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Highlights

- An axisymmetric backward heat equation with non-zero right hand side is introduced.
- Using the modified quasi-boundary value to regularize the nonlinear problem and to get some error estimates.
- Provides a numerical examples to illustrate the efficiency of our method for both deterministic and random error cases.

On the axisymmetric backward heat equation with non-zero right hand side: Regularization and Error estimates

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Abstract

In this paper we consider the time-inverse problem for the axisymmetric heat equation with non-zero right hand side. This problem is ill-posed: the solution (if it exists) does not depend continuously on the final data. We use a modified quasi-boundary value to regularize the nonlinear problem. Numerical results are presented to illustrate the accuracy and efficiency of the method.

Keyword. Axisymmetric inverse heat problem; Ill-posed problem; Nonhomogeneous heat; Error estimates.

1 Introduction

Partial differential equations (PDEs) such as heat equations have fundamental significance for natural sciences, and various boundary value problems for them were widely studied including inverse and ill-posed problems (see, e.g., Tikhonov and Arsenin [27] and Glasko [15]). A classical example can be named here is the backward heat conduction problem (BHCP). The BHCP is the time-inverse boundary value problem, i.e, given the information at a specific point of time, say $t = T$, the goal is to recover the corresponding structure at earlier time $t < T$. Furthermore, in many branches of engineering sciences, the

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BHCP plays an important role. For instance, an accurate estimation of pollutant source is essential to environmental safeguard in cities with high populations, and it can be applied to several practical areas such as physics, mathematical finance and image processing.

In mathematical models of physical phenomena and engineering problems, the importance of existence, uniqueness and stability theorems of solution in the study of initial or boundary value problems is well-known due to their relevance in establishing the well-posedness of the real-world problems arising in physical, engineering systems. Besides, uniqueness results play a significant role in continuation of solutions and in the theory of autonomous systems (see [13, 14]). It is well-known the BHCP is a classical ill-posed problem that is notoriously difficult to solve since, in general, the solution does not always exist. Furthermore, even if the solution does exist, the continuous dependence of solution on the data would not be guaranteed. As a result, it makes a number of difficulties to do numerical calculations. BHCP is a very famous problem and has been considered by many authors by different methods [8]-[10], [16]-[23], [29, 30]. For the BHCP with a constant coefficient, there are many nice literatures can be listed. For instance, Trong and Tuan in [30] used the method of integral equation to regularize the BHCP with a nonlinear right hand side. In [17], Hao, Duc and Lesnic gave a very nice approximation for this problem by using a non-local boundary value problem method. Later on, Hao and Duc in [18] used Tikhonov regularization method to give an approximation for this problem in Banach space. Tautenhahn in [26] established an optimal error estimate for a backward heat equation with constant coefficient. Fu and Xiong [31] applied a wavelet dual least squares method to investigate a BHCP with constant coefficient.

As mentioned in the above, the results available in the literature on BHCP are mainly devoted to heat equation with constant coefficient. In this article, we aim at investigating the inhomogeneous backward heat equation with variable coefficient, i.e., an axisymmetric backward heat equation in an infinitely long cylinder. In many physical applications, it is sometimes necessary to determine the history temperature in an infinite long cylinder from its surface temperature and a measured temperature at a specific point of time. This so-called axisymmetric backward heat conduction problem has been studied by many well-known papers (see, e.g., [4, 5, 7, 19]). The physical model considered in this paper is an infinitely long cylinder of radius r_0 and it is considered axisymmetric and surface temperature distribution holding zero [4]. Given the terminal temperature of the cylinder, we intent to reconstruct the historical temperature of this cylinder. The correspondingly mathematical model can be described by the following axisymmetric BHCP with non-zero right hand side

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} = f(r, t), & 0 < r < r_0, 0 < t < T, \\ u(r_0, t) = 0, & 0 \leq t \leq T, \\ u(r, t) \text{ bounded as } r \rightarrow 0, & 0 \leq t \leq T, \\ u(r, T) = g(r), & 0 \leq r \leq r_0 \end{cases} \quad (1.1)$$

where r is the radial coordinate, $g(r)$ denotes the final temperature history of the cylinder and $f(x, t)$ is a source function. As mentioned in the above, our goal is to reconstruct the historical temperature distribution $u(\cdot, t)$ for $0 \leq t < T$. Similar to the case of constant

coefficient, the axisymmetric BHCP is also an ill-posed problem: a small perturbation in the final data may cause dramatically large errors in the solution. Specifically, in the problem (1.1) the data exact $g \in L^2(0, r_0; r)$ can be measured and its errors may appear with some noisy data $g^\varepsilon \in L^2(0, r_0; r)$, for which $\|g^\varepsilon - g\| \leq \varepsilon$ where the positive constant ε represents a bound on measurement error. For the problem (1.1) with the zero right hand side, i.e, $f \equiv 0$, there are some seminal works given by the group of Cheng and Fu. In [5], Cheng, Fu and Quin used the spectral method with a regularizing filter function to approximate problem (1.1). Later on, Cheng, Ma and Fu in [7] applied the well-known Tikhonov regularization to stabilize this problem. Although there are several works on the homogeneous case, according to the best of the authors' knowledge, we did not find any papers dealing with inhomogeneous cases of the axisymmetric backward heat equation. In the present article, we regularize the above problem (1.1) by using modified quasi-boundary value. The quasi-boundary value method was originally given by Showalter in 1983. Its main idea is to add an appropriate "corrector term" into the boundary condition in order to obtain a stable approximation. In this paper, we will construct the "corrector term" by suitably perturbing the source function f and the final temperature g .

The rest of the paper is organized as follows. In Section 2, we state the backward problem for the axisymmetric BHCP with non-zero right side. In Section 3, we formulate the regularized problem and provide an error estimation between the solutions of these two problems. Finally, Section 4 provides some numerical examples to illustrate the efficiency of our method.

2 Statement of the problem.

Throughout this paper, we denote by $L^2(0, r_0; r)$ the Hilbert space of Lebesgue measurable functions f with weight r on $[0, r_0]$. $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote inner product and norm on $L^2(0, r_0; r)$ respectively. Specifically, the norm and inner product in $L^2(0, r_0; r)$ are defined as follow

$$\|f\| := \|f\|_{L^2(0, r_0; r)} = \left(\int_0^{r_0} r |f(r)|^2 dr \right)^{1/2}, \quad \langle f, g \rangle = \int_0^{r_0} r f(r) g(r) dr,$$

for $f, g \in L^2(0, r_0; r)$.

Let us first make clear what a solution of the problem (1.1) is. By a solution of (1.1), we imply a function $u(r, t)$ satisfying (1.1) in the classical sense and for every fixed $t \in [0, T]$ this function $u(r, t) \in L^2(0, r_0; r)$. In this class of functions, if the solution of problem (1.1) exists, then it must be unique (see [11]).

Theorem 0. The original Problem (1.1) is equivalent to the following representation:

$$u(r, t) = \sum_{n=1}^{\infty} \left(g_n e^{\left(\frac{\mu_n}{r_0}\right)^2 (T-t)} - \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-t)} f_n(s) ds \right) J_0 \left(\frac{\mu_n}{r_0} r \right), \quad (2.2)$$

where $J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$ and

$$g_n = \frac{2}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r g(r) J_0\left(\frac{\mu_n}{r_0} r\right) dr, \quad n = 1, 2, 3, \dots$$

$$f_n(t) = \frac{2}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r f(r, t) J_0\left(\frac{\mu_n}{r_0} r\right) dr, \quad n = 1, 2, 3, \dots$$

Proof. Firstly, it is clear that if u satisfies (2.2), then u automatically satisfies (1.1). Conversely, let u satisfies (1.1), we prove that u satisfies the integral equation of (2.2). By learning the idea from [5], let us find a solution in the form of

$$u(r, t) = P(t)Q(r). \quad (2.3)$$

By substituting (2.3) into (1.1), for $0 < r \leq r_0$, $Q(r)$ must satisfy

$$Q''(r) + \frac{1}{r}Q'(r) + \lambda Q(r) = 0, \quad (2.4)$$

$$Q(r_0) = 0, \quad (2.5)$$

$$|Q(0)| < \infty, \quad (2.6)$$

where λ is an unknown constant.

It is well-known that eigenvalue λ of problem (2.4)- (2.6) is nonnegative (see [1]). However, the eigenvalue can not be zero since if $\lambda = 0$, then $Q(r) = 0$. For $\lambda > 0$, regarding to [2], we obtain the general solution of equation (2.4) taking the form

$$Q(r) = c_1 J_0(r \sqrt{\lambda}) + c_2 Y_0(r \sqrt{\lambda}), \quad (2.7)$$

where $J_0(z)$ and $Y_0(z)$ denote Bessel function of order zero of the first kind and the second kind, respectively. Noted that $\lim_{x \rightarrow 0} Y_0(x) = -\infty$. Therefore, from boundary condition (2.5) and (2.6), $c_2 = 0$. In addition, the boundary condition $Q(r_0) = 0$ tell us that

$$c_1 J_0(r_0 \sqrt{\lambda}) = 0.$$

The sequence of roots of $J_0(x)$ is $\{\mu_n\}_{n=1}^{\infty}$ which satisfies

$$0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq \dots \quad (2.8)$$

and $\lim_{n \rightarrow \infty} \mu_n = +\infty$. Thus, the eigenvalues of problem (2.4) - (2.6) are sequence of

$$\frac{\mu_n}{r_0}, \quad n = 1, 2, 3, \dots \quad (2.9)$$

and the corresponding eigenfunctions are

$$Q_n(r) = J_0\left(\frac{\mu_n}{r_0} r\right) \quad n = 1, 2, 3, \dots \quad (2.10)$$

Since the eigenfunction system $Q_n(r)$ is complete, and orthogonal with weight r in $L^2(0, r_0; r)$, we are finding the solution which takes the form of

$$u(r, t) = \sum_{n=1}^{\infty} P_n(t) Q_n(r). \quad (2.11)$$

It is noted that

$$f(r, t) = \sum_{n=1}^{\infty} f_n(t) Q_n(r), \quad (2.12)$$

where

$$f_n(t) = \frac{\int_0^{r_0} r f(r, t) J_0\left(\frac{\mu_n}{r_0} r\right) dr}{\int_0^{r_0} r J_0^2\left(\frac{\mu_n}{r_0} r\right) dr} = \frac{2}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r f(r, t) J_0\left(\frac{\mu_n}{r_0} r\right) dr, \quad n = 1, 2, 3, \dots \quad (2.13)$$

Since $u(r, T) = \sum_{n=1}^{\infty} P_n(T) Q_n(r)$, it follows that $P_n(t)$ must satisfy

$$P_n'(t) + \left(\frac{\mu_n}{r_0}\right)^2 P_n(t) = f_n(t) \quad (2.14)$$

$$P_n(T) = g_n. \quad (2.15)$$

By some elementary calculations, we can obtain the solution of (2.14) - (2.15)

$$P_n(t) = g_n e^{\left(\frac{\mu_n}{r_0}\right)^2 (T-t)} - \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-t)} f_n(s) ds. \quad (2.16)$$

Finally, the representation of solution of problem (1.1) is

$$u(r, t) = \sum_{n=1}^{\infty} \left(g_n e^{\left(\frac{\mu_n}{r_0}\right)^2 (T-t)} - \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-t)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right). \quad (2.17)$$

It is clear that the solution $u(r, t)$ belongs to $L^2(0, r_0; r)$. Therefore, it is the unique solution of problem (1.1).

3 Regularization and error estimates

In this section, we use a modified quasi-boundary value method to regularize the original problem (1.1). Let us consider the following regularized problem:

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t}(r, t) - \frac{\partial^2 u^\varepsilon}{\partial r^2}(r, t) - \frac{1}{r} \frac{\partial u^\varepsilon}{\partial r}(r, t) = \sum_{n=1}^{\infty} \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} f_n(t) J_0\left(\frac{\mu_n}{r_0} r\right), \\ u^\varepsilon(r_0, t) = 0, \quad 0 \leq t \leq T, \\ u^\varepsilon(r, t) \text{ bounded as } r \rightarrow 0, \quad 0 \leq t \leq T, \\ u^\varepsilon(r, T) = \sum_{n=1}^{\infty} g_n \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \right) J_0\left(\frac{\mu_n}{r_0} r\right), \quad 0 \leq r \leq r_0. \end{cases} \quad (3.18)$$

Lemma 1. ([29]). Let $0 \leq t < s \leq T, \varepsilon > 0$ and $x > 0$. Then the following inequality holds:

$$\frac{e^{-xt}}{\varepsilon + e^{-xs}} \leq \varepsilon^{\frac{t}{s}-1}. \quad (3.19)$$

The following theorem shows the well-posedness of the regularized problem (3.18).

Theorem 1. Let $g(r) \in L^2(0, r_0; r)$, $f(r, t) \in C(0, T; L^2(0, r_0; r))$ and a given $\varepsilon > 0$. Then the regularized problem (3.18) has a unique solution $u^\varepsilon \in C^{2,1}((0, r_0) \times (0, T), L^2(0, r_0; r))$ which is represented by

$$u^\varepsilon(r, t) = \sum_{n=1}^{\infty} \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} g_n - \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right). \quad (3.20)$$

This solution depends continuously on g in $L^2(0, r_0; r)$.

Proof. The proof is divided into two steps. In Step 1, we prove the existence and the uniqueness of solution of the regularized problem (3.18). In Step 2, the stability of the solution is given.

Step 1. If u satisfies (3.20), then u is a solution of the regularized problem (3.18). We have

$$u^\varepsilon(r, t) = \sum_{n=1}^{\infty} \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} g_n - \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right).$$

It follows that

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t}(r, t) &= \sum_{n=1}^{\infty} -\left(\frac{\mu_n}{r_0}\right)^2 \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \left(g_n - \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right) \\ &\quad + \sum_{n=1}^{\infty} \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} f_n(t) J_0\left(\frac{\mu_n}{r_0} r\right) \\ &= \frac{\partial^2 u^\varepsilon}{\partial r^2}(r, t) + \frac{1}{r} \frac{\partial u^\varepsilon}{\partial r}(r, t) + \sum_{n=1}^{\infty} \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} f_n(t) J_0\left(\frac{\mu_n}{r_0} r\right). \end{aligned}$$

On the other hand, for all $0 < r \leq r_0$, one has

$$u^\varepsilon(r, T) = \sum_{n=1}^{\infty} g_n \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \right) J_0\left(\frac{\mu_n}{r_0} r\right). \quad (3.21)$$

Therefore, u^ε is clearly a solution of problem (3.18) and u^ε belongs to $L^2(0, r_0; r)$. Hence, u^ε is a unique solution of the regularized problem (3.18).

Step 2. Let v^ε and w^ε be two solutions of the regularized problem (3.18) which are corresponding to the data g and h , respectively. Then the following representation holds:

$$v^\varepsilon(r, t) = \sum_{n=1}^{\infty} \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} g_n - \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right),$$

$$w^\varepsilon(r, t) = \sum_{n=1}^{\infty} \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} h_n - \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right).$$

By directly computing, one has

$$\begin{aligned} \|v^\varepsilon(r, t) - w^\varepsilon(r, t)\| &= \left\| \sum_{n=1}^{\infty} |g_n - h_n| \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \right) J_0\left(\frac{\mu_n}{r_0} r\right) \right\| \\ &= \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \right) \left\| \sum_{n=1}^{\infty} |g_n - g_n^\varepsilon| J_0\left(\frac{\mu_n}{r_0} r\right) \right\|. \end{aligned}$$

Applying Lemma 1 directly, we get

$$\|v^\varepsilon(r, t) - w^\varepsilon(r, t)\| \leq \varepsilon^{\frac{t}{T}-1} \left\| \sum_{n=1}^{\infty} |g_n - h_n| J_0\left(\frac{\mu_n}{r_0} r\right) \right\| = \varepsilon^{\frac{t}{T}-1} \|g - h\|. \quad (3.22)$$

The proof is completed.

Until now, we already stated that our regularized problem is a well-posed problem in the sense of Jacques Hadamard. In the coming part, we will establish an error estimate between the exact solution and the regularized solution. Let μ_n be the sequence of roots of the Bessel function $J_0(x)$, the initial state has an expansion of

$$u(r, 0) = \sum_{n=1}^{\infty} u_n(0) J_0\left(\frac{\mu_n}{r_0} r\right),$$

where

$$u_n(0) = \frac{\langle u(r, 0), J_0\left(\frac{\mu_n}{r_0} r\right) \rangle}{\left\| J_0\left(\frac{\mu_n}{r_0} r\right) \right\|^2} = \frac{2}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r u(r, 0) J_0\left(\frac{\mu_n}{r_0} r\right) dr, \quad n = 1, 2, 3, \dots$$

Theorem 2. Let $g(r), f(r, t)$ as in Theorem 1 and given $\varepsilon > 0$. Assume further that f satisfies $\sqrt{\sum_{n=1}^{\infty} \left(\int_0^T e^{n^2} f_n(s) ds \right)^2} < \infty$. Suppose that the problem (1.1) has uniquely a solution u such that $\|u(\cdot, 0)\| < \infty$. Then the following estimation holds for all $t \in (0, T]$:

$$\|u(\cdot, t) - u^\varepsilon(\cdot, t)\| \leq C \varepsilon^{t/T}, \quad (3.23)$$

where $C = \left(\|u(0)\| + \frac{r_0 M}{\sqrt{2}} \right)$, $M = e^{\frac{1}{d^2}} \sqrt{\sum_{n=1}^{\infty} \left(\int_0^T e^{n^2} f_n(s) ds \right)^2}$, u^ε is the unique solution of problem (3.18) and d is some positive constant.

Proof. Denote

$$P_n(t) = g_n e^{\left(\frac{\mu_n}{r_0}\right)^2 (T-t)} - \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-t)} f_n(s) ds,$$

and

$$P_n^\varepsilon(t) = \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} g_n - \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds.$$

Suppose the problem (1.1) has uniquely a solution u , then u is represented by

$$u(r, t) = \sum_{n=1}^{\infty} P_n(t) J_0\left(\frac{\mu_n}{r_0} r\right). \quad (3.24)$$

It is noted that in terms of the initial condition $u(r, 0)$, we have

$$u(r, T) = \sum_{n=1}^{\infty} \left(u_n(0) e^{-\left(\frac{\mu_n}{r_0}\right)^2 T} + \int_0^T e^{-\left(\frac{\mu_n}{r_0}\right)^2 (T-s)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right).$$

Therefore,

$$g_n = u_n(0) e^{-\left(\frac{\mu_n}{r_0}\right)^2 T} + \int_0^T e^{-\left(\frac{\mu_n}{r_0}\right)^2 (T-s)} f_n(s) ds. \quad (3.25)$$

As a result, we have

$$\begin{aligned} \|u(r, t) - u^\varepsilon(r, t)\| &= \left\| \frac{\varepsilon e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \sum_{n=1}^{\infty} \left(g_n - \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right) \right\| \\ &\leq \left\| \frac{\varepsilon e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \sum_{n=1}^{\infty} \left(u_n(0) + \int_0^t e^{\left(\frac{\mu_n}{r_0}\right)^2 s} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right) \right\| \\ &\leq \varepsilon \varepsilon^{t/T-1} \left\| \sum_{n=1}^{\infty} u_n(0) J_0\left(\frac{\mu_n}{r_0} r\right) + \sum_{n=1}^{\infty} \left(\int_0^t e^{\left(\frac{\mu_n}{r_0}\right)^2 s} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right) \right\|. \end{aligned}$$

Thank to the properties the Bessel function of order zero $J_0(x)$, there exists a positive constant d such that, for all natural numbers n , $n^2 \geq d\mu_n^2$. Therefore, $e^{\left(\frac{\mu_n}{r_0}\right)^2} \leq e^{\frac{1}{dr_0^2}} e^{n^2}$ and as a result, the following estimation holds

$$\|u(\cdot, t) - u^\varepsilon(\cdot, t)\| \leq \varepsilon^{t/T} \left(\|u(0)\| + \frac{r_0 |J_1(\mu_n)|}{\sqrt{2}} M \right), \quad (3.26)$$

where $M = e^{\frac{1}{dr_0^2}} \sqrt{\sum_{n=1}^{\infty} \left(\int_0^T e^{n^2 s} f_n(s) ds \right)^2}$. Since $|J_1(\mu_n)| \leq 1$, we obtain

$$\|u(\cdot, t) - u^\varepsilon(\cdot, t)\| \leq C \varepsilon^{t/T}, \quad (3.27)$$

where $C = \left(\|u(0)\| + \frac{r_0 M}{\sqrt{2}} \right)$. The proof is completed.

Remark 1. We note that the estimation in Theorem 2 only holds for all $0 < t \leq T$. At $t = 0$, $\varepsilon^{t/T} = 1$. Therefore, it is necessary to approximate the initial time by a different way to obtain a good approximation at the initial time. In fact, the following theorem shows the estimate at $t = 0$.

Theorem 3. Let f, g be as in Theorem 2 and let u be a solution of problem (1.1) such that $u_t(r, t) \in L^2(0, r_0; r)$ for every $(r, t) \in (0, r_0] \times [0, T]$. Then for all $\varepsilon > 0$ there exists a t_ε such that

$$\|u(\cdot, 0) - u^\varepsilon(\cdot, t_\varepsilon)\| \leq E \left(\ln \frac{1}{\varepsilon} \right)^{-\frac{1}{4}}, \quad (3.28)$$

where u^ε is the unique solution of (3.18), C as in Theorem 2, $E = 2 \max\{N, C\} \sqrt[4]{T}$ and

$$N = \sqrt{\int_0^T \|u_s(\cdot, s)\|^2 ds}.$$

Proof. Using the idea in [28] with a little modification, we actually have

$$\begin{aligned} \|u(\cdot, 0) - u(\cdot, t)\|^2 &= \int_0^{r_0} \left(\int_0^t \sqrt{r} u_s(r, s) ds \right)^2 dr \\ &\leq t \int_0^T \int_0^{r_0} r (u_s(r, s))^2 dr ds \\ &\leq t \int_0^T \|u_s(\cdot, s)\|^2 ds = N^2 t. \end{aligned}$$

Therefore, for all $t \in [0, T]$, the following estimate holds:

$$\|u(\cdot, 0) - u(\cdot, t)\| \leq N \sqrt{t}.$$

Combining the above with the result of Theorem 2, we get

$$\|u(\cdot, 0) - u^\varepsilon(\cdot, t)\| \leq N \sqrt{t} + C \varepsilon^{t/T} \leq \max\{N, C\} \left(\sqrt{t} + C \varepsilon^{t/T} \right).$$

For every $\varepsilon > 0$, there exists a t_ε such that $\sqrt{t_\varepsilon} = \varepsilon^{\frac{t_\varepsilon}{T}}$, i.e. $\frac{\ln t_\varepsilon}{t_\varepsilon} = \frac{2 \ln \varepsilon}{T}$. Since for all $t > 0$, $\ln t \geq -\frac{1}{t}$, we get

$$\sqrt{t_\varepsilon} < \sqrt[4]{T} \left(\ln \frac{1}{\varepsilon} \right)^{-\frac{1}{4}}.$$

It follows that

$$\|u(\cdot, 0) - u^\varepsilon(\cdot, t_\varepsilon)\| \leq 2 \max\{N, C\} \sqrt[4]{T} \left(\ln \frac{1}{\varepsilon} \right)^{-\frac{1}{4}} = E \left(\ln \frac{1}{\varepsilon} \right)^{-\frac{1}{4}},$$

where $E = 2 \max\{N, C\} \sqrt[4]{T}$. The proof is completed.

It is now ready to state the main result of this paper which presents the error estimation between the exact and regularized solution under the measured data. In fact, we have the following theorem.

Theorem 4. Let f, g be as in Theorem 2 and let u be a solution of problem (1.1) corresponding to the exact data g such that $u_t(r, t) \in L^2(0, r_0; r)$ for every $(r, t) \in (0, r_0] \times [0, T]$. Let $g^\varepsilon \in L^2(0, r_0; r)$ be a measured data such that

$$\|g - g^\varepsilon\| \leq \varepsilon.$$

Then there exists an approximate solution U^ε , which links to the noisy data g^ε , satisfying

$$\|U^\varepsilon(\cdot, t) - u(\cdot, t)\| \leq (1 + C)\varepsilon^{t/T}, \quad (3.29)$$

for all $t \in (0, T]$ and

$$\|U^\varepsilon(\cdot, 0) - u(\cdot, 0)\| \leq \left(E + \sqrt[4]{T}\right) \left(\ln \frac{1}{\varepsilon}\right)^{-\frac{1}{4}}, \quad (3.30)$$

where the value of C is as in Theorem 2 and the value of E is as in Theorem 3.

Proof. Let u^ε be the solution of the problem (3.18) corresponding to the exact data g and let v^ε be the solution of the problem (3.18) corresponding to the noisy data g^ε . From the results of Theorem 3, there exists a t_ε such that

$$\sqrt{t_\varepsilon} = \varepsilon^{\frac{t_\varepsilon}{T}},$$

and

$$\|u(\cdot, 0) - u^\varepsilon(\cdot, t)\| \leq E \left(\ln \frac{1}{\varepsilon}\right)^{-\frac{1}{4}}.$$

Now let

$$U^\varepsilon(\cdot, t) = \begin{cases} v^\varepsilon(\cdot, t), & 0 < t < T, \\ v^\varepsilon(\cdot, t_\varepsilon), & t = 0. \end{cases}$$

Then we have for all $t \in (0, T]$,

$$\begin{aligned} \|U^\varepsilon(\cdot, t) - u(\cdot, t)\| &\leq \|v^\varepsilon(\cdot, t) - u^\varepsilon(\cdot, t)\| + \|u^\varepsilon(\cdot, t) - u(\cdot, t)\| \\ &\leq \varepsilon^{t/T} + C\varepsilon^{t/T} \\ &\leq (1 + C)\varepsilon^{t/T}. \end{aligned}$$

From (3.22) and (3.28), at the initial time $t = 0$, we have

$$\begin{aligned} \|U^\varepsilon(\cdot, 0) - u(\cdot, 0)\| &\leq \|v^\varepsilon(\cdot, t_\varepsilon) - u^\varepsilon(\cdot, t_\varepsilon)\| + \|u^\varepsilon(\cdot, t_\varepsilon) - u(\cdot, 0)\| \\ &\leq \varepsilon^{t_\varepsilon/T} + E \left(\ln \frac{1}{\varepsilon}\right)^{-\frac{1}{4}} \\ &\leq \left(E + \sqrt[4]{T}\right) \left(\ln \frac{1}{\varepsilon}\right)^{-\frac{1}{4}}. \end{aligned}$$

The proof is completed.

Remark 2. For all $0 < t \leq T$, the error estimate is in Hölder-type, which is quite good. However, at initial time $t = 0$, the error estimate is just in the logarithm-type, which is a weaker convergence rate in comparison to the Hölder-type. This is the weak point of this paper. The improving of convergence rate at initial time $t = 0$ can be regarded as a promising future work..

4 Numerical illustration

The aim of this section is to establish some numerical tests to visualize the theoretical results in the above sections. Let us consider the following two examples:

Example 1. Let us consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} = f(r, t), & 0 < r \leq r_0, 0 \leq t \leq T \\ u(r_0, t) = 0, & 0 \leq t \leq T, \\ u(r, t) \text{ bounded as } r \rightarrow 0 \\ u(r, T) = g^{ex}(r), & 0 \leq r \leq r_0 \end{cases} \quad (4.31)$$

where

$$\begin{aligned} f(r, t) &= \left(1 + \left(\frac{\mu_1}{r_0}\right)^2\right) e^t J_0\left(\frac{\mu_1}{r_0} r\right), \\ g^{ex}(r) &= e^T J_0\left(\frac{\mu_1}{r_0} r\right), \end{aligned}$$

and μ_n as in (2.8). Under the above assumptions, the exact solution is given by

$$u(r, t) = e^t J_0\left(\frac{\mu_1}{r_0} r\right). \quad (4.32)$$

Because of the error in measurement process, the measured data is given by

$$g^\varepsilon(r) = e^T J_0\left(\frac{\mu_1}{r_0} r\right) + \varepsilon J_0\left(\frac{\mu_2}{r_0} r\right). \quad (4.33)$$

It follows that the error in measurement process is bounded by ε , $\|g^\varepsilon - g\| \leq \varepsilon$. The regularized solution, which is obtained by (3.20) and corresponded to the data g^ε

$$u^\varepsilon(r, t) = \sum_{n=1}^{\infty} \left(\frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} g_n^\varepsilon - \frac{e^{-\left(\frac{\mu_n}{r_0}\right)^2 t}}{\varepsilon + e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \int_t^T e^{\left(\frac{\mu_n}{r_0}\right)^2 (s-T)} f_n(s) ds \right) J_0\left(\frac{\mu_n}{r_0} r\right), \quad (4.34)$$

where

$$g_n^\varepsilon = \frac{2}{r_0^2 J_1^2(\mu_n)} \int_0^{r_0} r \left[e^T J_0\left(\frac{\mu_1}{r_0} r\right) + \varepsilon J_0\left(\frac{\mu_2}{r_0} r\right) \right] J_0\left(\frac{\mu_n}{r_0} r\right) dr.$$

For each point of time, let's evaluate the "Relative error" between the exact solution and the regularized solution which is defined by

$$RE(\varepsilon, t) = \frac{\|u^{re}(\cdot, t) - u^{ex}(\cdot, t)\|}{\|u^{ex}(\cdot, t)\|}. \quad (4.35)$$

Relative error has a better representation of the difference between the exact solution and the approximate solution. Let say when the value of the exact solution is large, the difference between the exact solution and the approximate solution does not tell us much information about the goodness of the approximation. In this case, relative error is a better measurement to measure how fit the approximate method is.

Let $T = 1$, $r_0 = 3$, $\varepsilon_1 = 10^{-1}$, $\varepsilon_2 = 10^{-3}$, $\varepsilon_3 = 10^{-5}$. We have some figures of the exact solution and the regularized solution with various values of ε (see Figures 1 and 2). In addition, by fixing $t_* = 0.05$ and let ε runs from 10^{-1} to 10^{-5} , Figure 3 shows that the regularized

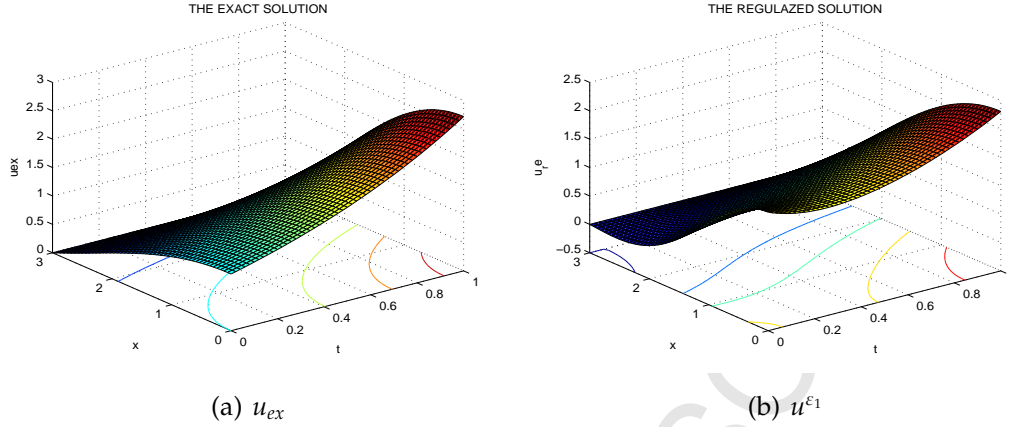


Figure 1: The exact solution (a) and the regularized solution with $\varepsilon_1 = 10^{-1}$ (b)

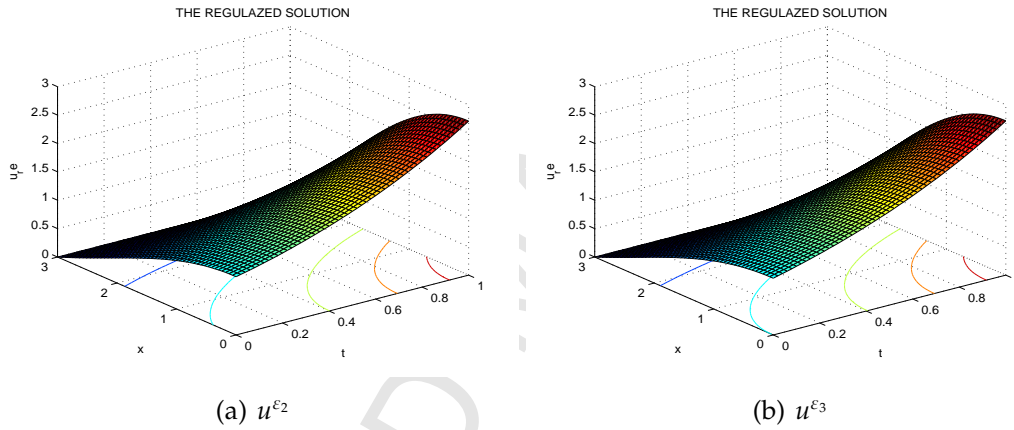


Figure 2: The regularized solution with $\varepsilon_2 = 10^{-3}$ (a) and $\varepsilon_3 = 10^{-5}$ (b)

solution is closed to exact solution as ε gets smaller and smaller. Error estimate at $t_* = 0.05$ is also provided in Table 1.

Remark 3. Beside the figures represented in the above, the authors also provide three videos which represents the variation of the exact solution and the regularized solution with three cases $\varepsilon_1 = 10^{-1}$, $\varepsilon_2 = 10^{-2}$ and $\varepsilon_3 = 10^{-3}$ as time goes by. At each point of time, the videos show the exact solution, the regularized solution and the relative error.

Example 2. Since until now, we did not find any paper which deals with the case $f \neq 0$, let us make an assimilation with previous results for the case $f \equiv 0$ through this specific

Table 1: The error of method under various values of ε .

ε	$\ u^\varepsilon(\cdot, t_*) - u^{ex}(\cdot, t_*)\ $	$RE(\varepsilon, t_*)$
$\varepsilon = 10^{-1}$	0.0393315594347542	0.424452355038714
$\varepsilon = 10^{-2}$	0.0112547210388897	0.121456990745181
$\varepsilon = 10^{-3}$	0.00141027193275177	0.0152191586528491
$\varepsilon = 10^{-4}$	0.000144711425761107	0.00156167480639065
$\varepsilon = 10^{-5}$	0.0000145090681012323	0.00015657675956635

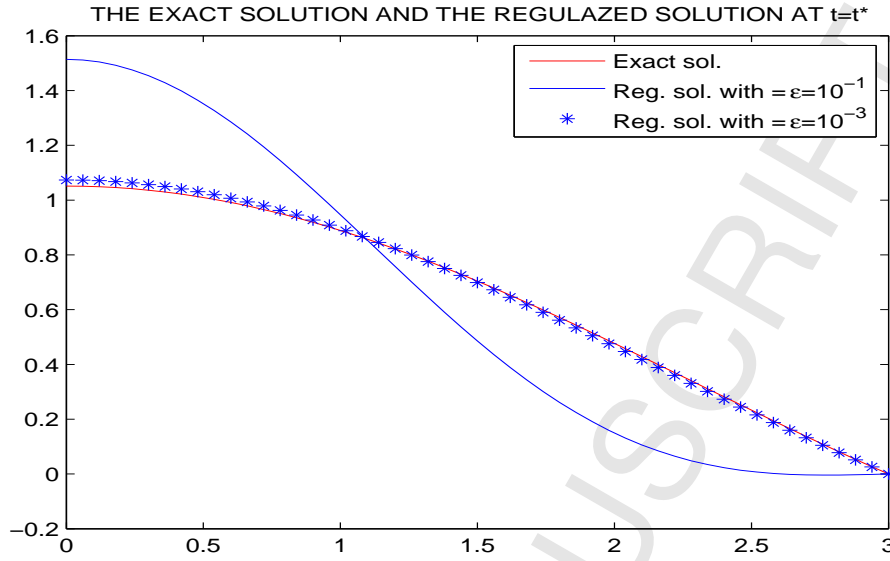


Figure 3: At time $t_0 = 0.1$: Exact solution (red) and regularized solution with $\varepsilon_1 = 10^{-1}$ (blue), $\varepsilon_3 = 10^{-3}$ (* in blue)

Table 2: The error $\|u^\varepsilon(\cdot, 0) - u^{ex}(\cdot, 0)\|$ under various values of ε in the specific Example 2

ε	Error of method in this paper	Error of method in [4]
$\varepsilon = 10^{-1}$	0.0475101758774219	593.756402590195
$\varepsilon = 10^{-2}$	0.0371567369541916	59.3756402590195
$\varepsilon = 10^{-3}$	0.0297696807682976	5.93756402590195
$\varepsilon = 10^{-4}$	0.0194999394458554	0.59375640259019
$\varepsilon = 10^{-5}$	0.0112779776144284	0.05937564025901

example. Consider the special source case $f \equiv 0$ attaching to the data $g(r) = e^T J_0\left(\frac{\mu_1}{r_0} r\right)$. Under these assumptions, the exact solution is given by

$$u(r, t) = e^{T + \left(\frac{\mu_1}{r_0}\right)^2 (T-t)} J_0\left(\frac{\mu_1}{r_0} r\right).$$

Assume that the measured data is given by

$$g^\varepsilon(r) = e^T J_0\left(\frac{\mu_1}{r_0} r\right) + \sum_{q=1}^Q \varepsilon a_i J_0\left(\frac{\mu_q}{r_0} r\right),$$

where Q is a natural number, ε is the noise-level and a_i is random uniform number in $[0, 1]$. Let $T = 1, r_0 = 5, Q = 5$, we present the Figure 4 to visualize the efficient of the regularization method in this paper in assimilation with the method used in the famous seminal work in [4] at initial time $t = 0$.

In addition, a table of error estimation for both methods used in this specific example is also provided in the Table 2.

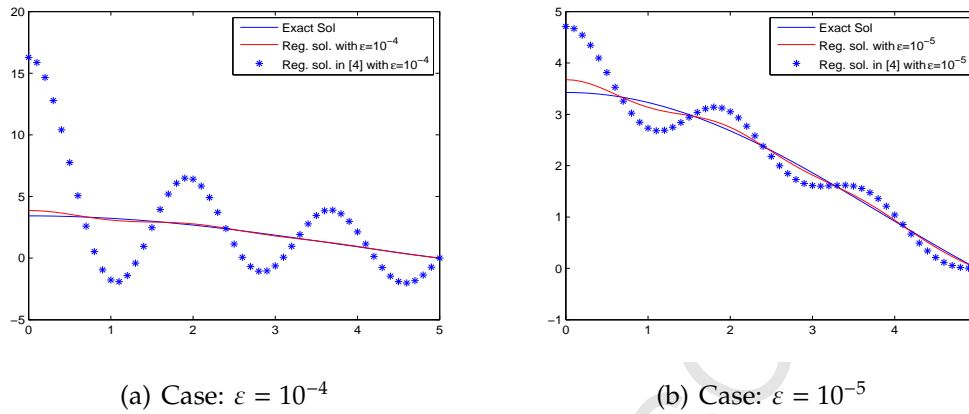


Figure 4: At initial time $t = 0$: The exact solution (blue), the regularized solution (red) and regularized solution due to [4] (* in blue) with $\varepsilon = 10^{-4}$ (a) and $\varepsilon = 10^{-5}$ (b).

Table 3: The computational time between the proposed method in this paper and the method in [4]

Unit: Second	Computational time with $Q = 100$		Computational time with $Q = 300$	
ε	Method in this paper	Method in [4]	Method in this paper	Method in [4]
10^{-1}	2.849945	2.735825	8.719662	3.954644
10^{-2}	2.842857	2.746649	8.682224	4.414593
10^{-3}	2.870219	2.772956	8.613752	5.938571
10^{-4}	2.871325	2.793383	8.520386	7.321766
10^{-5}	2.837246	2.844965	8.446833	8.360391

Next, let $Q = 100$ and $Q = 300$, we have the following Table 3 for the computational time of the proposed method in this paper and the proposed method in [4]. The code was run in a 64-bit Microsoft Window 7 operation system with the processor of Intel Core i5-460M and 4.00GB of RAM.

Remark 4. Regarding to Table 3, in this example, the computational time using the proposed method in this paper seems to be stationary while the computational time using the proposed method in [4] seems to be increase as the decreasing of ε . In fact, the method in [4] has a shorter computational time compared to the one in this paper. This kind of situation is quite reasonable because the method used in [4] is the Fourier truncation method, which will lead to a short computational time at a big value of ε . However, to get more accuracy approximate solution, Fourier truncation method requires more computation and it will make the computational time increase. It is also noted that the difference in computational time between two methods becomes smaller when the value of ε decrease. For instance, at $\varepsilon = 10^{-5}$, the computational time for two methods are almost the same.

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