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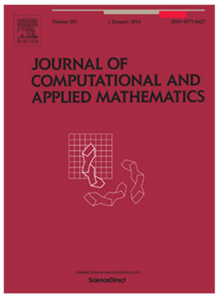

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CONVERGENCE OF A B-E BASED FINITE ELEMENT METHOD FOR MHD MODELS ON LIPSCHITZ DOMAINS

KAIBO HU, WEIFENG QIU, AND KE SHI

ABSTRACT. We discuss a class of magnetic-electric fields based finite element schemes for stationary magnetohydrodynamics (MHD) systems with two types of boundary conditions. We establish a key L^3 estimate for divergence-free finite element functions for a new type of boundary conditions. With this estimate and a similar one in [24], we rigorously prove the convergence of Picard iterations and the finite element schemes with weak regularity assumptions. These results demonstrate the convergence of the finite element methods for singular solutions.

1. INTRODUCTION

Magnetohydrodynamics (MHD) models have various important applications in liquid metal industry, controlled fusion and astronomy etc. There have been extensive discussions on numerical methods for MHD models. However, due to the nonlinear coupling and rich structures of MHD systems, the numerical simulation still remains a challenging and active research area. This paper is devoted to the analysis of a class of stable and structure-preserving finite element methods.

We consider the following stationary MHD system on a polyhedral domain Ω:

$$\begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - R_e^{-1}\Delta\boldsymbol{u} - S\boldsymbol{j}\times\boldsymbol{B} + \nabla p = \boldsymbol{f}, \\ \boldsymbol{j} - R_m^{-1}\nabla\times\boldsymbol{B} = \boldsymbol{0}, \\ \nabla\times\boldsymbol{E} = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{B} = 0, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{j} = \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}. \end{cases} \tag{1.1}$$

Here $\boldsymbol{u}$ is the fluid velocity, p is the fluid pressure, $\boldsymbol{j}$ is the current density, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields respectively. The system is characterized by three parameters: the hydrodynamic Reynolds number R_e, the magnetic Reynolds number R_m and the coupling number S.

We mainly consider the following type of boundary conditions:

$$\boldsymbol{u} = 0, \quad \boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{0}, \quad \text{on } \partial\Omega, \tag{1.2}$$

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where $\boldsymbol{n}$ is the unit normal vector of $\partial\Omega$. We also consider an alternative boundary condition c.f., [17]:

$$\boldsymbol{u} = 0, \quad \boldsymbol{B} \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{E} \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega. \tag{1.3}$$

This paper mainly focuses on the error estimates of the proposed FEM. For the sake of simplicity, we only consider above two homogeneous boundary conditions. We refer to [14, 27] for a more comprehensive discussion on the practical aspects such as boundary conditions and dimensionless parameters of the MHD systems. Finite element discretizations of the MHD system (1.1) have a long history. Based on the function and finite element spaces for the magnetic variable $\boldsymbol{B}$, these schemes can be classified as H^1-, $H(\mathrm{curl})$- and $H(\mathrm{div})$-based formulations. Gunzburger [17] studied a finite element method where $\boldsymbol{B}$ was discretized in H^1 with the Lagrange elements. With certain conditions on the boundary data and right hand side, Gunzburger [17] proved the existence and uniqueness of the weak solutions and established optimal error estimates for the finite element methods. The domain is assumed to be bounded in $\mathbb{R}^3$ which is either convex or has a $C^{1,1}$ boundary. Under this assumption, the true solution is smooth. And the convergence proof in [17] also relies on this assumption. To remove this restriction on the domain, Schötzau [32] proposed another variational formulation with the magnetic variable in $H(\mathrm{curl})$. In the finite element scheme based on this formulation, $\boldsymbol{B}$ is discretized in the $H(\mathrm{curl})$-conforming Nédélec spaces [28, 29] and the quasi-optimal convergence of the approximation solutions was shown in [32]. We refer to, e.g., [12, 16, 18, 19, 13, 4, 34, 36, 38] for some variants and the convergence analysis of iterative methods and finite element discretizations.

For MHD systems, magnetic Gauss's law plays an important role in both physics (nonexistence of magnetic monopole) and numerical simulations (c.f., [7, 11]). However, in the above $H(\mathrm{curl})$ based approach, magnetic Gauss's law is only preserved in the weak sense. One way to obtain schemes with precisely preserved magnetic Gauss's law is to use the vector potential of $\boldsymbol{B}$, see [1, 21, 22] and the references therein. Since the vector potential belongs to $H(\mathrm{curl})$, this method also falls in the category of $H(\mathrm{curl})$ based formulations.

To preserve magnetic Gauss's law precisely on the discrete level with electric and magnetic fields as variables, a class of finite element schemes was developed in [23, 24] for the time dependent and the stationary MHD systems respectively. The magnetic field $\boldsymbol{B}$ is discretized by the $H(\mathrm{div})$ conforming Raviart-Tomas [31] or BDM [9] elements. An electric variable, either the electric field $\boldsymbol{E}$ in [23] or the current density $\boldsymbol{j}$ in [24], is retained in the formulation and discretized by the $H(\mathrm{curl})$ conforming elements in the same discrete de Rham complex.

In this paper, we prove the convergence of the $H(\mathrm{div})$ based methods for stationary MHD problems with weak regularity assumptions. Several variants of this type of schemes exist, and we choose to consider a $\boldsymbol{B}$-$\boldsymbol{E}$ based formulation in the discussions below. This formulation is the stationary case of [23] and differs from the $\boldsymbol{B}$-$\boldsymbol{j}$ formulation in [24] by a projection of nonlinear terms (see Section 4 below for details). Therefore we do not claim the discretization studied in this paper as a brand new method, although the precise formulation has not appeared in the literature to the best of our knowledge.

To show the convergence with both types of boundary conditions, we extend the key Hodge mapping and L^3 estimates established in [24] to a new type of boundary condition. With an analysis based on the reduced systems, we show that the schemes are unconditional stable and well-posed.We prove the convergence of the finite element scheme by carefully choosing interpolation functions (see (5.6) below). Comparing with the convergence analysis in [24] for the $\boldsymbol{B}$-$\boldsymbol{j}$ based finite element methods, we adopt a new strategy and, as a result, only assume weak regularity of the solutions in this paper ((5.1) below). This demonstrates the convergence of the Picard iterations and the finite element schemes for singular solutions.

We also show another strategy to impose the strong divergence-free condition, instead of using Lagrange multipliers as in the previous work [24] by one of the authors and collaborator. We introduce an augmented term $(\nabla\cdot\boldsymbol{B},\nabla\cdot\boldsymbol{C})$ in the variational formulation. Thanks to the structure-preserving properties, these two approaches are actually equivalent and Faraday's law $\nabla\cdot\boldsymbol{B}=0$ also holds precisely on the discrete level.

The remaining part of this paper will be organized as follows. In Section 2, we provide some preliminary settings. In Section 3, we give two types of L^3 estimates for the discrete magnetic field. In Sections 4, 5 and 6, we formulate the numerical method for the MHD models with boundary condition (1.2), show that its Picard iterations are well-posed and convergent, and show the optimal convergence of approximations to the velocity field and magnetic field even for singular solutions. In Section 7, we generalize the numerical method for the MHD models with boundary condition (1.3), provide its basic properties and show the optimal convergence.

2. Preliminaries

We assume that Ω is a bounded Lipschitz polyhedron. For the ease of exposition, we further assume that Ω is contractable, i.e. there is no nontrivial harmonic form.

Using the standard notation for the inner product and the norm of the L^2 space

$$(u,v):=\int_\Omega u\cdot v\mathrm{d}x,\quad \|u\|:=\left(\int_\Omega |u|^2\mathrm{d}x\right)^{1/2}.$$

The scalar function space H^1 is defined by

$$H^1(\Omega):=\left\{v\in L^2(\Omega):\nabla\boldsymbol{v}\in L^2(\Omega)\right\}.$$

For a function $u\in W^{k,p}(\Omega)$, we use $\|u\|_{k,p}$ for the standard norm in $W^{k,p}(\Omega)$. When $p=2$ we drop the index p, i.e. $\|u\|_k:=\|u\|_{k,2}$ and $\|u\|:=\|u\|_{0,2}$. We define vector function spaces

$$H(\mathrm{curl},\Omega):=\{\boldsymbol{v}\in L^2(\Omega),\nabla\times\boldsymbol{v}\in L^2(\Omega)\},$$

and

$$H(\mathrm{div},\Omega):=\{\boldsymbol{w}\in L^2(\Omega),\nabla\cdot\boldsymbol{w}\in L^2(\Omega)\}.$$

With explicit boundary conditions, we define

$$H_0^1(\Omega):=\left\{v\in H^1(\Omega):v\,|_{\partial\Omega}=0\right\},$$

$$H_0(\mathrm{curl},\Omega):=\{\boldsymbol{v}\in H(\mathrm{curl},\Omega),\boldsymbol{v}\times\boldsymbol{n}=0\text{ on }\partial\Omega\},$$

and

$$H_0(\mathrm{div},\Omega):=\{\boldsymbol{w}\in H(\mathrm{div},\Omega),\boldsymbol{w}\cdot\boldsymbol{n}=0\text{ on }\partial\Omega\}.$$

We often use the following notation:

$$L_0^2(\Omega):=\left\{v\in L^2(\Omega):\int_\Omega v=0\right\}.$$

The corresponding norms in H^1, $H(\mathrm{curl})$ and $H(\mathrm{div})$ spaces are defined by

$$\|\boldsymbol{u}\|_1^2=\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2,$$

$$\|\boldsymbol{F}\|_{\mathrm{curl}}^2 := \|\boldsymbol{F}\|^2 + \|\nabla \times \boldsymbol{F}\|^2,$$
$$\|\boldsymbol{C}\|_{\mathrm{div}}^2 := \|\boldsymbol{C}\|^2 + \|\nabla \cdot \boldsymbol{C}\|^2.$$

For a general Banach space $\boldsymbol{Y}$ with a norm $\|\cdot\|_{\boldsymbol{Y}}$, the dual space $\boldsymbol{Y}^*$ is equipped with the dual norm defined by

$$\|\boldsymbol{h}\|_{\boldsymbol{Y}^*} := \sup_{0\neq \boldsymbol{y}\in \boldsymbol{Y}} \frac{\langle \boldsymbol{h}, \boldsymbol{y}\rangle}{\|\boldsymbol{y}\|_{\boldsymbol{Y}}}.$$

For the special case that $\boldsymbol{Y} = H_0^1(\Omega)$, the dual space $\boldsymbol{Y}^* = H^{-1}(\Omega)$ and the corresponding norm is denoted by $\|\cdot\|_{-1}$, which is defined by

$$\|\boldsymbol{f}\|_{-1} := \sup_{0\neq \boldsymbol{v}\in [H_0^1(\Omega)]^3} \frac{\langle \boldsymbol{f}, \boldsymbol{v}\rangle}{\|\nabla \boldsymbol{v}\|}.$$

In this paper, we will use C to denote a generic constant in inequalities which is independent of the exact solution and the mesh size. For instance, we will need the following Poincaré's inequality:

$$\|u\|_{0,6} \leq C\|\nabla u\|, \quad \forall\, u \in H_0^1(\Omega). \tag{2.1}$$

Since the fluid convection frequently appears in subsequent discussions, we introduce a trilinear form

$$L(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}[((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w}\cdot\nabla)\boldsymbol{v}, \boldsymbol{u})].$$

Considering $\boldsymbol{w}$ as a known function, $L(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})$ is a bilinear form of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Let $\mathcal{T}_h$ be a triangulation of Ω, and we assume that the mesh is regular and quasi-uniform, so that the inverse estimates hold [8]. We use $P_k(\mathcal{T}_h)$ to denote the piecewise polynomial space of degree k on $\mathcal{T}_h$. The finite element de Rham sequence is an abstract framework to unify the above spaces and their discretizations, see e.g. Arnold, Falk, Winther [2, 3], Hiptmair [20], Bossavit [6] for more detailed discussions. Figure 1 and Figure 2 show the commuting diagrams we will use. The electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ will be discretized in the middle two spaces respectively. Notice that though projections in Figure 1 can be different from corresponding ones in Figure 2, we don't need to distinguish them in any analysis in this paper.

$$\begin{array}{ccccccc}
H_0(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H_0(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H_0(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_0^2 \\
\downarrow \Pi^{\mathrm{grad}} & & \downarrow \Pi^{\mathrm{curl}} & & \downarrow \Pi^{\mathrm{div}} & & \downarrow \Pi^0 \\
H_0^h(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H_0^h(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H_0^h(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_0^{2,h}
\end{array}$$

FIGURE 1. Continuous and discrete de Rham sequence - homogeneous boundary conditions

$$\begin{array}{ccccccc}
H(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
\downarrow \Pi^{\mathrm{grad}} & & \downarrow \Pi^{\mathrm{curl}} & & \downarrow \Pi^{\mathrm{div}} & & \downarrow \Pi^0 \\
H^h(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & H^h(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H^h(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^{2,h}
\end{array}$$

FIGURE 2. Continuous and discrete de Rham sequence - no boundary condition

As we shall see, $H(\mathrm{div})$ functions with vanishing divergence will play an important role in the study. So we define on the continuous level

$$H_0(\mathrm{div}0, \Omega) := \{\boldsymbol{C} \in H_0(\mathrm{div}, \Omega) : \nabla \cdot \boldsymbol{C} = 0\},$$

and the finite element subspace

$$H_0^h(\mathrm{div}0, \Omega) := \{\boldsymbol{C}_h \in H_0^h(\mathrm{div}, \Omega) : \nabla \cdot \boldsymbol{C}_h = 0\}.$$

We use $\boldsymbol{V}_h$ to denote the finite element subspace of velocity $\boldsymbol{u}_h$, and Q_h for pressure p_h. There are many existing stable pairs for $\boldsymbol{V}_h$ and Q_h, for example, the Taylor-Hood elements [15, 5]. Spaces $H_0^h(\mathrm{div}, \Omega)$ and $L_0^{2,h}(\Omega)$ are finite elements from the discrete de Rham sequence. For these spaces we use the explicit names for clarity, and use the notation $\boldsymbol{V}_h$ and $\boldsymbol{Q}_h$ for the fluid part to indicate that they may be different from $H_0^h(\mathrm{grad}, \Omega)$ and $L_0^{2,h}(\Omega)$ in the de Rham sequence. We use $\boldsymbol{V}_h^0$ to denote the discrete velocity space, i.e.

$$\boldsymbol{V}_h^0 := \{\boldsymbol{v}_h \in \boldsymbol{V}_h : (\nabla \cdot \boldsymbol{v}_h, q_h) = 0, \ \forall q_h \in Q_h\}.$$

There is a unified theory for the discrete de Rham sequence of arbitrary order [5, 2, 3]. In the case $n = 3$, the lowest order elements can be represented as:

$$\mathbb{R} \xrightarrow{\subset} \mathcal{P}_3\Lambda^0 \xrightarrow{d} \mathcal{P}_2\Lambda^1 \xrightarrow{d} \mathcal{P}_1\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \longrightarrow 0,$$

$$\mathbb{R} \xrightarrow{\subset} \mathcal{P}_2\Lambda^0 \xrightarrow{d} \mathcal{P}_1\Lambda^1 \xrightarrow{d} \mathcal{P}_1^-\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \longrightarrow 0,$$

$$\mathbb{R} \xrightarrow{\subset} \mathcal{P}_2\Lambda^0 \xrightarrow{d} \mathcal{P}_2^-\Lambda^1 \xrightarrow{d} \mathcal{P}_1\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \longrightarrow 0,$$

$$\mathbb{R} \xrightarrow{\subset} \mathcal{P}_1\Lambda^0 \xrightarrow{d} \mathcal{P}_1^-\Lambda^1 \xrightarrow{d} \mathcal{P}_1^-\Lambda^2 \xrightarrow{d} \mathcal{P}_0\Lambda^3 \longrightarrow 0.$$

The correspondence between the language of differential forms and classical finite element methods is summarized in Table 1.

To obtain compatible finite element schemes, below we require that the discrete spaces $H_0^h(\mathrm{curl}, \Omega)$, $H_0^h(\mathrm{div}, \Omega)$ and $L_0^{2,h}(\Omega)$ belong to the same finite element de Rham sequence.

k	$\Lambda_h^k(\Omega)$	Classical finite element space
0	$\mathcal{P}_r\Lambda^0(\mathcal{T})$	Lagrange elements of degree $\leq r$
1	$\mathcal{P}_r\Lambda^1(\mathcal{T})$	Nedelec 2nd-kind $H(\mathrm{curl})$ elements of degree $\leq r$
2	$\mathcal{P}_r\Lambda^2(\mathcal{T})$	Nedelec 2nd-kind $H(\mathrm{div})$ elements of degree $\leq r$
3	$\mathcal{P}_r\Lambda^3(\mathcal{T})$	discontinuous elements of degree $\leq r$
0	$\mathcal{P}_r^-\Lambda^0(\mathcal{T})$	Lagrange elements of degree $\leq r$
1	$\mathcal{P}_r^-\Lambda^1(\mathcal{T})$	Nedelec 1st-kind $H(\mathrm{curl})$ elements of order $r-1$
2	$\mathcal{P}_r^-\Lambda^2(\mathcal{T})$	Nedelec 1st-kind $H(\mathrm{div})$ elements of order $r-1$
3	$\mathcal{P}_r^-\Lambda^3(\mathcal{T})$	discontinuous elements of degree $\leq r-1$

TABLE 1. Correspondences between finite element differential forms and the classical finite element spaces for $n = 3$ (from [2])

As we shall see, it is useful to group the spaces to define

$$\boldsymbol{X}_h := \boldsymbol{V}_h \times H_0^h(\mathrm{curl}, \Omega) \times H_0^h(\mathrm{div}, \Omega).$$

and group $Q_h \times L_0^{2,h}(\Omega)$ to define

$$\boldsymbol{Y}_h := Q_h \times L_0^{2,h}(\Omega).$$

For the analysis, we also need to define a reduced space, where $\boldsymbol{E}_h$ is eliminated:

$$\boldsymbol{W}_h := \boldsymbol{V}_h \times H_0^h(\mathrm{div}, \Omega).$$

Denote the kernel space

$$\boldsymbol{X}_h^{00} := \boldsymbol{V}_h^0 \times H_0^h(\mathrm{curl}, \Omega) \times H_0^h(\mathrm{div}0, \Omega),$$

and

$$\boldsymbol{W}_h^{00} := \boldsymbol{V}_h^0 \times H_0^h(\mathrm{div}0, \Omega).$$

By definition, any $(\boldsymbol{u}_h, \boldsymbol{B}_h) \in \boldsymbol{W}_h^{00}$ satisfies $(\nabla \cdot \boldsymbol{u}_h, q_h) = 0$, $\forall q_h \in Q_h$ and $\nabla \cdot \boldsymbol{B}_h = 0$.

In order to define appropriate norms, we introduce the discrete curl operator on the discrete level. For any $\boldsymbol{C}_h \in H_0^h(\mathrm{div}, \Omega)$, define $\nabla_h \times \boldsymbol{C}_h \in H_0^h(\mathrm{curl}, \Omega)$ by:

$$(\nabla_h \times \boldsymbol{C}_h, \boldsymbol{F}_h) = (\boldsymbol{C}_h, \nabla \times \boldsymbol{F}_h), \quad \forall \boldsymbol{F}_h \in H_0^h(\mathrm{curl}, \Omega). \tag{2.2}$$

For any $\boldsymbol{w}_h \in H_0^h(\mathrm{curl}, \Omega)$, we define $\nabla_h \cdot \boldsymbol{w}_h \in H_0^h(\mathrm{grad}, \Omega)$ by

$$(\nabla_h \cdot \boldsymbol{w}_h, v_h) = -(\boldsymbol{w}_h, \nabla v_h), \quad \forall v_h \in H_0^h(\mathrm{grad}, \Omega). \tag{2.3}$$

We define $\mathbb{P} : L^2(\Omega) \to H_0^h(\mathrm{curl}, \Omega)$ to be the L^2 projection

$$(\mathbb{P}\phi, \boldsymbol{F}_h) = (\phi, \boldsymbol{F}_h), \quad \forall \boldsymbol{F}_h \in H_0^h(\mathrm{curl}, \Omega), \phi \in L^2(\Omega).$$

We further define $\|\cdot\|_d$ as a modified norm of $H_0^h(\mathrm{div}, \Omega)$ by

$$\|\boldsymbol{C}_h\|_d^2 := \|\boldsymbol{C}_h\|^2 + \|\nabla \cdot \boldsymbol{C}_h\|^2 + \|\nabla_h \times \boldsymbol{C}_h\|^2.$$

Now we define the norms for various product spaces. For space $\boldsymbol{Y}_h$, we define

$$\|(q, r)\|_{\boldsymbol{Y}}^2 := \|q\|^2 + \|r\|^2. \tag{2.4}$$

For other product spaces, we define

$$\|(\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C})\|_{\boldsymbol{X}}^2 := \|\boldsymbol{v}\|^2 + \|\nabla \boldsymbol{v}\|^2 + \|\nabla \times \boldsymbol{F}\|^2 + \|\boldsymbol{F} + \boldsymbol{v} \times \boldsymbol{B}^-\|^2 + \|\boldsymbol{C}\|^2 + \|\nabla \cdot \boldsymbol{C}\|^2, \quad \forall (\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C}) \in \boldsymbol{X}_h, \tag{2.5}$$

and

$$\|(\boldsymbol{u}_h, \boldsymbol{B}_h)\|_{\boldsymbol{W}}^2 := \|\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{B}_h\|_d^2, \quad \forall (\boldsymbol{u}_h, \boldsymbol{B}_h) \in \boldsymbol{W}_h.$$

Here $\boldsymbol{B}^- \in H(\mathrm{div}, \Omega)$ is a given function.

The constant SR_m^{-1} will appear in the discussions below frequently, therefore we denote

$$\alpha := SR_m^{-1}.$$

3. Hodge mapping and L^p estimates for divergence-free finite elements

In this section we present some key L^3 embedding results which are crucial for our analysis in the following sections.

Theorem 1. *For any function $\boldsymbol{d}_h \in H_0^h(\mathrm{div}\,0, \Omega)$, we have*

$$\|\boldsymbol{d}_h\|_{0,3} \leq C\|\nabla_h \times \boldsymbol{d}_h\|,$$

where the generic constant C solely depends on Ω.

Theorem 1 and its proof can be found in [24, Theorem 1]. For the boundary condition given in (1.3), we have similar estimates.

Theorem 2. *For any function $\boldsymbol{d}_h \in H^h(\mathrm{div}\,0, \Omega)$, we have*

$$\|\boldsymbol{d}_h\|_{0,3} \leq C\|\tilde{\nabla}_h \times \boldsymbol{d}_h\|,$$

where $\tilde{\nabla}_h \times \boldsymbol{d}_h \in H^h(\mathrm{curl}, \Omega)$ satisfies

$$(\tilde{\nabla}_h \times \boldsymbol{d}_h, \boldsymbol{F}) = (\boldsymbol{d}_h, \nabla \times \boldsymbol{F}), \quad \forall \boldsymbol{F} \in H^h(\mathrm{curl}, \Omega).$$

The generic constant C solely depends on Ω.

Proof. We define $Z_0 = H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\,0, \Omega)$, $Z_0^h = H^h(\mathrm{div}, \Omega) \cap H(\mathrm{div}\,0, \Omega)$. Obviously, $\boldsymbol{d}_h \in Z_0^h$. We define an operator $H_d : Z_0^h \to Z_0$ by

$$(\nabla \times (H_d \boldsymbol{d}_h), \nabla \times \boldsymbol{v}) = (\tilde{\nabla}_h \times \boldsymbol{d}_h, \nabla \times \boldsymbol{v}), \quad \forall \boldsymbol{v} \in Z_0.$$

Obviously, H_d is well defined. Since $H_d \boldsymbol{d}_h \in Z_0$, we have

$$\|H_d \boldsymbol{d}_h\|_{\frac{1}{2}+\delta} \leq C\|\nabla \times (H_d \boldsymbol{d}_h)\| \leq C\|\tilde{\nabla}_h \times \boldsymbol{d}_h\|, \tag{3.1}$$

where $\delta \in (0, \frac{1}{2}]$.

We use the projections Π^{curl} and Π^{div} in the commuting diagram in Figure 2.

Since $\nabla \cdot (\boldsymbol{d}_h - \Pi^{\mathrm{div}}(H_d \boldsymbol{d}_h)) = 0$ in Ω, there exists $\boldsymbol{\phi}_h \in \{\boldsymbol{v} \in H^h(\mathrm{curl}, \Omega) : (\boldsymbol{v}, \nabla s) = 0, \quad \forall s \in H^h(\mathrm{grad}, \Omega)\}$, such that

$$\nabla \times \boldsymbol{\phi}_h = \boldsymbol{d}_h - \Pi^{\mathrm{div}}(H_d \boldsymbol{d}_h).$$

We consider the auxiliary problem:

$$\begin{aligned} \nabla \times \nabla \times \boldsymbol{\psi} &= \nabla \times \boldsymbol{\phi}_h \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\psi} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{\psi} \times \boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Since $\nabla \cdot (\nabla \times \boldsymbol{\phi}_h) = 0$ in Ω, the auxiliary problem (3.2) is well-posed. Obviously, $\nabla \times \boldsymbol{\psi}$ satisfies

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{\psi}) &= \nabla \times \boldsymbol{\phi}_h \quad \text{in } \Omega, \\ \nabla \cdot (\nabla \times \boldsymbol{\psi}) &= 0 \quad \text{in } \Omega, \\ (\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

According to [20, Lemma 4.2], we have

$$\|\nabla \times \boldsymbol{\psi}\|_{\frac{1}{2}+\delta} \leq C\|\nabla \times \boldsymbol{\phi}_h\| = C\|\boldsymbol{d}_h - \Pi^{\mathrm{div}}(H_d \boldsymbol{d}_h)\|. \tag{3.3}$$

We claim that

$$\|\nabla\times\boldsymbol{\psi}-\boldsymbol{\phi}_h\|\le Ch^{\frac{1}{2}+\delta}\|\boldsymbol{d}_h-\Pi^{\mathrm{div}}(H_d\boldsymbol{d}_h)\|. \tag{3.4}$$

Notice that by (3.2),

$$\nabla\times\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi})=\Pi^{\mathrm{div}}(\nabla\times\nabla\times\boldsymbol{\psi})=\Pi^{\mathrm{div}}(\nabla\times\boldsymbol{\phi}_h)=\nabla\times\boldsymbol{\phi}_h.$$

Since $\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi}),\boldsymbol{\phi}_h\in H^h(\mathrm{curl},\Omega)$, there exists $s_h\in H^h(\mathrm{grad},\Omega)$ such that

$$\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi})-\boldsymbol{\phi}_h=\nabla s_h\quad\text{ in }\Omega.$$

Since $(\nabla\times\boldsymbol{\psi})\cdot\boldsymbol{n}=0$ on $\partial\Omega$, we have

$$(\nabla\times\boldsymbol{\psi},\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi})-\boldsymbol{\phi}_h)=(\nabla\times\boldsymbol{\psi},\nabla s_h)=0.$$

By the construction of $\boldsymbol{\phi}_h$, we have

$$(\boldsymbol{\phi}_h,\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi})-\boldsymbol{\phi}_h)=(\boldsymbol{\phi}_h,\nabla s_h)=0.$$

Thus

$$(\nabla\times\boldsymbol{\psi}-\boldsymbol{\phi}_h,\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi})-\boldsymbol{\phi}_h)=0.$$

So, by the above identify and (3.3), we have

$$\begin{aligned}\|\nabla\times\boldsymbol{\psi}-\boldsymbol{\phi}_h\|\le&\|(\nabla\times\boldsymbol{\psi}-\boldsymbol{\phi}_h)-(\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi})-\boldsymbol{\phi}_h)\|\\=&\|\nabla\times\boldsymbol{\psi}-\Pi^{\mathrm{curl}}(\nabla\times\boldsymbol{\psi})\|\\\le&Ch^{\frac{1}{2}+\delta}\|\boldsymbol{d}_h-\Pi^{\mathrm{div}}(H_d\boldsymbol{d}_h)\|.\end{aligned}$$

Therefore, the claim (3.4) is correct.

By the construction of H_d and the fact that $\boldsymbol{\psi}\in Z_0$,

$$(\tilde{\nabla}_h\times\boldsymbol{d}_h,\nabla\times\boldsymbol{\psi})=(\nabla\times(H_d\boldsymbol{d}_h),\nabla\times\boldsymbol{\psi})=(H_d\boldsymbol{d}_h,\nabla\times\nabla\times\boldsymbol{\psi})=(H_d\boldsymbol{d}_h,\nabla\times\boldsymbol{\phi}_h).$$

By the fact that $\boldsymbol{\phi}_h\in H^h(\mathrm{curl},\Omega)$ and the above identity,

$$\begin{aligned}&(\boldsymbol{d}_h,\nabla\times\boldsymbol{\phi}_h)=(\tilde{\nabla}_h\times\boldsymbol{d}_h,\boldsymbol{\phi}_h)\\=&(\tilde{\nabla}_h\times\boldsymbol{d}_h,\boldsymbol{\phi}_h-\nabla\times\boldsymbol{\psi})+(\tilde{\nabla}_h\times\boldsymbol{d}_h,\nabla\times\boldsymbol{\psi})\\=&(\tilde{\nabla}_h\times\boldsymbol{d}_h,\boldsymbol{\phi}_h-\nabla\times\boldsymbol{\psi})+(H_d\boldsymbol{d}_h,\nabla\times\boldsymbol{\phi}_h).\end{aligned}$$

Thus we have

$$(\boldsymbol{d}_h-H_d\boldsymbol{d}_h,\boldsymbol{d}_h-\Pi^{\mathrm{div}}(H_d\boldsymbol{d}_h))=(\boldsymbol{d}_h-H_d\boldsymbol{d}_h,\nabla\times\boldsymbol{\phi}_h)=(\tilde{\nabla}_h\times\boldsymbol{d}_h,\boldsymbol{\phi}_h-\nabla\times\boldsymbol{\psi}).$$

So we have

$$\begin{aligned}&\|\boldsymbol{d}_h-H_d\boldsymbol{d}_h\|^2\\=&(\boldsymbol{d}_h-H_d\boldsymbol{d}_h,\boldsymbol{d}_h-\Pi^{\mathrm{div}}(H_d\boldsymbol{d}_h))+(\boldsymbol{d}_h-H_d\boldsymbol{d}_h,\Pi^{\mathrm{div}}(H_d\boldsymbol{d}_h)-H_d\boldsymbol{d}_h)\\=&(\tilde{\nabla}_h\times\boldsymbol{d}_h,\boldsymbol{\phi}_h-\nabla\times\boldsymbol{\psi})+(\boldsymbol{d}_h-H_d\boldsymbol{d}_h,\Pi^{\mathrm{div}}(H_d\boldsymbol{d}_h)-H_d\boldsymbol{d}_h)\\\le&\|\tilde{\nabla}_h\times\boldsymbol{d}_h\|\cdot\|\boldsymbol{\phi}_h-\nabla\times\boldsymbol{\psi}\|+\|\boldsymbol{d}_h-H_d\boldsymbol{d}_h\|\cdot\|\Pi^{\mathrm{div}}(H_d\boldsymbol{d}_h)-H_d\boldsymbol{d}_h\|.\end{aligned}$$

By applying (3.4) in the above inequality, we have

$$\|\boldsymbol{d}_h-H_d\boldsymbol{d}_h\|\le Ch^{\frac{1}{2}+\delta}\|\tilde{\nabla}_h\times\boldsymbol{d}_h\|. \tag{3.5}$$

Let k_0 be a positive integer such that $H_0^h(\operatorname{div} 0, \Omega) \subset [P_{k_0}(\mathcal{T}_h)]^3$. We denote by $\boldsymbol{\Pi}$ the standard $\boldsymbol{L}^2$-orthogonal projection onto $[P_{k_0}(\mathcal{T}_h)]^3$. Thus $\boldsymbol{\Pi}\boldsymbol{d}_h = \boldsymbol{d}_h$. So, by the discrete inverse inequality and the fact that $\|\boldsymbol{\Pi}\boldsymbol{v}\|_{0,3} \leq C\|\boldsymbol{v}\|_{0,3}$ for any $\boldsymbol{v} \in [L^3(\Omega)]^3$, we have

$$\begin{aligned}\|\boldsymbol{d}_h\|_{0,3} =& \|\boldsymbol{\Pi}\boldsymbol{d}_h\|_{0,3} \leq \|\boldsymbol{\Pi}(\boldsymbol{d}_h - H_d\boldsymbol{d}_h)\|_{0,3} + \|\boldsymbol{\Pi}(H_d\boldsymbol{d}_h)\|_{0,3} \\ \leq& C\big(\|h^{-\frac{1}{2}}\boldsymbol{\Pi}(\boldsymbol{d}_h - H_d\boldsymbol{d}_h)\| + \|H_d\boldsymbol{d}_h\|_{0,3}\big) \\ \leq& C\big(h^{\delta}\|\nabla_h \times \boldsymbol{d}_h\| + \|H_d\boldsymbol{d}_h\|_{\frac{1}{2}+\delta}\big).\end{aligned}$$

Since $H_d\boldsymbol{d}_h \in H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div} 0, \Omega)$,

$$\|H_d\boldsymbol{d}_h\|_{\frac{1}{2}+\delta} \leq C\|\nabla \times (H_d\boldsymbol{d}_h)\| \leq C\|\nabla_h \times \boldsymbol{d}_h\|.$$

So, we can conclude that

$$\|\boldsymbol{d}_h\|_{0,3} \leq C\|\nabla_h \times \boldsymbol{d}_h\|.$$

This completes the proof.

□

4. Variational formulations

4.1. **Nonlinear scheme.** We propose the following variational form for (1.1) with boundary condition (1.2):

Problem 1. *Find $(\boldsymbol{u}_h, \boldsymbol{E}_h, \boldsymbol{B}_h) \in \boldsymbol{X}_h$ and $(p_h, r_h) \in \boldsymbol{Y}_h$, such that for any $(\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C}) \in \boldsymbol{X}_h$ and $(q, s) \in \boldsymbol{Y}_h$,*

$$L(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) + R_e^{-1}(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}) - S(\boldsymbol{j}_h \times \boldsymbol{B}_h, \boldsymbol{v}) - (p_h, \nabla\cdot\boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \tag{4.1a}$$

$$S(\boldsymbol{j}_h, \boldsymbol{F}) - \alpha(\boldsymbol{B}_h, \nabla\times\boldsymbol{F}) = 0, \tag{4.1b}$$

$$\alpha(\nabla\times\boldsymbol{E}_h, \boldsymbol{C}) + (r_h, \nabla\cdot\boldsymbol{C}) = 0, \tag{4.1c}$$

$$-(\nabla\cdot\boldsymbol{u}_h, q) = 0, \tag{4.1d}$$

$$(\nabla\cdot\boldsymbol{B}_h, s) = 0, \tag{4.1e}$$

where $\boldsymbol{j}_h$ is given by Ohm's law: $\boldsymbol{j}_h = \boldsymbol{E}_h + \boldsymbol{u}_h \times \boldsymbol{B}_h$. Here r_h is the Lagrange multiplier which approximates $r = 0$.

We verify some properties of the variational form Problem 1:

Theorem 3. *Any solution for Problem 1 satisfies*

(1) *magnetic Gauss's law:*

$$\nabla\cdot\boldsymbol{B}_h = 0.$$

(2) *Lagrange multiplier $r_h = 0$, and the strong form*

$$\nabla\times\boldsymbol{E}_h = 0,$$

(3) *energy estimates:*

$$R_e^{-1}\|\nabla \boldsymbol{u}_h\|^2 + S\|\boldsymbol{j}_h\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}_h\rangle, \tag{4.2}$$

$$\frac{1}{2}R_e^{-1}\|\nabla \boldsymbol{u}_h\|^2 + S\|\boldsymbol{j}_h\|^2 \le \frac{R_e}{2}\|\boldsymbol{f}\|_{-1}^2, \tag{4.3}$$

$$R_m^{-1}\|\nabla_h \times \boldsymbol{B}_h\| \le \|\boldsymbol{j}_h\|, \tag{4.4}$$

$$\|\nabla_h \times \boldsymbol{B}_h\| \le C R_e^{\frac{1}{2}} R_m S^{-\frac{1}{2}}\|\boldsymbol{f}\|_{-1}, \tag{4.5}$$

$$\|\boldsymbol{E}_h\| \le C R_e^{\frac{3}{2}} R_m S^{-\frac{1}{2}}\|\boldsymbol{f}\|_{-1}^2. \tag{4.6}$$

Proof. The magnetic Gauss's law is a direct consequence of (4.1e).

Taking $\boldsymbol{C} = \nabla \times \boldsymbol{E}_h$ in (4.1c), we have $\nabla \times \boldsymbol{E}_h = 0$. Therefore (4.1c) reduces to

$$(r_h, \nabla \cdot \boldsymbol{C}) = 0, \quad \forall \boldsymbol{C} \in H_0^h(\mathrm{div}, \Omega).$$

Since $L^2_{0,h}(\Omega) = \nabla \cdot H_0^h(\mathrm{div}, \Omega)$, we get $r_h = 0$.

To obtain the first energy estimate, we take $\boldsymbol{v} = \boldsymbol{u}_h$, $\boldsymbol{F} = \boldsymbol{E}_h, \boldsymbol{C} = \boldsymbol{B}_h$ and $q = p_h$ in (4.1a) - (4.1d) and add the equations together. The second energy estimate follows from the Young's inequality

$$\langle \boldsymbol{f}, \boldsymbol{u}_h\rangle \le \|\boldsymbol{f}\|_{-1}\|\nabla \boldsymbol{u}_h\| \le \frac{1}{2R_e}\|\nabla \boldsymbol{u}_h\|^2 + \frac{1}{2}R_e\|\boldsymbol{f}\|_{-1}^2.$$

Taking $\boldsymbol{F} = \nabla_h \times \boldsymbol{B}_h$ in (4.1b) we have

$$R_m^{-1}\|\nabla_h \times \boldsymbol{B}_h\|^2 = R_m^{-1}(\boldsymbol{j}_h, \nabla_h \times \boldsymbol{B}_h) \le \|\boldsymbol{j}_h\|\|\nabla_h \times \boldsymbol{B}_h\|,$$

which implies (4.4). Obviously, the estimate (4.5) is due to estimates (4.3) and (4.4).

Next we take $\boldsymbol{F} = \boldsymbol{E}_h$ in (4.1b) and by the definition of $\boldsymbol{j}_h$ we have

$$(\boldsymbol{E}_h + \boldsymbol{u}_h \times \boldsymbol{B}_h, \boldsymbol{E}_h) - R_m^{-1}(\boldsymbol{B}_h, \nabla \times \boldsymbol{E}_h) = 0.$$

By the fact that $\nabla \times \boldsymbol{E}_h = 0$ and the generalized Hölder's inequality we have

$$\begin{aligned}\|\boldsymbol{E}_h\|^2 &= -(\boldsymbol{u}_h \times \boldsymbol{B}_h, \boldsymbol{E}_h) \le \|\boldsymbol{u}_h\|_{0,6}\|\boldsymbol{B}_h\|_{0,3}\|\boldsymbol{E}_h\| \\ &\le C\|\nabla \boldsymbol{u}_h\|\|\nabla_h \times \boldsymbol{B}_h\|\|\boldsymbol{E}_h\|,\end{aligned}$$

the last step is due to the Sobolev embedding results (2.1) and Theorem 1. The estimate (4.6) can be obtained by combining the above estimate with (4.3) and (4.5). This completes the proof. □

Remark 1. *From the above result we can see that the energy norm of the unknowns $\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{E}_h$ solely depends on $\|\boldsymbol{f}\|_{-1}$ and the physical constants R_m, R_e, S. In addition, it is easy to verify that the exact solution satisfies the same stability estimate*

$$\begin{aligned}\|\nabla \boldsymbol{u}\| &\le R_e\|\boldsymbol{f}\|_{-1}, \\ \|\boldsymbol{B}\|_{0,3} + \|\nabla \times \boldsymbol{B}\| &\le C R_e^{\frac{1}{2}} R_m S^{-\frac{1}{2}}\|\boldsymbol{f}\|_{-1}, \\ \|\boldsymbol{E}\| &\le C R_e^{\frac{3}{2}} R_m S^{-\frac{1}{2}}\|\boldsymbol{f}\|_{-1}^2.\end{aligned} \tag{4.7}$$

Theorem 4. *Problem 1 is well-posed.*

In the remaining part of this section we prove the well-posedness of Problem 1. We will first recast Problem 1 into an equivalent form ((4.9) and Problem 2) where $\boldsymbol{E}$ is formally eliminated. Then we demonstrate that this equivalent form is well-posed using the Brezzi theory and the key L^3 estimate (Theorem 5). Then we can conclude with the well-posedness of Problem 1.

Using (4.1b), we have

$$\boldsymbol{E}_h + \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h) = R_m^{-1}\nabla_h \times \boldsymbol{B}.$$

Now the Lorentz force has an equivalent form

$$\begin{aligned} -(\boldsymbol{j}_h \times \boldsymbol{B}_h, \boldsymbol{v}) &= (\boldsymbol{E}_h + \mathbb{P}(\boldsymbol{u}_h \times \boldsymbol{B}_h), \boldsymbol{v} \times \boldsymbol{B}_h) + ((I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{B}_h), \boldsymbol{v} \times \boldsymbol{B}_h) \\ &= R_m^{-1}(\nabla_h \times \boldsymbol{B}_h, \boldsymbol{v} \times \boldsymbol{B}_h) + ((I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{B}_h), (I-\mathbb{P})(\boldsymbol{v} \times \boldsymbol{B}_h)). \end{aligned} \tag{4.8}$$

Even though the velocity field $\boldsymbol{u}_h$ is smooth, the $H(\mathrm{div})$ conformality of the magnetic field $\boldsymbol{B}_h$ cannot guarantee $\boldsymbol{u}_h \times \boldsymbol{B}_h \in H(\mathrm{curl}, \Omega)$. The term $(I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{B}_h)$ on the right hand side of (4.8) measures the deviation of $\boldsymbol{u}_h \times \boldsymbol{B}_h$ from $H^h(\mathrm{curl})$ and $((I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{B}_h), (I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{B}_h))$ can be regarded as a penalty term.

Therefore (4.1) is equivalent to the following problem: Find $(\boldsymbol{u}_h, \boldsymbol{B}_h) \in \boldsymbol{W}_h$ and $(p_h, r_h) \in \boldsymbol{Y}_h$ such that for any $(\boldsymbol{v}_h, \boldsymbol{C}_h) \in \boldsymbol{W}_h$ and $(q_h, s_h) \in \boldsymbol{Y}_h$,

$$\begin{cases} L(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + R_e^{-1}(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - \alpha(\nabla_h \times \boldsymbol{B}_h, \boldsymbol{B}_h \times \boldsymbol{v}_h) \\ \qquad + S((I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{B}_h), (I-\mathbb{P})(\boldsymbol{v}_h \times \boldsymbol{B}_h)) - (p_h, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \\ -\alpha(\boldsymbol{u}_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h) + SR_m^{-2}(\nabla_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h) + (r_h, \nabla \cdot \boldsymbol{C}_h) = 0, \\ (\nabla \cdot \boldsymbol{u}_h, q_h) = 0, \\ (\nabla \cdot \boldsymbol{B}_h, s_h) = 0. \end{cases} \tag{4.9}$$

We note that the reduced system (4.9) has a similar form compared with the work by Gunzburger [17] and Schötzau [32]. However, this similarity is only formal. The magnetic field $\boldsymbol{B}$ is discretized as 0-forms with the Lagrange finite elements in [17] and treated as 1-forms with the Nédélec elements in [32]. In both approaches [17, 32], the curl operator can be evaluated on $\boldsymbol{B}$ in a straightforward way. In contrast, $\boldsymbol{B}$ is discretized as a 2-form in (4.9). As a result, the discrete curl operator $\nabla_h \times$ is globally defined by (2.2), which leads to a new mixed formulation. This also makes the analysis essentially different from [17] or [32]. Compared with the $\boldsymbol{B}$-$\boldsymbol{j}$ based scheme in [24], a quadratic term

$$S((I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{B}_h), (I-\mathbb{P})(\boldsymbol{v}_h \times \boldsymbol{B}_h)),$$

comes into the reduced variational formulation (4.9). This is due to the different choice of variables.

Denote $\boldsymbol{\psi}_h = (\boldsymbol{w}_h, \boldsymbol{G}_h)$, $\boldsymbol{\xi}_h = (\boldsymbol{u}_h, \boldsymbol{B}_h)$, $\boldsymbol{\eta}_h = (\boldsymbol{v}_h, \boldsymbol{C}_h)$ and $\boldsymbol{x}_h = (p_h, r_h)$, $\boldsymbol{y}_h = (q_h, s_h)$. Define

$$\begin{aligned} \boldsymbol{a}_s(\boldsymbol{\psi}_h; \boldsymbol{\xi}_h, \boldsymbol{\eta}_h) := &\frac{1}{2}[((\boldsymbol{w}_h \cdot \nabla)\boldsymbol{u}_h, \boldsymbol{v}_h) - ((\boldsymbol{w}_h \cdot \nabla)\boldsymbol{v}_h, \boldsymbol{u}_h)] + R_e^{-1}(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) \\ &- \alpha(\nabla_h \times \boldsymbol{B}_h, \boldsymbol{G}_h \times \boldsymbol{v}_h) + S((I-\mathbb{P})(\boldsymbol{u}_h \times \boldsymbol{G}_h), (I-\mathbb{P})(\boldsymbol{v}_h \times \boldsymbol{G}_h)) \\ &- \alpha(\boldsymbol{u}_h \times \boldsymbol{G}_h, \nabla_h \times \boldsymbol{C}_h) + SR_m^{-2}(\nabla_h \times \boldsymbol{B}_h, \nabla_h \times \boldsymbol{C}_h), \end{aligned}$$

and

$$\boldsymbol{b}_s(\boldsymbol{\xi}_h, \boldsymbol{y}_h) := -(\nabla \cdot \boldsymbol{u}_h, q_h) + (\nabla \cdot \boldsymbol{B}_h, s_h).$$

Equation (4.9) can be recast into a mixed system:

Problem 2. *Given $\boldsymbol{\theta} \in \boldsymbol{W}_h^*$ and $\boldsymbol{\psi} \in \boldsymbol{Y}_h^*$, find $(\boldsymbol{\xi}_h, \boldsymbol{x}_h) \in \boldsymbol{W}_h \times \boldsymbol{Y}_h$, such that*

$$\begin{cases} \boldsymbol{a}_s(\boldsymbol{\xi}_h; \boldsymbol{\xi}_h, \boldsymbol{\eta}_h) + \boldsymbol{b}_s(\boldsymbol{\eta}_h, \boldsymbol{x}_h) = \langle \boldsymbol{\theta}, \boldsymbol{\eta}_h \rangle, & \forall\, \boldsymbol{\eta}_h \in \boldsymbol{W}_h, \\ \boldsymbol{b}_s(\boldsymbol{\xi}_h, \boldsymbol{y}_h) = \langle \boldsymbol{\psi}, \boldsymbol{y}_h \rangle, & \forall \boldsymbol{y}_h \in \boldsymbol{Y}_h. \end{cases} \tag{4.10}$$

Theorem 5. *Problem 2 is well-posed.*

We prove the existence of solutions to the discrete variational form. To use the Brezzi theory and the fixed point theorem (see [15]), we need to show

- each term in (4.10) is bounded,
- the inf-sup condition for $\boldsymbol{b}_s$,
- coercivity of $\boldsymbol{a}_s$ on $\boldsymbol{W}_h^{00}$.

We establish these conditions in the subsequent lemmas.

The boundedness of the variational form is a direct consequence of the key L^3 estimate.

Lemma 1. *The trilinear form $\boldsymbol{a}_s(\cdot;\cdot,\cdot)$ and the bilinear form $\boldsymbol{b}_s(\cdot,\cdot)$ are bounded, i.e. there exists a positive constant C such that*

$$\boldsymbol{a}_s(\boldsymbol{\psi}_h;\boldsymbol{\xi}_h,\boldsymbol{\eta}_h) \leq C\|\boldsymbol{\psi}_h\|_{\boldsymbol{W}}\|\boldsymbol{\xi}_h\|_{\boldsymbol{W}}\|\boldsymbol{\eta}_h\|_{\boldsymbol{W}}, \quad \forall \boldsymbol{\psi}_h,\boldsymbol{\xi}_h,\boldsymbol{\eta}_h \in \boldsymbol{W}_h,$$

and

$$\boldsymbol{b}_s(\boldsymbol{\eta}_h,\boldsymbol{y}_h) \leq C\|\boldsymbol{\eta}_h\|_{\boldsymbol{W}}\|\boldsymbol{y}_h\|_{\boldsymbol{Y}}, \quad \forall \boldsymbol{\eta}_h \in W_h, \boldsymbol{y}_h \in \boldsymbol{Y}_h.$$

Since we have used a stronger norm for $\boldsymbol{B}_h, \boldsymbol{C}_h \in H_0^h(\mathrm{div},\Omega)$, the inf-sup condition for the bilinear form $\boldsymbol{b}_s(\cdot,\cdot)$ becomes more subtle. Following a similar proof as shown in [24] for the $\boldsymbol{B}$-$\boldsymbol{j}$ formulation, we get:

Lemma 2. (inf-sup conditions for $\boldsymbol{b}_s(\cdot,\cdot)$) *There exists a positive constant γ such that*

$$\inf_{\boldsymbol{y}_h\in\boldsymbol{Y}_h}\sup_{\boldsymbol{\eta}_h\in\boldsymbol{W}_h}\frac{\boldsymbol{b}_s(\boldsymbol{\eta}_h,\boldsymbol{y}_h)}{\|\boldsymbol{\eta}_h\|_{\boldsymbol{W}}\|\boldsymbol{y}_h\|_{\boldsymbol{Y}}} \geq \gamma > 0.$$

The coercivity of $\boldsymbol{a}_s(\cdot;\cdot,\cdot)$ holds on the kernel space $\boldsymbol{W}_h^{00}$.

Lemma 3. *On $\boldsymbol{W}_h^{00}$ we have*

$$\boldsymbol{a}_s(\boldsymbol{\xi}_h;\boldsymbol{\xi}_h,\boldsymbol{\xi}_h) \geq \gamma\|\boldsymbol{\xi}_h\|_{\boldsymbol{W}}^2,$$

where γ is a positive constant.

Proof. We note that

$$\boldsymbol{a}_s(\boldsymbol{\xi}_h;\boldsymbol{\xi}_h,\boldsymbol{\xi}_h) = R_e^{-1}\|\nabla\boldsymbol{u}_h\|^2 + S\|(I-\mathbb{P})(\boldsymbol{u}_h\times\boldsymbol{B}_h)\|^2 + SR_m^{-2}\|\nabla_h\times\boldsymbol{B}_h\|^2.$$

Discrete Poincaré's inequality holds on $\boldsymbol{W}_h^{00}$:

$$\|\boldsymbol{B}_h\| \lesssim \|\nabla_h\times\boldsymbol{B}_h\|.$$

This completes the proof. □

By Lemma 1, Lemma 2 and Lemma 3, the nonlinear variational form (4.10) is well-posed. Therefore (4.9) has at least one solution. For suitable source and boundary data, the solution is also unique.

4.2. **Picard iterations.** We propose the following Picard type iterations for Problem 1:

Algorithm 1 (Picard iterations for nonlinear schemes)**.** *Given* $(\boldsymbol{u}^{n-1}, \boldsymbol{B}^{n-1})$*, find* $(\boldsymbol{u}^n, \boldsymbol{E}^n, \boldsymbol{B}^n) \in \boldsymbol{X}_h$ *and* $(p^n, r^n) \in \boldsymbol{Y}_h$*, such that for any* $(\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C}) \in \boldsymbol{X}_h$ *and* $(q, s) \in \boldsymbol{Y}_h$*,*

$$L(\boldsymbol{u}^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}) + R_e^{-1}(\nabla \boldsymbol{u}^n, \nabla \boldsymbol{v}) - S(\boldsymbol{j}_{n-1}^n \times \boldsymbol{B}^{n-1}, \boldsymbol{v}) - (p^n, \nabla \cdot \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \tag{4.11}$$

$$S(\boldsymbol{j}_{n-1}^n, \boldsymbol{F}) - \alpha(\boldsymbol{B}^n, \nabla \times \boldsymbol{F}) = 0, \tag{4.12}$$

$$\alpha(\nabla \times \boldsymbol{E}^n, \boldsymbol{C}) + (r^n, \nabla \cdot \boldsymbol{C}) = 0, \tag{4.13}$$

$$-(\nabla \cdot \boldsymbol{u}^n, q) = 0, \tag{4.14}$$

$$(\nabla \cdot \boldsymbol{B}^n, s) = 0, \tag{4.15}$$

where $\boldsymbol{j}_{n-1}^n$ *is defined by* $\boldsymbol{j}_{n-1}^n = \boldsymbol{E}^n + \boldsymbol{u}^n \times \boldsymbol{B}^{n-1}$.

The convergence of Picard iterations is summarized in the following theorem:

Theorem 6. *If both* $R_e^2\|\boldsymbol{f}\|_{-1}$ *and* $R_e R_m^{\frac{3}{2}}\|\boldsymbol{f}\|_{-1}$ *are small enough, then the method* (4.1) *(Problem 1) with the boundary condition* (1.2) *has a unique solution, and the solution of the Picard iteration (Algorithm 1) converges to it with respect to the norms defined by* (2.5) *and* (2.4)*.*

We skip the proof of Theorem 6, since it is a simpler version of the proofs of the Theorem 13 in Section 5.

The divergence-free property, compatibility and energy estimates can be obtained in an analogous way:

Theorem 7. *For any possible solution to Algorithm 1:*

(1) *magnetic Gauss's law holds precisely:*

$$\nabla \cdot \boldsymbol{B}^n = 0.$$

(2) *the Lagrange multiplier* $r^n = 0$*, therefore* (4.13) *has the form*

$$\nabla \times \boldsymbol{E}^n = 0.$$

(3) *the energy estimates hold:*

$$R_e^{-1}\|\nabla \boldsymbol{u}^n\|^2 + S\|\boldsymbol{j}_{n-1}^n\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}^n\rangle,$$

and

$$\frac{1}{2}R_e^{-1}\|\nabla \boldsymbol{u}^n\|^2 + S\|\boldsymbol{j}_{n-1}^n\|^2 \leq \frac{1}{2}R_e\|\boldsymbol{f}\|_{-1}^2. \tag{4.16}$$

We will use the Brezzi theory to prove the well-posedness of the Picard iterations. We first recast Picard iterations (Algorithm 1) as follows. Given $(\boldsymbol{u}^-, \boldsymbol{B}^-) \in \boldsymbol{W}_h$. For $\mathfrak{U} = (\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{B})$, $\mathfrak{V} = (\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C}) \in \boldsymbol{X}_h$ and $(p, r), (q, s) \in \boldsymbol{Y}_h$, define bilinear forms $\boldsymbol{a}_{s,L}(\cdot, \cdot)$ and $\boldsymbol{b}(\cdot, \cdot)$:

$$\begin{aligned}\boldsymbol{a}_{s,L}(\mathfrak{U}, \mathfrak{V}) :=& \frac{1}{2}L\left(\boldsymbol{u}^-; \boldsymbol{u}, \boldsymbol{v}\right) + R_e^{-1}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + S(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}^-, \boldsymbol{F} + \boldsymbol{v} \times \boldsymbol{B}^-)\\ &- \alpha(\boldsymbol{B}, \nabla \times \boldsymbol{F}) + \alpha(\nabla \times \boldsymbol{E}, \boldsymbol{C}).\end{aligned}$$

Given a nonlinear iterative step, the mixed form of the iterative scheme in Algorithm 1 can be written as: for any $\boldsymbol{h} = (\boldsymbol{f}, \boldsymbol{r}, \boldsymbol{l}) \in \boldsymbol{X}_h^*$ and $\boldsymbol{g} \in \boldsymbol{Y}_h^*$, find $(\mathfrak{U}, \boldsymbol{x}) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$, such that for any

$(\mathfrak{V}, \boldsymbol{y}) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$,

$$\begin{cases} \boldsymbol{a}_{s,L}(\mathfrak{U}, \boldsymbol{\eta}) + \boldsymbol{b}_s(\mathfrak{V}, \boldsymbol{x}) = \langle \boldsymbol{h}, \boldsymbol{\eta} \rangle, \\ \boldsymbol{b}_s(\mathfrak{U}, \boldsymbol{y}) = \langle \boldsymbol{g}, \boldsymbol{y} \rangle. \end{cases} \tag{4.17}$$

To prove the well-posedness of (4.17) based on the Brezzi theory, we need to verify the boundedness of each term, the inf-sup condition of $\boldsymbol{b}_s(\cdot,\cdot)$ and the coercivity of $\boldsymbol{a}_{s,L}(\cdot,\cdot)$ on $\boldsymbol{X}_h^{00}$.

For the inf-sup condition of $\boldsymbol{b}_s(\cdot,\cdot)$, we have:

Lemma 4. (inf-sup conditions of $\boldsymbol{b}_s(\cdot,\cdot)$) *There exists a positive constant γ such that*

$$\inf_{\boldsymbol{y} \in \boldsymbol{Y}_h} \sup_{\mathfrak{V} \in \boldsymbol{X}_h} \frac{\boldsymbol{b}_s(\mathfrak{V}, \boldsymbol{y})}{\|\mathfrak{V}\|_{\boldsymbol{X}} \|\boldsymbol{y}\|_{\boldsymbol{Y}}} \geq \gamma > 0.$$

Proof. There exists a positive constant $\gamma_0 > 0$ such that

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{-(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1 \|q\|} \geq \alpha_0 > 0.$$

Consequently, for any $q \in Q_h$ there exists $\boldsymbol{v}_q \in \boldsymbol{V}_h$, such that

$$-(\nabla \cdot \boldsymbol{v}_q, q) \geq \gamma_0 \|q\|^2,$$

and

$$\|\boldsymbol{v}_q\|_1 = \|q\|.$$

For the magnetic multiplier, we have $\nabla \cdot H_0^h(\mathrm{div}, \Omega) = L_{0,h}^2(\Omega)$. For any $s \in L_{0,h}^2(\Omega)$, there exists $\boldsymbol{C}_s \in H_0^h(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{C}_s = s$, $\|\boldsymbol{C}_s\|_{\mathrm{div}} \leq C\|s\|$, where C is a positive constant.

For any $\mathfrak{V} = (q, s)$, take $\boldsymbol{y} = (\boldsymbol{v}_q, \boldsymbol{C}_s)$. Then

$$\boldsymbol{b}_s(\mathfrak{V}, \boldsymbol{y}) = -(\nabla \cdot \boldsymbol{v}_q, q) + (\nabla \cdot \boldsymbol{C}_s, s) \geq \gamma_0 \|q\|^2 + \|s\|^2 \geq \min(\gamma_0, 1) \|\boldsymbol{y}\|_{\boldsymbol{Y}}^2,$$

and

$$\|\boldsymbol{v}_q\|_1^2 + \|\boldsymbol{C}_s\|_{\mathrm{div}}^2 \leq \|q\|^2 + C^2 \|s\|^2 \leq \max(1, C^2) \|\boldsymbol{y}\|_{\boldsymbol{Y}}^2.$$

This completes the proof. □

Theorem 8. *Problem (4.17), therefore Algorithm 1, is well-posed with the norms defined by (2.5) and (2.4).*

Proof. The boundedness of the variational form is obvious from the definition of $\|\cdot\|_{\boldsymbol{X}}$. Moreover, we note that $\boldsymbol{a}_{s,L}(\mathfrak{U}, \mathfrak{U}) = R_e^{-1} \|\nabla \boldsymbol{u}\| + S\|\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}^-\|^2$. Therefore the bilinear form $\boldsymbol{a}_{s,L}(\cdot,\cdot)$ is coercive on $\boldsymbol{X}_h^{00}$.

Combining the boundedness of the variational form, the inf-sup condition of $\boldsymbol{b}_s(\cdot,\cdot)$ (Lemma 4) and the coercivity of $\boldsymbol{a}_{s,L}(\cdot,\cdot)$ on $\boldsymbol{X}_h^{00}$, we complete the proof.

□

From the triangular inequality and Hölder's inequality, we have

$$\|\boldsymbol{E}\| \leq \|\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}^-\| + \|\boldsymbol{u} \times \boldsymbol{B}^-\| \lesssim \|\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}^-\| + \|\boldsymbol{u}\|_1 \|\boldsymbol{B}^-\|_{0,3}.$$

In Picard iterations (Algorithm 1), function $\boldsymbol{B}^-$ is given by the magnetic field from the previous iterative step, i.e. $\boldsymbol{B}^- = \boldsymbol{B}^{n-1}$. We have the following estimate:

$$\|\boldsymbol{B}^-\|_{0,3} = \|\boldsymbol{B}^{n-1}\|_{0,3} \lesssim \|\nabla_h \times \boldsymbol{B}^{n-1}\| \lesssim \|\boldsymbol{f}\|_{-1}, \tag{4.18}$$

where the last equality is due to the energy law.

Therefore the L^2 norm of the electric field $\boldsymbol{E}$ can be bounded by $\|(\boldsymbol{u},\boldsymbol{E},\boldsymbol{B})\|_{\boldsymbol{X}}$ and given data, i.e., norm $\|(\boldsymbol{u},\boldsymbol{E},\boldsymbol{B})\|_{\boldsymbol{X}}$ is equivalent to the decoupled norm

$$\left(\|\boldsymbol{u}\|_1^2+\|\boldsymbol{E}\|_{\mathrm{curl}}^2+\|\boldsymbol{B}\|_{\mathrm{div}}^2\right)^{\frac{1}{2}}.$$

The constants involved in the equivalence depend on $\|\boldsymbol{B}^-\|_{0,3}$ which further depends on $\|\boldsymbol{f}\|_{-1}$.

4.3. **Schemes without magnetic Lagrange multipliers.** Thanks to the structure-preserving properties of the discrete de Rham complex, we can design a finite element scheme for stationary MHD problems without using magnetic multipliers. The resulting scheme is equivalent to (4.1), therefore magnetic Gauss's law is precisely preserved.

Consider the following weak form:

Problem 3. *Find $(\boldsymbol{u}_h,\boldsymbol{E}_h,\boldsymbol{B}_h)\in\boldsymbol{X}_h$ and $p_h\in Q_h$, such that for any $(\boldsymbol{v},\boldsymbol{F},\boldsymbol{C})\in\boldsymbol{X}_h$ and $q\in Q_h$,*

$$\begin{cases} L(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v})+R_e^{-1}(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v})-S(\boldsymbol{j}_h\times\boldsymbol{B}_h,\boldsymbol{v})-(p_h,\nabla\cdot\boldsymbol{v})=\langle\boldsymbol{f},\boldsymbol{v}\rangle,\\ S(\boldsymbol{j}_h,\boldsymbol{F})-\alpha(\boldsymbol{B}_h,\nabla\times\boldsymbol{F})=0,\\ \alpha(\nabla\times\boldsymbol{E}_h,\boldsymbol{C})+\alpha(\nabla\cdot\boldsymbol{B}_h,\nabla\cdot\boldsymbol{C})=0,\\ -(\nabla\cdot\boldsymbol{u}_h,q)=0,\end{cases}\tag{4.19}$$

where $\boldsymbol{j}_h$ is given from Ohm's law: $\boldsymbol{j}_h=\boldsymbol{E}_h+\boldsymbol{u}_h\times\boldsymbol{B}_h$.

Compared with Problem 1, the magnetic Lagrange multiplier has been removed and we augment the variational formulation by introducing $(\nabla\cdot\boldsymbol{B}_h,\nabla\cdot\boldsymbol{C})$ term. Next we verify some properties of the proposed schemes.

Theorem 9. *Any solution to Problem 3 satisfies*

(1) *magnetic Gauss's law in the strong sense:*

$$\nabla\cdot\boldsymbol{B}_h=0,$$

(2) *the discrete energy law:*

$$R_e^{-1}\|\nabla\boldsymbol{u}_h\|^2+S\|\boldsymbol{j}_h\|^2=\langle\boldsymbol{f},\boldsymbol{u}_h\rangle,$$

and

$$\frac{1}{2}R_e^{-1}\|\nabla\boldsymbol{u}_h\|^2+S\|\boldsymbol{j}_h\|^2\le\frac{R_e}{2}\|\boldsymbol{f}\|_{-1}^2.$$

Proof. The proof of the discrete energy law is almost the same as Problem 1. Therefore we only prove the magnetic Gauss's law.

Taking $\boldsymbol{C}=\nabla\times\boldsymbol{E}_h$ in (4.19), we have $\nabla\times\boldsymbol{E}_h=0$. Therefore

$$(\nabla\cdot\boldsymbol{B}_h,\nabla\cdot\boldsymbol{C}_h)=0,\quad\forall\boldsymbol{C}_h\in H_0^h(\mathrm{div}\,0,\Omega).$$

This implies that $\nabla\cdot\boldsymbol{B}_h=0$.

□

To verify the well-posedness, we can formally eliminate $\boldsymbol{E}_h$ to get a system with $\boldsymbol{u}_h$, p_h and $\boldsymbol{B}$. For the Lagrange multiplier p_h, one can verify the inf-sup condition of the $(\nabla\cdot\boldsymbol{u},q)$ pair. We can also verify the boundedness and coercivity in $\boldsymbol{V}_h^0\times H_0^h(\mathrm{curl},\Omega)\times H_0^h(\mathrm{div},\Omega)$ for other terms. Consequently, we have the well-posedness result:

Theorem 10. *Problem 3 has at least one solution* $(\boldsymbol{u}_h, \boldsymbol{E}_h, \boldsymbol{B}_h, p_h) \in \boldsymbol{X}_h \times Q_h$. *With suitable data, the solution is unique.*

We can similarly define Picard iterations: For $n = 1, 2, \cdots$, given $\left(\boldsymbol{u}^{n-1}, \boldsymbol{B}^{n-1}\right) \in \boldsymbol{W}_h$, find $(\boldsymbol{u}^n, \boldsymbol{E}^n, \boldsymbol{B}^n) \in \boldsymbol{X}_h$ and $p^n \in Q_h$, such that for any $(\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C}) \in \boldsymbol{X}_h$ and $q \in Q_h$,

$$\begin{cases} L(\boldsymbol{u}^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}) + R_e^{-1}(\nabla \boldsymbol{u}^n, \nabla \boldsymbol{v}) - S(\boldsymbol{j}_{n-1}^n \times \boldsymbol{B}^{n-1}, \boldsymbol{v}) - (p^n, \nabla \cdot \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ \hfill S(\boldsymbol{j}_{n-1}^n, \boldsymbol{F}) - \alpha(\boldsymbol{B}^n, \nabla \times \boldsymbol{F}) = 0, \\ \hfill \alpha(\nabla \times \boldsymbol{E}^n, \boldsymbol{C}) + \alpha(\nabla \cdot \boldsymbol{B}^n, \nabla \cdot \boldsymbol{C}) = 0, \\ \hfill -(\nabla \cdot \boldsymbol{u}^n, q) = 0, \end{cases} \tag{4.20}$$

where $\boldsymbol{j}_{n-1}^n$ is given by Ohm's law: $\boldsymbol{j}_{n-1}^n = \boldsymbol{E}^n + \boldsymbol{u}^n \times \boldsymbol{B}^{n-1}$. one can similarly verify the following properties:

Theorem 11. *Any solution to Problem 4.20 satisfies*

(1) *magnetic Gauss's law in the strong sense:*

$$\nabla \cdot \boldsymbol{B}^n = 0, \quad n = 1, 2, \cdots,$$

(2) *the discrete energy law:*

$$R_e^{-1} \|\nabla \boldsymbol{u}^n\|^2 + S \|\boldsymbol{j}_{n-1}^n\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}^n \rangle,$$

and

$$\frac{1}{2} R_e^{-1} \|\nabla \boldsymbol{u}^n\|^2 + S \|\boldsymbol{j}_{n-1}^n\|^2 \le \frac{R_e}{2} \|\boldsymbol{f}\|_{-1}^2.$$

Analogous to Theorem 5, we can verify the well-posedness:

Theorem 12. *Variational form (4.20) has a unique solution* $(\boldsymbol{u}^n, \boldsymbol{E}^n, \boldsymbol{B}^n, p^n) \in \boldsymbol{X}_h \times Q_h$.

5. Convergence of finite element methods

In this section, we present the error estimates of the method (4.1), which is for the boundary condition (1.2). Our analysis is based on the weak regularity assumption on the exact solutions (c.f. [32]). Namely, we assume

$$\boldsymbol{u} \in [H^{1+\sigma}(\Omega)]^3, \quad \boldsymbol{B}, \nabla \times \boldsymbol{B}, \boldsymbol{E} \in [H^\sigma(\Omega)]^3, \quad p \in H^\sigma(\Omega) \cap L_0^2(\Omega), \tag{5.1}$$

here $\sigma > \frac{1}{2}$. Next we introduce notations used in the analysis. For a generic unknown $\mathcal{U}$ and its numerical counterpart $\mathcal{U}_h$ we split the error as:

$$\mathcal{U} - \mathcal{U}_h = (\mathcal{U} - \Pi\mathcal{U}) + (\Pi\mathcal{U} - \mathcal{U}_h) := \delta_U + e_U.$$

Here $\Pi\mathcal{U}$ is a projection of $\mathcal{U}$ into the corresponding discrete space that $\mathcal{U}_h$ belongs to. Namely, for $(\boldsymbol{E}, r)$ we use the projections $(\Pi^{\mathrm{curl}} \boldsymbol{E}, \Pi^0 r)$ in the commuting diagram in Figure 1. For $\boldsymbol{B}$ and p we define the L^2 projection $\Pi^D \boldsymbol{B}, \Pi^Q p$ into $H_0^h(\mathrm{div}\, 0, \Omega), Q_h$ respectively. Notice here $r = 0$ implies that $\Pi^0 r = 0$ and hence $\delta_r = 0$. Finally, for the velocity $\boldsymbol{u}$ we define $(\Pi^V \boldsymbol{u}, \tilde{p}_h) \in \boldsymbol{V}_h \times Q_h$ be the unique numerical solution of the Stokes equation:

$$(\nabla \Pi^V \boldsymbol{u}, \nabla \boldsymbol{v}) + (\tilde{p}_h, \nabla \cdot \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \tag{5.2}$$

$$(\nabla \cdot \Pi^V \boldsymbol{u}, q) = 0, \tag{5.3}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h$. Notice that $(\boldsymbol{u}, 0)$ is the exact solution of the Stokes equations:

$$-\Delta \tilde{\boldsymbol{u}} + \nabla \tilde{p} = -\Delta \boldsymbol{u},$$
$$\nabla \cdot \tilde{\boldsymbol{u}} = 0,$$

with $\tilde{\boldsymbol{u}} = \boldsymbol{0}$ on $\partial\Omega$. Hence, if $\boldsymbol{V}_h \times Q_h$ is a stable Stokes pair, we should have optimal approximation for the above equation:

$$\|\boldsymbol{u} - \Pi^V \boldsymbol{u}\|_1 \leq C \inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_1. \tag{5.4}$$

Immediately we can see that

$$(\delta_{\boldsymbol{u}}, q) = 0 \quad \text{for all} \quad q \in Q_h. \tag{5.5}$$

Since $\boldsymbol{B}, \Pi^D \boldsymbol{B}, \boldsymbol{B}_h \in H_0(\operatorname{div} 0, \Omega)$ and $\boldsymbol{E}, \boldsymbol{E}_h, \Pi^{\text{curl}} \boldsymbol{E} \in H_0(\operatorname{curl} 0, \Omega)$ we have

$$\nabla \cdot e_{\boldsymbol{B}} = \nabla \cdot \delta_{\boldsymbol{B}} = 0, \quad \nabla \times e_{\boldsymbol{E}} = \nabla \times \delta_{\boldsymbol{E}} = 0. \tag{5.6}$$

In addition, since $\nabla \times H_0^h(\operatorname{curl}, \Omega) \subset H_0^h(\operatorname{div} 0, \Omega)$ we have

$$(\delta_{\boldsymbol{B}}, \nabla \times \boldsymbol{F}) = 0 \quad \text{for all} \quad \boldsymbol{F} \in H_0^h(\operatorname{curl}, \Omega). \tag{5.7}$$

Let Π^{div} be the $H(\text{div})$-conforming projection in the commuting diagram in Figure 1. Obviously, $\Pi^{\text{div}} \boldsymbol{B} \in H_0^h(\operatorname{div} 0, \Omega)$. Then, due to the construction of Π^D, we have

$$\|\Pi^D \boldsymbol{B} - \boldsymbol{B}\| = \inf_{\boldsymbol{C} \in H_0^h(\operatorname{div} 0, \Omega)} \|\boldsymbol{B} - \boldsymbol{C}\| \leq \|\Pi^{\text{div}} \boldsymbol{B} - \boldsymbol{B}\| \leq C \inf_{\boldsymbol{C} \in H_0^h(\operatorname{div}, \Omega)} \|\boldsymbol{B} - \boldsymbol{C}\|. \tag{5.8}$$

Now we are ready to present the error equations for the error estimates. Notice that the exact solution $(\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{B}, r, p)$ also satisfies the discrete formulation (4.1). Subtracting two systems, with the spliting of the errors and above properties of the projections (5.5), (5.6) and (5.7), we arrive at:

$$\begin{aligned}(L(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) - L(\boldsymbol{u}_h; \boldsymbol{u}_{\boldsymbol{h}}, \boldsymbol{v})) + R_e^{-1}(\nabla e_{\boldsymbol{u}}, \nabla v) - S(\boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{j}_h \times \boldsymbol{B}_h, \boldsymbol{v}) - (e_p, \nabla \cdot \boldsymbol{v}) \\ = -R_e^{-1}(\nabla \delta_{\boldsymbol{u}}, \nabla \boldsymbol{v}) + (\delta_p, \nabla \cdot \boldsymbol{v}),\end{aligned} \tag{5.9}$$

$$S(\boldsymbol{j} - \boldsymbol{j}_h, \boldsymbol{F}) - \alpha(e_{\boldsymbol{B}}, \nabla \times \boldsymbol{F}) = 0, \tag{5.10}$$

$$\alpha(\nabla \times e_{\boldsymbol{E}}, \boldsymbol{C}) + (e_r, \nabla \cdot \boldsymbol{C}) = -(\delta_r, \nabla \cdot \boldsymbol{C}), \tag{5.11}$$

$$-(\nabla \cdot e_{\boldsymbol{u}}, q) = 0, \tag{5.12}$$

$$(\nabla \cdot e_{\boldsymbol{B}}, s) = 0, \tag{5.13}$$

for all $(\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C}) \in \boldsymbol{X}_h$ and $(q, s) \in \boldsymbol{Y}_h$.

Lemma 5. *We have the energy identity:*

$$\begin{aligned}R_e^{-1}\|\nabla e_{\boldsymbol{u}}\|^2 + \alpha\|\nabla_h \times e_{\boldsymbol{B}}\|^2 = &- (L(\boldsymbol{u}; \boldsymbol{u}, e_{\boldsymbol{u}}) - L(\boldsymbol{u}_h; \boldsymbol{u}_{\boldsymbol{h}}, e_{\boldsymbol{u}})) + (\delta_p, \nabla \cdot e_{\boldsymbol{u}}) - R_e^{-1}(\nabla \delta_{\boldsymbol{u}}, \nabla e_{\boldsymbol{u}}) \\ &+ S(\boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{j}_h \times \boldsymbol{B}_h, e_{\boldsymbol{u}}) + S(\boldsymbol{j} - \boldsymbol{j}_h, \nabla_h \times e_{\boldsymbol{B}}).\end{aligned}$$

Proof. Taking $\boldsymbol{v} = e_{\boldsymbol{u}}, \boldsymbol{F} = -\nabla_h \times e_{\boldsymbol{B}}, q = e_p$ in (5.9), (5.10) and (5.12) and adding these equations, we can obtain the above identity by rearranging terms in the equation. □

From the above result we can see that it suffices to bound the terms on the right hand side of the energy identity to get the error estimates in the energy norm. The first four terms can be handled with standard tools for Navier-Stokes equations, see [15, 37] for instance. In particular, we need the following continuity result for the advection term, see [37]:

Lemma 6. *For any* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in [H_0^1(\Omega)]^3$*, we have*

$$L(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) \leq C\|\nabla \boldsymbol{w}\|\|\nabla \boldsymbol{u}\|\|\nabla \boldsymbol{v}\|,$$

where C *solely depends on the domain* Ω.

In order to bound the last two terms, we need the following auxiliary results:

Lemma 7. *If the regularity assumption* (5.1) *is satisfied, we have*

$$\|\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{u}_h \times \boldsymbol{B}_h\| \leq C\big(\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\| + R_e\|\boldsymbol{f}\|_{-1}\|\nabla_h \times e_{\boldsymbol{B}}\| + R_e^{\frac{1}{2}} R_m S^{-\frac{1}{2}}\|\boldsymbol{f}\|_{-1}(\|e_{\boldsymbol{u}}\|_1 + \|\delta_{\boldsymbol{u}}\|_1)\big),$$
$$\|e_{\boldsymbol{E}}\| \leq \|\delta_{\boldsymbol{E}}\| + \|\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{u}_h \times \boldsymbol{B}_h\|.$$

Proof. For $\|\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{u}_h \times \boldsymbol{B}_h\|$, we have

$$\begin{aligned}\|\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{u}_h \times \boldsymbol{B}_h\| &= \|\boldsymbol{u} \times \delta_{\boldsymbol{B}} + \boldsymbol{u} \times e_{\boldsymbol{B}} + (\delta_{\boldsymbol{u}} + e_{\boldsymbol{u}}) \times \boldsymbol{B}_h\| \\ &\leq \|\boldsymbol{u} \times \delta_{\boldsymbol{B}}\| + \|\boldsymbol{u} \times e_{\boldsymbol{B}}\| + \|(\delta_{\boldsymbol{u}} + e_{\boldsymbol{u}}) \times \boldsymbol{B}_h\| \\ &\leq \|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\| + \|\boldsymbol{u}\|_{0,6}\|e_{\boldsymbol{B}}\|_{0,3} + (\|\delta_{\boldsymbol{u}}\|_{0,6} + \|e_{\boldsymbol{u}}\|_{0,6})\|\boldsymbol{B}_h\|_{0,3},\end{aligned}$$

the last step is due to Hölder's inequality. By (2.1) and Theorem 1, we have

$$\|\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{u}_h \times \boldsymbol{B}_h\| \leq C(\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\| + \|\boldsymbol{u}\|_1\|\nabla_h \times e_{\boldsymbol{B}}\| + (\|\nabla\delta_{\boldsymbol{u}}\| + \|\nabla e_{\boldsymbol{u}}\|)\|\nabla_h \times \boldsymbol{B}_h\|).$$

Finally we can obtain the estimate for this term by the stability result in Theorem 3 and Remark 1. Next, taking $\boldsymbol{F} = e_{\boldsymbol{E}}$ in (5.10), by (5.6), we have

$$(\boldsymbol{j} - \boldsymbol{j}_h, e_{\boldsymbol{E}}) = 0.$$

By the definition of $\boldsymbol{j}, \boldsymbol{j}_h$, we obtain:

$$\|e_{\boldsymbol{E}}\|^2 = -(\delta_{\boldsymbol{E}}, e_{\boldsymbol{E}}) - (\boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{u}_h \times \boldsymbol{B}_h, e_{\boldsymbol{E}}).$$

The proof is completed by Cauchy-Schwarz inequality. □

Now we are ready to give our first error estimate:

Theorem 13. *If the regulartity assumtion* (5.1) *holds, in addition, both* $R_e^2\|\boldsymbol{f}\|_{-1}$ *and* $R_e R_m^{\frac{3}{2}}\|\boldsymbol{f}\|_{-1}$ *are small enough, then we have*

$$R_e^{-\frac{1}{2}}\|\nabla e_{\boldsymbol{u}}\| + \alpha^{\frac{1}{2}}\|e_{\boldsymbol{B}}\|_{0,3} + \alpha^{\frac{1}{2}}\|\nabla_h \times e_{\boldsymbol{B}}\| \leq \mathbb{C}(\|\delta_p\| + \|\nabla\delta_{\boldsymbol{u}}\| + (\|\boldsymbol{u}\|_{1+\sigma} + \|\nabla \times \boldsymbol{B}\|_\sigma)\|\delta_{\boldsymbol{B}}\| + \|\delta_{\boldsymbol{E}}\|),$$

where $\mathbb{C}$ *depends on all the parameters* R_m, R_e, S *and* $\|\boldsymbol{f}\|_{-1}$.

Proof. Since $\nabla \cdot e_{\boldsymbol{B}} = 0$ by (5.6), we can apply Theorem 1 to obtain

$$\|e_{\boldsymbol{B}}\|_{0,3} \leq C\|\nabla_h \times e_{\boldsymbol{B}}\|.$$

By Lemma 5, it suffices to bound terms on the right hand side in the energy identity. The two bilinear terms can be bounded by using Cauchy-Schwarz inequality as,

$$(\delta_p, \nabla \cdot e_{\boldsymbol{u}}) \leq \|\delta_p\|\|\nabla e_{\boldsymbol{u}}\|,$$
$$R_e^{-1}(\nabla\delta_{\boldsymbol{u}}, \nabla e_{\boldsymbol{u}}) \leq R_e^{-1}\|\nabla\delta_{\boldsymbol{u}}\|\|\nabla e_{\boldsymbol{u}}\|.$$

For the convection term, by Lemma 6 we have

$$\begin{aligned}L(\boldsymbol{u};\boldsymbol{u},e_{\boldsymbol{u}})-L(\boldsymbol{u}_h;\boldsymbol{u_h},e_{\boldsymbol{u}})&=L(\boldsymbol{u}-\boldsymbol{u}_h;\boldsymbol{u},e_{\boldsymbol{u}})+L(\boldsymbol{u}_h;\boldsymbol{u}-\boldsymbol{u}_h,e_{\boldsymbol{u}})\\&\leq C(\|\nabla\delta_{\boldsymbol{u}}\|+\|\nabla e_{\boldsymbol{u}}\|)\|\nabla\boldsymbol{u}\|\|\nabla e_{\boldsymbol{u}}\|+C(\|\nabla\delta_{\boldsymbol{u}}\|+\|\nabla e_{\boldsymbol{u}}\|)\|\nabla\boldsymbol{u}_h\|\|\nabla e_{\boldsymbol{u}}\|\\&\leq CR_e^2\|\boldsymbol{f}\|_{-1}\big(R_e^{-1}\|\nabla e_{\boldsymbol{u}}\|^2+R_e^{-1}\|\nabla\delta_{\boldsymbol{u}}\|\cdot\|\nabla e_{\boldsymbol{u}}\|\big),\end{aligned}$$

the last step is by the stability result (4.7) in Remark 1. In order to obtain the convergent result, we need $R_e^2\|\boldsymbol{f}\|_{-1}$ to be small enough.

Next we need to bound the last two terms in Lemma 5. By Cauchy-Schwarz inequality we have

$$\begin{aligned}S(\boldsymbol{j}-\boldsymbol{j}_h,\nabla_h\times e_{\boldsymbol{B}})&\leq S\|\boldsymbol{j}-\boldsymbol{j}_h\|\|\nabla_h\times e_{\boldsymbol{B}}\|\\&=S\|\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}-(\boldsymbol{E}_h+\boldsymbol{u}_h\times\boldsymbol{B}_h)\|\|\nabla_h\times e_{\boldsymbol{B}}\|\\&\leq S(\|\delta_{\boldsymbol{E}}\|+\|e_{\boldsymbol{E}}\|+\|\boldsymbol{u}\times\boldsymbol{B}-\boldsymbol{u}_h\times\boldsymbol{B}_h\|)\|\nabla_h\times e_{\boldsymbol{B}}\|\\&\leq CS\big(\|\delta_{\boldsymbol{E}}\|+\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\|+R_e\|\boldsymbol{f}\|_{-1}\|\nabla_h\times e_{\boldsymbol{B}}\|\\&\qquad+R_e^{\frac12}R_mS^{-\frac12}\|\boldsymbol{f}\|_{-1}(\|e_{\boldsymbol{u}}\|_1+\|\delta_{\boldsymbol{u}}\|_1)\big)\|\nabla_h\times e_{\boldsymbol{B}}\|\\&=CS\big(\|\delta_{\boldsymbol{E}}\|+\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\|+R_e^{\frac12}R_mS^{-\frac12}\|\boldsymbol{f}\|_{-1}\|\delta_{\boldsymbol{u}}\|_1\big)\|\nabla_h\times e_{\boldsymbol{B}}\|\\&\qquad+C\big(R_eS\|\boldsymbol{f}\|_{-1}\|\nabla_h\times e_{\boldsymbol{B}}\|^2+R_e^{\frac12}R_mS^{\frac12}\|\boldsymbol{f}\|_{-1}\|e_{\boldsymbol{u}}\|_1\|\nabla_h\times e_B\|\big)\\&\leq CS\big(\|\delta_{\boldsymbol{E}}\|+\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\|+R_e^{\frac12}R_mS^{-\frac12}\|\boldsymbol{f}\|_{-1}\|\delta_{\boldsymbol{u}}\|_1\big)\|\nabla_h\times e_{\boldsymbol{B}}\|\\&\qquad+C\big(R_eR_m\|\boldsymbol{f}\|_{-1}(\alpha\|\nabla_h\times e_{\boldsymbol{B}}\|^2)+R_eR_m^{\frac32}\|\boldsymbol{f}\|_{-1}(R_e^{-1}\|\nabla e_{\boldsymbol{u}}\|^2+\alpha\|\nabla_h\times e_{\boldsymbol{B}}\|^2)\big).\end{aligned}$$

In order to obtain the convergent result, we need $R_eR_m^{\frac32}\|\boldsymbol{f}\|_{-1}$ to be small enough. Finally, for the last term we begin by spliting the term into three terms and applying the generalized Hölder's inequality to have

$$\begin{aligned}S(\boldsymbol{j}\times\boldsymbol{B}-\boldsymbol{j}_h\times\boldsymbol{B}_h,e_{\boldsymbol{u}})&=S(\boldsymbol{j}\times\delta_{\boldsymbol{B}},e_{\boldsymbol{u}})+S(\boldsymbol{j}\times e_{\boldsymbol{B}},e_{\boldsymbol{u}})+S((\boldsymbol{j}-\boldsymbol{j}_h)\times\boldsymbol{B}_h,e_{\boldsymbol{u}})\\&=T_1+T_2+T_3.\end{aligned}$$

By the fact that $\boldsymbol{j}=R_m^{-1}\nabla\times\boldsymbol{B}$, we can further apply the generalized Hölder's inequalities, Sobolev embedding inequalities, $H^\sigma(\Omega)\hookrightarrow L^3(\Omega)$, (2.1) and Theorem 1 for T_1, T_2 and T_3 as:

$$\begin{aligned}T_1&\leq S\|\boldsymbol{j}\|_{0,3}\|\delta_{\boldsymbol{B}}\|\|e_{\boldsymbol{u}}\|_{0,6}\leq CSR_m^{-1}\|\nabla\times\boldsymbol{B}\|_\sigma\|\delta_{\boldsymbol{B}}\|\|\nabla e_{\boldsymbol{u}}\|,\\T_2&\leq S\|\boldsymbol{j}\|\|e_{\boldsymbol{B}}\|_{0,3}\|e_{\boldsymbol{u}}\|_{0,6}\leq CSR_m^{-1}\|\nabla\times\boldsymbol{B}\|\|\nabla_h\times e_{\boldsymbol{B}}\|\|\nabla e_{\boldsymbol{u}}\|\leq CR_e^{\frac12}S^{\frac12}\|\boldsymbol{f}\|_{-1}\|\nabla_h\times e_{\boldsymbol{B}}\|\|\nabla e_{\boldsymbol{u}}\|\\&\quad\leq CR_eR_m^{\frac12}\|\boldsymbol{f}\|_{-1}(R_e^{-1}\|\nabla e_{\boldsymbol{u}}\|^2+\alpha\|\nabla_h\times e_{\boldsymbol{B}}\|^2),\\T_3&\leq S\|\boldsymbol{j}-\boldsymbol{j}_h\|\|\boldsymbol{B}_h\|_{0,3}\|e_{\boldsymbol{u}}\|_{0,6}\leq CS\|\boldsymbol{j}-\boldsymbol{j}_h\|\|\nabla_h\times\boldsymbol{B}_h\|\|\nabla e_{\boldsymbol{u}}\|\leq CR_e^{\frac12}R_mS^{\frac12}\|\boldsymbol{f}\|_{-1}\|\boldsymbol{j}-\boldsymbol{j}_h\|\|\nabla e_{\boldsymbol{u}}\|\\&\quad\leq CR_e^{\frac12}R_mS^{\frac12}\|\boldsymbol{f}\|_{-1}\big(\|\delta_{\boldsymbol{E}}\|+\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\|+R_e\|\boldsymbol{f}\|_{-1}\|\nabla_h\times e_{\boldsymbol{B}}\|+R_e^{\frac12}R_mS^{-\frac12}\|\boldsymbol{f}\|_{-1}(\|e_{\boldsymbol{u}}\|_1+\|\delta_{\boldsymbol{u}}\|_1)\big)\|\nabla e_{\boldsymbol{u}}\|\\&\quad\leq CR_e^{\frac12}R_mS^{\frac12}\|\boldsymbol{f}\|_{-1}\big(\|\delta_{\boldsymbol{E}}\|+\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\|+R_e^{\frac12}R_mS^{-\frac12}\|\boldsymbol{f}\|_{-1}\|\delta_{\boldsymbol{u}}\|_1\big)\\&\qquad+CR_e^{\frac32}R_mS^{\frac12}\|\boldsymbol{f}\|_{-1}^2\|e_{\boldsymbol{u}}\|_1\|\nabla_h\times e_{\boldsymbol{B}}\|+CR_eR_m^2\|\boldsymbol{f}\|_{-1}^2\|e_{\boldsymbol{u}}\|_1^2\\&\quad\leq CR_e^{\frac12}R_mS^{\frac12}\|\boldsymbol{f}\|_{-1}\big(\|\delta_{\boldsymbol{E}}\|+\|\boldsymbol{u}\|_{0,\infty}\|\delta_{\boldsymbol{B}}\|+R_e^{\frac12}R_mS^{-\frac12}\|\boldsymbol{f}\|_{-1}\|\delta_{\boldsymbol{u}}\|_1\big)\\&\qquad+R_eR_m^{\frac32}\|\boldsymbol{f}\|_{-1}^2(R_e^{-1}\|\nabla e_{\boldsymbol{u}}\|^2+\alpha\|\nabla_h\times e_{\boldsymbol{B}}\|^2)+CR_e^2R_m^2\|\boldsymbol{f}\|_{-1}^2(R_e^{-1}\|\nabla e_{\boldsymbol{u}}\|^2).\end{aligned}$$

Referring to T_2 and T_3, we need $R_e R_m^{\frac{1}{2}} \|\boldsymbol{f}\|_{-1}$, $R_e R_m^{\frac{3}{2}} \|\boldsymbol{f}\|_{-1}^2$ and $R_e^2 R_m^2 \|\boldsymbol{f}\|_{-1}^2$ to be small enough such that convergent results can be obtained.

So, if $R_e^2 \|\boldsymbol{f}\|_{-1}$ and $R_e R_m^{\frac{3}{2}} \|\boldsymbol{f}\|_{-1}$ are both small enough, we have

$$R_e^{-\frac{1}{2}} \|\nabla e_{\boldsymbol{u}}\| + \alpha^{\frac{1}{2}} \|\nabla_h \times e_{\boldsymbol{B}}\| \le \mathbb{C}(\|\delta_p\| + \|\nabla \delta_{\boldsymbol{u}}\| + (\|\boldsymbol{u}\|_{1+\sigma} + \|\nabla \times \boldsymbol{B}\|_\sigma)\|\delta_{\boldsymbol{B}}\| + \|\delta_{\boldsymbol{E}}\|).$$

Here $\mathbb{C}$ depends on all the parameters R_m, R_e, s and $\|\boldsymbol{f}\|_{-1}$. This completes the proof. $\square$

6. Nonlinear scheme for the alternative boundary condition

We propose the following variational form for (1.1) with boundary condition (1.3):

Problem 4. *Find $(\boldsymbol{u}_h, \boldsymbol{E}_h, \boldsymbol{B}_h) \in \tilde{\boldsymbol{X}}_h$ and $(p_h, r_h) \in \boldsymbol{Y}_h$, such that for any $(\boldsymbol{v}, \boldsymbol{F}, \boldsymbol{C}) \in \tilde{\boldsymbol{X}}_h$ and $(q, s) \in \boldsymbol{Y}_h$,*

$$L(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) + R_e^{-1}(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}) - S(\boldsymbol{j}_h \times \boldsymbol{B}_h, \boldsymbol{v}) - (p_h, \nabla \cdot \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \tag{6.1a}$$

$$S(\boldsymbol{j}_h, \boldsymbol{F}) - \alpha(\boldsymbol{B}_h, \nabla \times \boldsymbol{F}) = 0, \tag{6.1b}$$

$$\alpha(\nabla \times \boldsymbol{E}_h, \boldsymbol{C}) + (r_h, \nabla \cdot \boldsymbol{C}) = 0, \tag{6.1c}$$

$$-(\nabla \cdot \boldsymbol{u}_h, q) = 0, \tag{6.1d}$$

$$(\nabla \cdot \boldsymbol{B}_h, s) = 0, \tag{6.1e}$$

where $\boldsymbol{j}_h$ is given by Ohm's law: $\boldsymbol{j}_h = \boldsymbol{E}_h + \boldsymbol{u}_h \times \boldsymbol{B}_h$ and r_h is the Lagrange multiplier which approximates $r = 0$, and $\tilde{\boldsymbol{X}}_h = \boldsymbol{V}_h \times H^h(\mathrm{curl}, \Omega) \times H^h(\mathrm{div}, \Omega)$.

Similar to Theorem 3, we have Theorem 14, whose proof is the same as that of Theorem 3.

Theorem 14. *Any solution for Problem 4 satisfies*

(1) *magnetic Gauss's law:*

$$\nabla \cdot \boldsymbol{B}_h = 0.$$

(2) *Lagrange multiplier $r = 0$, and the strong form*

$$\nabla \times \boldsymbol{E}_h = 0,$$

(3) *energy estimates:*

$$R_e^{-1} \|\nabla \boldsymbol{u}_h\|^2 + S\|\boldsymbol{j}_h\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}_h \rangle, \tag{6.2}$$

$$\frac{1}{2} R_e^{-1} \|\nabla \boldsymbol{u}_h\|^2 + S\|\boldsymbol{j}_h\|^2 \le \frac{R_e}{2} \|\boldsymbol{f}\|_{-1}^2, \tag{6.3}$$

$$R_m^{-1} \|\nabla_h \times \boldsymbol{B}_h\| \le \|\boldsymbol{j}_h\|, \tag{6.4}$$

$$\|\nabla_h \times \boldsymbol{B}_h\| \le C R_e^{\frac{1}{2}} R_m S^{-\frac{1}{2}} \|\boldsymbol{f}\|_{-1}, \tag{6.5}$$

$$\|\boldsymbol{E}_h\| \le C R_e^{\frac{3}{2}} R_m S^{-\frac{1}{2}} \|\boldsymbol{f}\|_{-1}. \tag{6.6}$$

Similar to the argument in Section 4.1, we can conclude that Problem 4 is well-posed.

We define $e_{\boldsymbol{u}}$, $\delta_{\boldsymbol{u}}$, e_p, δ_p, e_r, δ_r the same as those in Section 5. We use Π^{curl} in Figure 2 for the electric field $\boldsymbol{E}$. We define $e_{\boldsymbol{E}} = \Pi^{\mathrm{curl}} \boldsymbol{E} - \boldsymbol{E}_h$ and $\delta_{\boldsymbol{E}} = \boldsymbol{E} - \Pi^{\mathrm{curl}} \boldsymbol{E}$. For the magnetic field $\boldsymbol{B}$, we define the L^2-projection $\Pi^{\tilde{D}}$ into $H^h(\operatorname{div} 0,\Omega)$. We denote $e_{\boldsymbol{B}} = \Pi^{\tilde{D}}\boldsymbol{B} - \boldsymbol{B}_h$ and $\delta_{\boldsymbol{B}} = \boldsymbol{B} - \Pi^{\tilde{D}}\boldsymbol{B}$. It is easy to see that

$$\begin{aligned}
&\nabla\cdot e_{\boldsymbol{B}} = 0,\\
&(\boldsymbol{B} - \Pi^{\tilde{D}}\boldsymbol{B}, \nabla\times\boldsymbol{F}) = 0 \quad \text{ for all } F\in H^h(\mathrm{curl},\Omega),\\
&\|\boldsymbol{B} - \Pi^{\tilde{D}}\boldsymbol{B}\| \le C \inf_{\boldsymbol{C}\in H^h(\mathrm{div},\Omega)} \|\boldsymbol{B}-\boldsymbol{C}\|.
\end{aligned}$$

Thus by using Theorem 2 to replace Theorem 1, we can use the same argument in Section 5 to obtain Theorem 15.

Theorem 15. *If the regulartity assumtion (5.1) holds, in addition, both $R_e^2\|\boldsymbol{f}\|_{-1}$ and $R_e R_m^{\frac{3}{2}}\|\boldsymbol{f}\|_{-1}$ are small enough, then we have*

$$R_e^{-\frac{1}{2}}\|\nabla e_{\boldsymbol{u}}\| + \alpha^{\frac{1}{2}}\|e_{\boldsymbol{B}}\|_{0,3} + \alpha^{\frac{1}{2}}\|\nabla_h\times e_{\boldsymbol{B}}\| \le \mathbb{C}(\|\delta_p\| + \|\nabla\delta_{\boldsymbol{u}}\| + (\|\boldsymbol{u}\|_{1+\sigma} + \|\nabla\times\boldsymbol{B}\|_\sigma)\|\delta_{\boldsymbol{B}}\| + \|\delta_{\boldsymbol{E}}\|),$$

where $\mathbb{C}$ depends on all the parameters R_m, R_e, S and $\|\boldsymbol{f}\|_{-1}$.

7. Conclusion

We analyzed a mixed finite element scheme for the stationary MHD system where both the electric and the magnetic fields were discretized on a discrete de Rham complex. Two types of boundary conditions were considered. We rigorously established the well-posedness and proved the convergence of the finite element schemes based on weak regularity assumptions.

The electric-magnetic mixed formulation (also see [23, 24]) and the technical tools developed in this paper may also be useful for a broader class of plasma models and numerical methods, for example, compressible MHD models and discontinuous Galerkin methods (c.f. [30, 25, 35, 33]).

The theoretical analysis in this paper also lays a foundation for further investigation of block preconditioners for stationary MHD systems (c.f. [26, 10]).

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