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A class of tests for the two-sample problem for count data

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Abstract

A class of tests for the two-sample problem for count data whose test statistic is an L_2 -norm of the difference between the empirical probability generating functions associated with each sample is considered. The tests can be applied to count data of any arbitrary fixed dimension. Since the null distribution of the test statistic is unknown, some approximations are investigated. Specifically, the bootstrap, permutation and weighted bootstrap estimators are examined. All of them provide consistent estimators. A simulation study analyzes the performance of these approximations for small and moderate sample sizes. This study also includes a comparison with other two-sample tests whose test statistic is a weighted integral of the difference between the empirical characteristic functions of the samples.

Keywords: two-sample problem, count data, probability generating function, simulation

1. Introduction

The two-sample problem, which consists on testing whether two samples come from the same population, is a statistical issue of great interest and many different approaches have been proposed to deal with it (see, for example, Baringhaus and Kolbe [1] for a recent paper on this topic and the references therein). One of them is related to the use of the characteristic function (CF) and its empirical counterpart (ECF) by means of an L_2 -norm between the ECFs associated with each sample. This kind of tests can be applied to all sort of data, continuous, discrete or mixed of any arbitrary fixed dimension. Since the resultant test statistic is not

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distribution free, Meintanis [2] (see also Alba-Fernández et al. [3]) suggested that it could be approximated by means of permutation and bootstrap procedures. Jiménez-Gamero et al. [4] studied the use of a weighted bootstrap estimator, in the sense of Burke [5].

This paper studies the two-sample problem for count data. When dealing with this type of data, Nakamura and Pérez-Abreu [6] argue in favor of applying inferential methods based on the empirical probability generating function (EPGF). The motivation for using the probability generating function (PGF) is that it is usually much simpler than the corresponding probability mass function (PMF), fully characterizes the distribution and possesses convenient features not shared by the CF or the moment generating function such as being a real valued continuous analytic function which always exists in the range $[0, 1]^d$, where d is the dimension of the random vector under study. In fact, Nakamura and Pérez-Abreu [6] proposed an exploratory procedure for the univariate two-sample problem for count data, which consists in plotting the EPGF of both samples; if the resulting curves are not close, this would be an evidence that the samples come from different populations. In this paper, we formalize such a procedure and derive a test for the two-sample problem for count data with any arbitrary dimension. Specifically, the closeness between the EPGFs will be assessed by means of an L_2 -norm.

Therefore, motivated by Nakamura and Pérez-Abreu [6] and Alba-Fernández et al. [3], a class of tests based on the L_2 -norm between the EPGFs associated with both samples is considered. The limiting distribution of the test statistic under the null hypothesis is derived. It is not distribution free, so some approximations, such as bootstrap, permutation and weighted bootstrap, are studied.

Since the tests based on the EPGFs and those based on ECFs have similar asymptotic properties, a simulation study is carried out in order to investigate the performance of both approaches for small or moderate sample sizes. As expected from the results in Janssen [7] (the author states that every test has a preference for a finite dimensional space of alternatives; apart from this space, the power function is almost flat on balls of alternatives), there is no test yielding the highest power against all considered alternatives.

The paper is organized as follows. Section 2 defines the test statistic, derives its asymptotic null distribution and studies the consistency of the resulting test against fixed alternatives. Because the asymptotic null distribution of the test statistic depends on the unknown common PGF, Section 3 discusses three approaches to determine a critical point for the test or equivalently to approximate the p -value of the observed value of the test statistic. Specifically the bootstrap, the permutation and the weighted bootstrap are considered in order to consistently estimate the null distribution. Section 4 describes some computational issues related to the

calculation of the test statistic, as well as the bootstrap, permutation and weighted bootstrap approximations. To investigate the finite sample performance of the considered approximations, the power of the proposed test and to compare it with the tests based on an L_2 -norm of the difference between the ECFs, a simulation study was carried out. A summary of the obtained results is reported in Section 5. Section 6 concludes summarizing the main findings. All proofs are deferred to Section 7.

Before ending this section, some notation is introduced. Throughout this paper all vectors are column vectors; $1_n \in \mathbb{R}^n$ has all its components equal to 1; for any vector v , v_j denotes its j th coordinate and v' its transpose; I_A denotes the indicator function of the set A ; $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$; for $a = (a_1, \dots, a_d)' \in \mathbb{N}_0^d$ and $t = (t_1, \dots, t_d)' \in \mathbb{R}^d$, $t^a = t_1^{a_1} \cdot \dots \cdot t_d^{a_d}$; P_0 , E_0 and Cov_0 denote probability, expectation and covariance, respectively, by assuming that the null hypothesis is true; P_* , E_* and Cov_* denote the conditional probability law, conditional expectation and conditional covariance, given the data, respectively; $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution; \xrightarrow{P} denotes convergence in probability; $\xrightarrow{a.s.}$ denotes the almost sure convergence; an unspecified integral denotes integration over $[0, 1]^d$; $L_2(w) = \{f : [0, 1]^d \rightarrow \mathbb{R} : \|f\|_w^2 = \int |f(t)|^2 w(t) dt < \infty\}$, for some nonnegative function w satisfying $0 < \int w(t) dt < \infty$; $\langle \cdot, \cdot \rangle_w$ denotes the scalar product in the Hilbert space $L_2(w)$.

2. The test statistic

Let X and Y be two random vectors taking values in \mathbb{N}_0^d , for some fixed $d \in \mathbb{N}$, with cumulative distribution functions (CDF) F_X and F_Y , respectively, which are assumed to be unknown. Let us consider the problem of testing for the equality of both distributions. The null hypothesis is stated as

$$H_0 : F_X(x) = F_Y(x), \quad \forall x \in \mathbb{N}_0^d \iff C_X(t) = C_Y(t), \quad \forall t \in [0, 1]^d, \quad (1)$$

where C_X and C_Y are the PGFs of X and Y , respectively, that is,

$$C_X(t) = \int t^x dF_X(x), \quad C_Y(t) = \int t^x dF_Y(x).$$

Let X_1, \dots, X_n and Y_1, \dots, Y_m be two independent random samples from X and Y , with sizes n and m , respectively, and let $C_{X,n}$ and $C_{Y,m}$ denote the EPGF associated with the samples,

$$C_{X,n}(t) = \frac{1}{n} \sum_{j=1}^n t^{X_j}, \quad C_{Y,m}(t) = \frac{1}{m} \sum_{l=1}^m t^{Y_l}.$$

Motivated by Sim and Ong [8] and Ng et al. [9], to measure the closeness between two populations defined on \mathbb{N}_0^d , with PGFs C_X and C_Y , we consider

$$D^2 = \int \{C_X(t) - C_Y(t)\}^2 w(t) dt = \|C_X - C_Y\|_w^2, \quad (2)$$

where w is a probability density function (PDF) defined on $[0, 1]^d$, that will be assumed to be positive for almost all (with respect to the Lebesgue measure) $[0, 1]^d$. Note that $D = 0$ if and only if $C_X(t) = C_Y(t)$, $\forall t \in [0, 1]^d$, and therefore, the associated populations coincide.

Taking into account the a.s. convergence of the EPGF to the population PGF (see for example Novoa-Muñoz and Jiménez-Gamero [10]), for testing (1), we consider the following test function

$$\Psi = \begin{cases} 1, & \text{if } D_{n,m} \geq d_{n,m,\alpha}, \\ 0, & \text{otherwise,} \end{cases}$$

where $D_{n,m}$ is the sample version of D^2 ,

$$D_{n,m} = \|C_{X,n} - C_{Y,m}\|_w^2,$$

and $d_{n,m,\alpha}$ is the $1 - \alpha$ percentile of the null distribution of $D_{n,m}$.

In order to give a sound justification of $D_{n,m}$ as a test statistic for testing H_0 we next derive its limit.

Theorem 1. *Let X_1, \dots, X_n and Y_1, \dots, Y_m be two independent random samples from X and Y , respectively. Then*

$$D_{n,m} \xrightarrow{a.s.} D^2, \text{ when } n, m \rightarrow \infty,$$

where D^2 is as defined in (2).

Note that $D^2 \geq 0$ with, as observed before, $D^2 = 0$ if and only if H_0 is true. Thus, a reasonable test for testing H_0 should reject the null hypothesis for large values of $D_{n,m}$.

To decide when to reject H_0 , that is, to calculate $d_{n,m,\alpha}$ or, equivalently, to calculate the p -value of the observed value of the test statistic, we need to know the null distribution of $D_{n,m}$, which is clearly unknown, so one has to approximate it. We first try to estimate the null distribution of $D_{n,m}$ by means of its asymptotic null distribution. Let $N = n + m$. Note that under the null hypothesis $C_X(t) = C_Y(t) = C(t)$, $\forall t \in [0, 1]^d$.

Theorem 2. *Let X_1, \dots, X_n and Y_1, \dots, Y_m be two independent random samples having common PGF C . Suppose that $n/N \rightarrow \tau \in (0, 1)$, as $n, m \rightarrow \infty$. Then*

$$\frac{nm}{N} D_{n,m} \xrightarrow{\mathcal{L}} \|Z\|_w^2,$$

where $\{Z(t), t \in [0, 1]^d\}$ is a centered Gaussian process on $L_2(w)$ with covariance kernel $\varrho_0(t, s) = C(ts) - C(t)C(s)$.

The asymptotic null distribution of $D_{n,m}$ does not provide a useful approximation to its null distribution since it depends on the unknown common PGF, C . In the next section, we will study other ways of approximating it.

Remark 1. Before ending this section we have to mention that similar comments to that given by Alba-Fernández et al. [3] for the role of the weight function w in the expression of the test statistic $D_{n,m}$ hold. Specifically, it affects the calculation of $D_{n,m}$ (this fact will become evident later in Section 4), the consistency of Ψ (which is a direct consequence of Theorems 1 and 2 previously stated in this paper) and the null distribution of $D_{n,m}$ (which follows from Theorem 2 above).

3. Approximations of the null distribution

Since the asymptotic null distribution is unknown, we consider other ways of approximating the null distribution of the test statistic. Specifically, the bootstrap, permutation and weighted bootstrap methods are studied in this section.

The bootstrap approximation. Let $X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_m^*$ be independent and identically distributed (IID) from the empirical CFD of the pooled sample $X_1, \dots, X_n, Y_1, \dots, Y_m$. The bootstrap version of $D_{n,m}$, say $D_{n,m}^*$, is defined as $D_{n,m}^* = \|C_{X,n}^* - C_{Y,m}^*\|_w^2$, where $C_{X,n}^*$ is the EPGF associated with X_1^*, \dots, X_n^* and $C_{Y,m}^*$ is the EPGF associated with Y_1^*, \dots, Y_m^* . The next result gives the weak limit of the conditional distribution of $D_{n,m}^*$, given the data $X_1, \dots, X_n, Y_1, \dots, Y_m$.

Theorem 3. Let X_1, \dots, X_n and Y_1, \dots, Y_m be two independent random samples from populations with PGFs C_X and C_Y , respectively. Suppose that $n/N \rightarrow \tau \in (0, 1)$, as $n, m \rightarrow \infty$. Then,

$$\sup_x \left| P_* \left\{ \frac{nm}{N} D_{n,m}^* \leq x \right\} - P \left\{ \|Z_\tau\|_w^2 \leq x \right\} \right| \xrightarrow{a.s.} 0,$$

where $\{Z_\tau(t), t \in [0, 1]^d\}$ is a centered Gaussian process on $L_2(w)$ with covariance kernel $\varrho_\tau(t, s) = C_\tau(ts) - C_\tau(t)C_\tau(s)$, $C_\tau(t) = \tau C_X(t) + (1 - \tau)C_Y(t)$.

It is important to note that the result in Theorem 3 holds whether or not the null hypothesis H_0 is true. Note that if H_0 is true, then $C_\tau = C_X = C_Y$ and in such a case the processes $\{Z(t), t \in [0, 1]^d\}$ and $\{Z_\tau(t), t \in [0, 1]^d\}$ appearing in Theorems 2 and 3, respectively, both

have the same distribution. Therefore, if H_0 is true, a direct consequence of Theorems 2 and 3 is that $D_{n,m}^*$ provides a consistent estimator of the distribution of $D_{n,m}$. This is formally stated in the next corollary.

Corollary 1. *If H_0 is true and the assumptions in Theorem 2 hold, then*

$$\sup_x \left| P_* \left\{ \frac{nm}{N} D_{n,m}^* \leq x \right\} - P_0 \left\{ \frac{nm}{N} D_{n,m} \leq x \right\} \right| \xrightarrow{a.s.} 0.$$

Let $\alpha \in (0, 1)$ and

$$\Psi_* = \begin{cases} 1, & \text{if } D_{n,m} \geq d_{n,m,\alpha}^*, \\ 0, & \text{otherwise,} \end{cases}$$

where $d_{n,m,\alpha}^*$ is the $1 - \alpha$ percentile of the conditional distribution of $D_{n,m}^*$, or equivalently, $\Psi_* = 1$ if $p^* \leq \alpha$, where $p^* = P_* \{ D_{n,m}^* \geq D_{n,m,obs} \}$ and $D_{n,m,obs}$ is the observed value of the test statistic. The result in Corollary 1 states that Ψ_* is asymptotically correct, in the sense that its type I error probability is asymptotically equal to the nominal value α .

Corollary 2. *If H_0 is not true and the assumptions in Theorem 3 hold, then $P(\Psi_* = 1) \rightarrow 1$.*

Corollary 2 shows that the test Ψ_* is consistent in the sense of being able to asymptotically detect any (fixed) alternative.

The permutation approximation. Let $X_1^\dagger, \dots, X_n^\dagger, Y_1^\dagger, \dots, Y_m^\dagger$ be the sample obtained by randomly permuting the pooled sample. The permutation version of $D_{n,m}$, say $D_{n,m}^\dagger$, is defined as $D_{n,m}^\dagger = \|C_{X,n}^\dagger - C_{Y,m}^\dagger\|_w^2$, where $C_{X,n}^\dagger$ is the EPGF associated with $X_1^\dagger, \dots, X_n^\dagger$ and $C_{Y,m}^\dagger$ is the EPGF associated with $Y_1^\dagger, \dots, Y_m^\dagger$. The next result gives the weak limit of the conditional distribution of $D_{n,m}^\dagger$, given the data $X_1, \dots, X_n, Y_1, \dots, Y_m$.

Theorem 4. *Let X_1, \dots, X_n and Y_1, \dots, Y_m be two independent random samples from populations with PGFs C_X and C_Y , respectively. Suppose that $n/N \rightarrow \tau \in (0, 1)$, as $n, m \rightarrow \infty$. Then,*

$$\sup_x \left| P_* \left\{ \frac{nm}{N} D_{n,m}^\dagger \leq x \right\} - P \left\{ \|Z_\tau\|_w^2 \leq x \right\} \right| \xrightarrow{a.s.} 0,$$

where $\{Z_\tau(t), t \in [0, 1]^d\}$ is the centered Gaussian process on $L_2(w)$ defined in Theorem 3.

From Theorems 3 and 4, it follows that the bootstrap and permutation approximations are asymptotically equivalent, since both null distribution estimators converge to the same law. As a consequence, similar results to those stated in Corollaries 1 and 2 can be given for the

permutation approximation (to save space, we omit them). Therefore, for $0 < \alpha < 1$, the test function

$$\Psi_{\dagger} = \begin{cases} 1, & \text{if } D_{n,m} \geq d_{n,m,\alpha}^{\dagger} \Leftrightarrow P_*(D_{n,m}^{\dagger} \geq D_{n,m,obs}) \leq \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

is asymptotically correct and able to asymptotically detect any (fixed) alternative, where $d_{n,m,\alpha}^{\dagger}$ is the $1 - \alpha$ percentile of the conditional distribution of $D_{n,m}^{\dagger}$.

The weighted bootstrap approximation. Let $\xi_{X,1}, \dots, \xi_{X,n}$ and $\xi_{Y,1}, \dots, \xi_{Y,m}$ be IID random variates with mean 0 and variance 1, which are independent of X_1, \dots, X_n and Y_1, \dots, Y_m . The weighted bootstrap version of $D_{n,m}$, say $\tilde{D}_{n,m}$, is defined as $\tilde{D}_{n,m} = \|\tilde{C}_{X,n} - \tilde{C}_{Y,m}\|_w^2$, where

$$\tilde{C}_{X,n}(t) = \frac{1}{n} \sum_{j=1}^n \xi_j t^{X_j} - \tilde{C}(t), \quad \tilde{C}_{Y,m}(t) = \frac{1}{m} \sum_{l=1}^m \xi_l t^{Y_l} - \tilde{C}(t),$$

and

$$\tilde{C}(t) = \frac{n}{N} C_{X,n}(t) + \frac{m}{N} C_{Y,m}(t),$$

which estimates the common PGF under the null hypothesis. The conditional asymptotic distribution of $\tilde{D}_{n,m}$, given $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_m$, is stated in the next result.

Theorem 5. *Let X_1, \dots, X_n and Y_1, \dots, Y_m be two independent random samples from populations with PGFs C_X and C_Y , respectively. Assume that the sample sizes are such that $n/N \rightarrow \tau \in (0, 1)$, as $n, m \rightarrow \infty$ then*

$$\sup_x \left| P_* \left\{ \frac{nm}{N} \tilde{D}_{n,m} \leq x \right\} - P \left\{ \|Z_{\tau}\|_w^2 \leq x \right\} \right| \xrightarrow{P} 0,$$

where $\{Z_{\tau}(t), t \in [0, 1]^d\}$ is the centered Gaussian process on $L_2(w)$ defined in Theorem 3.

From Theorems 3 and 5, it follows that the bootstrap and weighted bootstrap approximations are asymptotically equivalent. As a consequence, similar results to those stated in Corollaries 1 and 2 can be given for the weighted bootstrap approximation (to save space, we omit them). Therefore, for $0 < \alpha < 1$, the test function

$$\tilde{\Psi} = \begin{cases} 1, & \text{if } D_{n,m} \geq \tilde{d}_{n,m,\alpha} \Leftrightarrow P_*(\tilde{D}_{n,m} \geq D_{n,m,obs}) \leq \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

is asymptotically correct and able to asymptotically detect any (fixed) alternative, where $\tilde{d}_{n,m,\alpha}$ is the $1 - \alpha$ percentile of the conditional distribution of $\tilde{D}_{n,m}$.

Remark 2. *The results so far stated keep on being true if instead of using the raw multipliers, $\xi_{X,1}, \dots, \xi_{X,n}, \xi_{Y,1}, \dots, \xi_{Y,m}$, we use the centered multipliers, $\xi_{X,1} - \bar{\xi}_X, \dots, \xi_{X,n} - \bar{\xi}_X, \xi_{Y,1} - \bar{\xi}_Y, \dots, \xi_{Y,m} - \bar{\xi}_Y$, where $\bar{\xi}_X = \frac{1}{n} \sum_{j=1}^n \xi_{X,j}$ and $\bar{\xi}_Y = \frac{1}{m} \sum_{l=1}^m \xi_{Y,l}$, as suggested in Burke [5] and [11, 12] for goodness-of-fit tests.*

Remark 3. As observed in the Introduction, the class of tests introduced in [2], which are based on comparing the ECFs associated with the samples, can be also used for testing the equality of two count populations. The tests in [2] satisfy: (a) they are not distribution free, so their null distribution must be also approximated; (b) the results in [3] are similar to those stated in Theorems 3 and 4 in this paper, but referred to the asymptotic behaviour of the bootstrap and permutation approximations for the test statistics proposed in [2]; (c) the paper [4] contains a result which is similar to that in Theorem 5 in this paper, but referred to the asymptotic behaviour of the weighted bootstrap approximation for the test statistics proposed in [2]. Therefore, the tests Ψ_* , Ψ_{\dagger} and $\tilde{\Psi}$ and their analogues based on comparing the ECFs associated with the samples share some properties, specifically, all of them are asymptotically correct and able to asymptotically detect any (fixed) alternative.

4. Practical calculations

This section describes some computational issues related to the calculation of the test statistic, as well as the bootstrap, permutation and weighted bootstrap approximations.

For computational purposes, the test statistic $D_{n,m}$ can be expressed as

$$D_{n,m} = \frac{1}{n^2} \sum_{i,r=1}^n m_{ir}^X + \frac{1}{m^2} \sum_{j,s=1}^m m_{js}^Y - \frac{2}{nm} \sum_{i=1}^n \sum_{j=1}^m m_{ij}^{XY} = v' M v = 1_{n+m} M_1 1'_{n+m}, \quad (3)$$

where v is the vector of \mathbb{R}^{n+m} with the first n components equal to $1/n$ and the rest equal to $-1/m$, while M_1 is the $(n+m) \times (n+m)$ matrix obtained as the Hadamard product, denoted by \odot , of the matrices M and A , i.e. $M_1 = M \odot A$, where

$$M = \begin{pmatrix} M_{XX} & M_{XY} \\ M_{YX} & M_{YY} \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{n^2} 1_n 1'_n & \frac{-1}{nm} 1_n 1'_m \\ \frac{-1}{nm} 1_m 1'_n & \frac{1}{m^2} 1_m 1'_m \end{pmatrix},$$

$M_{XX} = (m_{ir}^X)$, $M_{YY} = (m_{js}^Y)$ and $M_{XY} = (m_{ij}^{XY})$, with $m_{ir}^X = W(X_i + X_r)$, $1 \leq i, r \leq n$ (note that $m_{ir}^X = m_{ri}^X$), $m_{js}^Y = W(Y_j + Y_s)$, $1 \leq j, s \leq m$ (note that $m_{js}^Y = m_{sj}^Y$), $m_{ij}^{XY} = W(X_i + Y_j)$, $1 \leq i \leq n$, $1 \leq j \leq m$, and for $c = (c_1, c_2, \dots, c_d) \in \mathbb{N}_0^d$,

$$W(c) = \int t^c w(t) dt.$$

Many options are possible for the weight function w . In our simulations we considered $w(t) = \prod_{i=1}^d w_i(t_i)$, where w_i is the PDF of a univariate law. Specifically, if w_i is the PDF of a beta law, $\beta(a_i, b_i)$, then

$$W(c) = \prod_{i=1}^d \frac{\beta(a_i + c_i, b_i)}{\beta(a_i, b_i)}.$$

The calculation of the exact bootstrap, permutation and weighted bootstrap distribution of $D_{n,m}$ is, from a practical point of view, unaffordable. The corresponding p -values can be approximated by simulation by using the algorithms below.

Algorithm 1: Bootstrap approximation.

1. Compute the value of the test statistic at the observed data according to (3). Let $D_{n,m,obs}$ denote the observed value of $D_{n,m}$.
2. For some large integer B , repeat for every $b \in \{1, \dots, B\}$ the following steps:
 - (a) Generate v^{*b} , where $v^{*b} = \frac{1}{n}n_1 - \frac{1}{m}n_2$, n_1, n_2 being two independent multinomial random vectors with $n_1 = (n_{1,1}, \dots, n_{1,n+m}) \sim \mathcal{M}(n; \frac{1}{n+m}, \dots, \frac{1}{n+m})$ and $n_2 = (n_{2,1}, \dots, n_{2,n+m}) \sim \mathcal{M}(m; \frac{1}{n+m}, \dots, \frac{1}{n+m})$.
 - (b) Calculate $D_{n,m}^{*b} = v^{*b'} M v^{*b}$.
3. Approximate the p -value of the observed value of the test statistic by $\hat{p}^* = \frac{1}{B} \sum_{b=1}^B I\{D_{n,m}^{*b} > D_{n,m,obs}\}$.

Algorithm 2: Permutation approximation.

1. Compute the value of the test statistic at the observed data according to (3). Let $D_{n,m,obs}$ denote the observed value of $D_{n,m}$.
2. For some large integer B , repeat for every $b \in \{1, \dots, B\}$ the following steps:
 - (a) Generate $v^{\dagger,b}$ as a random permutation of the components of v .
 - (b) Calculate $D_{n,m}^{\dagger,b} = v^{\dagger,b'} M v^{\dagger,b}$.
3. Approximate the p -value of the observed value of the test statistic by $\hat{p}^\dagger = \frac{1}{B} \sum_{b=1}^B I\{D_{n,m}^{\dagger,b} > D_{n,m,obs}\}$.

The weighted bootstrap version of $D_{n,m}$, $\tilde{D}_{n,m}$, can be expressed as

$$\tilde{D}_{n,m} = \xi' M_2 \xi$$

with

$$M_2 = (M - \omega 1'_{n+m} - 1_{n+m} \omega' - c 1_{n+m} 1'_{n+m}) \odot A,$$

$$\omega = \frac{1}{n+m} M 1'_{n+m}, \quad c = \frac{1}{(n+m)^2} 1_{n+m} M 1'_{n+m}$$

and

$$\xi = (\xi_{X,1}, \dots, \xi_{X,n}, \xi_{Y,1}, \dots, \xi_{Y,m})'.$$

The weighted bootstrap null distribution estimator can be also approximated by simulation, similarly to the two above algorithms. However, the exact weighted bootstrap approximation

can be easily calculated if the multipliers have a normal distribution. In this case, conditional on the data, $\tilde{D}_{n,m}$ is distributed as $\Omega = \sum_{j=1}^{n+m} \lambda_j \chi_{1,j}^2$, where $\lambda_1, \dots, \lambda_{n+m}$ are the eigenvalues of M_2 and $\chi_{1,1}^2, \dots, \chi_{1,n+m}^2$ are independent variables having a chi-squared distribution with 1 degree of freedom. The law of Ω can be numerically approximated by using, for example, Imhof's [13] method. In this case, the weighted bootstrap estimator of the p -value can be calculated as follows.

Algorithm 3: Weighted bootstrap approximation.

1. Compute the value of the test statistic at the observed data according to (3). Let $D_{n,m,obs}$ denote the observed value of $D_{n,m}$.
2. Calculate the eigenvalues of M_2 , $\lambda_1, \dots, \lambda_{n+m}$.
3. Approximate the p -value by $\hat{p} = P_*(\Omega > D_{n,m,obs})$.

5. Simulation results

The properties studied in Sections 2 and 3 are asymptotic, that is, they describe the behaviour of the proposed class of tests and of the proposed approximations to its null distribution for rather large samples. In this section we empirically investigate the finite, in the sense of small or moderate sample sizes, properties of these approximations by means of several numerical simulation experiments. In this context, the aim of this section is threefold: first, the evaluation of the bootstrap, permutation and weighted bootstrap approximations to the null distribution of the test statistic $D_{n,m}$; second, to compare the previously mentioned approximations in terms of the power; and third, to compare the proposed tests to the ones studied in [2, 3], which are based on the ECF and can be also applied to count data, as observed in Remark 3. All computations in this paper have been performed by using programs written in the R language [14]. In order to calculate the \hat{p} -value for Algorithm 3 we used the function `imhof` of the package `CompQuadForm` [15].

To study the goodness of the bootstrap, permutation and weighted bootstrap approximations to the null distribution of the test statistic $D_{n,m}$, we first generated two independent samples with equal sample sizes, $n = m = 20$, from a univariate Poisson distribution with mean $\theta = 2$, $P(2)$; the approximation of the p -value, \hat{p} , was calculated following Algorithms 1 and 2 with $B = 1000$ replications and Algorithm 3 for the weighted bootstrap approximation. As weight function w we considered the PDF of a $\beta(1, 1)$ distribution and the PDF of a $\beta(2, 2)$ distribution. This was repeated 1000 times and we calculated the fraction of estimated p -values, \hat{p} , less than or equal to 0.05, which is the estimated type I error probability for $\alpha = 0.05$. The experiment was repeated for the bivariate Poisson distribution, $BP(\theta)$, $\theta = (\theta_1, \theta_2, \theta_3)$, which is

Table 1: Estimated type I error probabilities based on 1000 Monte Carlo samples from: (a) $P(2)$; (b) $BP(2, 3, 1)$.

		<i>EPGF</i>						<i>ECF</i>					
		$w = \beta(1, 1)$			$w = \beta(2, 2)$			$w = U(-1, 1)$			$w = N(0, 1)$		
	$n = m$	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB
(a)	20	.050	.049	.048	.050	.048	.049	.053	.054	.053	.053	.053	.052
	50	.054	.054	.053	.053	.055	.052	.052	.053	.051	.055	.053	.053
	100	.052	.049	.049	.053	.052	.050	.050	.050	.047	.054	.052	.051
(b)	20	.038	.052	.039	.036	.054	.039	.051	.055	.049	.039	.053	.036
	50	.038	.046	.039	.041	.048	.044	.048	.047	.045	.041	.047	.041
	100	.049	.055	.049	.050	.055	.051	.051	.052	.050	.050	.051	.049

the joint distribution of the variates $Z_1 = W_1 + W_2$ and $Z_2 = W_1 + W_3$, where W_1 , W_2 and W_3 are mutually independent univariate Poisson variates with means $\theta_1 = 2$, $\theta_2 = 3$ and $\theta_3 = 1$, respectively, $BP(2, 3, 1)$ (see Johnson et al. [16, p. 124]). In this bivariate case, as weight function w we considered the products of two PDFs of a $\beta(1, 1)$ distribution and products of two PDFs of a $\beta(2, 2)$ distribution.

As competitor, in the simulation experiments, we have also included the test statistic studied in [2, 3]. As in the simulations in [3], we took as weight function the PDF of a uniform distribution (products of, in the bivariate case) in the interval $(-1, 1)$, $U(-1, 1)$, and the PDF of a standard normal distribution (or products of, in the bivariate case), $N(0, 1)$. Observe that the integrals involved in the definition of the test statistics studied in this paper and the ones in [2, 3] are different: $[0, 1]^d$ and \mathbb{R}^d , respectively. Because of this reason, the weight functions considered in each case necessarily differ. The whole experiment was repeated for $n = m = 50, 100$. Table 1 shows the obtained results. In all tables, the results related to the bootstrap, permutation and weighted bootstrap approximations are denoted by “Boot”, “Perm” and “WB”, respectively. Moreover, we also compared all the approximations in terms of CPU-time (Intel(R) Core(TM) i7-4710MQ, 2.5Ghz). Table 2 shows the CPU-time consumed in seconds to get a p -value.

Looking at Table 1 we can see that the estimated type I error probabilities are quite close to the nominal values for the univariate and bivariate cases. With respect to the type of approximations, for the univariate case all of them exhibit quite similar results, while when sampling from the bivariate Poisson distribution the permutation method gives values closer to the nominal value for small sample sizes ($n = m = 20, 50$). The weight functions considered do not affect the estimated type I error probabilities. With respect to the CPU time required for each approximation, the bootstrap procedure is a little more time consuming than the

Table 2: CPU-time to get a p -value for: (a) $P(2)$; (b) $BP(2, 3, 1)$.

		<i>EPGF</i>						<i>ECF</i>					
		$w = \beta(1, 1)$			$w = \beta(2, 2)$			$w = U(-1, 1)$			$w = N(0, 1)$		
	$n = m$	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB
(a)	20	.025	.016	.021	.024	.014	.022	.027	.022	.038	.020	.018	.003
	50	.070	.060	.026	.068	.053	.024	.089	.068	.056	.070	.053	.007
	100	.214	.185	.033	.209	.180	.047	.293	.260	.160	.215	.186	.025
(b)	20	.026	.021	.021	.027	.015	.023	.029	.024	.009	.029	.017	.004
	50	.069	.059	.026	.070	.057	.024	.111	.095	.046	.072	.055	.007
	100	.214	.189	.034	.210	.182	.048	.347	.343	.181	.219	.189	.024

permutation one. Anyway, both approximations require much more time than the weighted bootstrap approximation.

Next, we compared the tests Ψ_* , Ψ_{\dagger} and $\tilde{\Psi}$ in terms of the power. We also compared them with the ones based on the ECF. With this aim, we repeated the above experiment for some alternative distributions. In all studied cases, the estimated power is calculated for the nominal significance level $\alpha = 0.05$. For the univariate case we considered several families of distributions as alternatives. The first group consists of two Poisson laws with different means. In the second one we took a Poisson law, $P(\theta)$, and a Geometric law with parameter p , $Ge(p)$. In this case, if the means of both distributions are lower than 1, the corresponding PGFs are very similar. We explored this situation by considering several values of θ and p . We also analyzed as alternatives the Poisson and the discrete Lindley distribution introduced by Gómez-Déniz and Calderín-Ojeda [17], which is obtained by discretizing the continuous Lindley distribution. The resultant count model is over-dispersed and competitive with the Poisson distribution and it is denoted by $DL(\gamma)$, $0 < \gamma < 1$. Additionally, we took the count distribution obtained by discretizing the continuous two parameter Lindley distribution studied in Shanker et al. [18], denoted by $TL(\theta, \tilde{\alpha})$. Finally, the univariate Lagrangian Poisson, $LGP(\theta, \lambda)$, was included in the simulation study because when $\lambda = 0$, this model reduces to the ordinary Poisson with mean equal to θ (see Consul and Famoye [19] for further details). The alternatives examined in the univariate case are listed in Table 3. Table 4 displays the obtained results.

For the bivariate case, we studied as alternatives bivariate Poisson laws with different parameter values, mixtures of them, denoted as $\nu BP(\theta) + (1 - \nu)BP(\delta)$, $0 < \nu < 1$, and bivariate Lagrangian Poisson laws, denoted as $BLP(\theta, \delta)$. The models considered in the bivariate case are listed in Table 5. Table 6 displays the results.

Looking at Tables 3 and 5, we can highlight some findings: (i) although the choice of the

Table 3: Univariate alternatives.

(a) $P(1)$ vs. $P(1.5)$	(b) $P(1)$ vs. $P(2)$	(c) $P(7)$ vs. $P(8)$
(d) $P(7)$ vs. $P(9)$	(e) $P(1)$ vs. $Ge(0.6)$	(f) $P(2)$ vs. $Ge(0.4)$
(g) $P(0.75)$ vs. $DL(0.25)$	(h) $P(0.5)$ vs. $DL(0.1)$	(i) $P(2.7)$ vs. $TL(0.3, 0.1)$
(j) $P(1)$ vs. $TL(0.6, 0.3)$	(k) $P(5)$ vs. $LP(5, 0.2)$	(l) $P(7)$ vs. $LP(7, 0.2)$
(m) $P(10)$ vs. $LP(10, 0.2)$		

weight function affects the power of the tests, it is observed that there is no weight function yielding the highest power against all considered alternatives; (ii) with respect to the type of null distribution estimator, the differences observed among the three examined approximations do not follow a specific pattern and none outperforms the others, in fact, the results are very close, specially for larger sample sizes, as expected from the theory (all of them satisfy the result in Corollary 2); (iii) comparing the performance of the tests based on the EPGF and on the ECF we have to mention that there is no test yielding the highest power against all analyzed alternatives. However, in some cases the tests based on the EPGF give better results than the tests based on the ECF. Because of these reasons, we conclude that the class of tests studied in this paper is a useful tool that deserves to be considered when comparing count data with arbitrary dimension.

6. Conclusions

A new class of tests for the two-sample problem when dealing with count data of any arbitrary fixed dimension has been proposed. This class of tests extends and formalizes the exploratory method for the univariate two-sample problem for count data proposed by Nakamura and Pérez-Abreu [6]. The test statistic of any member in this class is a weighted integral of the difference between the EPGFs of the samples. It is proved that the tests are consistent against any fixed alternative. Since the null distribution of the test statistic is unknown, three different approaches for approximating it were investigated. Bootstrap, permutation and weighted bootstrap algorithms were given to consistently estimate the null distribution of the test statistics in this class. We have also shown that these procedures are asymptotically equivalent. Based on simulations experiments, the performance of these approximations for small and moderate sample sizes were analyzed. The behaviour of the new class of tests, relative to the nominal level and the power, was also compared with the class of tests studied in [2, 3], which are based on the ECF and can be also applied to count data of any arbitrary fixed dimension. The simulations reveal that the proposed tests compete very satisfactory with those based on the ECF and thus they are a valuable addition to the existing literature.

Table 4: Estimated power based on 1000 Monte Carlo samples from the univariate cases in Table 3.

		<i>EPGF</i>						<i>ECF</i>					
		$w = \beta(1, 1)$			$w = \beta(2, 2)$			$w = U(-1, 1)$			$w = N(0, 1)$		
	$n = m$	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB
(a)	20	.223	.220	.249	.228	.227	.250	.257	.258	.293	.218	.223	.243
	50	.477	.481	.500	.504	.501	.532	.549	.553	.571	.487	.482	.499
(b)	20	.614	.612	.644	.638	.630	.669	.676	.668	.701	.600	.593	.630
	50	.939	.936	.939	.945	.947	.949	.971	.971	.972	.947	.946	.950
(c)	20	.172	.184	.172	.152	.180	.163	.100	.100	.101	.093	.094	.092
	50	.380	.391	.384	.360	.368	.368	.210	.206	.208	.189	.195	.186
(d)	20	.524	.544	.525	.491	.522	.500	.309	.311	.313	.286	.289	.278
	50	.900	.909	.903	.877	.883	.880	.660	.665	.664	.640	.640	.637
(e)	20	.311	.320	.344	.302	.314	.340	.277	.287	.303	.309	.319	.352
	50	.621	.625	.642	.607	.610	.623	.564	.566	.577	.632	.633	.642
(f)	20	.477	.467	.531	.468	.466	.515	.456	.451	.485	.480	.476	.510
	50	.852	.850	.858	.836	.841	.847	.841	.841	.846	.856	.863	.871
(g)	20	.345	.347	.393	.297	.285	.330	.123	.122	.148	.115	.118	.138
	50	.752	.750	.769	.638	.639	.654	.282	.289	.295	.289	.291	.302
(h)	20	.333	.321	.352	.245	.238	.277	.095	.092	.114	.148	.149	.182
	50	.676	.664	.682	.494	.493	.507	.127	.134	.136	.402	.404	.420
(i)	20	.359	.391	.411	.364	.378	.408	.235	.234	.272	.190	.197	.224
	50	.811	.819	.823	.801	.808	.815	.599	.604	.610	.510	.518	.524
(j)	20	.926	.928	.937	.933	.933	.941	.843	.852	.866	.812	.814	.837
(k)	20	.172	.188	.176	.157	.169	.155	.156	.153	.152	.130	.133	.128
	50	.336	.346	.337	.283	.299	.287	.345	.350	.349	.319	.321	.320
(l)	20	.232	.244	.234	.193	.216	.204	.144	.149	.150	.145	.146	.136
	50	.508	.521	.512	.434	.439	.437	.398	.401	.401	.382	.389	.378
(m)	20	.405	.409	.409	.357	.377	.370	.203	.210	.207	.202	.219	.196
	50	.756	.756	.764	.679	.690	.676	.514	.511	.503	.496	.505	.494

Table 5: Bivariate alternatives.

(a) $BP(2, 3, 1)$ vs. $BP(3, 2, 1)$
(b) $BP(2, 3, 1)$ vs. $BP(2.5, 2.5, 1)$
(c) $BP(5, 5, 1)$ vs. $BLP((5, 5, 1), (0.2, 0.2, 0.2))$
(d) $BP(5, 7, 1)$ vs. $BLP((5, 7, 1), (0.2, 0.2, 0.2))$
(e) $BP(0.4, 0.4, 0.1)$ vs. $0.8BP(0.7, 0.7, 0.1) + 0.2BP(0.4, 0.4, 0.1)$
(f) $BP(0.2, 0.8, 0.1)$ vs. $0.8BP(0.7, 1.3, 0.1) + 0.2BP(0.2, 0.8, 0.1)$
(g) $BP(0.2, 0.8, 0.1)$ vs. $0.8BP(0.4, 1, 0.1) + 0.2BP(0.2, 0.8, 0.1)$

Table 6: Estimated power based on 1000 Monte Carlo samples from the bivariate cases in Table 5.

		<i>EPGF</i>						<i>ECF</i>					
		$w = \beta(1, 1)$			$w = \beta(2, 2)$			$w = U(-1, 1)$			$w = N(0, 1)$		
	$n = m$	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB	Boot	Perm	WB
(a)	20	.132	.183	.166	.073	.110	.106	.406	.433	.461	.304	.297	.280
	50	.344	.374	.357	.147	.171	.162	.849	.860	.741	.752	.748	.733
(b)	20	.410	.485	.474	.413	.482	.469	.486	.499	.526	.358	.405	.398
	50	.841	.853	.855	.816	.830	.837	.904	.902	.907	.802	.816	.810
(c)	20	.289	.325	.322	.244	.285	.277	.148	.181	.174	.091	.156	.118
	50	.587	.624	.507	.485	.516	.510	.433	.457	.429	.352	.417	.346
(d)	20	.389	.411	.355	.324	.366	.377	.152	.181	.151	.093	.173	.090
	50	.726	.751	.798	.617	.637	.624	.467	.490	.462	.403	.474	.397
(e)	20	.480	.474	.480	.442	.436	.446	.240	.238	.230	.242	.238	.234
	50	.855	.857	.861	.814	.815	.816	.525	.509	.512	.548	.561	.558
(f)	20	.727	.720	.727	.696	.692	.692	.435	.449	.440	.452	.465	.448
	50	.989	.990	.989	.983	.982	.982	.903	.907	.904	.902	.901	.901
(g)	20	.302	.282	.284	.238	.238	.239	.110	.112	.112	.114	.146	.138
	50	.666	.668	.667	.592	.588	.594	.224	.224	.220	.296	.290	.294

7. Proofs

Proof of Theorem 1 The proof is similar to that of Theorem 2 in [3], so we omit it. \square

Proof of Theorem 2 Note that $\frac{nm}{N}D_{n,m} = \|\sqrt{\frac{m}{N}}U_n - \sqrt{\frac{n}{N}}V_m\|_w^2$, with $U_n(t) = \sqrt{n}\{C_{X,n}(t) - C(t)\}$ and $V_m(t) = \sqrt{m}\{C_{Y,m}(t) - C(t)\}$. By the central limit theorem for IID random elements in Hilbert spaces (see, for example, van der Vaart and Wellner [20, p. 50]), $\{U_n(t), t \in [0, 1]^d\}$ converges to a centered Gaussian process U on $L^2(w)$ with covariance structure $\varrho_0(t, s)$. By the independence of the two samples, $\{V_m(t), t \in [0, 1]^d\}$ converges in distribution to an independent copy V of U . As, for constants a and b satisfying $a^2 + b^2 = 1$, the centered process $Z(t) = aU(t) + bV(t)$ has covariance structure $\varrho_0(t, s)$, and since $(\sqrt{1 - n/N})^2 + (\sqrt{n/N})^2 = 1$ and n/N converges to τ , it follows that $\{\sqrt{\frac{m}{N}}U_n(t) - \sqrt{\frac{n}{N}}V_m(t), t \in [0, 1]^d\}$ converges in law to $\{Z(t), t \in [0, 1]^d\}$, under H_0 . Finally, the result follows from the continuous mapping theorem. \square

Proof of Theorem 3 The proof is similar to that of Theorem 4 in [3], so we omit it. \square

Proof of Theorem 4 The proof is similar to that of Theorem 5 in [3], so we omit it. \square

Proof of Theorem 5 The proof follows similar steps to the proof of Theorem 1 in [4], so we only sketch it. First, by applying Theorem 1.1 in Kundu et al. [21], it can be shown that, conditional on X_1, \dots, X_n , $\{U_n^*(t), t \in [0, 1]^d\}$ converges in law to $\{U_\tau(t), t \in [0, 1]^d\}$ on $L_2(w)$, where

$$U_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \xi_{X,j} f_\tau(X_j; t),$$

with

$$f_\tau(x; t) = t^x - C_\tau(t),$$

and $\{U_\tau(t), t \in [0, 1]^d\}$ is a centered Gaussian process on $L_2(w)$ with covariance kernel $K_{X,\tau}(t, s) = C_X(ts) - C_X(t)C_\tau(s) - C_X(s)C_\tau(t) + C_\tau(t)C_\tau(s)$. Second, proceeding similarly, it can be shown that, conditional on Y_1, \dots, Y_m , $\{V_m^*(t), t \in [0, 1]^d\}$ converges in law to $\{V_\tau(t), t \in [0, 1]^d\}$ on $L_2(w)$, where

$$V_m^*(t) = \frac{1}{\sqrt{m}} \sum_{l=1}^m \xi_{Y,l} f_\tau(Y_l; t),$$

and $\{V_\tau(t), t \in [0, 1]^d\}$ is a centered Gaussian process on $L_2(w)$ with covariance kernel $K_{Y,\tau}(t, s) = C_Y(ts) - C_Y(t)C_\tau(s) - C_Y(s)C_\tau(t) + C_\tau(t)C_\tau(s)$. As a consequence, conditional on the data, the

process $\{Z_{n,m}^*(t) = \sqrt{\frac{m}{n+m}} U_n^*(t) - \sqrt{\frac{n}{n+m}} V_m^*(t), t \in [0, 1]^d\}$ converges in law to $\{Z_\tau(t), t \in [0, 1]^d\}$ on $L_2(w)$. Finally, it is easy to prove that $\tilde{D}_{n,m} = \|Z_{n,m}^*\|_w^2 + o_{P_*}(1)$ a.s., which implies the result. \square

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