to $k = 5$, the mesh step sizes are for $k = 1 : \mathbf{h} = H^{3/2}$, $\mathbf{k} = K^{3/2}$, for $k = 2 : \mathbf{h} = H^2$, $\mathbf{k} = K^2$, $\ldots$, for $k = 5 : \mathbf{h} = H^{7/2}$, $\mathbf{k} = K^{7/2}$. At iteration $k = 1$ the interpolation for $u_H^2(a_2, y_j)$ is linear, while for $\mathring{u}_h^2(a_2, y_j)$ at $k > 1$ a cubic spline interpolation is used. The results, errors in different discrete norms and convergence rates, are shown in Table 4. In this case the statement of Theorem 9 is also validated.

Conclusions

The main advantage of Algorithms 1, 2 is that we reach a high accuracy of the numerical solution, solving the whole (global) problem (in $\Omega_1 \cup \Omega_2$) only once—on the coarse mesh, then the problem is separated into two local problems and we solve them *separately* and consecutively on the fine mesh, increasing the convergence. Therefore, on the base of this technique one can design parallel and adaptive algorithms for such problems. This study can obviously extended to elliptic and parabolic problems with polygonal and curvilinear domains. Also, a nonlinear two-grid decoupling method can be developed to nonlinear differential problems on disjoint domains of type [1,2], see [4] for some results in this direction.

Based on the above approach, we can design the following type of *parallel algorithms*: first solve the standard finite element discretization of the whole problem (on $\Omega_1 \cup \Omega_1 \cup \cdots \cup \Omega_n$) on a relatively coarse grid, and then correct the residual by some discretizations on fine grigs of the local problems separately, each on its own domain $\Omega_i,\ i = 1, \ldots, n$. As a consequence, the global problem only needs to be solved on a coarse grid once and it does not to have to be coupled with the subsequence parallel local solvers. However, the local solvers will interact each other because of the boundary conditions (4)–(5).

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