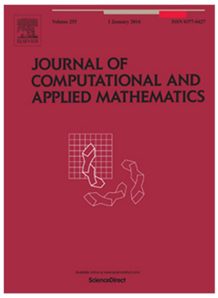
Accepted Manuscript



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PII: S0377-0427(18)30292-9
DOI: https://doi.org/10.1016/j.cam.2018.05.021
Reference: CAM 11686

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 14 April 2017
Revised date : 10 March 2018

A new weak Galerkin finite element scheme for general second-order elliptic problems

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Abstract: A new weak Galerkin(WG) finite element scheme is presented for general second-order elliptic problems in this paper. In this new scheme, a skew symmetric form has been used for handling the convection term. The advantage of the new scheme is that the system of linear equations from the scheme is positive definite and one might easily get the well-postedness of the system. In this scheme, the WG elements are designed to have the form of $(P_k(T), P_{k-1}(e))$. That is, we choose the polynomials of degree $k \geq 1$ on each element and the polynomials of degree $k-1$ on the edge/face of each element. As a result, fewer degrees of freedom are generated in the new WG finite element scheme. It is also worth pointing out that the WG finite element scheme is established on finite element partitions consisting of arbitrary shape of polygons/polyhedra which are shape regular. Optimal-order error estimates are presented for the corresponding numerical approximation in various norms. Some numerical results are reported to confirm the theory.

Key words: weak Galerkin, finite element methods, discrete gradient, general second-order elliptic problems

AMS Subject Classification: Primary, 65N15, 65N30; Secondary, 35J50.

1 Introduction

The weak Galerkin (WG) finite element method is a newly developed and efficient numerical technique for solving partial differential equations. The central idea of the WG finite element method is to interpret partial differential operators as generalized distributions, called weak differential operators, over the space of discontinuous functions including boundary information. Since its contribution, the WG finite element method has been applied successfully to the discretization of several classes of partial differential equations, e.g., (general)second-order elliptic problems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], the Biharmonic equations [11, 12, 13, 14], the Stokes equations [15, 16, 17], the Helmholtz equations [18, 19] and the Oseen equations [20].

In this paper, we are concerned with the WG finite element method for general second-order elliptic problems which seeks an unknown function $u = u(\boldsymbol{x})$ satisfying

$$-\nabla \cdot (\mathcal{A}\nabla u) + \nabla \cdot (\boldsymbol{b}u) + cu = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

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where Ω is a polygonal/plolyhedral domain in $\mathbb{R}^d (d = 2, 3)$, $\mathcal{A} = (a_{ij}(\boldsymbol{x})) \in [L^\infty(\Omega)]^{d^2}$ is a symmetric matrix-valued function, $\boldsymbol{b} = (b_i(\boldsymbol{x}))_{d\times 1} \in [L^\infty(\Omega)]^d$ is a vector-valued function, and $c = c(\boldsymbol{x}) \in L^\infty(\Omega)$ is a scalar function on Ω. Assume that the matrix $\mathcal{A}$ satisfies the following property: there exists a constant $\lambda > 0$ such that

$$\xi^t \mathcal{A} \xi \geq \lambda \xi^t \xi, \ \forall \xi \in \mathbb{R}^d,$$

where ξ is understood as a column vector and ξ^t is the transpose of ξ.

A natural variational formulation for the problem (1.1)-(1.2) is to seek $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$(\mathcal{A}\nabla u, \nabla v) - (\boldsymbol{b}u, \nabla v) + (cu, v) = (f, v), \ \forall v \in H_0^1(\Omega). \tag{1.3}$$

Based on the formulation (1.3), one may design various conforming and non-conforming finite element schemes for the problem (1.1)-(1.2). In [1], a WG finite element scheme was designed for the problem (1.1)-(1.2) by replacing the gradient operator with weak gradient operator in formulation (1.3). For (general) second-order elliptic problems, the weak function has the form of $v = \{v_0, v_b\}$ with $v = v_0$ inside of each element and $v = v_b$ on the boundary of the element. In the WG finite element method, both v_0 and v_b can be correspondingly approximated by polynomials in $P_l(T)$ and $P_s(e)$, and the weak gradient operators can be approximated in the polynomial space $[P_m(T)]^d$, where l, s, m are non-negative integers with possibly different values, T stands for an element and e stands for the edge/face of T. Various combinations of $(P_l(T), P_s(e), [P_m(T)]^d)$ lead to different class of WG methods tailored for specific partial differential equations. In [1], the WG finite element space was established arising from BDM element or RT element. More specifically, one takes the combination $(P_{k-1}(T), P_k(e), [P_k(T)]^d)$ for BDM element and $(P_{k-1}(T), P_{k-1}(e), [P_{k-1}(T)]^d + \widehat{P}_{k-1}(T)\boldsymbol{x})$ for RT element, where $\boldsymbol{x}$ is a column vector and $\widehat{P}_{k-1}(T)$ is the set of homogeneous polynomials of order $k-1$ in the variable $\boldsymbol{x}$. Due to the use of the BDM element and RT element, the WG finite element formulation in [1] was limited to classical finite element partitions of triangles ($d = 2$) or tetrahedra ($d = 3$).

In [3], a WG scheme was established by using the configuration of $(P_k(T), P_{k-1}(e), [P_{k-1}(T)]^d)$. With a suitable stabilization operator, the WG scheme can be applied on general polytopal meshes. In order to get the well-postedness, the method in [3] needed the WG formulation to be positive definite. Since the system of linear equations from variational formulation (1.3) is non-positive definite, we introduce a new variational form for (1.1)-(1.2) as follows: seek $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$(\mathcal{A}\nabla u, \nabla v) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla u, v) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla v, u) + (c_0 u, v) = (f, v), \ \forall v \in H_0^1(\Omega). \tag{1.4}$$

Here $c_0 = \frac{1}{2}(\nabla \cdot b) + c$ and we always suppose $c_0 \geq 0$ for all $x \in \Omega$. It is easy to check that the system of linear equations from (1.4) is positive definite. Based on the variational formulation (1.4), we can describe a new WG finite element scheme for the problem (1.1)-(1.2). The advantage of the new scheme is that the system of linear equations from the scheme is positive definite and one might easily get the well-postedness of the system. The new WG method can be applied on general polytopal meshes. The goal of the paper is to specify all the details for the new WG finite element scheme, and further justify the rigorousness of the method by establishing a mathematical convergence theory.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries and notations for Sobolev spaces. In Section 3, we define some weak differential operators and the corresponding discrete weak differential operators. In Section 4, we establish a new WG finite element scheme for the problem (1.1)-(1.2). In Section 5, we present some technical estimates which are useful for the error analysis. In Section 6, we derive optimal-order error estimates for

WG approximation in both H^1 and L^2 norms. In Section 7, we present some numerical results which confirm the theory developed in earlier sections. In Section 8, we show our discussion on convection-dominated problems. Finally, conclusions are drawn in Section 9.

2 Preliminaries and notations

Let K be any domain in $\mathbb{R}^d$, $d = 2, 3$. We use the standard definition for the Sobolev space $H^s(K)$ and their associated inner products $(\cdot,\cdot)_{s,K}$, norms $\|\cdot\|_{s,K}$, and seminorms $|\cdot|_{s,K}$ for any $s \geq 0$. For instance, for any integer $s \geq 0$, the seminorm $|\cdot|_{s,K}$ is defined by

$$|v|_{s,K} = \left(\sum_{|\alpha|=s} \int_K |\partial^\alpha v|^2 dK \right)^{1/2}$$

with the usual notation

$$\alpha = (\alpha_1, \cdots, \alpha_d), \ |\alpha| = \alpha_1 + \cdots + \alpha_d, \ \partial^\alpha = \Pi_{j=1}^d \partial_{x_j}^{\alpha_j}.$$

The Sobolev norm $\|\cdot\|_{m,K}$ is given by

$$\|v\|_{m,K} = \left(\sum_{j=0}^m |v|_{j,K}^2 \right)^{1/2}.$$

The space $H^0(K)$ coincides with $L^2(K)$, for which norm and inner product are denoted by $\|\cdot\|_K$ and $(\cdot,\cdot)_K$, respectively. If $K = \Omega$, we shall drop the subscript K in the L^2 norm and the L^2 inner product notations.

The space $H(div;K)$ is given by the set of vector-valued functions on K which, together with their divergence, are square integrable, i.e.,

$$H(div;K) = \{\boldsymbol{v} : \boldsymbol{v} \in [L^2(K)]^d, \nabla \cdot \boldsymbol{v} \in L^2(K)\}.$$

The norm in $H(div;K)$ is defined as

$$\|\boldsymbol{v}\|_{H(div;K)} = \left(\|\boldsymbol{v}\|_K^2 + \|\nabla \cdot \boldsymbol{v}\|_K^2 \right)^{1/2}.$$

3 Weak differential operators

The goal of this section is to introduce the weak gradient operator and the weak convective operator defined on a space of weak functions. These weak differential operators will be employed to discretize general second-order elliptic problems. To this end, let K be any polygonal/polyhedral domain with boundary ∂K. A weak function on the region K refers to a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$. The first component v_0 can be understood as the value of v inside K, and the second component v_b represents v on the boundary of K. Note that v_b may not necessarily be related to the trace of v_0 on ∂K. Denote by $W(K)$ the space of weak functions on K, i.e.,

$$W(K) = \{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K)\}.$$

Define $(v, w)_K = \int_K vw dx$ and $\langle v, w\rangle_\gamma = \int_\gamma vw ds$. The weak gradient operator, as was introduced in [1], is defined as follows.

Definition 3.1 *([1]) The dual of $L^2(K)$ can be identified with itself by using the standard L^2 inner product as the action of linear functionals. With a similar interpretation, for any $v \in W(K)$, the weak gradient of $v = \{v_0, v_b\}$ is defined as a linear functional $\nabla_w v$ in the dual space of $H(div, K)$ whose action on each $\boldsymbol{q} \in H(div, K)$ is given by*

$$(\nabla_w v, \boldsymbol{q})_K = -(v_0, \nabla \cdot \boldsymbol{q})_K + \langle v_b, \boldsymbol{q} \cdot \boldsymbol{n} \rangle_{\partial K}, \tag{3.1}$$

where $\boldsymbol{n}$ is the outward normal direction to ∂K, $(v_0, \nabla \cdot \boldsymbol{q})_K = \int_K v_0(\nabla \cdot \boldsymbol{q}) dK$ is the action of v_0 on $\nabla \cdot \boldsymbol{q}$, and $\langle v_b, \boldsymbol{q} \cdot \boldsymbol{n} \rangle_{\partial K}$ is the action of $\boldsymbol{q} \cdot \boldsymbol{n}$ on $v_b \in H^{\frac{1}{2}}(\partial K)$.

Consider the inclusion map $i_w : H^1(K) \to W(K)$ defined as follows

$$i_w(\phi) = \{\phi|_K, \phi|_{\partial K}\}, \ \phi \in H^1(K),$$

by which the Sobolev space $H^1(K)$ is embedded into the space $W(K)$. With the help of the inclusion map i_w, the Sobolev space $H^1(K)$ can be viewed as a subspace of $W(K)$ by identifying each $\phi \in H^1(K)$ with $i_w(\phi)$. For smooth function $v \in H^1(K)$, it is not hard to see that the weak gradient is identical with the strong gradient (i.e.,$\nabla_w v = \nabla v$).

Let $\boldsymbol{b} \in H(div, K)$. For any $v \in W(K)$, we define by $\boldsymbol{b} \cdot \nabla_w v$ the weak convective operator of $v = \{v_0, v_b\}$. By (3.1), we can derive

$$(\boldsymbol{b} \cdot \nabla_w v, \phi)_K = -(\boldsymbol{b} \cdot \nabla \phi, v_0)_K - ((\nabla \cdot \boldsymbol{b})\phi, v_0) + \langle \boldsymbol{b} \cdot \boldsymbol{n}, v_b \phi \rangle_{\partial K}, \ \forall \phi \in H^1(K). \tag{3.2}$$

In fact,

$$\begin{aligned} (\boldsymbol{b} \cdot \nabla_w v, \phi)_K &= (\nabla_w v, \boldsymbol{b}\phi) = -(v_0, \nabla \cdot (\boldsymbol{b}\phi))_K + \langle v_b, (\boldsymbol{b}\phi) \cdot \boldsymbol{n} \rangle_{\partial K} \\ &= -(\boldsymbol{b} \cdot \nabla \phi, v_0)_K - ((\nabla \cdot \boldsymbol{b})\phi, v_0) + \langle \boldsymbol{b} \cdot \boldsymbol{n}, v_b \phi \rangle_{\partial K}, \end{aligned}$$

which implies (3.2).

Denote by $P_r(K)$ the set of polynomials on K with degree no more than r. We can define a discrete weak gradient operator by approximating ∇_w in a polynomial subspace of the dual of $H(div, K)$.

Definition 3.2 *([1]) The discrete weak gradient operator, denoted by $\nabla_{w,r,K}$, is defined as the unique polynomial $\nabla_{w,r,K} v \in [P_r(K)]^d$ satisfying the following equation*

$$(\nabla_{w,r,K} v, \boldsymbol{q})_K = -(v_0, \nabla \cdot \boldsymbol{q})_K + \langle v_b, \boldsymbol{q} \cdot \boldsymbol{n} \rangle_{\partial K}, \ \forall \boldsymbol{q} \in [P_r(K)]^d. \tag{3.3}$$

By applying the usual integration by part to the first term on the right hand side of (3.3), we can rewrite the equation (3.3) as

$$(\nabla_{w,r,K} v, \boldsymbol{q})_K = (\nabla v_0, \boldsymbol{q})_K + \langle v_b - v_0, \boldsymbol{q} \cdot \boldsymbol{n} \rangle_{\partial K}, \ \forall \boldsymbol{q} \in [P_r(K)]^d. \tag{3.4}$$

Similarly, we can define a discrete weak convective operator as follows by approximating $\boldsymbol{b} \cdot \nabla_w$ in a polynomial subspace of the dual of $H^1(K)$.

Definition 3.3 *The discrete weak convective operator, denoted by $\boldsymbol{b} \cdot \nabla_{w,r,K}$, is defined as the unique polynomial $\boldsymbol{b} \cdot \nabla_{w,r,K} v \in P_r(K)$ satisfying the following equation*

$$(\boldsymbol{b} \cdot \nabla_{w,r,K} v, \phi)_K = -(\boldsymbol{b} \cdot \nabla \phi, v_0)_K - ((\nabla \cdot \boldsymbol{b})\phi, v_0) + \langle \boldsymbol{b} \cdot \boldsymbol{n}, v_b \phi \rangle_{\partial K}, \ \forall \phi \in P_r(K). \tag{3.5}$$

4 Weak Galerkin finite element schemes

The aim of this section is to establish a new WG finite element scheme for the problem (1.1)-(1.2) and further derive some characterizations of the WG finite element scheme. To this end, we give the corresponding variational form of the problem (1.1)-(1.2) by seeking $u \in H^1(\Omega)$ satisfying $u|_{\partial\Omega} = g$ and

$$(\mathcal{A}\nabla u, \nabla v) + (\nabla \cdot (\boldsymbol{b}u), v) + (cu, v) = (f, v), \ \forall v \in H_0^1(\Omega), \tag{4.1}$$

where $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions with vanishing value on $\partial\Omega$. In fact, for $v \in H_0^1(\Omega)$, we can get

$$\begin{aligned} (\nabla \cdot (\boldsymbol{b}u), v) &= ((\nabla \cdot \boldsymbol{b})u, v) + (\boldsymbol{b} \cdot \nabla u, v) \\ &= \frac{1}{2}((\nabla \cdot \boldsymbol{b})u, v) + (\boldsymbol{b} \cdot \nabla u, v) + \frac{1}{2}((\nabla \cdot \boldsymbol{b})u, v) \\ &= -\frac{1}{2}(\boldsymbol{b}, \nabla(uv)) + (\boldsymbol{b} \cdot \nabla u, v) + \frac{1}{2}((\nabla \cdot \boldsymbol{b})u, v) \\ &= \frac{1}{2}(\boldsymbol{b} \cdot \nabla u, v) - \frac{1}{2}(\boldsymbol{b} \cdot \nabla v, u) + \frac{1}{2}((\nabla \cdot \boldsymbol{b})u, v). \end{aligned} \tag{4.2}$$

Therefore, by (4.1) and (4.2), we obtain a new variational form of the problem (1.1)-(1.2) as follows

$$(\mathcal{A}\nabla u, \nabla v) + \frac{1}{2}(\boldsymbol{b} \cdot \nabla u, v) - \frac{1}{2}(\boldsymbol{b} \cdot \nabla v, u) + (c_0 u, v) = (f, v), \ v \in H_0^1(\Omega), \tag{4.3}$$

where $c_0 = \frac{1}{2}(\nabla \cdot \boldsymbol{b}) + c$. Note that (4.3) will be used to establish the WG finite element formulation. In the rest of the paper, we always suppose that $c_0 \geq 0$ for all $x \in \Omega$.

Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in 2D or polyhedra in 3D. Assume that $\mathcal{T}_h$ is shape regular in the sense as defined in [2]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and mesh size $h = max_{T \in \mathcal{E}_h} h_T$ for $\mathcal{T}_h$.

For a given integer $k \geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined by

$$V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_{k-1}(e), e \in \partial T, T \in \mathcal{T}_h\}.$$

We would like to emphasize that any function $v \in V_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$. Denote by V_h^0 the subspace of V_h with vanishing boundary values on $\partial\Omega$, i.e.,

$$V_h^0 = \{v = \{v_0, v_b\} \in V_h, v_b|_{\partial T \cap \partial\Omega} = 0, \forall T \in \mathcal{T}_h\}.$$

For each element $T \in \mathcal{T}_h$, denote by Q_0 the L^2 projection from $L^2(T)$ onto $P_k(T)$ and by Q_b the L^2 projection operator from $L^2(e)$ onto $P_{k-1}(e)$. Denote by $\mathbb{Q}_h$ the L^2 projection onto the local discrete gradient space $[P_{k-1}(T)]^d$. For any $v \in H^1(\Omega)$, we define the projection operator $Q_h : H^1(\Omega) \to V_h$ such that for each element $T \in \mathcal{T}_h$, we have

$$Q_h v = \{Q_0 v_0, Q_b v_b\}, \ \{v_0, v_b\} = i_w(v) \in W(T).$$

Denote by $\nabla_{w,k-1}$ the discrete weak gradient operator on the finite element space V_h computed by using(3.3) on each element T, i.e.,

$$(\nabla_{w,k-1} v)|_T = \nabla_{w,k-1,T}(v|_T), \quad \forall v \in V_h.$$

With an abuse of notation, from now on we shall drop the subscript $k-1$ in the notation $\nabla_{w,k-1}$ for the discrete weak gradient.

Now we introduce two forms on V_h as follows

$$\begin{aligned} a(v,w) &= (\mathcal{A}\nabla_w v, \nabla_w w) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w v, w_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w w, v_0) + (c_0 v_0, w_0), \\ s(v,w) &= \rho \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b v_0 - v_b, Q_b w_0 - w_b\rangle_{\partial T}, \end{aligned}$$

where $c_0 = \frac{1}{2}(\nabla\cdot\boldsymbol{b}) + c \geq 0$ for all $x \in \Omega$, ρ can be any positive number and the usual L^2 inner product can be written locally on each element by

$$\begin{aligned} (\mathcal{A}\nabla_w v, \nabla_w w) &= \sum_{T\in\mathcal{T}_h} (\mathcal{A}\nabla_w v, \nabla_w w)_T,\ (\boldsymbol{b}\cdot\nabla_w v, w_0) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{b}\cdot\nabla_w v, w_0)_T, \\ (\boldsymbol{b}\cdot\nabla_w w, v_0) &= \sum_{T\in\mathcal{T}_h} (\boldsymbol{b}\cdot\nabla_w w, v_0)_T,\ (c_0 u_0, v_0) = \sum_{T\in\mathcal{T}_h} (c_0 u_0, v_0)_T. \end{aligned}$$

In practical computation, one might set $\rho = 1$. Denote by $a_s(\cdot,\cdot)$ a stabilization of $a(\cdot,\cdot)$ given by

$$a_s(v,w) = a(v,w) + s(v,w). \tag{4.4}$$

A weak Galerkin algorithm based on the variational form (4.3) is given as follows

Algorithm 4.1 *A numerical approximation for the problem* (1.1)-(1.2) *can be obtained by seeking* $u_h = \{u_0, u_b\} \in V_h$ *satisfying both* $u_b = Q_b g$ *on* $\partial\Omega$ *and the following equation*

$$a_s(u_h, v) = (f, v_0),\ \forall v = \{v_0, v_b\} \in V_h^0. \tag{4.5}$$

Note that the system (4.5) is positive definite for any parameter value of $\rho > 0$. Next, we justify the well-postedness of the scheme (4.5). For any $v \in V_h$, let

$$|\!|\!| v |\!|\!| := \sqrt{a_s(v,v)}. \tag{4.6}$$

It is not hard to see that $|\!|\!|\cdot|\!|\!|$ defines a seminorm in the finite element space V_h. We claim that this seminorm becomes a full norm in the finite element space V_h^0. It suffices to check the positivity property for $|\!|\!|\cdot|\!|\!|$. To this end, we need check that for $v \in V_h^0$, $v = 0$ if $|\!|\!| v |\!|\!| = 0$. In fact, $|\!|\!| v |\!|\!| = 0$ implies that $\nabla_w v = 0$ on each element T and $Q_b v_0 = v_b$ on ∂T. It follows from $\nabla_w v = 0$ and the equation (3.4) that for any $\boldsymbol{q} \in [P_{k-1}(T)]^d$,

$$\begin{aligned} 0 &= (\nabla_w v, \boldsymbol{q})_T \\ &= (\nabla v_0, \boldsymbol{q})_T - \langle v_0 - v_b, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial T} \\ &= (\nabla v_0, \boldsymbol{q})_T - \langle Q_b v_0 - v_b, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial T} \\ &= (\nabla v_0, \boldsymbol{q})_T. \end{aligned}$$

Letting $q = \nabla v_0$ in the equation above yields $\nabla v_0 = 0$ on $T \in \mathcal{T}_h$. Thus, $v_0 = const$ on every $T \in \mathcal{T}_h$. This, together with the fact that $Q_b v_0 = v_b$ on ∂T and $v_b = 0$ on $\partial\Omega$, implies that $v_0 = v_b = 0$.

Lemma 4.2 *The weak Galerkin finite element scheme* (4.5) *has a unique solution.*

Proof. Let $u_h^{(1)}$ and $u_h^{(2)}$ be two solutions of (4.5). It is clear that the difference $w_h = u_h^{(1)} - u_h^{(2)}$ is a finite element function in V_h^0 satisfying

$$a_s(w_h, v) = 0, \ \forall v \in V_h^0. \tag{4.7}$$

By setting $v = w_h$ in (4.7) we obtain

$$|\!|\!| w_h |\!|\!| = a_s(w_h, w_h) = 0.$$

It follows that $w_h \equiv 0$, or equivalently $u_h^{(1)} = u_h^{(2)}$, which completes the proof.

5 Some technical estimates

In this section, we shall present some technical results useful for the forthcoming error analysis. To this end, we firstly introduce the trace inequality and inverse inequality on shape regular partitions. For more details we refer the reader to [2]. For simplicity of notation, we shall use $\lesssim$ denote less than or equal to up to a constant independent of the mesh size, variables, or other parameters appearing in the inequality.

Lemma 5.1 *([2]) Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Then, there exists a constant C such that for any $T \in \mathcal{T}_h$ and edge/face $e \in \partial T$, we have*

$$\|\varphi\|_e^2 \le C(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2), \tag{5.1}$$

where $\varphi \in H^1(T)$ is any function.

Lemma 5.2 *([2]) Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Then, there exists a constant $C(n)$ such that*

$$\|\nabla\varphi\|_T \le C(n) h_T^{-1}\|\varphi\|_T, \quad \forall T \in \mathcal{T}_h, \tag{5.2}$$

for any piecewise polynomial φ of degree n on $\mathcal{T}_h$.

We shall present a useful property which indicates the discrete weak gradient operator is good approximation to the gradient operator in the classical sense.

Lemma 5.3 *([1]) Let Q_h and $\mathbb{Q}_h$ be the L^2 projection operators defined in previous sections. Then, on each element $T \in \mathcal{T}_h$, one has the following commutative property*

$$\nabla_w(Q_h\phi) = \mathbb{Q}_h(\nabla\phi), \quad \forall \phi \in H^1(T). \tag{5.3}$$

The following lemma provides some estimates for the projection operators Q_0 and $\mathbb{Q}_h$.

Lemma 5.4 *([2, 3]) Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Then, for any $\phi \in H^{k+1}(\Omega)$, one has*

$$\sum_{T\in\mathcal{T}_h} \|\phi - Q_0\phi\|_T^2 \lesssim h^{2(k+1)}\|\phi\|_{k+1}^2, \tag{5.4}$$

$$\sum_{T\in\mathcal{T}_h} \|\nabla(\phi - Q_0\phi)\|_T^2 \lesssim h^{2k}\|\phi\|_{k+1}^2, \tag{5.5}$$

$$\sum_{T\in\mathcal{T}_h} \|\mathcal{A}(\nabla\phi - \mathbb{Q}_h(\nabla\phi))\|_T^2 \lesssim h^{2k}\|\phi\|_{k+1}^2, \tag{5.6}$$

$$\sum_{T\in\mathcal{T}_h} \left(\|\phi - Q_0\phi\|_T^2 + h_T^2\|\nabla(\phi - Q_0\phi)\|_T^2\right) \lesssim h^{2(k+1)}\|\phi\|_{k+1}^2. \tag{5.7}$$

By using the trace inequality (5.1) and Lemma 5.4, one can obtain the following lemma.

Lemma 5.5 *Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Then, for any $\phi \in H^{k+1}(\Omega)$, one obtains*

$$\sum_{T\in\mathcal{T}_h} \|\phi - Q_0\phi\|_{\partial T}^2 \lesssim h^{2k+1}\|\phi\|_{k+1}^2. \tag{5.8}$$

Proof. According to the trace inequality (5.1) and Lemma 5.4, we have

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \|\phi - Q_0\phi\|_{\partial T}^2 &\lesssim \sum_{T\in\mathcal{T}_h} \left(h_T^{-1}\|\phi - Q_0\phi\|_T^2 + h_T\|\nabla(\phi - Q_0\phi)\|_T^2\right) \\
&= h_T^{-1}\sum_{T\in\mathcal{T}_h} \left(\|\phi - Q_0\phi\|_T^2 + h_T^2\|\nabla(\phi - Q_0\phi)\|_T^2\right) \\
&\lesssim h^{2k+1}\|\phi\|_{k+1}^2,
\end{aligned}$$

which finishes the proof.

In the finite element space V_h, we define a discrete H^1 seminorm by

$$\|v\|_{1,h} = \left(\sum_{T\in\mathcal{T}_h} (\|\nabla v_0\|_T^2 + h_T^{-1}\|Q_b v_0 - v_b\|_{\partial T}^2\right)^{1/2}. \tag{5.9}$$

Lemma 5.6 *For any $v = \{v_0, v_b\} \in V_h$, we have*

$$\|v\|_{1,h} \lesssim |||v|||. \tag{5.10}$$

Proof. For any $v = \{v_0, v_b\} \in V_h$, it follows from (3.4) and the definition of Q_b that

$$(\nabla_w v, \boldsymbol{q})_T = (\nabla v_0, \boldsymbol{q})_T + \langle v_b - Q_b v_0, \boldsymbol{q}\cdot n\rangle_{\partial T}, \ \forall \boldsymbol{q} \in [P_{k-1}(T)]^d. \tag{5.11}$$

By setting $\boldsymbol{q} = \nabla v_0$ in (5.11), we obtain

$$(\nabla_w v, \nabla v_0)_T = (\nabla v_0, \nabla v_0)_T + \langle v_b - Q_b v_0, \nabla v_0 \cdot \boldsymbol{n}\rangle_{\partial T}.$$

By the trace inequality (5.1) and inverse inequality (5.2), we have

$$\|\nabla v_0\|_T^2 \lesssim \|\nabla_w v\|_T\|\nabla v_0\|_T + h_T^{-1/2}\|v_b - Q_b v_0\|_{\partial T}\|\nabla v_0\|_T.$$

Thus,

$$\|\nabla v_0\|_T \lesssim \|\nabla_w v\|_T + h_T^{-1/2}\|v_b - Q_b v_0\|_{\partial T}.$$

This leads to

$$\|\nabla v_0\|_T \lesssim \left(\|\nabla_w v\|_T^2 + h_T^{-1}\|v_b - Q_b v_0\|_{\partial T}^2\right)^{1/2} \lesssim |||v|||,$$

which completes the proof.

In the rest of the paper, we always suppose $\boldsymbol{b} \in H(div, \Omega)$. The following lemma may be used for the forthcoming error analysis.

Lemma 5.7 *Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Then, for any $w \in H^{k+1}(\Omega)$ and $v = \{v_0, v_b\} \in V_h$, one has*

$$\begin{aligned}
|s(Q_h w, v)| &\lesssim h^k |||v|||\|w\|_{k+1}, &(5.12)\\
|l_w(v)| &\lesssim h^k |||v|||\|w\|_{k+1}, &(5.13)\\
|t_{w,\boldsymbol{b}}(v)| &\lesssim h^{k+1} |||v|||\|w\|_{k+1}, &(5.14)\\
|\lambda_{w,\boldsymbol{b}}(v)| &\lesssim h^{k+1} |||v|||\|w\|_{k+1}. &(5.15)
\end{aligned}$$

where

$$\begin{aligned}
l_w(v) &= \sum_{T\in\mathcal{T}_h} \langle v_0 - v_b, (\mathcal{A}\nabla w - \mathcal{A}\mathbb{Q}_h(\nabla w))\cdot \boldsymbol{n}\rangle_{\partial T},\\
t_{w,\boldsymbol{b}}(v) &= \frac{1}{2}\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{b}\cdot\boldsymbol{n}, (w - Q_b w)(v_0 - v_b)\rangle_{\partial T},\\
\lambda_{w,\boldsymbol{b}}(v) &= \frac{1}{2}\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{b}\cdot\boldsymbol{n}, (w - Q_0 w)(v_0 - v_b)\rangle_{\partial T}.
\end{aligned}$$

Proof. For the detailed verification of (5.12) and (5.13), we refer the reader to [3]. We are left with the task of proving (5.14) and (5.15). With the Cauchy-Schwarz inequality and the definition of Q_b, one can derive

$$\begin{aligned}
t_{w,\boldsymbol{b}}(v) &= \frac{1}{2}\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{b}\cdot\boldsymbol{n}, (w - Q_b w)(v_0 - v_b)\rangle_{\partial T}\\
&\le \frac{1}{2}\Big|\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{b}\cdot\boldsymbol{n}, (w - Q_b w)(v_0 - Q_b v_0)\rangle_{\partial T}\Big| + \frac{1}{2}\Big|\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{b}\cdot\boldsymbol{n}, (w - Q_b w)(Q_b v_0 - v_b)\rangle_{\partial T}\Big|\\
&\le \frac{1}{2}\Big(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{b}\|_{L^\infty(\partial T)}\Big)\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - Q_b v_0\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h} h_T\|w - Q_b w\|_{\partial T}^2\Big)^{1/2}\\
&\quad + \frac{1}{2}\Big(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{b}\|_{L^\infty(\partial T)}\Big)\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_b v_0 - v_b\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h} h_T\|w - Q_b w\|_{\partial T}^2\Big)^{1/2}\\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - Q_b v_0\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h} h_T\|w - Q_0 w\|_{\partial T}^2\Big)^{1/2}\\
&\quad + \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}|Q_b v_0 - v_b\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h} h_T\|w - Q_0 w\|_{\partial T}^2\Big)^{1/2}.
\end{aligned} \tag{5.16}$$

We claim that the following estimate

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - Q_b v_0\|_{\partial T}^2 \lesssim |\!|\!| v |\!|\!|. \tag{5.17}$$

Indeed, by using the trace inequality (5.1) and the approximation property of L^2 projection operator we can derive

$$h_T^{-1}\|v_0 - Q_b v_0\|_{\partial T}^2 \lesssim \|\nabla v_0\|_T^2.$$

Thus, by (5.9) and Lemma 5.6, we have

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - Q_0 v_0\|_{\partial T}^2 \lesssim \sum_{T\in\mathcal{T}_h} \|\nabla v_0\|^2 \lesssim \|v\|_{1,h} \lesssim |\!|\!| v |\!|\!|,$$

which implies (5.17). Applying (5.8), (5.16), (5.17) and the definition of $|\!|\!| \cdot |\!|\!|$, we obtain

$$t_{w,\boldsymbol{b}}(v) \lesssim h^{k+1}|\!|\!| v |\!|\!| \|w\|_{k+1},$$

which verifies the desired estimate (5.14). Similarly, we can obtain (5.15).

6 Error analysis

The goal of this section is to establish some error estimates for the WG finite element solution u_h arising from (4.5). The error will be measured in the triple-bar norm as defined in (4.6), the discrete H^1 seminorm defined in (5.9) and the standard L^2 norm.

6.1 Error equation

Let $u_h = \{u_0, u_b\} \in V_h$ be the WG finite element solution arising from the numerical scheme (4.5). Assume that the exact solution of the problem (1.1)-(1.2) is given by u. The L^2 projection of u in the finite element space V_h is given by

$$Q_h u = \{Q_0 u, Q_b u\}.$$

Let

$$e_h = \{e_0, e_b\} = u_h - Q_h u = \{u_0 - Q_0 u, u_b - Q_b u\}$$

be the error between the WG finite element solution and the L^2 projection of the exact solution.

Lemma 6.1 *Let e_h be the error of the weak Galerkin finite element solution arising from* (4.5)*. Then, for any $v \in V_h^0$, one has*

$$\begin{aligned} a_s(e_h, v) &= \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0 u - u)v_0) + (\boldsymbol{b}\cdot\nabla v_0, Q_0 u - u) + (c_0(u - Q_0 u), v_0) \\ &\quad + t_{u,\boldsymbol{b}}(v) + \lambda_{u,\boldsymbol{b}}(v) - l_u(v) - S(Q_h u, v), \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} l_u(v) &= \sum_{T\in\mathcal{T}_h} \langle v_0 - v_b, (\mathcal{A}\nabla u - \mathcal{A}\mathbb{Q}_h(\nabla u))\cdot\boldsymbol{n}\rangle_{\partial T}, \\ t_{u,\boldsymbol{b}}(v) &= \frac{1}{2}\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{b}\cdot\boldsymbol{n}, (u - Q_b u)(v_0 - v_b)\rangle_{\partial T}, \\ \lambda_{u,\boldsymbol{b}}(v) &= \frac{1}{2}\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{b}\cdot\boldsymbol{n}, (u - Q_0 u)(v_0 - v_b)\rangle_{\partial T}. \end{aligned}$$

Proof. Testing (1.1) by using v_0 of $v = \{v_0, v_b\} \in V_h^0$, we arrive at

$$\begin{aligned} (f, v_0) &= (\mathcal{A}\nabla u, \nabla v_0) + \frac{1}{2}((\nabla\cdot\boldsymbol{b})u, v_0) + (\boldsymbol{b}\cdot\nabla u, v_0) \\ &\quad + (c_0 u, v_0) - \sum_{T\in\mathcal{T}_h} \langle (\mathcal{A}\nabla u)\cdot\boldsymbol{n}, v_0 - v_b\rangle_{\partial T}, \end{aligned} \tag{6.2}$$

where we have used the fact that $\sum_{T\in\mathcal{T}_h} \langle (\mathcal{A}\nabla u)\cdot\boldsymbol{n}, v_b\rangle_{\partial T} = 0$ and $c_0 = \frac{1}{2}(\nabla\cdot\boldsymbol{b}) + c$. We first deal with the form $(\mathcal{A}\nabla u, \nabla v_0)$ in (6.2). In fact, for any $v \in V_h^0$, it follows from Lemma 5.3 and (3.4) that

$$\begin{aligned} (\mathcal{A}\nabla_w(Q_h u), \nabla_w v)_T &= (\mathcal{A}\mathbb{Q}_h(\nabla u), \nabla_w v)_T \\ &= (\nabla v_0, \mathcal{A}\mathbb{Q}_h(\nabla u))_T - \langle v_0 - v_b, (\mathcal{A}\mathbb{Q}_h(\nabla u))\cdot\boldsymbol{n}\rangle_{\partial T} \\ &= (\mathcal{A}\nabla u, \nabla v_0)_T - \langle (\mathcal{A}\mathbb{Q}_h(\nabla u))\cdot\boldsymbol{n}, v_0 - v_b\rangle_{\partial T}. \end{aligned}$$

Thus,

$$(\mathcal{A}\nabla u, \nabla v_0) = (\mathcal{A}\nabla_w(Q_h u), \nabla_w v) + \sum_{T\in\mathcal{T}_h} \langle (\mathcal{A}\mathbb{Q}_h(\nabla u))\cdot\boldsymbol{n}, v_0 - v_b\rangle_{\partial T}. \tag{6.3}$$

Then, we handle the term $\frac{1}{2}((\nabla\cdot\boldsymbol{b})u, v_0)+(\boldsymbol{b}\cdot\nabla u, v_0)$ in (6.2). For $w=\{w_0,w_b\}\in V_h$, according to the definition of discrete weak convective operator (3.5) and the fact $\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, w_b v_b\rangle_{\partial T}=0$, we have

$$(\boldsymbol{b}\cdot\nabla_w w, v_0) = -(\boldsymbol{b}\cdot\nabla v_0, w_0) - (\nabla\cdot\boldsymbol{b}, w_0 v_0) + \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, w_b(v_0-v_b)\rangle_{\partial T} \tag{6.4}$$

and

$$\begin{aligned}(\boldsymbol{b}\cdot\nabla_w v, w_0) &= -(\boldsymbol{b}\cdot\nabla w_0, v_0) - (\nabla\cdot\boldsymbol{b}, w_0 v_0) + \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, v_b w_0\rangle_{\partial T}\\ &= (\boldsymbol{b}\cdot\nabla v_0, w_0) - \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, (v_0-v_b)w_0\rangle_{\partial T}.\end{aligned} \tag{6.5}$$

By letting $w=Q_hu$ in (6.4) and $w_0=Q_0u$ in (6.5), we obtain the following equations

$$(\boldsymbol{b}\cdot\nabla_w(Q_hu), v_0) = -(\boldsymbol{b}\cdot\nabla v_0, Q_0u) - (\nabla\cdot\boldsymbol{b}, (Q_0u)v_0) + \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, Q_bu(v_0-v_b)\rangle_{\partial T}, \tag{6.6}$$

$$(\boldsymbol{b}\cdot\nabla_w v, Q_0u) = (\boldsymbol{b}\cdot\nabla v_0, Q_0u) - \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, Q_0u(v_0-v_b)\rangle_{\partial T}. \tag{6.7}$$

Thus, by using the integration by parts and (6.6)-(6.7) , we obtain

$$\begin{aligned}&(\boldsymbol{b}\cdot\nabla u, v_0)+\frac{1}{2}((\nabla\cdot\boldsymbol{b})u, v_0)\\ =\;& -(\boldsymbol{b}\cdot\nabla v_0, u) - \frac{1}{2}(\nabla\cdot\boldsymbol{b}, uv_0) + \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, u(v_0-v_b)\rangle_{\partial T}\\ =\;& -(\boldsymbol{b}\cdot\nabla v_0, Q_0u) - \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0u)v_0) + (\boldsymbol{b}\cdot\nabla v_0, Q_0u-u)\\ & +\frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0u-u)v_0) + \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, u(v_0-v_b)\rangle_{\partial T}\\ =\;& -\frac{1}{2}(\boldsymbol{b}\cdot\nabla v_0, Q_0u) - \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0u)v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla v_0, Q_0u) + (\boldsymbol{b}\cdot\nabla v_0, Q_0u-u)\\ & +\frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0u-u)v_0) + \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, u(v_0-v_b)\rangle_{\partial T}\\ =\;& \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w(Q_hu), v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w v, Q_0u) + \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0u-u)v_0)\\ & +(\boldsymbol{b}\cdot\nabla v_0, Q_0u-u) + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, (u-Q_bu)(v_0-v_b)\rangle_{\partial T}\\ & +\frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, (u-Q_0u)(v_0-v_b)\rangle_{\partial T}.\end{aligned} \tag{6.8}$$

By the definition of $a(u,v)$, we can get

$$\begin{aligned}a(Q_hu, v) &= (\mathcal{A}\nabla_w(Q_hu), \nabla_w v) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w(Q_hu), v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w v, Q_0u)\\ &\quad +(c_0Q_0u, v_0).\end{aligned} \tag{6.9}$$

According to (4.5) and the definition of e_h, we can get

$$a_s(e_h, v) = a_s(u_h, v) - a_s(Q_h u, v) = (f, v_0) - a(Q_h u, v) - s(Q_h u, v). \tag{6.10}$$

Combining (6.2), (6.3) and (6.8)-(6.10), we obtain

$$\begin{aligned} a_s(e_h, v) \quad = \quad & \frac{1}{2}(\nabla \cdot \boldsymbol{b}, (Q_0 u - u)v_0) + (\boldsymbol{b} \cdot \nabla v_0, Q_0 u - u) + (c_0(u - Q_0 u), v_0) \\ & + t_{u,\boldsymbol{b}}(v) + \lambda_{u,\boldsymbol{b}}(v) - l_u(v) - S(Q_h u, v), \end{aligned}$$

which completes the proof.

6.2 Error estimates

The error equation (6.1) can be used to derive the following error estimates for the WG finite element solution.

Theorem 6.2 *Let $u_h \in V_h$ be the weak Galerkin finite element solution of the problem* (1.1)-(1.2) *arising from* (4.5)*. Assume the exact solution $u \in H^{k+1}(\Omega)$. Then*

$$|||u_h - Q_h u||| \lesssim h^k \|u\|_{k+1}. \tag{6.11}$$

Proof. By letting $v = e_h$ in (6.1), we have

$$\begin{aligned} |||e_h|||^2 \quad = \quad & \frac{1}{2}(\nabla \cdot \boldsymbol{b}, (Q_0 u - u)e_0) + (\boldsymbol{b} \cdot \nabla e_0, Q_0 u - u) + (c_0(u - Q_0 u), e_0) \\ & + t_{u,\boldsymbol{b}}(e_h) + \lambda_{u,\boldsymbol{b}}(e_h) - l_u(e_h) - S(Q_h u, e_h). \end{aligned} \tag{6.12}$$

Applying the Lemma 5.6, the definition of $|||\cdot|||$ and the approximation property of the L^2- projection, we derive

$$\begin{aligned} & \left|\frac{1}{2}(\nabla \cdot \boldsymbol{b}, (Q_0 u - u)e_0) + (\boldsymbol{b} \cdot \nabla e_0, Q_0 u - u) + (c_0(u - Q_0 u), e_0)\right| \\ \leq \; & \sum_{T \in \mathcal{T}_h} \Big(\|\nabla \cdot \boldsymbol{b}\|_{L^\infty(T)} \|Q_0 u - u\|_T \|e_0\|_T + \|\boldsymbol{b}\|_{L^\infty(T)} \|Q_0 u - u\|_T \|\nabla e_0\|_T \\ & + \|c_0\|_{L^\infty(T)} \|Q_0 u - u\|_T \|e_0\|_T \Big) \\ \lesssim \; & \Big(\sum_{T \in \mathcal{T}_h} \|Q_0 u - u\|_T^2 \Big)^{1/2} |||e_h||| + \Big(\sum_{T \in \mathcal{T}_h} \|Q_0 u - u\|_T^2 \Big)^{1/2} \|e_h\|_{1,h} \\ \lesssim \; & h^{k+1} |||e_h||| \|u\|_{k+1}. \end{aligned} \tag{6.13}$$

By Lemma 5.7 and (6.13), we have

$$|||e_h|||^2 \lesssim h^k \|u\|_{k+1} |||e_h||| + h^{k+1} \|u\|_{k+1} |||e_h||| \lesssim h^k \|u\|_{k+1} |||e_h|||,$$

which completes the proof.

Next, we will measure the difference between u and u_h in the discrete H^1 seminorm $\|\cdot\|_{1,h}$ as defined in (5.9).

Corollary 6.3 *Let $u_h \in V_h$ be the weak Galerkin finite element solution of the problem* (1.1)-(1.2) *arising from* (4.5)*. Assume the exact solution $u \in H^{k+1}(\Omega)$. Then*

$$\|u - u_h\|_{1,h} \lesssim h^k \|u\|_{k+1}. \tag{6.14}$$

Proof. Applying Lemma 5.6 and Theorem 6.2, we obtain

$$\|u_h - Q_h u\|_{1,h} \lesssim |||u_h - Q_h u||| \lesssim h^k \|u\|_{k+1}. \tag{6.15}$$

With the definition of discrete H^1 seminorm and the estimates (5.5) and (5.8), we derive

$$\begin{aligned}
& \|u - Q_h u\|_{1,h}^2 \\
= & \sum_{T\in\mathcal{T}_h} \left(\|\nabla(u - Q_0 u)\|_T^2 + h^{-1}\|Q_b(u - Q_0 u) - (u - Q_b u)\|_{\partial T}^2 \right) \\
\le & \sum_{T\in\mathcal{T}_h} \left(\|\nabla(u - Q_0 u)\|_T^2 + h^{-1}\|u - Q_0 u\|_{\partial T}^2 \right) \\
\lesssim & \; h^{2k}\|u\|_{k+1}^2.
\end{aligned}$$

Thus,

$$\|u - Q_h u\|_{1,h} \lesssim h^k \|u\|_{k+1}. \tag{6.16}$$

It follows from the triangle inequality, and the estimates (6.15) and (6.16) that

$$\|u - u_h\|_{1,h} \le \|u - Q_h u\|_{1,h} + \|u_h - Q_h u\|_{1,h} \lesssim h^k \|u\|_{k+1}.$$

This completes the proof.

In the rest of the section, we shall establish an optimal-order error for the weak Galerkin finite element scheme (4.5) in the usual L^2 norm by using a duality argument. To this end, we consider a dual problem that seeks $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$-\nabla \cdot (\mathcal{A}\nabla\psi) - \boldsymbol{b}\cdot\nabla\psi + c\psi = e_0 \quad \text{in } \Omega. \tag{6.17}$$

Assume that the above dual problem has the usual H^2-regularity. This means that

$$\|\psi\|_2 \lesssim \|e_0\|. \tag{6.18}$$

Theorem 6.4 *Let $u_h \in V_h$ be weak Galerkin finite element solution of the problem* (1.1)-(1.2) *arising from* (4.5)*. Assume the exact solution $u \in H^{k+1}(\Omega)$. In addition, assume that the dual problem* (6.17) *has the usual H^2-regularity. Then*

$$\|u - u_0\| \lesssim h^{k+1}\|u\|_{k+1}. \tag{6.19}$$

Proof. By testing (6.17) with e_0, we obtain

$$\begin{aligned}
\|e_0\|^2 &= -(\nabla\cdot(\mathcal{A}\nabla\psi), e_0) - (\boldsymbol{b}\cdot\nabla\psi, e_0) + (c\psi, e_0) \\
&= (\mathcal{A}\nabla\psi, \nabla e_0) - \sum_{T\in\mathcal{T}_h} \langle(\mathcal{A}\nabla\psi)\cdot\boldsymbol{n}, e_0 - e_b\rangle_{\partial T} - (\boldsymbol{b}\cdot\nabla\psi, e_0) \\
&\quad -\frac{1}{2}((\nabla\cdot\boldsymbol{b})\psi, e_0) + (c_0\psi, e_0),
\end{aligned} \tag{6.20}$$

where we have used the fact that $e_b = 0$ on $\partial\Omega$ and $c_0 = \frac{1}{2}(\nabla\cdot\boldsymbol{b}) + c$. Setting $u = \psi$ and $v_0 = e_0$

in (6.3) and (6.8), we can obtain the following equations

$$(\mathcal{A}\nabla\psi, \nabla e_0) = (\mathcal{A}\nabla_w(Q_h\psi), \nabla_w e_h) + \sum_{T\in\mathcal{T}_h} \langle(\mathcal{A}\mathbb{Q}_h(\nabla\psi))\cdot\boldsymbol{n}, e_0 - v_b\rangle_{\partial T}, \tag{6.21}$$

$$\begin{aligned}
&(\boldsymbol{b}\cdot\nabla\psi, e_0) + \frac{1}{2}((\nabla\cdot\boldsymbol{b})\psi, e_0) \\
= \;& \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w(Q_h\psi), e_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w e_h, Q_0\psi) \\
&+\frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0\psi-\psi)e_0) + (\boldsymbol{b}\cdot\nabla e_0, Q_0\psi-\psi) \\
&+\frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, (\psi - Q_b\psi)(e_0-e_b)\rangle_{\partial T} \\
&+\frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n}, (\psi - Q_0\psi)(e_0-e_b)\rangle_{\partial T}.
\end{aligned} \tag{6.22}$$

According the definition of $a_s(v,w)$ in (4.4) and the symmetry of $\mathcal{A}$, we can get

$$\begin{aligned}
a_s(e_h, Q_h\psi) \;=\;& (\mathcal{A}\nabla_w(Q_h\psi), \nabla_w e_h) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w(Q_h\psi), e_0) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w e_h, Q_0\psi) \\
&+(c_0e_0, Q_0\psi) + s(e_h, Q_h\psi).
\end{aligned} \tag{6.23}$$

By combining the (6.20)-(6.23), we can obtain

$$\begin{aligned}
\|e_0\|^2 \;=\;& a_s(e_h, Q_h\psi) - \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0\psi-\psi)e_0) - (\boldsymbol{b}\cdot\nabla e_0, Q_0\psi-\psi) \\
&-l_\psi(e_h) - t_{\psi,\boldsymbol{b}}(e_h) - \lambda_{\psi,\boldsymbol{b}}(e_h) - s(e_h, Q_h\psi) + (c_0e_0, \psi - Q_0\psi).
\end{aligned} \tag{6.24}$$

By using the (6.1), we can get

$$\begin{aligned}
\|e_0\|^2 \;=\;& \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0u-u)Q_0\psi) + (\boldsymbol{b}\cdot\nabla(Q_0\psi), Q_0u-u) + (c_0(u-Q_0u), Q_0\psi) \\
&+t_{u,\boldsymbol{b}}(Q_h\psi) + \lambda_{u,\boldsymbol{b}}(Q_h\psi) - l_u(Q_h\psi) - S(Q_hu, Q_h\psi) \\
&-\frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0\psi-\psi)e_0) - (\boldsymbol{b}\cdot\nabla e_0, Q_0\psi-\psi) + (c_0e_0, \psi-Q_0\psi) \\
&-l_\psi(e_h) - t_{\psi,\boldsymbol{b}}(e_h) - \lambda_{\psi,\boldsymbol{b}}(e_h) - s(e_h, Q_h\psi).
\end{aligned} \tag{6.25}$$

According to the L^2 error estimates in [3], we have the following estimates

$$|l_u(Q_h\psi)| \;\lesssim\; h^{k+1}\|u\|_{k+1}\|\psi\|_2, \tag{6.26}$$
$$|s(Q_hu, Q_h\psi)| \;\lesssim\; h^{k+1}\|u\|_{k+1}\|\psi\|_2, \tag{6.27}$$
$$|s(e_h, Q_h\psi) \;\lesssim\; h^{k+1}\|u\|_{k+1}\|\psi\|_2, \tag{6.28}$$
$$|l_\psi(e_h)| \;\lesssim\; h^{k+1}\|u\|_{k+1}\|\psi\|_2. \tag{6.29}$$

Applying the Cauchy-Schwarz inequality, the definition of Q_b and (5.8), one can derive

$$
\begin{aligned}
& t_{u,\boldsymbol{b}}(Q_h\psi) \\
= \;& \frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, (u-Q_bu)(Q_0\psi-Q_b\psi)\rangle_{\partial T} \\
\le \;& \frac{1}{2}\Big|\sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, (u-Q_bu)(Q_0\psi-\psi)\rangle_{\partial T}\Big| \\
& +\frac{1}{2}\Big|\sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, (u-Q_bu)(Q_b\psi-\psi)\rangle_{\partial T}\Big| \\
\le \;& \frac{1}{2}\Big(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{b}\|_{L^\infty(\partial T)}\Big)\Big(\sum_{T\in\mathcal{T}_h}\|u-Q_bu\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|Q_0\psi-\psi\|_{\partial T}^2\Big)^{1/2} \\
& +\frac{1}{2}\Big(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{b}\|_{L^\infty(\partial T)}\Big)\Big(\sum_{T\in\mathcal{T}_h}\|u-Q_bu\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|Q_b\psi-\psi\|_{\partial T}^2\Big)^{1/2} \\
\lesssim \;& \Big(\sum_{T\in\mathcal{T}_h}\|u-Q_0u\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|Q_0\psi-\psi\|_{\partial T}^2\Big)^{1/2} \\
\lesssim \;& h^{k+2}\|u\|_{k+1}\|\psi\|_2.
\end{aligned}
\tag{6.30}
$$

Analogously, we can get

$$
\lambda_{u,\boldsymbol{b}}(Q_h\psi) \;\lesssim\; h^{k+2}\|u\|_{k+1}\|\psi\|_2. \tag{6.31}
$$

With the Cauchy-Schwarz inequality and Lemma 5.4, we can derive the following estimate

$$
\begin{aligned}
& \Big|\frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0u-u)Q_0\psi)+(\boldsymbol{b}\cdot\nabla(Q_0\psi), Q_0u-u)+(c_0(u-Q_0u), Q_0\psi)\Big| \\
\le & \sum_{T\in\mathcal{T}_h}\Big(\|\nabla\cdot\boldsymbol{b}\|_{L^\infty(T)}\|Q_0u-u\|_T\|Q_0\psi\|_T+\|\boldsymbol{b}\|_{L^\infty(T)}\|\nabla(Q_0\psi)\|_T\|Q_0u-u\|_T \\
& \quad +\|c_0\|_{L^\infty(T)}\|Q_0u-u\|_T\|Q_0\psi\|_T\Big) \\
\lesssim & \sum_{T\in\mathcal{T}_h}\|Q_0u-u\|_T\|\psi-Q_0\psi\|_T+\sum_{T\in\mathcal{T}_h}\|Q_0u-u\|_T\|\psi\|_T \\
& +\sum_{T\in\mathcal{T}_h}\|\nabla\psi-\nabla(Q_0\psi)\|_T\|Q_0u-u\|_T+\sum_{T\in\mathcal{T}_h}\|\nabla\psi\|_T\|Q_0u-u\|_T \\
\le & \big(\sum_{T\in\mathcal{T}_h}\|Q_0u-u\|_T^2\big)^{1/2}\big(\sum_{T\in\mathcal{T}_h}\|\psi-Q_0\psi\|_T^2\big)^{1/2} \\
& +\big(\sum_{T\in\mathcal{T}_h}\|Q_0u-u\|_T^2\big)^{1/2}\big(\sum_{T\in\mathcal{T}_h}\|\psi\|_T^2\big)^{1/2} \\
& +\big(\sum_{T\in\mathcal{T}_h}\|\nabla\psi-\nabla(Q_0\psi)\|_T^2\big)^{1/2}\big(\sum_{T\in\mathcal{T}_h}\|Q_0u-u\|_T^2\big)^{1/2} \\
& +\big(\sum_{T\in\mathcal{T}_h}\|\nabla\psi\|_T^2\big)^{1/2}\big(\sum_{T\in\mathcal{T}_h}\|Q_0u-u\|_T^2\big)^{1/2} \\
\lesssim & h^{k+3}\|\psi\|_2\|u\|_{k+1}+h^{k+2}\|\psi\|_2\|u\|_{k+1}+h^{k+1}\|\psi\|_2\|u\|_{k+1} \\
\lesssim & h^{k+1}\|\psi\|_2\|u\|_{k+1}.
\end{aligned}
\tag{6.32}
$$

By (5.14), (5.15) and Theorem 6.2, we obtain

$$t_{\psi,\boldsymbol{b}}(e_h) \lesssim h^2 |||e_h||| \|\psi\|_2 \lesssim h^{k+2}\|\psi\|_2\|u\|_{k+1}, \tag{6.33}$$

$$\lambda_{\psi,\boldsymbol{b}}(e_h) \lesssim h^2 |||e_h||| \|\psi\|_2 \lesssim h^{k+2}\|\psi\|_2\|u\|_{k+1}. \tag{6.34}$$

With (5.4), Lemma 5.6 and Theorem 6.2, we can obtain the following estimate

$$\begin{aligned}
&\left| -\frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0\psi-\psi)e_0) - (\boldsymbol{b}\cdot\nabla e_0, Q_0\psi-\psi) + (c_0 e_0, \psi - Q_0\psi)\right| \\
\le& \sum_{T\in\mathcal{T}_h} \big(\|\nabla\cdot\boldsymbol{b}\|_{L^\infty(T)}\|Q_0\psi-\psi\|_T\|e_0\|_T + \|\boldsymbol{b}\|_{L^\infty(T)}\|\nabla e_0\|_T\|Q_0\psi-\psi\|_T \\
&\quad + \|c_0\|_{L^\infty(T)}\|Q_0\psi-\psi\|_T\|e_0\|_T\big) \\
\lesssim& \sum_{T\in\mathcal{T}_h} \big(\|Q_0\psi-\psi\|_T\|e_0\|_T + \|\nabla e_0\|_T\|Q_0\psi-\psi\|_T\big) \\
\lesssim& \left(\sum_{T\in\mathcal{T}_h}\|Q_0\psi-\psi\|_T^2\right)^{1/2} |||e_h||| + \left(\sum_{T\in\mathcal{T}_h}\|Q_0\psi-\psi\|_T^2\right)^{1/2}\|e_h\|_{1,h} \\
\lesssim& h^{k+2}\|\psi\|_2\|u\|_{k+1}.
\end{aligned} \tag{6.35}$$

Combining (6.25) with the estimates (6.26)-(6.35), we obtain

$$\|e_0\|^2 \lesssim h^{k+1}\|u\|_{k+1}\|\psi\|_2,$$

which, combined with the regularity assumption (6.18) and the triangle inequality, gives the desired optimal-order error estimate (6.19).

7 Numerical experiments

The goal of this section is to report some numerical results for the new weak Galerkin finite element scheme proposed and analyzed in previous sections. For simplicity, we consider a rectangular domain $\Omega = [0,1]\times[0,1]$ with uniform triangulation in this section. The triangular mesh is constructed by: 1) uniformly partitioning the domain into $n\times n$ sub-rectangles; 2) dividing each rectangular element by diagonal line with a negative slope. The mesh size is denoted by $h = 1/n$.

Let $u_h = \{u_0, u_b\}$ and u be the solutions to the weak Galerkin equation (4.5) and the original problem (1.1)-(1.2), respectively. Define the error by $e_h = u_h - Q_h u = \{e_0, e_b\}$ where $Q_h u$ is the L^2 projection of u onto appropriately defined spaces. The following norms will be measured in all the numerical experiments:

Discrete H^1 norm: $|||e_h||| = \left(\sum_{T\in\mathcal{T}_h}\left(\int_T |\nabla_w e_h|^2 dT + h^{-1}\int_{\partial T}|Q_b e_0 - e_b|^2 ds\right)\right)^{\frac{1}{2}}$,

Element-based L^2 norm: $\|e_0\| = \left(\sum_{T\in\mathcal{T}_h}\int_T |e_0|^2 dx\right)^{\frac{1}{2}}$.

7.1 Case 1: a model problem with ordinary coefficient

In this case, we consider $\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$, $\boldsymbol{b} = (1,1)^T$, $c = 1$ and the boundary condition $u|_{\partial\Omega}$ and f are chosen such that the exact solution is given by $u = sin(\pi x)cos(\pi y)$. In the test of convergence, the first ($k=1$) and second ($k=2$) order of weak Galerkin elements are used

Table 1: Case 1. Numerical results for first-order WG ($k = 1$).

h	$\|\|\|e_h\|\|\|$	order	$\|e_0\|$	order
1/4	1.0466e+00		1.3612e-01	
1/8	5.2317e-01	1.0003e+00	3.4362e-02	1.9860e+00
1/16	2.6150e-01	1.0004e+00	8.6000e-03	1.9967e+00
1/32	1.3074e-01	1.0001e+00	2.1537e-03	1.9992e+00
1/64	6.5368e-02	1.0000e+00	5.3850e-04	1.9998 e+00
1/128	3.2684e-02	1.0000e+00	1.3463e-04	2.0000 e+00

Table 2: Case 1. Numerical results for second-order WG ($k = 2$).

h	$\|\|\|e_h\|\|\|$	order	$\|e_0\|$	order
1/4	1.9511e-01		1.6626e-02	
1/8	4.9543e-02	1.9776e+00	2.1126e-03	2.9763e+00
1/16	1.2436e-02	1.9941e+00	2.6511e-04	2.9944e+00
1/32	3.1125e-03	1.9984e+00	3.3168e-05	2.9987e+00
1/64	7.7838e-04	1.9995e+00	4.1469e-06	2.9997e+00
1/128	1.9461e-04	1.9998e+00	5.1838e-07	2.9999e+00

in the construction of the finite element space V_h. For simplicity, these two elements shall be referred to as $(P_1(T), P_0(e))$ and $(P_2(T), P_1(e))$.

The numerical results on Table 1 and Table 2 show the rate of convergence for the WG solutions in H^1 and L^2 norms associated with $k = 1$ and $k = 2$, respectively. The numerical results indicate that the WG solution is convergent with rate $O(h^k)$ in H^1 norm and $O(h^{k+1})$ in L^2 norm, which are same as the theoretical results shown in Theorems 6.2 and 6.4.

7.2 Case 2: a model problem with variable coefficient

In this case, we consider the test problems where the coefficient $\mathcal{A} = \begin{pmatrix} x+y & 0 \\ 0 & x+y \end{pmatrix}$, the coefficient $\boldsymbol{b} = (x, y)^t$ and $c = 1$. We set the exact solution to be $u = sin(\pi x)sin(\pi y)$ in the homogeneous boundary case and $u = sin(\pi x)sin(\pi y) + x + y$ in the nonhomogeneous boundary case, respectively. In these tests, we only consider the linear weak Galerkin elements ($k = 1$) in the finite element space V_h. The numerical results are shown in Table 3 for homogeneous boundary case and Table 4 for homogeneous boundary case. From Tables 3 and 4, we can see that the WG solution is convergent with rate $O(h)$ in H^1 norm and $O(h^2)$ in L^2 norm.

All the numerical examples given above are in good agreement with the theoretical analysis in Section 6, which demonstrate that the new weak Galerkin finite element scheme (4.5) is accurate and robust.

8 Discussion on convection-dominated problems

Letting $\mathcal{A} = \varepsilon$, where ε is a constant coefficient and $0 < \varepsilon \ll 1$, one can rewrite the problem (1.1)-(1.2) by the following convection-dominated problem: seek an unknown function $u = u(\boldsymbol{x})$ such that

$$-\varepsilon\Delta u + \nabla \cdot (\boldsymbol{b}u) + cu = f \quad \text{in } \Omega, \tag{8.1}$$

$$u = g \quad \text{on } \partial\Omega. \tag{8.2}$$

Table 3: Case 2. Numerical results for homogeneous boundary problems.

h	$\|\|\|e_h\|\|\|$	order	$\|\|e_0\|\|$	order
1/4	1.1639e+00		1.3831e-01	
1/8	5.8995e-01	9.8027e-01	3.5787e-02	1.9504e+00
1/16	2.9597e-01	9.9513e-01	9.0347e-03	1.9859e+00
1/32	1.4811e-01	9.9878e-01	2.2648e-03	1.9961e+00
1/64	7.4072e-02	9.9969e-01	5.6662e-04	1.9989e+00
1/128	3.7038e-02	9.9992e-01	1.4168e-04	1.9997e+00

Table 4: Case 2. Numerical results for nonhomogeneous boundary problems.

h	$\|\|\|e_h\|\|\|$	order	$\|\|e_0\|\|$	order
1/4	1.1560e+00		1.3820e-01	
1/8	5.8644e-01	9.7913e-01	3.5930e-02	1.9436e+00
1/16	2.9429e-01	9.9473e-01	9.0888e-03	1.9830e+00
1/32	1.4728e-01	9.9866e-0	2.2803e-03	1.9949e+00
1/64	7.3659e-02	9.9965e-01	5.7073e-04	1.9983e+00
1/128	3.6832e-02	9.9990e-01	1.4275e-04	1.9993e+00

It is well-known that standard finite element methods often suffer from the deterioration of numerical accuracy for convection-dominated problems due to local singularities arising from interior or boundary layers. A lot of research has been devoted to solving such kinds of problems properly, such as stabilized methods [21, 22], discontinuous Galerkin(DG) methods [23, 24] and WG methods [25].

From the numerical examples in Section 7, we can see that the WG scheme (4.5) is accurate. However, considering $0 < \varepsilon \ll h$ in the convection-dominated problem (8.1)-(8.2), one can find that the numerical accuracy is deteriorating. This disappointing behaviour occurs because the numerical solutions can not adequately approximate the solutions inside layers. One remedy to this problem is to modify the stabilization operator $s(\cdot,\cdot)$ in the scheme (4.5). To this end, we may give the stabilization operator having the form of

$$\sum_{T\in\mathcal{T}_h} \kappa\langle Q_b u_0 - u_b, Q_b v_0 - v_b\rangle_{\partial T},$$

where κ is a stabilization parameter and need to be suitably selected. That is, the WG scheme for the convection-dominated problem (8.1)-(8.2) can be established as follows: seek $u_h = \{u_0, u_b\} \in V_h$ satisfying both $u_b = Q_b g$ on $\partial\Omega$ and the following equation

$$a_h(u_h, v) + \sum_{T\in\mathcal{T}_h} \kappa\langle Q_b u_0 - u_b, Q_b v_0 - v_b\rangle_{\partial T} = (f, v_0),\ \forall v = \{v_0, v_b\} \in V_h^0, \tag{8.3}$$

where

$$a_h(u_h, v) = \varepsilon(\nabla_w u_h, \nabla_w v) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w u_h, v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w v, u_0) + (c_0 u_0, v_0).$$

From [25], we see that it is difficult to choose such stabilization parameter κ. Different problems lead to different choices of κ. For most (general) second-order elliptic problems, it is suitable to choose $\kappa = h_T^{-1}$ to establish the corresponding WG scheme (see [2, 3]). However, for convection-dominated problems, the choice of the stabilization parameter κ becomes more complicated. For instance, Chen, Feng and Xie select κ related to ε, $\boldsymbol{b}$ and h_T in [25]. Since the choice of the stabilization parameter κ for convection-dominated problems is complicated and the different κ corresponds to the different stabilization operator which involves the different mathematical convergence theory for the corresponding WG method, we leave this to our next work.

9 Conclusions

We have presented a new weak Galerkin finite element scheme for general second-order elliptic problems. The advantage of the new scheme is that the system of linear equations from the scheme is positive definite and one might easily get the well-postedness of the system. The new WG method has improved the method in [1] for general second-order elliptic equations in two ways. First, the new WG method can be applied on general polytopal meshes while the method in [1] only works for simplicial elements. Second, the error analysis has been simplified by using a skew symmetric form for handling the convection term. The theoretical analysis and the numerical results indicate that the new WG scheme is accurate and robust.

Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grant No.11671157, No.91430104 and No.91430213.

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