



# Improved local convergence of Newton's method under weak majorant condition

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## ABSTRACT

We provide a local convergence analysis for Newton's method under a weak majorant condition in a Banach space setting. Our results provide under the same information a larger radius of convergence and tighter error estimates on the distances involved than before [14]. Special cases and numerical examples are also provided in this study.

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## 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution  $x^*$  of equation

$$F(x) = 0, \quad (1.1)$$

where  $F$  is a Fréchet-differentiable operator defined on an open, convex subset  $\mathcal{D}$  of a Banach space  $\mathcal{X}$  with values in a Banach space  $\mathcal{Y}$ .

The field of computational sciences has seen a considerable development in mathematics, engineering sciences and economic equilibrium theory. For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation  $\dot{x} = T(x)$ , for some suitable operator  $T$ , where  $x$  is the state. Then the equilibrium states are determined by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for

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solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. We note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

Newton's method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \quad (1.2)$$

is undoubtedly the most popular method for generating a sequence  $\{x_n\}$  approximating  $x^*$  [1–25]. Here,  $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  ( $x \in \mathcal{D}$ ) the space of bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$  denotes the Fréchet-derivative of operator  $F$  [3]. A local as well as a semilocal convergence analysis of NM has been given by many authors under various conditions. A survey of such results can be found in [3,12] and the references there (see also [2,20]). Recently, some results using convex majorants are provided in [4–8], which improved and extended the corresponding ones given in [15,16,25].

In this study we are motivated by a recent paper of Ferreira [14], which weakened earlier convergence conditions [15,16,25] for the local convergence analysis of NM under general majorant condition (see (2.2)). The information used is  $I(x^*, F, f)$ , where  $f$  is a majorant function (to be precised later in (2.2)). Using  $I(x^*, F, f)$ , Ferreira [14] provided error estimates on the distances  $\|x_n - x^*\|$  ( $n \geq 1$ ) as well as what he claimed to be the best possible convergence radius.

In our analysis we are also motivated by optimization considerations and the work in [14]. Using the same information  $I(x^*, F, f)$ , we show that in general the radius of convergence given in [14] is not as the best possible but it can be enlarged. We also show that the upper bounds on the distances  $\|x_n - x^*\|$  ( $n \geq 1$ ) can be tighter. These observations are very important in computational mathematics, since they allow a wider choice of initial guesses  $x_0$  and fewer iterations to obtain a desired error tolerance  $\epsilon > 0$ . Note that similar improvements in both the local and semilocal case of the works in [15,16,21,25], have already been obtained by us in [4,11,12] under stronger than (2.2) majorant-type conditions.

The paper is organized as follows: Section 2 contains the local convergence analysis of NM under weak majorant conditions, whereas in Section 3 we provide special cases and numerical examples further validating the theoretical results.

## 2. Local convergence analysis for NM

We denote by  $U(z, \alpha)$  the open ball centered at  $z \in \mathcal{X}$  and of radius  $\alpha > 0$ , whereas  $\bar{U}(z, \alpha)$  denotes its closure.

We state the main local convergence result for NM under the majorant condition.

**Theorem 2.1.** Let  $\mathcal{D}$  be an open and convex set; let  $\mathcal{X}$  and  $\mathcal{Y}$  be Banach spaces and let  $F : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{Y}$  be Fréchet-differentiable. Let  $x^* \in \mathcal{D}$  such that  $F(x^*) = 0$ ,  $R > 0$  and  $\kappa := \sup\{t \in [0, R) : U(x^*, t) \subset \mathcal{D}\}$ . Suppose  $F'(x^*)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  and there exist  $f, f_0 : [0, R) \longrightarrow (-\infty, +\infty)$  continuously differentiable such that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq f'_0(\|x - x^*\|) - f'_0(0), \quad (2.1)$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x_\theta))\| \leq f'(\|x - x^*\|) - f'(\theta \|x - x^*\|), \quad (2.2)$$

for all  $x \in U(x^*, \kappa)$  and  $x_\theta = x^* + \theta(x - x^*)$ ,  $\theta \in [0, 1]$ ,

( $\mathcal{H}_1$ )  $f_0(0) = f(0) = 0$  and  $f'_0(0) = f'(0) = -1$ ;

( $\mathcal{H}_2$ )  $f'_0, f'$  are strictly increasing,

$$f_0(t) \leq f(t) \quad \text{and} \quad f'_0(t) \leq f'(t) \quad t \in [0, R). \quad (2.3)$$

Define: parameter  $\nu_0$ , function  $f_1$  on  $(0, \nu_0)$ , parameters  $\nu, \rho_0, r_0$  and scalar iteration  $\{s_n\}$  by

$$\nu_0 := \sup\{t \in [0, R) : f'_0(t) < 0\},$$

$$f_1(t) := \frac{f'(t)}{f'_0(t)}, \quad (2.4)$$

$$\nu := \sup\{t \in [0, R) : f'(t) < 0\},$$

$$\rho_0 := \sup \left\{ \delta \in [0, \nu) : \left( \frac{f(t)}{f'(t)} - t \right) \frac{f_1(t)}{t} < 1, \quad t \in [0, \delta) \right\}$$

$$r_0 := \min\{\kappa, \rho_0, \}$$

$$s_0 = \|x_0 - x^*\|, \quad s_{n+1} = \left| \left( s_n - \frac{f(s_n)}{f'(s_n)} \right) f_1(s_n) \right| \quad (n \geq 0). \quad (2.5)$$

Then, the following assertions hold:

(a)  $\{s_n\}$  is well defined; strictly decreasing; contained in  $(0, r_0)$ ; converges to zero and

$$\lim_{n \rightarrow 0} \frac{s_{n+1}}{s_n} = 0. \quad (2.6)$$

(b)  $\{x_n\}$  generated by NM, starting from  $x_0 \in U(x^*, r_0) \setminus \{x^*\}$  is well defined; remains in  $U(x^*, r_0)$  for all  $n \geq 0$  and converges to  $x^*$ , which is the unique solution of Eq. (1.1) in  $U(x^*, \sigma_0)$ , where,

$$\sigma_0 := \sup\{t \in [0, \kappa) : f_0(t) < 0\}$$

and

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x_n\|}{\|x_n - x^*\|} = 0. \quad (2.7)$$

(c) If

$$\left( \frac{f(\rho_0)}{\rho_0 f'(\rho_0)} - 1 \right) f_1(\rho_0) = 1 \quad \text{and} \quad \rho_0 < \kappa$$

then  $r_0 = \rho_0$  is the possible convergence radius.

(d) If scalar sequence  $\{t_n\}$  is given by

$$t_0 = \|x_0 - x^*\|, \quad t_{n+1} = \left| t_n - \frac{f(t_n)}{f'(t_n)} \right| \quad (n \geq 0) \quad (2.8)$$

then

$$s_n \leq t_n \quad (n \geq 0) \quad (2.9)$$

and strict inequality holds for  $n > 1$  in (2.9), if  $f'_0(t) < f'(t)$ ,  $t \in [0, R)$ .

If additionally, given  $0 \leq p \leq 1$

( $\mathcal{H}_3$ ) The function  $t \rightarrow \left( \frac{f(t)}{f'(t)} - t \right) \frac{f_1(t)}{t^{p+1}}$  is strictly increasing on  $(0, \nu_0)$ ,

then,

(e) The sequence  $\left\{ \frac{s_{n+1}}{s_n^{p+1}} \right\}$  is strictly decreasing so that

$$\|x_{n+1} - x^*\| \leq \frac{s_{n+1}}{s_n^{p+1}} \|x_n - x^*\|^{p+1} \quad (n \geq 0). \quad (2.10)$$

Furthermore, for  $n \geq 0$ ,

$$\|x_n - x^*\| \leq \begin{cases} s_0 \left[ \frac{s_1}{s_0} \right]^n & \text{if } p = 0 \\ s_0 \left( \frac{s_1}{s_0} \right)^{((p+1)n-1)/p} & \text{if } p \neq 0. \end{cases} \quad (2.11)$$

**Proof of Theorem 2.1.** We shall break down the proof into 10 pieces called lemmas.

First we shall show the statements of the theorem involving sequence  $\{s_n\}$ .  $\square$

**Lemma 2.2.** The constants  $\kappa, \nu, \sigma_0$  are positive and  $(t - \frac{f(t)}{f'(t)})f_1(t) < 0$  for all  $t \in (0, \nu)$ .

**Proof.** The set  $\mathcal{D}$  is open and  $x^* \in \mathcal{D}$ , so we deduce  $\kappa$  is positive. Since  $f'$  is continuous in 0 with  $f'(0) = -1$ , there exists  $\delta > 0$  such that  $f'(t) < 0$  for all  $t \in (0, \delta)$ . That is  $\nu > 0$ . Now, because  $f(0) = 0$  and  $f'(0) = -1$ , there exists  $\delta > 0$  such that  $f(t) < 0$  for all  $t \in (0, \delta)$ . Hence, we have  $\sigma = \sup\{t \in [0, \kappa) : f(t) < 0\} > 0$  and by ( $\mathcal{H}_2$ ):  $\sigma_0 \geq \sigma > 0, f_0(t) < 0$ ,  $t \in (0, \sigma_0)$ .

It also follows from ( $\mathcal{H}_1$ ) and ( $\mathcal{H}_2$ ) that  $0 = f(0) > f(t) - t f'(t)$  for  $t \in (0, R)$ . If  $t \in (0, \nu)$  then  $f'(t) < 0$ , which together with (2.4) complete the proof of Lemma 2.2.  $\square$

According to ( $\mathcal{H}_2$ ), the definition of  $\nu_0$  and  $\nu$ , we have  $f'_0(t) < 0, f'(t) < 0$  for all  $t \in [0, \nu)$ , since  $\nu \leq \nu_0$ . Moreover, function  $f_1$  is well defined on  $(0, \nu_0)$ . Therefore, Newton's iteration function

$$\begin{aligned} \eta_{f, f_0} : [0, \nu) &\longrightarrow (-\infty, 0] \\ t &\longrightarrow \left( t - \frac{f(t)}{f'(t)} \right) f_1(t) \end{aligned} \quad (2.12)$$

is well defined.

**Lemma 2.3.** *The following assertions hold:*

$$\lim_{t \rightarrow 0} \frac{\eta_{f, f_0}(t)}{t} = 0, \quad (2.13)$$

$$\rho_0 > 0 \quad (2.14)$$

and

$$|\eta_{f, f_0}(t)| < t \quad \text{for all } t \in (0, \rho_0). \quad (2.15)$$

**Proof.** Using definition (2.12), Lemma 2.2,  $(\mathcal{H}_1)$  and the definition of  $\nu$ , a simple algebraic manipulation leads to

$$\begin{aligned} \frac{|\eta_{f, f_0}(t)|}{t} &= \left( \frac{f(t)}{f'(t)} - t \right) \frac{f_1(t)}{t} \\ &= \left( \frac{1}{f'(t)} \frac{f(t) - f(0)}{t - 0} - 1 \right) f_1(t) \quad \text{for all } t \in (0, \nu), \end{aligned} \quad (2.16)$$

which leads to (2.13) if we let  $t \rightarrow 0$  in (2.16). It then follows from (2.13) and the first equality in (2.16) that there exists  $\delta > 0$  such that

$$0 < \left( \frac{f(t)}{f'(t)} - t \right) \frac{f_1(t)}{t} < 1 \quad \text{for all } t \in (0, \delta). \quad (2.17)$$

Hence, we deduce that  $\rho_0 > 0$ . Finally, the first equality in (2.16) together with the definition of  $\rho_0$  imply (2.15). That completes the proof of Lemma 2.3.

In view of (2.12), sequence  $\{s_n\}$  can be defined as:

$$s_0 = \|x_0 - x^*\|, \quad s_{n+1} = |\eta_{f, f_0}(s_n)| \quad (n \geq 0). \quad (2.18)$$

Replace  $\eta_f$  by  $\eta_{f, f_0}$  in the proof of [14, Corollary 5] to obtain.

**Lemma 2.4.** *Sequence  $\{s_n\}$  is well defined, strictly decreasing and contained in  $(0, \rho_0)$ . Moreover,  $\{s_n\}$  converges to zero with superlinear rate, i.e.,  $\lim_{n \rightarrow \infty} \frac{s_{n+1}}{s_n} = 0$ . Furthermore, if  $(\mathcal{H}_3)$  holds, then sequence  $\{\frac{s_{n+1}}{s_n}\}$  is strictly decreasing.*

Secondly, we need relationships between the majorant function  $f$  and nonlinear operator  $F$ . We provide in the following lemma a perturbation result.

**Lemma 2.5.** *If  $x \in U(x^*, t)$ ,  $t \in [0, \min\{\kappa, \nu_0\})$ ,  $\|x - x^*\| \leq \min\{\kappa, \nu_0\}$ , then the following assertions hold*

$$F'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$$

and

$$\|F'(x)^{-1} F'(x^*)\| \leq -\frac{1}{f'_0(\|x - x^*\|)} \leq -\frac{1}{f'_0(t)}. \quad (2.19)$$

**Proof.** Let  $x \in U(x^*, t)$ ,  $t \in [0, \min\{\kappa, \nu_0\})$ . Using  $f'_0(0) = -1$ , (2.1) and the fact that  $f'_0$  is strictly increasing, we obtain in turn

$$\begin{aligned} \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| &\leq f'_0(\|x - x^*\|) - f'_0(0) \\ &= f'_0(\|x - x^*\|) + 1 \leq f'_0(t) + 1 < 1. \end{aligned} \quad (2.20)$$

The last inequality in (2.20) holds by the definitions of  $\kappa$ ,  $\nu_0$  and the choice of  $t$ . It then follows from (2.20) and the Banach Lemma on invertible operators [3, 12, 20], that  $F'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  so that (2.19) holds.

That completes the proof of Lemma 2.5.  $\square$

Newton's iteration at a point is a zero of the linearization of  $F$  at such a point. Hence, we shall study the linearization error at a point in  $\mathcal{D}$ :

$$e_F(x, y) := F(y) - (F(x) + F'(x)(y - x)) \quad \text{for all } x, y \in \mathcal{D}. \quad (2.21)$$

We shall bound this error by the error in linearization of the majorant function  $f$ :

$$e_f(t, u) := f(u) - (f(t) + f'(t)(u - t)) \quad \text{for all } t, u \in [0, R]. \quad (2.22)$$

**Lemma 2.6.** *If  $\|x^* - x\| < \kappa$ , then the following assertion holds*

$$\|F'(x^*)^{-1} E_F(x, x^*)\| \leq e_f(\|x - x^*\|, 0).$$

**Proof.** The proof of Lemma 2.6 is given in [13, Lemma 2.10] or [14, Lemma 7].  $\square$

Lemma 2.5 guarantees the invertibility of  $F'$  and consequently

$$\begin{aligned} N_f : U(x^*, r_0) &\longrightarrow \mathcal{Y} \\ x &\longrightarrow x - F'(x)^{-1} F(x) \end{aligned} \quad (2.23)$$

is a well defined operator.

**Lemma 2.7.** *If  $\|x - x^*\| < r_0$ , then the following assertions hold*

$$\|N_F(x) - x^*\| \leq |\eta_{f, f_0}(\|x - x^*\|)| \quad (2.24)$$

and

$$N_F(U(x^*, r_0)) \subset U(x^*, r_0). \quad (2.25)$$

**Proof.** It follows from  $F(x^*) = 0$  that the inequality is trivial for  $x = x^*$ . If  $0 < \|x - x^*\| < r_0$ , Lemma 2.5 implies that  $F'(x)$  is invertible. Using  $F(x^*) = 0$  and (2.23), we obtain the approximation

$$\begin{aligned} x^* - N_F(x) &= -F'(x)^{-1} (F(x^*) - F(x) - F'(x)(x^* - x)) \\ &= -F'(x)^{-1} E_F(x, x^*). \end{aligned} \quad (2.26)$$

Using Lemmas 2.5, 2.6 and (2.26), we get in turn

$$\begin{aligned} \|x^* - N_F(x)\| &\leq \|F'(x)^{-1} F'(x^*)\| \|F'(x^*)^{-1} E_F(x, x^*)\| \\ &\leq \frac{e_f(\|x - x^*\|, 0)}{|f'_0(\|x - x^*\|)|}. \end{aligned} \quad (2.27)$$

By the definition of  $e_f$ ,  $\eta_{f, f_0}$  and hypothesis  $f(0) = 0$ , we have

$$\frac{e_f(\|x - x^*\|, 0)}{|f'_0(\|x - x^*\|)|} = |\eta_{f, f_0}(\|x - x^*\|)|, \quad (2.28)$$

which together with (2.27) show (2.24).

Let  $x \in U(x^*, r_0)$ . It follows from  $\|x - x^*\| < r_0 \leq \rho_0$ , (2.24) and Lemma 2.3 that

$$\|N_F(x) - x^*\| \leq |\eta_{f, f_0}(\|x - x^*\|)| < \|x - x^*\|,$$

which shows (2.25).

That completes the proof of Lemma 2.7.  $\square$

**Lemma 2.8.** *If  $(\mathcal{H}_3)$  holds and*

$$\|x - x^*\| \leq t < r_0, \quad (2.29)$$

then the following assertion holds

$$\|N_F(x) - x^*\| \leq \frac{|\eta_{f, f_0}(t)|}{t^{p+1}} \|x - x^*\|^{p+1}. \quad (2.30)$$

**Proof.** Estimate (2.30) is trivial, if  $x = x^*$ . Assume  $\|x - x^*\| \leq t$ , then  $(\mathcal{H}_3)$  and (2.12) imply

$$\frac{|\eta_{f, f_0}(\|x - x^*\|)|}{\|x - x^*\|^{p+1}} \leq \frac{|\eta_{f, f_0}(t)|}{t^{p+1}}. \quad (2.31)$$

The result follows from Lemma 2.7 and (2.31).

That completes the proof of Lemma 2.8.  $\square$

Next, we shall establish the uniqueness and optimal convergence radius. The proof of the next two results can be found in the analogous [13, Lemma 2.13] and [13, Lemma 2.15], respectively.

**Lemma 2.9.** *The point  $x^*$  is the unique zero of operator  $F$  in  $U(x^*, \sigma_0)$ .*

**Lemma 2.10.** *If*

$$\left( \frac{f(\rho_0)}{\rho_0 f'(\rho_0)} - 1 \right) f_1(\rho_0) = 1$$

and  $\rho_0 < \kappa$ , then  $r_0 = \rho_0$  is the optimal convergence radius.

Finally, we shall show the statements of [Theorem 2.1](#) involving Newton's method sequence  $\{x_n\}$ .

It follows from (1.2) and (2.23) that Newton's method can be written as:

$$x_{n+1} = N_F(x_n) \quad (n \geq 0). \quad (2.32)$$

**Lemma 2.11.** *Sequence  $\{x_n\}$  is well defined, is contained in  $U(x^*, r_0)$  and converges to the point  $x^*$ , which is the unique zero of  $F$  in  $U(x^*, \sigma_0)$ . Moreover,*

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} = 0. \quad (2.33)$$

Furthermore, if  $(\mathcal{H}_3)$  holds, then so do (2.10) and (2.11).

**Proof.** Let  $x_0 \in U(x^*, r_0)$  and  $r_0 \leq \nu_0$ . Using [Lemmas 2.5, 2.7](#) and (2.32), we deduce that sequence  $\{x_n\}$  is well defined and remains in  $U(x^*, r_0)$  for all  $n \geq 0$ .

Using [Lemmas 2.3, 2.7](#) and (2.32), we obtain in turn

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|N_F(x_n) - x^*\| \\ &\leq |\eta_{f, f_0}(\|x_n - x^*\|)| < \|x_n - x^*\| \quad (n \geq 0). \end{aligned} \quad (2.34)$$

Hence  $\{\|x_n - x^*\|\}$  is strictly decreasing and converges to some  $\alpha$ . Since  $\|x_n - x^*\|$  is inside  $(0, \rho_0)$  and strictly decreasing, we obtain  $0 \leq \alpha < \rho_0$ . It then follows from (2.34) and the continuity of  $\eta_{f, f_0}$  in  $[0, \rho_0)$  that  $0 \leq \alpha = |\eta_{f, f_0}(\alpha)|$  and from [Lemma 2.3](#), we get  $\alpha = 0$ .

The uniqueness part was shown in [Lemma 2.9](#).

Next we shall show (2.33) that

$$\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} \leq \frac{|\eta_{f, f_0}(\|x_n - x^*\|)|}{\|x_n - x^*\|} \quad (n \geq 0) \quad (2.35)$$

since,  $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ , (2.33) follows from [Lemma 2.3](#).

We shall show by induction

$$\|x_n - x^*\| \leq s_n \quad (n \geq 0). \quad (2.36)$$

Since  $s_0 = \|x_0 - x^*\|$ , (2.36) holds as equality for  $n = 0$ . Assume  $\|x_k - x^*\| \leq s_k$ . In view of (2.32), [Lemma 2.8](#), the induction hypothesis and (2.18), we obtain in turn

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|N_F(x_k) - x^*\| \\ &\leq \frac{|\eta_{f, f_0}(s_k)|}{s_k^{p+1}} \|x_k - x^*\|^{p+1} \leq |\eta_{f, f_0}(s_k)| = s_{k+1}. \end{aligned} \quad (2.37)$$

which completes the induction for (2.36). Hence (2.10) follows from (2.32), (2.36), [Lemma 2.8](#) and (2.18).

Finally, to show (2.11) notice that since sequence  $\{\frac{s_{k+1}}{s_k^{p+1}}\}$  is strictly decreasing, we have

$$\frac{s_{k+1}}{s_k^{p+1}} \leq \frac{s_1}{s_0^{p+1}} \quad (k \geq 0). \quad (2.38)$$

It then follows from (2.10) that

$$\|x_{k+1} - x^*\| \leq \frac{s_1}{s_0^{p+1}} \|x_k - x^*\|^{p+1} \quad (k \geq 0). \quad (2.39)$$

The first inequality in (2.11) follows from (2.39) if  $p = 0$ , whereas the second inequality is also derived from (2.39) if  $0 < p \leq 1$ .

That completes the proof of [Lemma 2.11](#).  $\square$

The proof of [Theorem 2.1](#) now follows the above lemmas.

**Remark 2.12.** (a) If  $f_0(t) = f(t)$  ( $t \in [0, R)$ ), then our Theorem 2.1 reduces to [14, Theorem 2, p. 1516]. Moreover, in this case, we have

$$s_n = t_n \quad (n \geq 0), \quad \rho_0 = \rho, \quad \sigma_0 = \sigma$$

and

$$r_0 = r,$$

where  $\rho, r$  are defined as  $\rho_0, r_0$ , respectively by replacing  $(f_0, f'_0)$  by  $(f, f')$ . Otherwise, it constitutes an improvement with advantages as already stated in the introduction of this study (see also (2.9)).

(b) Theorem 2.1 uses the same information  $(x^*, F, f)$  as [14, Theorem 2, p. 1516], since  $f_0$  is a special case of  $f$ . In practice, the computation of  $f$  requires that of  $f_0$ . Note also that the existence of function  $f_0$  is implied by (2.2). Hence, (2.1) is not an additional hypothesis. We also have:

$$f'_0(t) \leq f'(t) \quad t \in [0, R)$$

hold in general and  $\frac{f'(t)}{f'_0(t)}$  can be arbitrarily large [3,12]. The proof of [14, Theorem 2, p. 1516] uses (2.2) to obtain the estimate

$$\|F'(x)^{-1} F'(x^*)\| \leq -\frac{1}{f'(\|x - x^*\|)} \leq -\frac{1}{f'(t)}$$

corresponding to (2.19). However, we note that (2.19) is a tighter estimate than the above.

In the next section we provide special cases and numerical examples.

### 3. Special cases and application

#### 3.1. Convergence under Hölder-type condition

**Proposition 3.1.** Let  $\mathcal{D}$  be an open and convex set; let  $\mathcal{X}$  and  $\mathcal{Y}$  be Banach spaces and let  $F : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{Y}$  be a Fréchet-differentiable operator. Let  $x^* \in \mathcal{D}$  such that  $F(x^*) = 0$ ,  $R > 0$  and  $\kappa := \sup\{t \in [0, R) : U(x^*, t) \subset \mathcal{D}\}$ . Suppose  $F'(x^*)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  and there exist  $L_0 > 0$ ,  $L > 0$  and  $0 < p \leq 1$  such that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|^p,$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x_\theta))\| \leq L(1 - \theta^p) \|x - x^*\|^p,$$

for all  $x \in U(x^*, \kappa)$  and  $x_\theta = x^* + \theta(x - x^*)$ ,  $\theta \in [0, 1]$ .

Let

$$r_0 := \min \left\{ \kappa, \left( \frac{p+1}{L + L_0(p+1)} \right)^{1/p} \right\},$$

$$x_0 \in U(x^*, r_0) \setminus \{x^*\}, \quad s_0 = \|x_0 - x^*\|, \quad s_{n+1} = \frac{L p s_n^{p+1}}{(p+1)(1 - L_0 s_n^p)}.$$

Then, the following assertions hold:

(a)  $\{s_n\}$  is well defined; strictly decreasing; contained in  $(0, r_0)$ ; converges to zero and

$$\lim_{n \rightarrow \infty} \frac{s_{n+1}}{s_n} = 0.$$

(b) Sequence  $\{x_n\}$  given by NM, starting from  $x_0 \in U(x^*, r_0) \setminus \{x^*\}$  is well defined; remains in  $U(x^*, r_0)$  for all  $n \geq 0$  and converges to  $x^*$ , which is the unique solution of (1.1) in  $U(x^*, (\frac{p+1}{L_0})^{1/p})$ , so that for  $n \geq 0$ :

$$\|x_{n+1} - x^*\| \leq \frac{L p}{(p+1)(1 - L_0 s_n^p)} \|x_n - x^*\|^{p+1}$$

and

$$\|x_n - x^*\| \leq \left( \frac{L p \|x_0 - x^*\|^p}{(p+1)(1 - L_0 \|x_0 - x^*\|^p)} \right)^{((p+1)^n - 1)/p} \|x_0 - x^*\|.$$

Furthermore, if

$$\varrho_0 = \left( \frac{p+1}{L p + L_0(p+1)} \right)^{1/p} < \kappa,$$

then  $r = \varrho_0$  is the best possible convergence radius.

**Proof.** Use [Theorem 2.1](#) for functions  $f_0, f : [0, \kappa] \rightarrow \mathbb{R}$  defined by

$$f_0(t) = \frac{L_0 t^{p+1}}{p+1} - t \quad \text{and} \quad f(t) = \frac{L t^{p+1}}{p+1} - t.$$

That completes the proof of [Proposition 3.1](#).  $\square$

**Remark 3.2.** If  $L = L_0$ , our results reduce to the ones in [\[14, Theorem 13\]](#) (see also [\[13,15,16\]](#)). Moreover, if  $L_0 < L$ , we have

$$\varrho = \left( \frac{p+1}{(2p+1)L} \right)^{1/p} < \varrho_0$$

and

$$\|x_{n+1} - x^*\| \leq \frac{Lp}{(p+1)(1-L_0 t_n^p)} \|x_n - x^*\|^{p+1} \quad (n \geq 0).$$

That is our results provide a larger convergence radius and tighter error bounds than the ones in [\[14\]](#).

Note also that we have

$$\frac{\varrho}{\varrho_0} \rightarrow \left( \frac{p}{2p+1} \right)^{1/p} \quad \text{as} \quad \frac{L_0}{L} \rightarrow 0.$$

So, our approach provides a radius of convergence at most  $(\frac{p}{2p+1})^{-1/p}$  times larger than the one in [\[13–16\]](#).

If the Lipschitz condition

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L \|x - y\|,$$

holds for  $x, y \in \mathcal{D}$ , then  $p = 1$  and we have

$$\varrho = \frac{2}{3L} \leq \varrho_0 = \frac{2}{2L_0 + L}.$$

The radius of convergence  $\varrho$  was obtained in [\[24\]](#). Note however that the results in [\[13–16\]](#) were given in non-affine invariant form. The advantages of affine invariant over non-affine invariant results for Newton-type methods have been explained in [\[2,3,5–8,10–12\]](#).

**Example 3.3** ([\[3,12\]](#)). Let  $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ . Define function  $F$  on  $\mathcal{D} = (-1, 1)$ , given by

$$F(x) = e^x - 1. \tag{3.1}$$

Then, for  $x^* = 0$ , using [\(3.1\)](#), we have  $F(x^*) = 0$  and  $F'(x^*) = e^0 = 1$ . Moreover, hypotheses of [Proposition 3.1](#) hold for  $p = 1, L = e > L_0 = e - 1$ . Note that

$$\frac{L}{L_0} = \frac{e}{e-1} = 1.581976707$$

and

$$\varrho = \frac{2}{3L} = .2452529608 < \varrho_0 = \frac{2}{2L_0 + L} = .3249472314.$$

We also can provide the comparison table using the software Maple 13. Using [\(2.5\)](#) and [\(2.8\)](#) for  $x_0 = .7158$ .

Comparison table			
$k$	<a href="#">(1.2)</a> $\ x_{k+1} - x_k\ $	<a href="#">(2.5)</a> $s_k$	<a href="#">(2.8)</a> $t_k$
0	.2473838936	.2842	.2842
1	.03614663422	.2145495033	.4826134043
2	.0006692478074	.09909547154	1.015025071
3	2.239999498e-7	.01608560415	.7960154923
4	0	.0003616695761	.7399991923
5	~	1.778927982e-7	.7357830417
6	~	4.299999235e-14	.7357588833
7	~	0	.7357588824
8	~	~	.7357588825



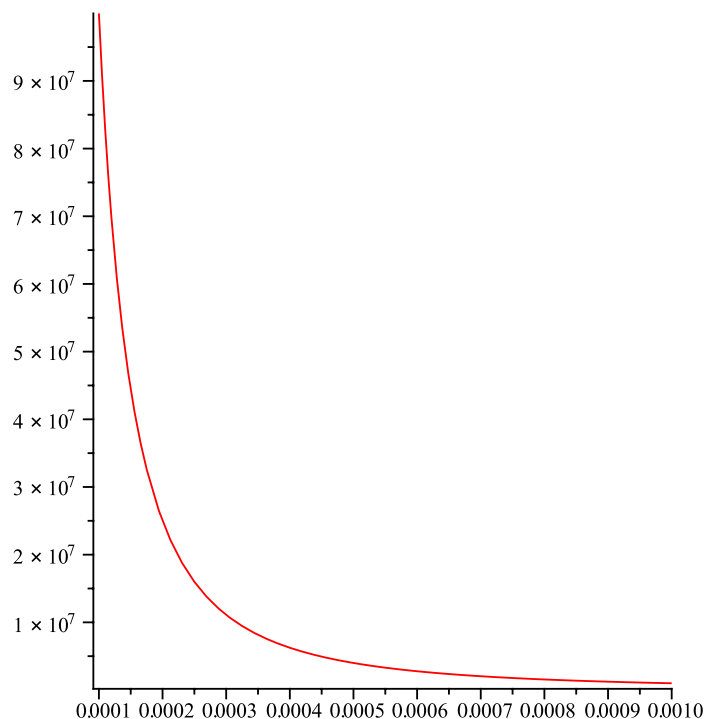


Fig. 1. Function  $\Gamma$  on interval  $(.0001, .001)$ .

The table shows that our error bounds (2.5) are tighter than (2.8). Note that hypothesis  $(\mathcal{H}_3)$  of Theorem 2.1 does not hold, since  $\Gamma$  is not increasing on  $(0, \nu_0)$  for all  $\nu_0 > 0$  (see Fig. 1), where,

$$f_0(t) = \frac{(e-1)t^2}{2} - t, \quad f(t) = \frac{et^2}{2} - t$$

and

$$\Gamma(t) = \left( \frac{f(t)}{f'(t)} - t \right) \frac{f_1(t)}{t^2} = \left( \frac{.5et - 1}{.5(e-1)t - 1} - t \right) \frac{et - 1}{(e-1)t^3 - t^2}.$$

**Example 3.4** ([12,14]). Let  $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ . Define function  $F$  on  $\mathcal{D} = (1, 3)$ , given by

$$F(x) = \frac{2}{3}x^{3/2} - x. \quad (3.2)$$

Then, the zero of  $F$  is  $x^* = \frac{9}{4} = 2.25$ . Using (3.2) and hypotheses of Proposition 3.1,  $F'(x^*) = .5$ ,  $L = 2 > L_0 = 1$  and  $p = .5$ . Moreover, we have

$$\varrho = .1406250000 < \varrho_0 = .1836734694.$$

**Example 3.5.** Let  $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ , the space of continuous functions defined on  $[0, 1]$ , equipped with the max norm and  $\mathcal{D} = U(0, 1)$ . Define function  $F$  on  $\mathcal{D}$ , given by

$$F(h)(x) = h(x) - 5 \int_0^1 x\theta h(\theta)^3 d\theta. \quad (3.3)$$

Then, we have:

$$F'(h[u])(x) = u(x) - 15 \int_0^1 x\theta h(\theta)^2 u(\theta) d\theta \quad \text{for all } u \in \mathcal{D}.$$

Using (3.3) and hypotheses of Proposition 3.1 for  $x^*(x) = 0$  for all  $x \in [0, 1]$ , we get

$$p = 1, \quad L = 15, \quad L_0 = 7.5$$

and

$$\varrho = .04444444444 < \varrho_0 = .06666666667.$$

### 3.2. Convergence under generalized Lipschitz condition

**Proposition 3.6.** Let  $\mathcal{D}$  be an open and convex set; let  $\mathcal{X}$  and  $\mathcal{Y}$  be Banach spaces and let  $F : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{Y}$  be a Fréchet-differentiable operator. Let  $x^* \in \mathcal{D}$  such that  $F(x^*) = 0$ ,  $R > 0$  and  $\kappa := \sup\{t \in [0, R) : U(x^*, t) \subset \mathcal{D}\}$ . Suppose  $F'(x^*)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  and there exist positive integrable functions  $L_0, L : [0, R) \longrightarrow \mathbb{R}$  such that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \int_0^{\|x-x^*\|} L_0(u) du,$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x_\theta))\| \leq \int_{\theta \|x-x^*\|}^{\|x-x^*\|} L(u) du,$$

for all  $x \in U(x^*, \kappa)$  and  $x_\theta = x^* + \theta(x - x^*)$ ,  $\theta \in [0, 1]$ .

Let  $\nu_0 > 0$ ,  $\rho_0 > 0$  and  $r_0 > 0$  be the constants defined by

$$\nu_0 = \sup \left\{ t \in [0, R) : \int_0^t L_0(u) du - 1 < 0 \right\},$$

$$\rho_0 = \sup \left\{ t \in [0, \nu_0) : \frac{\int_0^t L(u) du}{t \left( 1 - \int_0^t L_0(u) du \right)} < 1 \right\},$$

$$r_0 = \min\{\kappa, \rho_0\}.$$

Let

$$x_0 \in U(x^*, r_0) \setminus \{x^*\}, \quad s_0 = \|x_0 - x^*\|, \quad s_{n+1} = \frac{\int_0^{s_n} L(u) u du}{1 - \int_0^{s_n} L_0(u) du}.$$

Then, the following assertions hold:

(a) Sequence  $\{s_n\}$  is well defined; strictly decreasing; contained in  $(0, r_0)$ ; converges to zero and

$$\lim_{n \rightarrow \infty} \frac{s_{n+1}}{s_n} = 0.$$

(b) Sequence  $\{x_n\}$  given by NM, starting from  $x_0 \in U(x^*, r_0) \setminus \{x^*\}$  is well defined; remains in  $U(x^*, r_0)$  for all  $n \geq 0$  and converges to  $x^*$ , which is the unique solution of (1.1) in  $U(x^*, s_0)$ , so that

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} = 0,$$

where,

$$s_0 = \sup \left\{ t \in [0, \kappa) : \int_0^t L_0(u)(t-u) du - t < 0 \right\}.$$

Moreover, if

$$\varrho_0 = \frac{\int_0^{\rho_0} L(u) u du}{\rho_0 \left( 1 - \int_0^{\rho_0} L_0(u) du \right)} = 1$$

and  $\rho_0 < \kappa$ , then  $r_0 = \rho_0$  is the best possible convergence radius.

Furthermore, if  $(\mathcal{H}_3)$  of Theorem 2.1 holds for

$$f(t) = \int_0^t L(u)(t-u) du - t$$

$$f_0(t) = \int_0^t L_0(u)(t-u) du - t$$

then estimate (2.10) and (2.11) also hold.

**Proof.** Hypotheses  $(\mathcal{H}_1)$ – $(\mathcal{H}_3)$  can easily be verified with the above choices of functions  $f_0$  and  $f$ .  $\square$

**Remark 3.7.** If  $L = L_0$ , the results of Proposition 3.6 reduce the ones [14, Theorem 14]. Otherwise they constitute an improvement with advantages as already stated in Remark 3.2.

The results obtained in this study can be extended to include equation with a nondifferentiable term, or inexact Newton-type methods along the lines of our relevant works [2,3,5–8,10–12].

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## References

- [1] J. Appel, E. De Pascale, J.V. Lysenko, P.P. Zabrejko, New results on Newton–Kantorovich approximations with applications to nonlinear integral equations, *Numer. Funct. Anal. Optim.* 18 (1997) 1–17.
- [2] I.K. Argyros, A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space, *J. Math. Anal. and Appl.* 298 (2004) 374–397.
- [3] I.K. Argyros, Computational theory of iterative methods, in: *Studies in Computational Mathematics*, vol. 15, Elsevier, New York, USA, 2007.
- [4] I.K. Argyros, Concerning the semilocal convergence of Newton's method and convex majorants, *Rend. Circ. Mat. Palermo* 2 (57) (2008) 331–341.
- [5] I.K. Argyros, On the semilocal convergence of inexact Newton methods in Banach spaces, *J. Comput. Appl. Math.* 228 (2009) 434–443.
- [6] I.K. Argyros, Concerning the convergence of Newton's method and quadratic majorants, *J. Appl. Math. Comput.* 29 (2009) 391–400.
- [7] I.K. Argyros, A semilocal convergence analysis for directional Newton methods, in: *Mathematics of Computation*, vol. 80, AMS, 2011, pp. 327–343.
- [8] I.K. Argyros, Local convergence of Newton's method using Kantorovich's convex majorants, *Rev. Anal. Numér. Théor. Approx.* 39 (2010) 97–106.
- [9] I.K. Argyros, S. Hilout, On the convergence of inexact Newton-type methods using recurrent functions, *Panamer. Math. J.* 19 (2009) 79–96.
- [10] I.K. Argyros, S. Hilout, Inexact Newton methods and recurrent functions, *Appl. Math.* 37 (2010) 113–126.
- [11] I.K. Argyros, S. Hilout, Extending the Newton–Kantorovich hypothesis for solving equations, *J. Comput. Appl. Math.* 234 (2010) 2993–3006.
- [12] I.K. Argyros, S. Hilout, M.A. Tabatabai, *Mathematical Modelling with Applications in Biosciences and Engineering*, Nova Science Pub., New York, 2011.
- [13] O.P. Ferreira, Local convergence of Newton's method in Banach space from the viewpoint of the majorant principle, *IMA J. Numer. Anal.* 29 (2009) 746–759.
- [14] O.P. Ferreira, Local convergence of Newton's method under majorant condition, *J. Comput. Appl. Math.* 235 (2011) 1515–1522.
- [15] O.P. Ferreira, M.L.N. Gonçalves, Local convergence analysis of inexact Newton-like methods under majorant condition, *Comput. Optim. Appl.* 48 (2011) 1–21.
- [16] O.P. Ferreira, B.F. Svaiter, Kantorovich's majorants principle for Newton's method, *Comput. Optim. Appl.* 42 (2009) 213–229.
- [17] J.M. Gutiérrez, M.A. Hernández, Newton's method under weak Kantorovich conditions, *IMA J. Numer. Anal.* 20 (2000) 521–532.
- [18] J.M. Gutiérrez, M.A. Hernández, M.A. Salanova, Accessibility of solutions by Newton's method, *Int. J. Comput. Math.* 57 (1995) 239–247.
- [19] Z. Huang, Newton method under weak Lipschitz continuous derivative in Banach spaces, *Appl. Math. Comput.* 140 (2003) 115–126.
- [20] L.V. Kantorovich, G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1982.
- [21] C. Li, W.P. Shen, Local convergence of inexact methods under the Hölder condition, *J. Comput. Appl. Math.* 222 (2008) 544–560.
- [22] P.D. Proinov, General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity* 25 (2009) 38–62.
- [23] P.D. Proinov, New general convergence theory for iterative processes and its applications to Newton–Kantorovich type theorems, *J. Complexity* 26 (2010) 3–42.
- [24] W.C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, *Polish Academy of Science, Banach Center Publ.* 3 (1977) 129–142.
- [25] X. Wang, Convergence on Newton's method and inverse function theorem in Banach space, *Math. Comput.* 68 (1999) 169–186.