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Option pricing under regime-switching jump–diffusion models

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ABSTRACT

We present an explicit formula and a multinomial approach for pricing contingent claims under a regime-switching jump–diffusion model. The explicit formula, obtained as an expectation of Merton-type formulae for jump–diffusion processes, allows to compute the price of European options in the case of a two-regime economy with lognormal jumps, while the multinomial approach allows to accommodate an arbitrary number of regimes and a generic jump size distribution, and is suitable for pricing American-style options. The latter algorithm discretizes log-returns in each regime independently, starting from the highest volatility regime where a recombining multinomial lattice is established. In the remaining regimes, lattice nodes are the same but branching probabilities are adjusted. Derivative prices are computed by a backward induction scheme.

1. Introduction

Several empirical studies show that financial returns exhibit volatility with a stochastic pattern and fatter tails than the standard normal model, which has been shown not to be suitable for capturing the asset price dynamics. Consequently, many alternative approaches have been proposed in order to capture the dynamics of financial returns. Examples are the jump–diffusion process introduced by Merton [1], and the regime-switching model introduced by Hamilton [2,3] that has become more and more attractive for researchers who, starting from the first contribution presented by Naik [4] in 1993, have developed a wide range of option pricing models in this framework.

In this paper, we propose an explicit formula and a multinomial approach for evaluating contingent claims when the underlying asset dynamics evolves according to a regime-switching model with jumps. The choice of this framework is motivated by two main aspects:

- regime-switching models represent a simple way to capture stochastic volatility and, hence, fat tails, thus overcoming the drawback of the classical lognormality assumption characterized by constant volatility;
- the addition of a jump component to the regime-switching context contributes to explain accurately some of the empirical biases evidenced by the classical lognormal model.

In financial literature, regime-switching models have been prevalently applied in order to allow Lévy processes to switch in a finite state space. Among the contributions in the field of option pricing, it is worth mentioning Konikov and Madan [5], who introduce an extension of the variance-gamma model in which the parameters switch, according to a two-state Markov chain, between two fixed sets of values at infinitesimal time intervals. Furthermore, they evidence that more than two states should be considered for the Lévy process, but the mathematical approach they use cannot easily accommodate more than two states for option valuations. To overcome this limit, Elliot and Osakwe [6] extend their work to more than two

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states introducing a multi-state Markov switching model where the underlying process is a jump process with parameters that may switch among drift/compensator pairs. Albanese et al. [7] develop a model similar to the one of Konikov and Madan [5] except that switches occur only at finite time intervals, deriving as well closed form formulae for European options. Jackson et al. [8] propose a Fourier space time-stepping algorithm to derive option prices in a regime-switching Lévy process. Jiang and Pistorius [9] evaluate perpetual American put options in an exponential regime-switching Lévy model deriving analytically tractable results. A different approach, which uses a Markov chain with a progressively denser state space to approximate a continuous time stochastic volatility model with jumps, has been proposed by Chourdakis [10] who in this way obtains option prices in semi-closed form.

Among the contributions in option pricing that consider a regime-switching model with jumps, it is worth mentioning Yuen and Yang [11] who, after generalizing the Naik [4] model to more than two regimes, provide a trinomial lattice to price options under a jump–diffusion Markov regime-switching model. Indeed, as in Naik [4], the underlying asset process presents jumps only during the switches among states with the jump size depending upon the state before and after the switching and the current asset price. A more general framework is proposed in Ramponi [12], who presents a Fourier transform method to compute the price of European contingent claims when the underlying asset behavior is described by a jump–diffusion dynamics with parameters driven by a continuous time and stationary Markov chain on a finite state space. His method is suitable for pricing European-style contingent claims but may not be applicable to evaluate American options.

In this paper, we work in the framework proposed by Ramponi [12] where a regime-switching model for the underlying asset embedding a jump component that may switch among different regimes is considered. After a preliminary econometric analysis that supports the choice of a regime-switching jump–diffusion dynamics to model the equity price, we present an explicit formula to compute the price of European options in the case of a two-regime economy with jumps in the asset price process following a lognormal distribution. The formula is obtained as an expectation of Merton-type formulae for jump–diffusion processes by conditioning the asset distribution on the occupation time in one of the two regimes and on the number of jumps occurring in each regime. It does not require possibly cumbersome inversions and represents an alternative approach for option pricing with respect to the Ramponi's [12] formula, which is obtained applying Fourier methods. To complete the treatment of the pricing problem, we also propose a discrete multinomial approach that is flexible enough to accommodate an arbitrary distribution for the jump component and an arbitrary number of regimes both for the diffusion and the jump component, and presents the advantage of being easily applied to price American-style options. We develop a multinomial lattice which is needed to capture both the diffusion and the jump component in the underlying asset process associated to the highest volatility regime. Indeed, we approximate the diffusion part by a trinomial tree and add more branches to capture jumps. For the other regimes, instead of generating new lattices, we simply adjust branching probabilities as suggested by Yuen and Yang [13]. Then, option prices are computed via backward induction. Numerical results within a two-state regime-switching version of the Merton [1] jump–diffusion model are also provided to support the model.

The rest of the paper is organized as follows. After a preliminary section presenting the framework and the econometric analysis aimed at validating regime-switching models with jumps (Section 2), we develop the explicit formula to evaluate European options in the presence of a two-regime economy when jumps follow a lognormal distribution (Section 3), and the multinomial lattice for more general cases (Section 4). Section 5 presents numerical results confirming the accuracy of the proposed model for European, American, and barrier options, and provides a numerical discussion concerning the behavior of the option prices computed by the multinomial approach. Finally, Section 6 concludes.

2. Framework and econometric analysis

We divide this section into two parts. In the first one, we analyze the framework in which we will develop our model; then, in the second part, we provide an econometric analysis supporting the choice of the underlying framework. For the sake of simplicity, we limit the analysis to a Markov chain with only two states but its extension to a greater number of states is straightforward.

2.1. The framework

On a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is the real-world probability measure, we consider a risky asset in a security market where trading takes place in the interval $[0, T]$. The parameters of the asset dynamics may switch according to a continuous time, homogeneous and stationary Markov process, $\epsilon(t)$, on the state space $\mathcal{L} = \{0, 1\}$ with generator $A \in \mathbb{R}^{2\times 2}$,

$$A = \begin{pmatrix} -a_{0,1} & a_{0,1} \\ a_{1,0} & -a_{1,0} \end{pmatrix}, \tag{1}$$

governing the transition probabilities of the process from the current state to the other. The transition probability matrix in the interval $[t, t + \Delta t]$ is given by

$$P = e^{A\Delta t} = \sum_{n=0}^{\infty} \frac{(A\Delta t)^n}{n!} = I + A\Delta t + o(\Delta t),$$

where I is the identity matrix. Hence, ignoring terms of order superior to Δt, if at time t we are in regime 0, then with probability $a_{0,1}\Delta t$ there will be a switch to regime 1 at time $t+\Delta t$; the probability of remaining in regime 0 is $1-a_{0,1}\Delta t$. Transition probabilities from regime 1 are constructed similarly.

In order to capture the typical features of market data, i.e., higher picks and heavier tails than the standard normal model, the risky asset evolves according to the following stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu_{\epsilon(t)}dt + \sigma_{\epsilon(t)}d\widetilde{W}(t) + \int_E \gamma(y, \epsilon(t-))p^{\epsilon}(dy, dt),$$

where, conditional on the Markov state $\epsilon(t)$ observed at time t, $\mu : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is the instantaneous rate of return, $\sigma : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is the asset volatility, $\widetilde{W}(t)$ is a Brownian motion, $\gamma : E \times \mathcal{L} \to \mathbb{R}$ is a given function assuming values greater than -1, $(E, \mathcal{E})$ is a measurable mark space, and $p^{\epsilon}(dy, ds)$ is a marked point process[1] with intensity $\lambda(\epsilon(t), dy) = \lambda_{\epsilon(t)}\vartheta(\epsilon(t), dy)$. The variables $\lambda_{\epsilon(t)}$, i.e., the intensity of the counting process, $\vartheta(\epsilon(t), dy)$, i.e., a set of probability measures on $(E, \mathcal{E})$, and $\gamma(y, \epsilon(t))$, i.e., the jump size relative to the mark y in regime $\epsilon(t)$, depend upon the regime observed at time t and, consequently, they are defined for each state in the chain. It is worth evidencing that we assume $\epsilon(t)$ and $\widetilde{W}(t)$ being independent and $\widetilde{W}(t)$ and $p^{\epsilon}(dy, dt)$ conditionally independent given $\epsilon(t)$. All the processes are adapted to the filtration $\mathcal{G}_t = \mathcal{F}_t^{\epsilon} \vee \mathcal{F}_t^{S}$ where $\mathcal{F}_t^{\epsilon} = \sigma\{\epsilon(s), 0 \le s \le t\}$, and $\mathcal{F}_t^{S} = \sigma\{S(s), 0 \le s \le t\}$.

In view of our pricing application, we specify our model in a probability space where, denoting by $r_{\epsilon(t)}$ the instantaneous risk-free rate of return conditional on the Markov chain being in regime $\epsilon(t)$ at time t, the discounted process $S(t)e^{-\int_0^t r_{\epsilon(s)}ds}$ is a martingale. Consequently, the risky asset dynamics is specified as

$$\frac{dS(t)}{S(t-)} = (r_{\epsilon(t)} - \lambda_{\epsilon(t)}m_{\epsilon(t)})dt + \sigma_{\epsilon(t)}dW(t) + \int_E \gamma(y, \epsilon(t-))p^{\epsilon}(dy, dt), \tag{2}$$

where $m_{\epsilon(t)} = \int_E \gamma(y, \epsilon(t-))\vartheta(\epsilon(t-), dy)$ is finite for each regime, and $W(t)$ is a Brownian motion under the pricing measure. Further, we assume that the regime risk is not priced in the market, hence the rate matrix A in (1) is the same under both the physical measure and the pricing measure.

To fix the jump size distribution, throughout the paper we suppose $\gamma(Y, \epsilon(t)) = Y_{\epsilon(t)} - 1$, with $Y_{\epsilon(t)}$ random variable associated to the measure $\vartheta(\epsilon(t), dy)$. Consequently, the size of the asset price jump at time t in regime $\epsilon(t)$ is

$$S(t) - S(t-) = \begin{cases} 0 & \text{in case of no jump,} \\ S(t-)(Y_{\epsilon(t)} - 1) & \text{if a jump occur.} \end{cases}$$

Being $Y_{\epsilon(t)}$ a non-negative random variable, we ensure that $S(t)$ can never become negative, and on a logarithmic scale we have $\log S(t) = \log S(t-) + \log Y_{\epsilon(t)}$, which evidences that the jumps are additive in the logarithm of the price. The solution of (2), on a logarithmic scale, is given by

$$X(T) = \log\frac{S(T)}{S(0)} = \alpha_0 T_0 + \alpha_1 T_1 + \sigma_0 Z(T_0) + \sigma_1 Z(T_1) + \sum_{k=1}^{N(T_0)} \log Y_0(k) + \sum_{k=1}^{N(T_1)} \log Y_1(k), \tag{3}$$

where for $l = 0, 1$, T_l is the occupation time of the Markov chain in regime l, $\alpha_l = r_l - \sigma_l^2/2 - \lambda_l m_l$, $N(T_l)$ is distributed as a Poisson variable, $\mathrm{Poiss}(\lambda_l T_l)$, and $Z(T_l)$ is distributed as a normal variable, $\mathrm{N}(0, T_l)$. For option pricing, we also assume that the regime is observable and, consequently, we will derive the contingent claim price conditional on the observed regime at the option inception.

2.2. An econometric analysis

In this section, we provide an econometric analysis that supports the choice of a regime-switching jump–diffusion process (2) to describe the underlying asset dynamics.

Consider the following discretization of the regime-switching jump–diffusion model over the interval $[t-1, t]$

$$y(t)|\epsilon(t) = \log(S(t)/S(t-1))|\epsilon(t) = \mu_{\epsilon(t)} + \sigma_{\epsilon(t)}Z(t) + \sum_{k=1}^{N_{\epsilon(t)}} \Upsilon_{\epsilon(t)}(k), \tag{4}$$

where $\epsilon(t)$ is a state variable switching according to the transition matrix P, $\mu_{\epsilon(t)}$ and $\sigma_{\epsilon(t)}$ are the drift and the volatility of the diffusion part, $Z(t)$ are independent standard normal variates, $\Upsilon_{\epsilon(t)}(k) \sim \mathrm{N}(\varpi_{\epsilon(t)}, \nu^2_{\epsilon(t)})$, and $N_{\epsilon(t)} \sim \mathrm{Poiss}(\lambda_{\epsilon(t)})$.

Here we follow the setup and notation of Hamilton [15] to estimate the discrete time model (4) by maximum likelihood. Let θ be the vector of model parameters and $\Omega_t = \{y(t), \ldots, y(0)\}$ the observations up to time t. Inference of model (4) requires defining the 2×1 vectors $\hat{\xi}_t$ whose elements are $P(\epsilon(t) = l|\Omega_t, \theta)$, $l = 0, 1$, and η_t whose elements are

$$f(y(t)|\epsilon(t) = l, \Omega_t; \theta) = \sum_{n=0}^{+\infty} \frac{e^{-\lambda_l}\lambda_l^n}{n!}\phi(\mu_{n,l}, \sigma_{n,l}), \quad l = 0, 1,$$

[1] See Runggaldier [14] for further details.

Table 1
Daily estimated parameters and standard errors for the CAC 40 index.

	RS		LNJ		RSJ1		RSJ2	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ			−0.0003	0.0003				
μ_0	0.0009	0.0002			0.0012	0.0007	0.0019	0.0003
μ_1	−0.0016	0.0009			−0.0017	0.0016	0.0008	1.0136E−05
σ			0.0096	0.0004				
σ_0	0.0086	0.0003			0.0058	0.0033	0.0118	0.0025
σ_1	0.0223	0.0008			0.0235	0.0015	0.0053	0.0006
λ			0.1402	0.0357	1.0713	0.3515		
λ_0							1.0145	0.7329
λ_1							1.5442	0.6593
ϖ			−0.0016	0.0020	−0.0005	0.0015		
ϖ_0							−0.0031	0.0015
ϖ_1							−2.8948E−05	0.0002
ν			0.0274	0.0032	0.0074	0.0042		
ν_0							0.0183	0.0046
ν_1							0.0058	0.0007
p_{00}	0.9896	0.0039			0.9949	0.0024	0.9843	0.0062
p_{11}	0.9741	0.0088			0.9785	0.0100	0.9949	0.0025
LogL	6041.1822		5923.8901		6057.0785		6077.5327	
AIC	−12,070.3643		−11,837.7802		−12,096.1570		−12,131.0654	
BIC	−12,036.7589		−11,809.7757		−12,045.7489		−12,063.8545	

where $\phi(\mu, \sigma)$ denotes the density function of a normal random variable with mean μ, and standard deviation σ, and

$$\mu_{n,l} = \mu_l + n\varpi_l,$$
$$\sigma_{n,l} = \sqrt{\sigma_l^2 + n\nu_l^2}.$$

The loglikelihood for a sample of M observations is given by

$$\ell(\theta) = \sum_{t=1}^{M} \log f(y(t)|\Omega_t; \theta),$$

where the densities $f(y(t)|\Omega_t; \theta)$ are updated recursively as follows:

$$f(y(t)|\Omega_t; \theta) = \iota'(P\hat{\xi}_{t-1} \odot \eta_t),$$
$$\hat{\xi}_t = \frac{P\hat{\xi}_{t-1} \odot \eta_t}{f(y(t)|\Omega_t; \theta)},$$

where $\iota = (1, 1)'$ and $\odot$ is the Hadamard product. Typically, iterations are started setting the vector $\hat{\xi}$ to the unconditional probabilities, i.e., $\hat{\xi}_0$ satisfies

$$\hat{\xi}_0' = \hat{\xi}_0' P.$$

We fit model (4) to equity indices returns. We consider daily data covering the period from 01/07/2003 to 02/05/2011 for the CAC 40, DAX 30, FTSE 100, Nikkei 225 and S&P 500 indices. We compare a regime-switching (RS) model (i.e., without jumps) a lognormal with jumps (LNJ) model (i.e., without regime-switching), and two regime-switching models with jumps (RSJ1 and RSJ2). In RSJ1 model, we assume that in both regimes jump intensities as well as jump distribution parameters are equal, i.e., $\lambda_0 = \lambda_1$ and $\varpi_0 = \varpi_1$ and $\nu_0 = \nu_1$. In the RSJ2 model, instead, jump intensities and magnitudes are allowed to vary across regimes.

Estimation results, reported in Tables 1–5, show that for daily data using only a regime-switching or only a lognormal with jumps model is not enough to capture the return dynamics. Indeed, according to the information criteria used, models that incorporate both regime-switching features and jumps seem to have the best fit. In particular, if we consider the AIC criterion the "extended" RSJ2 model is always the preferred model. On the other hand, if we base our conclusions on the BIC criterion, the RSJ2 is the best model for three indices out of five (CAC 40, FTSE 100 and S&P 500), while the more parsimonious RSJ1 is the best model for the remaining two indices (DAX 30 and Nikkei 225). For both criteria, the LNJ model offers always the worst fit.

As far as the estimated parameters are concerned, in the RS model we find that for all the considered indices the high-volatility regime has associated a smaller, negative mean. We notice also that the jump component of the RSJ2 model is such that the high-volatility regime ($\sigma_0 > \sigma_1$) has associated larger jump intensities ($\lambda_0 > \lambda_1$), smaller jump magnitude mean ($\varpi_0 < \varpi_1$) and larger jump magnitude variance ($\nu_0^2 > \nu_1^2$). This is true for all the series examined with the exception of the FTSE 100 index.

Table 2
Daily estimated parameters and standard errors for the DAX 30 index.

	RS		LNJ		RSJ1		RSJ2	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ			0.0010	0.0003				
μ_0	0.0010	0.0003			0.0013	0.0003	0.0004	0.0010
μ_1	−0.0019	0.0013			−0.0005	0.0006	0.0011	0.0004
σ			0.0097	0.0004				
σ_0	0.0096	0.0003			0.0076	0.0004	0.0110	0.0013
σ_1	0.0241	0.0013			0.0152	0.0009	0.0082	0.0001
λ			0.1366	0.0449	0.0263	0.0107		
λ_0							0.8329	0.3914
λ_1							1.1717	0.2711
ϖ			−0.0044	0.0022	−0.0036	0.0071		
ϖ_0							−0.0014	0.0012
ϖ_1							9.3327E−05	5.0223E−05
ν			0.0258	0.0036	0.0428	0.0076		
ν_0							0.0166	0.0026
ν_1							0.0012	0.0018
p_{00}	0.9925	0.0032			0.9847	0.0056	0.9816	0.0069
p_{11}	0.9699	0.0116			0.9826	0.0064	0.9870	0.0047
LogL	6031.8759		5945.9511		6049.7692		6059.9261	
AIC	−12,051.7518		−11,881.9021		−12,081.5384		−12,095.8523	
BIC	−12,018.1463		−11,853.8976		−12,031.1302		−12,028.6414	

Table 3
Daily estimated parameters and standard errors for the FTSE 100 index.

	RS		LNJ		RSJ1		RSJ2	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ			0.0008	0.0002				
μ_0	0.0008	0.0002			0.0012	0.0003	0.0032	0.0008
μ_1	−0.0012	0.0008			−0.0009	0.0012	0.0014	0.0003
σ			0.0071	0.0003				
σ_0	0.0071	0.0002			0.0055	0.0009	0.0103	0.0006
σ_1	0.0200	0.0009			0.0215	0.0013	0.0057	0.0010
λ			0.2437	0.0445	0.6278	0.4917		
λ_0							0.8067	0.3593
λ_1							0.3888	0.2975
ϖ			−0.0023	0.0011	−0.0009	0.0009		
ϖ_0							−0.0042	0.0014
ϖ_1							−0.0018	0.0021
ν			0.0198	0.0017	0.0071	0.0020		
ν_0							0.0171	0.0027
ν_1							0.0063	0.0017
p_{00}	0.9902	0.0034			0.9939	0.0025	0.9869	0.0055
p_{11}	0.9762	0.0087			0.9761	0.0095	0.9944	0.0026
LogL	6382.0276		6240.9338		6395.9159		6417.8350	
AIC	−12,752.0552		−12,471.8677		−12,773.8318		−12,811.6699	
BIC	−12,718.4498		−12,443.8631		−12,723.4237		−12,744.4591	

3. Explicit formula for European options

In this section, we present an explicit formula to compute the price of European options in the case of a two-regime economy with jumps in the asset price process following a lognormal distribution. Following the lines of Naik [4], we exploit the fact that in the above model the log-return distribution is conditionally normal given the occupation time of the first of the two regimes and the number of jumps in both the regimes and, hence, the conditional European option price is of the Black–Scholes type. We show that, in the particular case of a jump component which presents the same distribution in both the regimes, the formula reduces to an expectation of the usual Merton formula with the expectation taken over the average future variance of the underlying asset price. The formula represents an alternative approach to the one proposed by Ramponi [12], in that we do not use Fourier based methods to obtain the evaluation formula. The main drawback of the latter approach is that it requires the choice of damping coefficients and upper integration limits that are not always straightforward to determine. On the contrary, as explained later, the proposed formula is not affected by such issues because it involves an integration over a finite interval.

The following proposition documents the call price at time zero conditional on the Markov chain being in the first state in a two-regime economy characterized by a constant risk-free rate (the proof is given in the Appendix).

Table 4
Daily estimated parameters and standard errors for the Nikkei 225 index.

	RS		LNJ		RSJ1		RSJ2	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ			0.0007	0.0003				
μ_0	0.0005	0.0003			0.0012	0.0007	0.0025	0.0040
μ_1	−0.0032	0.0022			−0.0020	0.0026	0.0020	0.0005
σ			0.0116	0.0004				
σ_0	0.0119	0.0003			0.0029	0.0010	0.0179	0.0134
σ_1	0.0331	0.0023			0.0324	0.0024	0.0068	0.0010
λ			0.1229	0.0345	2.5462	0.4932		
λ_0							0.9254	0.2792
λ_1							1.3627	0.3307
ϖ			−0.0063	0.0027	−0.0003	0.0003		
ϖ_0							−0.0053	0.0096
ϖ_1							−0.0012	0.0004
ν			0.0311	0.0039	0.0076	0.0007		
ν_0							0.0270	0.0197
ν_1							0.0086	0.0005
p_{00}	0.9932	0.0028			0.9974	0.0016	0.9830	0.0098
p_{11}	0.9515	0.0199			0.9767	0.0148	0.9975	0.0015
LogL	5710.2058		5615.7119		5730.4242		5735.7494	
AIC	−11,408.4116		−11,221.4238		−11,442.8483		−11,447.4988	
BIC	−11,374.8061		−11,193.4193		−11,392.4402		−11,380.2880	

Table 5
Daily estimated parameters and standard errors for the S&P 500 index.

	RS		LNJ		RSJ1		RSJ2	
	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.	Estimate	s.e.
μ			0.0009	0.0002				
μ_0	0.0008	0.0002			0.0016	0.0003	0.0018	0.0007
μ_1	−0.0016	0.0011			−0.0006	0.0017	0.0014	0.0003
σ			0.0067	0.0003				
σ_0	0.0074	0.0002			0.0029	0.0005	0.0113	0.0037
σ_1	0.0236	0.0010			0.0271	0.0020	0.0024	0.0006
λ			0.2688	0.0413	1.3861	0.2444		
λ_0							0.8272	0.4086
λ_1							1.6695	0.8272
ϖ			−0.0025	0.0011	−0.0008	0.0003		
ϖ_0							−0.0033	0.0017
ϖ_1							−0.0005	0.0003
ν			0.0211	0.0016	0.0067	0.0006		
ν_0							0.0218	0.0051
ν_1							0.0056	0.0007
p_{00}	0.9920	0.0029			0.9967	0.0019	0.9885	0.0054
p_{11}	0.9736	0.0101			0.9787	0.0102	0.9969	0.0017
LogL	6352.2513		6208.5450		6405.3982		6419.4778	
AIC	−12,692.5026		−12,407.0899		−12,792.7964		−12,814.9557	
BIC	−12,658.8972		−12,379.0854		−12,742.3882		−12,747.7448	

Proposition 1. *In the regime-switching jump–diffusion model (3) with risk-free rate r in both regimes, and* $\log Y_l(k) \sim \mathrm{N}(\eta_l, \delta_l)$, $l = 0, 1$, *the time zero call price conditional on the Markov chain being in the first state*[2] *is*

$$c(0|\epsilon(0) = 0) = \int_0^T \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C(S(0), K, T, r, A_{n,m}(t), B_{n,m}(t)) \frac{e^{-\lambda_0 t - \lambda_1 (T-t)} (\lambda_0 t)^n (\lambda_1 (T-t))^m}{n!m!} f_{T_0}(t|\epsilon(0) = 0) dt, \quad (5)$$

where $f_{T_0}(t|\epsilon(0) = 0)$ *is the conditional density of the occupation time of the first regime (cfr., Proposition 1 in [4]),*

$$C(S(0), K, T, r, A, B) = S(0)e^{-rT} e^{A+B^2/2} \Phi(d_1(A, B)) - Ke^{-rT} \Phi(d_2(A, B)),$$

$$d_1(A, B) = \frac{\log(S(0)/K) + A + B^2}{B},$$

2 The time zero call price conditional on the Markov chain being in the second regime is the same as above except that the conditional density $f_{T_0}(t|\epsilon(0) = 1)$ needs to be used in Eq. (5).

$$d_2(A,B) = \frac{\log(S(0)/K) + A}{B} = d_1(A,B) - B,$$
$$A_{n,m}(t) = rT + (-\sigma_0^2/2 - \lambda_0 m_0)t + (-\sigma_1^2/2 - \lambda_1 m_1)(T-t) + n\eta_0 + m\eta_1,$$
$$B_{n,m}(t) = \sqrt{\sigma_0^2 t + \sigma_1^2(T-t) + n\delta_0^2 + m\delta_1^2}.$$

We remark that the series inside the integral in (5) converges very quickly especially for small values of T, and this aspect reduces substantially the computational time for option pricing.

In the next proposition, we simplify matters and derive a result valid for the case in which only the diffusion part differs across the two regimes (the proof is given in the Appendix).

Proposition 2. *In the regime-switching jump–diffusion model (3) with risk-free rate r in both regimes,* $\lambda_0 = \lambda_1 = \lambda$, $\eta_0 = \eta_1 = \eta$, $\delta_0 = \delta_1 = \delta$, *and, consequently,* $\log Y_l(k) \sim \mathrm{N}(\eta, \delta)$, $l = 0, 1$, *and* $m_0 = m_1 = \overline{m} = \exp(\eta + \delta^2/2) - 1$, *the time zero call price conditional on the Markov chain being in the first state is*

$$c(0|\epsilon(0) = 0) = \int_0^T C_{Merton}(S(0), K, T, r, \sigma(t), \lambda, \eta, \delta) \times f_{T_0}(t|\epsilon(0) = 0)dt, \tag{6}$$

where again $f_{T_0}(t|\epsilon(0) = 0)$ *is the conditional density of the occupation time of the first regime,* $\sigma(t) = \sqrt{\sigma_0^2 t + \sigma_1^2(T-t)}$, *and*

$$C_{Merton}(S(0), K, T, r, \sigma, \lambda, \eta, \delta) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} C_{BS}(S_n, K, T, r, \sigma_n)$$

with

$$S_n = S(0)\exp[(\eta + \delta^2/2)n - \lambda\overline{m}T]$$
$$\sigma_n = \sqrt{\sigma^2 + n\delta^2/T}.$$

$C_{BS}(S, K, T, r, \sigma)$ *is the usual Black–Scholes formula with risk-free r, for a call with time to maturity T, strike K, written on a stock with price S and volatility* σ.

The time zero call price conditional on the Markov chain being in the second regime is the same as above except that the conditional density $f_{T_0}(t|\epsilon(0) = 1)$ needs to be used in Eq. (6). It is worth noting that in the latter proposition, in the case of no jumps that is $\lambda = 0$ and/or jumps that are zero with probability 1, i.e., $\eta = \delta = 0$, the Merton [1] formula inside the integral in (6) reduces to a Black–Scholes–Merton formula and, consequently, formula (6) coincides with the Naik's formula. Additionally, our formula is applicable also to the case of deterministic jumps, i.e., zero variance in the jump size distribution, for which Naik [4] provides only a numerical procedure and no explicit formula.

4. The multinomial approach

Here we present the discrete version of the continuous time framework of Section 2.1, which is based on a multinomial representation of the risky asset dynamics. We detail the case of two regimes but the algorithm is easily applicable to the case of more than two regimes. The proposed method establishes a recombining multinomial lattice for the highest volatility regime and, then, adjusts the probability measure to describe the underlying asset dynamics in the other regime. Then, it computes the option values through the expected value of their payoffs along the lattice branches. The model is flexible enough to allow the valuation of both European and American-style options thus making, in the latter case, a significant practical contribution given the wide trade volume registered in several important markets.

We work on the logarithmic version of the regime-switching jump–diffusion process reported in Eq. (3), where regime 0 is the highest volatility regime and regime 1 is the lowest volatility one, and the logarithm of the jump-magnitude, $\log Y_l(k)$, has known distribution function $F_l(\cdot)$, $l = 0, 1$, not necessarily of the normal type.

We establish a multinomial recombining grid based on n time steps of length $\Delta t = T/n$, with T being the option maturity, where the number of nodes at each time step is bounded by fixing a tolerance level ε. Let d_l and u_l, with $l = 0, 1$, be the smallest integer numbers satisfying

$$F_l(\alpha_l \Delta t + (-d_l + 0.5)\Delta y) < \varepsilon, \quad \text{and} \tag{7}$$
$$1 - F_l(\alpha_l \Delta t + (u_l - 0.5)\Delta y) < \varepsilon, \quad \text{respectively,} \tag{8}$$

where $\Delta y = \sigma_0\sqrt{3/2\Delta t}$. Clearly, by choosing $d = \max\{d_l : l = 0, 1\}$ and $u = \max\{u_l : l = 0, 1\}$, we have that in both regimes the probability in the right and in the left tail of the jump size distribution is smaller than ε. Starting from inception (node $(0, 0)$) where the discrete process has value $X(0, 0) = 0$, the value assumed by the logarithm of the asset price return at each node (i, j) of the grid at the i-th time step is given by $X(i, j) = i\alpha_0\Delta t + j\Delta y$, $i = 0, \ldots, n$; $j = -id, \ldots, iu$, where $\alpha_0 = r_0 - \sigma_0^2/2 - \lambda_0 m_0$.

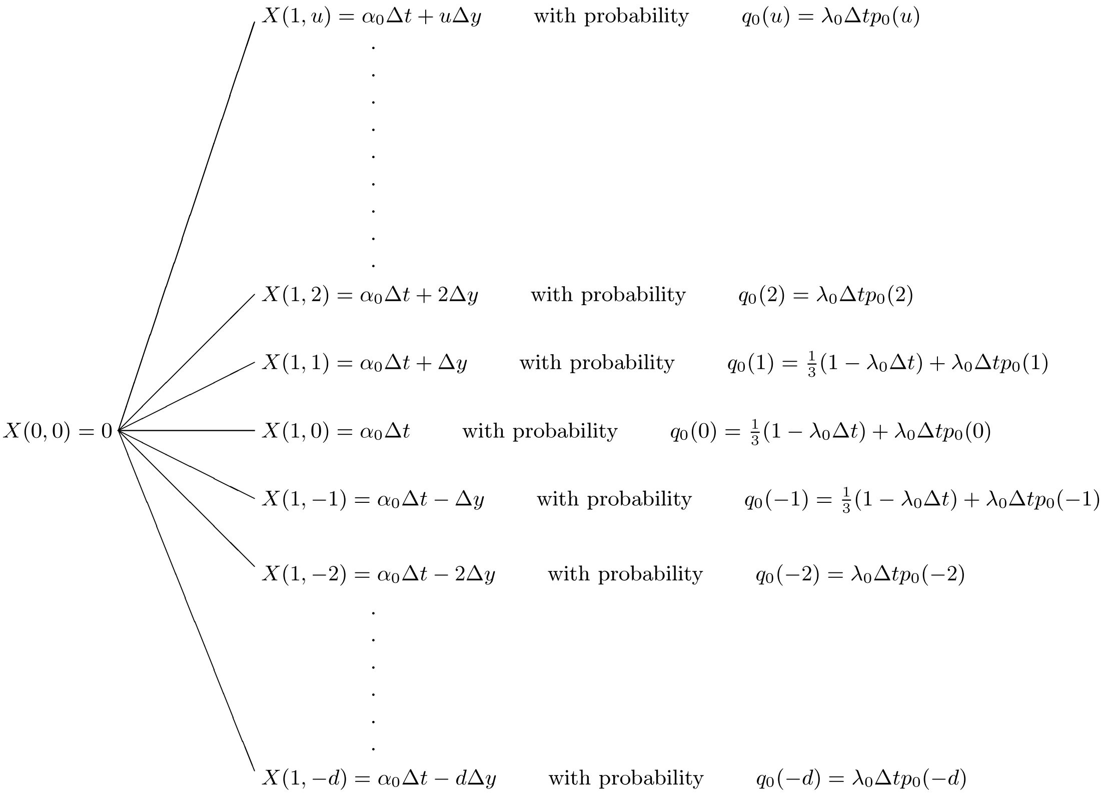

Fig. 1. The approximation of the process (3) at the first time step in regime 0.

Assuming that at inception we observe regime 0, to guarantee that the discrete approximating process has the same local mean and the same local variance of the diffusion component in the continuous process, we assign transition probability equal to 1/3 to the values $X(1,1) = \alpha_0 \Delta t + \Delta y$, $X(1,0) = \alpha_0 \Delta t$, and $X(1,-1) = \alpha_0 \Delta t - \Delta y$. To take properly into account the effect of the jump component, we approximate the jump distribution on the entire real line by breaking it down over nonoverlapping intervals of equal width Δy. The probability mass over each one of these intervals is assigned to the proper node in the interval. Note that, in regime 0, the probability that the process shows a single jump in a small interval Δt is given by $\lambda_0 \Delta t + o(\Delta t)$ while the probability of multiple jumps is $o(\Delta t)$. We assume that the probability of a jump in the discrete approximating model at each time step $i\Delta t$, $i = 1, \ldots, n$, is $\lambda_0 \Delta t$ and that multiple jumps cannot occur during each time interval, hence the probability that the process does not jump is $1 - \lambda_0 \Delta t$. Consequently, the probability assigned to each node $X(1,j) = \alpha_0 \Delta t + j\Delta y$, $j = -d, \ldots, u$, is given by

$$q_0(j) = \begin{cases} \dfrac{1}{3}(1-\lambda_0\Delta t) + p_0(j)\lambda_0\Delta t & \text{if } j = -1, 0, 1; \\ p_0(j)\lambda_0\Delta t & \text{otherwise;} \end{cases}$$

where

$$p_0(j) = \begin{cases} F_0\left(\alpha_0\Delta t + (-d+0.5)\Delta y\right) & \text{if } j = -d; \\ F_0\left(\alpha_0\Delta t + (j+0.5)\Delta y\right) - F_0\left(\alpha_0\Delta t + (j-0.5)\Delta y\right) & \text{if } j = -d+1, \ldots, u-1; \\ 1 - F_0\left(\alpha_0\Delta t + (u-0.5)\Delta y\right) & \text{if } j = u. \end{cases}$$

In Fig. 1, we illustrate the discretization defined above for the first time step.

At a generic node (i,j) of the grid, if regime 0 is observed at time $i\Delta t$, the successor points of $X(i,j)$ at the $(i+1)$-th time step are $X(i+1, j+x) = (i+1)\alpha_0\Delta t + (j+x)\Delta y$ with $x = -d, \ldots, u$, and probabilities

$$q_0(x) = \begin{cases} \dfrac{1}{3}(1-\lambda_0\Delta t) + p_0(x)\lambda_0\Delta t & \text{if } x = j-1, j, j+1; \\ p_0(x)\lambda_0\Delta t & \text{otherwise;} \end{cases}$$

where

$$p_0(x) = \begin{cases} F_0\left(\alpha_0\Delta t + (-d+0.5)\Delta y\right) & \text{if } x = -d; \\ F_0\left(\alpha_0\Delta t + (x+0.5)\Delta y\right) - F_0\left(\alpha_0\Delta t + (x-0.5)\Delta y\right) & \text{if } x = -d+1, \ldots, u-1; \\ 1 - F_0\left(\alpha_0\Delta t + (u-0.5)\Delta y\right) & \text{if } x = u. \end{cases}$$

Contrary, if regime 1 is observed, branching probabilities are defined as follows. Let π_1^u be the probability assigned to $X(i+1,j+1) = (i+1)\alpha_0\Delta t + (j+1)\Delta y$, π_1^m the transition probability assigned to $X(i+1,j) = (i+1)\alpha_0\Delta t + j\Delta y$, and π_1^d the transition probability assigned to $X(i+1,j-1) = (i+1)\alpha_0\Delta t + (j-1)\Delta y$. The probabilities π_1^u, π_1^m, π_1^d are obtained by solving the following system

$$\begin{cases} \pi_1^u + \pi_1^m + \pi_1^d = 1, \\ \pi_1^u(\alpha_0\Delta t + \Delta y) + \pi_1^m\alpha_0\Delta t + \pi_1^d(\alpha_0\Delta t - \Delta y) = \alpha_1\Delta t, \\ \pi_1^u(\alpha_0\Delta t + \Delta y)^2 + \pi_1^m\alpha_0^2\Delta t^2 + \pi_1^d(\alpha_0\Delta t - \Delta y)^2 = \sigma_1^2\Delta t + \alpha_1^2\Delta t^2, \end{cases}$$

in order to assure that the local drift and the local second moment of the discrete process approach the continuous time ones, and they are given by

$$\pi_1^u = \frac{\sigma_1^2\Delta t + (\alpha_1-\alpha_0)^2\Delta t^2 + (\alpha_1-\alpha_0)\Delta t\Delta y}{2\Delta y^2};$$

$$\pi_1^m = 1 - \frac{\sigma_1^2\Delta t + (\alpha_1-\alpha_0)^2\Delta t^2}{\Delta y^2};$$

$$\pi_1^d = \frac{\sigma_1^2\Delta t + (\alpha_1-\alpha_0)^2\Delta t^2 - (\alpha_1-\alpha_0)\Delta t\Delta y}{2\Delta y^2}.$$

To take also into account the jump component, we assign to the possible successor points of $X(i,j)$ at the $(i+1)$-th time step, $X(i+1, j+x)$ with $x = -d, \dots, u$, probabilities

$$q_1(x) = \begin{cases} \pi_1^u(1-\lambda_1\Delta t) + p_1(x)\lambda_1\Delta t & \text{if } x = j+1; \\ \pi_1^m(1-\lambda_1\Delta t) + p_1(x)\lambda_1\Delta t & \text{if } x = j; \\ \pi_1^d(1-\lambda_1\Delta t) + p_1(x)\lambda_1\Delta t & \text{if } x = j-1; \\ p_1(x)\lambda_1\Delta t & \text{otherwise;} \end{cases}$$

where

$$p_1(x) = \begin{cases} F_1\left(\alpha_0\Delta t + (-d+0.5)\Delta y\right) & \text{if } x = -d; \\ F_1\left(\alpha_0\Delta t + (x+0.5)\Delta y\right) - F_1\left(\alpha_0\Delta t + (x-0.5)\Delta y\right) & \text{if } x = -d+1, \dots, u-1; \\ 1 - F_1\left(\alpha_0\Delta t + (u-0.5)\Delta y\right) & \text{if } x = u. \end{cases}$$

We are in the position now to establish the backward procedure that allows us to evaluate contingent claims by forming expectations of their payoffs over the grid branches. We model the regime persistence or transition through the probabilities arising from matrix A reported in (1). Here, we illustrate the computation of the price of a European call option having payoff at maturity (the n-th time step), in regime $l = 0, 1$,

$$c_l(n,j) = \max(S(0)e^{X(n,j)} - K, 0), \quad \text{with } j = -nd, \dots, nu.$$

At node (i,j), if regime 0 is observed at time $i\Delta t$, the option price is computed by backward recursion, that is

$$c_0(i,j) = e^{-r_0\Delta t}\sum_{x=-d}^{u} q_0(x)\left[\left(1 - a_{0,1}\Delta t\right)c_0(i+1, j+x) + a_{0,1}\Delta t c_1(i+1, j+x)\right]. \tag{9}$$

Similarly it happens if regime 1 is observed at time $i\Delta t$.

The price of the corresponding American-style counterpart contracts may be easily computed by the lattice algorithm described above by simply considering in Eq. (9) the maximum between the option continuation value and its early exercise value.

5. Numerical results

We test the pricing model presented in Section 4 by computing the prices of European options and by proposing a comparison with the Fourier transform method of Ramponi [12] and the explicit formulae derived in Section 3, chosen as the benchmark. To generate numerical results, we assume $Y_l(k)$, $l = 0, 1$ lognormally distributed, i.e., $\log Y_l(k) \sim \mathrm{N}(\eta_l, \delta_l)$, and set the tolerance level in Eqs. (7) and (8) to $\varepsilon = 10^{-10}$.

At first, to assess the goodness of the proposed model, in Table 6 we report the prices of European call options with maturity $T = 1$ year in a two-regime economy for different numbers of time steps n. The risk-free rate is $r = 0.1$ in both regimes, while the high-volatility regime is characterized by $\sigma_0 = 0.6$ and the low-volatility one by $\sigma_1 = 0.2$. We consider different initial values for the underlying asset, $S(0)$, the strike price is fixed at level $K = 10$, and the parameters governing the regime transition or persistence are $a_{0,1} = a_{1,0} = 1$ (1/year). The lognormal distribution of the jump size has mean $\eta_0 = -0.02$ and standard deviation $\delta_0 = 0.2$ in regime 0, and $\eta_1 = -0.01125$ and $\delta_1 = 0.15$ in regime 1, while the jump intensity is the same in both regimes, $\lambda_0 = \lambda_1 = 7$. In the last two columns, we report the option prices computed in both regimes by the Ramponi's [12] (R) method and by the explicit formula (Explicit) detailed in Section 3. The proposed formula

Table 6
European call option prices in a two-regime economy with $a_{0,1} = a_{1,0} = 1$ (1/year). The table presents a comparison among the European call option prices provided by the lattice model proposed in Section 4, for different numbers of time steps n, and the ones generated by the Fourier transform method proposed by Ramponi [12] (R), and by the explicit formula (Explicit) detailed in Section 3, chosen as the benchmark. A two-regime economy is considered and, in the first row and in the first column, we report the input parameters.

$r = 0.1$, $T = 1$ year, $K = 10$, $a_{0,1} = a_{1,0} = 1$, $\lambda_0 = \lambda_1 = 7$				
$S(0)$	$n = 1000$	$n = 2000$	R	Explicit
High-volatility regime $\sigma_0 = 0.6$, $\eta_0 = -0.02$, $\delta_0 = 0.2$				
6	0.8772	0.8797	0.8826	0.8742
8	1.8523	1.8569	1.8613	1.8542
10	3.1157	3.1216	3.1277	3.1199
12	4.5876	4.5951	4.6028	4.5925
14	6.2055	6.2148	6.2239	6.2108
Low-volatility regime $\sigma_1 = 0.2$, $\eta_1 = -0.01125$, $\delta_1 = 0.15$				
6	0.5414	0.5429	0.5447	0.5446
8	1.3670	1.3705	1.3738	1.3737
10	2.5686	2.5733	2.5784	2.5783
12	4.0464	4.0529	4.0597	4.0596
14	5.7066	5.7151	5.7234	5.7234

is implemented in such a way that all the summations involving the probability function of a Poisson random variable, say N, are truncated so that the upper limit of the summation is $n^* = \inf\{n \in \mathbb{N} | F_N(n) \geq 1 - \epsilon\}$, with $\epsilon = 10^{-10}$. It is worth noting that in all the examined cases, the prices computed by the discrete time model are close to the benchmark, while the option values obtained by the Ramponi method seem to be less accurate especially in the high-volatility regime and when the option is deep-out-of-the-money.

To complete the numerical treatment of the pricing problem, in Table 7 we provide also the prices of American put options and, for comparison, we report the prices of the European counterpart of the contract with the corresponding benchmark. In detail, in Table 7, we present the results for European and American put options with maturity $T = 1$ year in a two-regime economy for different numbers of time steps n. The risk-free rate is $r = 0.08$ in both regimes, while the high-volatility regime is characterized by $\sigma_0 = 0.3$ and the low-volatility one by $\sigma_1 = 0.1$. We consider different initial values for the strike price, K, the underlying asset has initial value $S(0) = 40$, and the parameters governing the regime transition or persistence are $a_{0,1} = a_{1,0} = 0.5$ (1/year). The lognormal distribution of the jump size has the same mean $\eta_0 = \eta_1 = -0.025$, the same standard deviation $\delta_0 = \delta_1 = \sqrt{0.05}$, and the same jump intensity $\lambda_0 = \lambda_1 = 5$ in both regimes. Again, in the last two columns, we report the option prices computed in both regimes by the Ramponi's [12] (R) method and by the explicit formula (Explicit) for the European case detailed in Section 3. In all the examined cases, as expected, the American put prices are greater than the corresponding European ones, which once again are accurate with respect to the prices computed using the explicit formula.

In order to show the numerical behavior of the option prices provided by the proposed multinomial approach, we study the values of European and American options in a regime switching model with and without jumps. For each case, we provide a numerical analysis showing the pattern followed by option prices when increasing the number of time steps in the lattice. In Table 8, we show the pattern followed by the at-the-money European call option prices reported in Table 6, while in Table 9 we consider the at-the-money American put option reported in Table 7. "Difference" stays for the difference between two consecutive option prices, while "Ratio" stays for the ratio between two consecutive differences. It is worth evidencing that, in both the cases, the ratios are closed to 0.5 even if they are different on the two regimes due to the approximation errors, which differ in each regime. Furthermore, the absolute value of the differences decreases and the changes in option prices is close to zero as the number of time steps is doubled.

To assess further the goodness of the proposed multinomial approach, in Tables 10 and 11, we provide a comparison among our model (Jump), our model without jumps (No Jump), and the Yuen–Yang [13] (YY) model that evaluates options under regime-switching. In Table 10, we show the price of an at-the-money European call option and, in Table 11, the price of an at-the-money American put option both with maturity $T = 1$ year in a two-regime economy when doubling the number of time steps n. The risk-free rate is $r_0 = 0.06$ in regime 0 and $r_1 = 0.04$ in regime 1, while the high-volatility regime is characterized by $\sigma_0 = 0.35$ and the low-volatility one by $\sigma_1 = 0.25$. The underlying asset is fixed at $S(0) = 100$, the strike price is fixed at level $K = 100$, and the parameters governing the regime transition or persistence are $a_{0,1} = a_{1,0} = 0.5$ (1/year). The lognormal distribution of the jump size has mean $\eta_0 = \eta_1 = -0.025$, standard deviation $\delta_0 = \delta_1 = \sqrt{0.05}$, and the jump intensity is $\lambda_0 = \lambda_1 = 7$ in both regimes. From both the tables, it is evident that our model without jumps provides option prices very close to the YY ones in both regimes.

Table 7
European and American put option prices in a two-regime economy with $a_{0,1} = a_{1,0} = 0.5$ (1/year). The table presents a comparison among the European and American put option prices provided by the lattice model proposed in Section 4, for different numbers of time steps n, and the ones generated by the Fourier transform method proposed by Ramponi [12] (R), and by the explicit formula (Explicit) detailed in Section 3, chosen as the benchmark. A two-regime economy is considered and, in the first row and in the first column, we report the input parameters.

$r = 0.08$, $T = 1$ year, $S(0) = 40$, $a_{0,1} = a_{1,0} = 0.5$, $\eta_0 = \eta_1 = -0.025$, $\delta_0 = \delta_1 = \sqrt{0.05}$, $\lambda_0 = \lambda_1 = 5$					
K	Type	$n = 1000$	$n = 2000$	R	Explicit
High-volatility regime: $\sigma_0 = 0.3$					
30	European	2.8518	2.8522	2.8529	2.8526
	American	2.9571	2.9577		
35	European	4.7070	4.7072	4.7077	4.7074
	American	4.9081	4.9086		
40	European	7.0365	7.0369	7.0372	7.0369
	American	7.3803	7.3810		
45	European	9.7876	9.7874	9.7877	9.7873
	American	10.3287	10.3290		
50	European	12.8957	12.8952	12.8952	12.8948
	American	13.6944	13.6946		
Low-volatility regime: $\sigma_1 = 0.1$					
30	European	2.3830	2.3825	2.3821	2.3819
	American	2.4707	2.4703		
35	European	4.0937	4.0926	4.0918	4.0915
	American	4.2690	4.2682		
40	European	6.3189	6.3177	6.3165	6.3162
	American	6.6312	6.6304		
45	European	9.0173	9.0155	9.0141	9.0137
	American	9.5273	9.5259		
50	European	12.1198	12.1175	12.1158	12.1154
	American	12.8968	12.8953		

Table 8
Convergence analysis for the at-the-money European call option in Table 6. The table presents the pattern followed by the at-the-money European call option in Table 6 when doubling the number of time steps n.

$S(0) = 10$, $r = 0.1$, $T = 1$ year, $K = 10$, $a_{0,1} = a_{1,0} = 1$, $\lambda_0 = \lambda_1 = 7$			
n	Price	Difference	Ratio
High-volatility regime $\sigma_0 = 0.6$, $\eta_0 = -0.02$, $\delta_0 = 0.2$			
40	2.9324	0.0919	0.5783
80	3.0244	0.0532	0.4520
160	3.0775	0.0240	0.5539
320	3.1016	0.0133	0.4712
640	3.1149	0.0063	0.5310
1280	3.1212	0.0033	
2560	3.1245		
Low-volatility regime $\sigma_1 = 0.2$, $\eta_1 = -0.01125$, $\delta_1 = 0.15$			
40	2.4370	0.0635	0.6453
80	2.5006	0.0410	0.4158
160	2.5416	0.0170	0.5999
320	2.5586	0.0102	0.4496
640	2.5688	0.0046	0.5560
1280	2.5734	0.0026	
2560	2.5760		

Finally, in Table 12, we apply our model to a down-and-out European call option with barrier level at $H = 90$. The parameters are set as follows: $S(0) = 100$, $K = 100$, $T = 1$ year, the risk-free rate is $r_0 = 0.06$ in regime 0 and $r_1 = 0.04$ in regime 1, the high-volatility regime is characterized by $\sigma_0 = 0.35$ and the low-volatility one by $\sigma_1 = 0.25$, the parameters

Table 9
Convergence analysis for the at-the-money American put option in Table 7. The table presents the pattern followed by the at-the-money American option in Table 7 when doubling the number of time steps n.

$S(0) = 40$, $r = 0.08$, $T = 1$ year, $K = 40$, $a_{0,1} = a_{1,0} = 0.5$, $\lambda_0 = \lambda_1 = 5$, $\eta_0 = \eta_1 = -0.025$, $\delta_0 = \delta_1 = \sqrt{0.05}$			
n	Price	Difference	Ratio
High-volatility regime $\sigma_0 = 0.3$			
40	7.3780	−0.0138	0.3550
80	7.3642	−0.0049	0.5181
160	7.3593	−0.0025	0.5809
320	7.3567	−0.0015	0.5544
640	7.3553	−0.0008	0.5695
1280	7.3544	−0.0005	
2560	7.3540		
Low-volatility regime $\sigma_1 = 0.1$			
40	6.7303	−0.0559	0.4530
80	6.6744	−0.0253	0.4994
160	6.6490	−0.0127	0.5164
320	6.6364	−0.0065	0.5128
640	6.6298	−0.0034	0.5191
1280	6.6265	−0.0017	
2560	6.6248		

Table 10
Comparison for an at-the-money European call option of the multinomial approach with no jump models. The table presents the pattern followed by an at-the-money European call option when doubling the number of time steps n and provides a comparison among the multinomial approach with (Jump) and without jumps (No Jump), and the Yuen–Yang's [13] (YY) model.

$S(0) = 100$, $T = 1$ year, $K = 100$, $a_{0,1} = a_{1,0} = 0.5$, $\lambda_0 = \lambda_1 = 5$, $\eta_0 = \eta_1 = -0.025$, $\delta_0 = \delta_1 = \sqrt{0.05}$			
n	Jump	No Jump	YY
High-volatility regime $r_0 = 0.06$, $\sigma_0 = 0.35$			
40	24.7370	15.7541	15.7603
80	25.1236	15.7611	15.7627
160	25.3196	15.7643	15.7640
320	25.4183	15.7658	15.7646
640	25.4677	15.7664	15.7650
1280	25.4924	15.7666	15.7651
2560	25.5047	15.7666	15.7652
Low-volatility regime $r_0 = 0.04$, $\sigma_0 = 0.25$			
40	23.2022	12.7075	12.6936
80	23.4744	12.7363	12.7260
160	23.6136	12.7502	12.7422
320	23.6840	12.7569	12.7503
640	23.7193	12.7601	12.7543
1280	23.7369	12.7616	12.7563
2560	23.7457	12.7622	12.7573

governing the regime transition or persistence are $a_{0,1} = a_{1,0} = 0.5$ (1/year), the lognormal distribution of the jump size has mean $\eta_0 = -0.02$ and standard deviation $\delta_0 = 0.2$ in regime 0, and mean $\eta_1 = -0.01125$ and standard deviation $\delta_1 = 0.15$ in regime 1, while the jump intensity is $\lambda_0 = \lambda_1 = 5$ in both regimes. To obtain unbiased option values in the presence of the barrier, we need a layer of lattice nodes such that the barrier is hit exactly. This aspect may not be achieved using the discretization proposed in Section 4 due to the presence of the drift in the discrete values considered for the underlying asset. Nevertheless, the discretization may be easily adjusted to satisfy the requirement. Indeed, referring to the considered

Table 11
Comparison for an at-the-money American put option of the multinomial approach with no jump models. The table presents the pattern followed by an at-the-money American put option when doubling the number of time steps n, and provides a comparison among the multinomial approach with (Jump) and without jumps (No Jump), and the Yuen–Yang's [13] (YY) model.

$S(0) = 100$, $T = 1$ year, $K = 100$, $a_{0,1} = a_{1,0} = 0.5$, $\lambda_0 = \lambda_1 = 5$, $\eta_0 = \eta_1 = -0.025$, $\delta_0 = \delta_1 = \sqrt{0.05}$			
n	Jump	No Jump	YY
High-volatility regime $r_0 = 0.06$, $\sigma_0 = 0.35$			
40	20.3627	10.8905	10.8949
80	20.4394	10.8954	10.8967
160	20.4756	10.8969	10.8969
320	20.4932	10.8975	10.8970
640	20.5018	10.8976	10.8970
1280	20.5061	10.8976	10.8970
2560	20.5081	10.8976	10.8970
Low-volatility regime $r_0 = 0.04$, $\sigma_0 = 0.25$			
40	19.7969	8.8634	8.8555
80	19.8262	8.8880	8.8823
160	19.8394	8.8997	8.8953
320	19.8455	8.9052	8.9016
640	19.8484	8.9076	8.9047
1280	19.8498	8.9090	8.9063
2560	19.8504	8.9095	8.9070

Table 12
Convergence analysis for an at-the-money down-and-out barrier European call option in presence of jumps, and comparison between the proposed approach without jumps and the YY model. The table presents the pattern followed by an at-the-money down-and-out barrier European call option in the presence of jumps when doubling the number of time steps n, and provides a comparison between the proposed approach without jumps and the Yuen–Yang's [13] (YY) model.

$S(0) = 100$, $T = 1$ year, $K = 100$, $H = 90$, $a_{0,1} = a_{1,0} = 0.5$, $\lambda_0 = \lambda_1 = 5$					
n	Jump	Difference	Ratio	No Jump	YY
High-volatility regime $r_0 = 0.06$, $\sigma_0 = 0.35$, $\eta_0 = -0.02$, $\delta_0 = 0.2$					
40	12.4078	0.3535	0.2586	9.6719	9.7118
80	12.7613	0.0914	0.5120	9.6993	9.7104
160	12.8527	0.0468	0.3611	9.6998	9.7036
320	12.8995	0.0169	0.5207	9.7001	9.7010
640	12.9164	0.0088	0.5227	9.6994	9.7002
1280	12.9252	0.0046		9.6991	9.6991
2560	12.9298			9.6990	9.6990
Low-volatility regime $r_1 = 0.04$, $\sigma_1 = 0.25$, $\eta_1 = -0.01125$, $\delta_1 = 0.15$					
40	11.1326	0.4002	0.2541	8.8956	8.9662
80	11.5328	0.1017	0.5152	8.9588	8.9748
160	11.6345	0.0524	0.3454	8.9656	8.9706
320	11.6869	0.0181	0.5193	8.9691	8.9701
640	11.7050	0.0094	0.5213	8.9693	8.9703
1280	11.7144	0.0049		8.9695	8.9693
2560	11.7193			8.9696	8.9696

case of a down-and-out European call option with knock-out barrier H, the value assumed by the logarithm of the asset price return at each node (i, j) of the grid at the i-th time step is now computed as $X(i, j) = ij\omega\Delta y$, $i = 0, \ldots, n$; $j = -d, \ldots, u$, where $\Delta y = \sigma_0\sqrt{\Delta t}$ and ω is a convenient parameter chosen as in Ritchken [16]. Let h be the number of consecutive down moves leading to the lowest layer of nodes above the barrier H, that is the largest integer smaller than $\eta = \frac{\log(S/H)}{\Delta y}$. If η is an integer, ω assumes value one. On the contrary, if η is not an integer, ω is chosen so that $\eta = h\omega$. With this construction, a layer of nodes of the multinomial tree coincides with the barrier H. It is worth noting that starting from a generic node (i, j) where regime 0 is observed, to guarantee that the discrete approximating process has the same local mean and the same local variance of the diffusion component in the continuous process, $\pi_0^u = \frac{1}{2\omega^2} + \frac{\alpha_0\sqrt{t}}{2\omega\sigma_0}$ is the transition probability assigned to $X(i+1, j+1) = (j+1)\omega\Delta y$, $\pi_0^m = 1 - \frac{1}{\omega^2}$ is the transition probability assigned to $X(i+1, j) = j\omega\Delta y$, and $\pi_0^d = \frac{1}{2\omega^2} - \frac{\alpha_0\sqrt{t}}{2\omega\sigma_0}$ is the transition probability assigned to $X(i+1, j-1) = (j-1)\omega\Delta y$. The probabilities when regime 1 is observed are computed following the lines in Section 4 taking into account the different discretization described above.

In order to show the behavior of the prices provided by the multinomial approach in the case of barrier option, in Table 12, we report the pattern followed by option prices when increasing the number of time steps in the lattice. It is worth evidencing that the ratios are close to 0.5, and that the absolute value of the differences decreases and the changes in option prices is close to zero as the number of time steps is doubled. We provide also a comparison among our model without jumps and the Yuen–Yang's [13] model, which evidences that the two models provide very closed values in both regimes even in the presence of the barrier.

6. Conclusions

We have proposed an explicit formula and a more general multinomial approach for pricing contingent claims when the underlying asset process follows a regime-switching jump–diffusion model. The choice of this framework is supported by an econometric analysis based on daily equity index data. The explicit formula allows to compute European option prices in the case of a two-regime economy with lognormal jumps characterized by the same risk-free rate in both regimes, while the discrete time model has the nice feature to accommodate an arbitrary number of regimes and generic jump size distributions. In the latter approach, all regime share the same lattice nodes and only branching probabilities are separately adjusted in order to match the first two order moments of the continuous time distribution in each regime. Using a backward induction scheme, the model allows to compute the price of both European and American-style contingent claims.

To support the model, we have provided some numerical examples showing that the proposed algorithm computes accurate values in comparison to the benchmark.

As future research, we plan to extend both the explicit formula by considering alternative underlying asset processes. For instance, the proposed formula may be easily adapted to all those cases in which the derivatives prices conditional to the occupation time in one of the two regimes is available in explicit form. Further researches will address the extension of the multinomial approach for pricing path-dependent derivatives, like lookback or Asian options.

Appendix

Preliminary results

Here, we report some preliminary results useful to prove Propositions 1 and 2. Defining $\Pi(y) = (y-K)^+$ with $K > 0$ and being $X \sim \mathrm{N}(A, B)$, we recall that for any $r, T, S(0) > 0$,

$$\mathbb{E}\left[e^{-rT}\Pi(e^{X+\log S(0)})\right] = e^{-rT}S(0)e^{A+B^2/2}\Phi(d_1(A,B)) - Ke^{-rT}\Phi(d_2(A,B)), \tag{A.1}$$

with

$$d_1(A,B) = \frac{\log(S(0)/K) + A + B^2}{B}, \qquad d_2(A,B) = \frac{\log(S(0)/K) + A}{B} = d_1(A,B) - B.$$

Suppose, instead, that

$$X(T) = \log(S(T)/S(0)) = (r - \sigma^2/2 - \lambda\overline{m})T + \sigma\sqrt{T}Z + \sum_{k=1}^{N}\log Y(k),$$

with $Z \sim \mathrm{N}(0,1)$, $N \sim \mathrm{Poiss}(\lambda T)$, $\log Y(k) \sim \mathrm{N}(\eta, \delta)$, and $\overline{m} = \exp(\eta + \delta^2/2) - 1$. Then,

$$X(T)|N = n \sim \mathrm{N}(A_n, B_n),$$

with

$$A_n = (r - \sigma^2/2 - \lambda\overline{m})T + n\eta,$$
$$B_n = \sqrt{\sigma^2 T + n\delta^2},$$

and

$$\mathbb{E}[e^{-rT}\Pi(e^{X(T)+\log S(0)})|N = n] = C(S(0), K, T, r, A_n, B_n). \tag{A.2}$$

Eq. (A.2) can be also expressed as

$$\mathbb{E}[e^{-rT}\Pi(e^{X(T)+\log S(0)})|N = n] = C_{BS}(S_n, K, T, r, \sigma_n),$$

where

$$S_n = S(0)\exp[(\eta + \delta^2/2)n - \lambda\overline{m}T] = S(0)\exp[A_n + B_n^2/2 - rT],$$
$$\sigma_n = \sqrt{\sigma^2 + n\delta^2/T} = B_n/\sqrt{T},$$

and $C_{BS}(S, K, T, r, \sigma)$ is the usual Black–Scholes formula with risk-free r, for a call with time to maturity T, strike K, written on a stock with price S and volatility σ. Hence,

$$\mathbb{E}\left[e^{-rT}\Pi(e^{X(T)+\log S(0)})\right] = \sum_{n=0}^{\infty} C_{BS}(S_n, K, T, r, \sigma_n)p_N(n). \tag{A.3}$$

We remark that (A.3) is the usual Merton formula. □

Proof of Proposition 1. Considering a two-regime economy characterized by a constant risk-free rate, r, log-returns in (3) may be represented (cfr., Proposition 2.1 in [12]) as

$$X(T) = \log\frac{S(T)}{S(0)} = \left(r - \frac{1}{2}\sigma_0^2 - \lambda_0 m_0\right)T_0 + \sigma_0\sqrt{T_0}Z_0 + \sum_{k=1}^{N(T_0)}\log Y_0(k)$$
$$+ \left(r - \frac{1}{2}\sigma_1^2 - \lambda_1 m_1\right)T_1 + \sigma_1\sqrt{T_1}Z_1 + \sum_{k=1}^{N(T_1)}\log Y_1(k),$$

where T_0 denotes the occupation time of the first of the two states of the Markov chain, $T_1 = T - T_0$, Z_0 and Z_1 are i.i.d $\mathrm{N}(0, 1)$, $N(T_l) \sim \mathrm{Poiss}(\lambda_l T_l)$, and $\log Y_l(k) \sim \mathrm{N}(\eta_l, \delta_l)$ with $l = 0, 1$. Consequently,

$$X(T)|T_0 = t, \qquad N(T_0) = n, \qquad N(T_1) = m \sim \mathrm{N}(A_{n,m}(t), B_{n,m}(t)).$$

The results follow from

$$c(0|\epsilon(0) = 0) = \int_0^T \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{E}[e^{-rT}(e^{X(T)+\log S(0)} - K)^+|T_0 = t, N(T_0) = n, N(T_1) = m]$$
$$\times p_{N(T_0)}(n|T_0 = t, \epsilon(0) = 0)p_{N(T_1)}(m|T_0 = t, \epsilon(0) = 0)f_{T_0}(t|\epsilon(0) = 0)dt,$$

and (A.1). □

Proof of Proposition 2. In this case,

$$X(T) = \log(S(T)/S(0)) = (-\sigma_0^2/2)T_0 + \sigma_0\sqrt{T_0}Z_0 + (-\sigma_1^2/2)(T_1) + \sigma_1\sqrt{T_1}Z_1 + (r - \lambda\overline{m})T + \sum_{k=1}^{N(T_0)+N(T_1)}\log Y(k),$$

where $\log Y_k \sim \mathrm{N}(\eta, \delta)$. Hence,

$$X(T) = \log(S(T)/S(0)) = (-\sigma_0^2/2)T_0 + \sigma_0\sqrt{T_0}Z_0 + (-\sigma_1^2/2)(T_1) + \sigma_1\sqrt{T_1}Z_1 + (r - \lambda\overline{m})T + \sum_{k=1}^{N}\log Y_k,$$

where $N \sim \mathrm{Poiss}(\lambda T)$. Consequently,

$$X(T)|T_0 = t, N = n \sim \mathrm{N}(A_n(t), B_n(t)),$$

with

$$A_n(t) = (r - \lambda\overline{m})T + (-\sigma_0^2/2)t + (-\sigma_1^2/2)(T - t) + n\eta,$$
$$B_n(t) = \sqrt{\sigma_0^2 t + \sigma_1^2(T - t) + n\delta^2}.$$

The desired results follow from (A.3) reported in the Appendix.

An alternative proof is based on the fact that Proposition 2 can be obtained as a particular case of Proposition 1. We note that when $\lambda_0 = \lambda_1 = \lambda$, $\eta_0 = \eta_1 = \eta$, $\delta_0 = \delta_1 = \delta$,

$$A_{n,m}(t) = A_{n+m}(t) \quad \text{and} \quad B_{n,m}(t) = B_{n+m}(t).$$

Denoting by $p(m; \lambda)$ the probability function of a Poisson random variable with intensity λ evaluated at point m, the quantity inside the integral of Proposition 1 can be written as

$$\sum_{h=0}^{\infty}\sum_{m=0}^{h} C(S(0), K, T, r, A_{h-m,m}(t), B_{h-m,m}(t))p(h - m; \lambda t)p(m; \lambda(T - t))$$

$$= \sum_{h=0}^{\infty} \sum_{m=0}^{h} C(S(0), K, T, r, A_h(t), B_h(t)) p(h-m; \lambda t) p(m; \lambda(T-t))$$

$$= \sum_{h=0}^{\infty} C(S(0), K, T, r, A_h(t), B_h(t)) \sum_{m=0}^{h} p(h-m; \lambda t) p(m; \lambda(T-t))$$

$$= \sum_{h=0}^{\infty} C(S(0), K, T, r, A_h(t), B_h(t)) p(h; \lambda T),$$

where the last equality follows from the convolution formula and from the fact that the sum of two independently distributed Poisson random variables is Poisson with intensity given by the sum of the intensities of the Poisson random variables. □

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