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Hierarchical multiscale finite element method for multi-continuum media

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ABSTRACT

Simulation in media with multiple continua where each continuum interacts with every other is often challenging due to multiple scales and high contrast. One needs some type of model reduction. One of the approaches is a multi-continuum technique, where every process in each continuum is modeled separately and an interaction term is added. Direct numerical simulation in multiscale media is usually not practicable. For this reason, one constructs the corresponding homogenized equations which approximate the solutions to the multiscale equations when the microscopic scales tend to 0. Computing the effective coefficients of the homogenized equations can be expensive because one needs to solve local cell problems for a large number of macroscopic points. The paper considers a two scale two continuum system where the interaction terms between the continua are scaled as $O(1/\epsilon^2)$ where ϵ is the microscopic scale. We prove that in the homogenization limit, we obtain the same limit for both continua. We derive the homogenized equation for the limit function; and prove the homogenization convergence rigorously. The homogenized coefficients are established from solutions of the cell problems which are systems of equations of a similar form as the two continuum system. We develop a hierarchical approach for solving these cell problems at a dense network of macroscopic points with an essentially optimal computation cost. The method employs the fact that neighboring representative volume elements (RVEs) share similar features; and effective properties of the neighboring RVEs are close to each other. The hierarchical approach reduces computation cost by using different levels of resolution for cell problems at different macroscopic points. Solutions of the cell problems which are solved with a higher level of resolution are employed to correct the solutions at neighboring macroscopic points that are computed by approximation spaces with a lower level of resolution. The method requires a hierarchy of macroscopic grid points and corresponding nested approximation spaces with different levels of resolution. Each level of macroscopic points is assigned to an approximation finite element (FE) space which is used to solve the cell problems at the macroscopic points from that level. We prove rigorously that this hierarchical method achieves the same level of accuracy as that of the full solve where cell problems at every macroscopic point are solved using FE spaces with the highest level of resolution, but at the essentially optimal computation cost. Numerical implementation that computes effective permeabilities of a two-scale multicontinuum system via the numerical solutions of the cell problems supports the analytical results.

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1. Introduction

Media with multiple continua where each continuum interacts with other continua often entail multiple scales and high contrast. For example, fractured media can have multiscale and high contrast due to complex material properties and geography of fractures. Therefore, numerical simulations in this type of media can be expensive and require model reduction. This can be achieved by computing effective properties in each coarse block using the solutions of local representative volume element (RVE) problems. In multi-continuum approaches [1–5], equations for each continuum are written separately with so-called interaction terms. Therefore, one deals with a system of coupled multiscale equations.

There have been several methods to solve multiscale equations without computing effective properties and establishing homogenized equations. The multiscale finite element method (MsFEM) [6] solves local cell problems in coarse blocks with fine mesh to obtain basis functions that capture small-scale information. The generalized multiscale finite element method (GMsFEM) [7–9] follows the outline of MsFEM but adds some degrees of freedom in each coarse block by building the snapshot spaces and solving local spectral problems in the spaces. There have been significant attempt to employ these methods and their variations to solve multiscale multicontinuum systems. We mention the applications of the GMsFEM in [10], of the constraint energy minimizing (CEM) in [11], and of the non-local multi-continuum method (NLMC) in [12–14]. These methods are known to effectively handle the multiple scales and high contrast features in multi-continuum media. However, they do not take into account periodic or local periodic structures of the media of interest and sometimes introduce high computation cost when the microscopic scale requires very small fine mesh sizes. In this paper, we establish an efficient algorithm for obtaining homogenized equations taking into account microscopic periodicity for a two-scale dual-continuum system with optimal computation complexity.

Homogenization of multiscale multicontinuum systems have not been paid much attention. We contribute in this paper the first rigorous results on homogenization of a two scale two continuum system where the interaction between the continua is scaled as $1/\epsilon^2$ where ϵ represents the microscopic scale of the medium. We derive the homogenized problem from the two-scale asymptotic expansion [15–17]. We show that for this scale of the interaction term, we obtain the same limit for both continua. Other scaling regimes of this term give rise to other limiting behaviors which will be studied in our forthcoming publications. The effective coefficients of the homogenized equation are established via the solutions of cell problems which are systems of equations of a similar form as the two continuum system. Since the two scale coefficients depend on both macro- and micro-scale variables, a different set of cell equations needs to be solved at each macroscopic point. The number of equations to be solved is thus very large. Solving them using the same fine mesh at every macroscopic point is extremely expensive.

Another main contribution of the paper is the development of a hierarchical approach to solve these cell problems to obtain the effective coefficients for the multi-continuum system for a large number of macroscopic points, using an optimal number of degrees of freedom, without sacrificing the accuracy. It solves cell problems for a dense hierarchical network of macroscopic points with different levels of resolution. The problems at those points belonging to a lower level in the hierarchy are solved with a higher level of accuracy. For the solution at a macroscopic point at a higher level in the hierarchy which is obtained with a lower level of accuracy, we use solutions at nearby macroscopic points that are solved with a higher level of accuracy to correct the error. We show that this hierarchical FE approach obtains the same level of accuracy at every macroscopic point as that obtained when every cell problem is solved with the highest level of resolution (we will refer to this as the full reference solve below), but uses only an essentially optimal number of degrees of freedom that is equal to that required to solve only one cell problem at the finest level of resolution (apart from a possible logarithmic factor).

The hierarchical finite element method has been developed to solve the cell problems and compute the effective coefficients for other multiscale equations. In [18], the method was developed for the effective coefficients of deterministic two-scale Stokes–Darcy systems in a slowly varying porous medium. In [19], they use the hierarchical algorithm for a two-scale ergodic random homogenization problem without assuming microscopic periodicity. In this paper, we follow the framework of these papers, but we utilize the hierarchical approach to compute homogenization coefficients for a two-scale dual-continuum system where the interaction terms are scaled as $\mathcal{O}(\frac{1}{\epsilon^2})$. The interaction terms give the interesting cell problems (2.4) in the form of a system of coupled equations.

The paper is organized as follows. In Section 2, we set up the multiscale multi-continuum system; and we derive the homogenized equations from two-scale asymptotic expansion. In Section 3, we outline the hierarchical finite element algorithm for solving the cell problems at a dense network of macroscopic points. We give a rigorous error estimate that shows the algorithm has the equivalent accuracy as the full reference solve, at essentially optimal computation cost. In Section 4, we present numerical examples that verify the theoretical results. We compute the effective permeabilities using the hierarchical solve and the full solve. We find that the effective permeabilities obtained from these two approaches are essentially equal, with a very small relative error between each other. Finally, in Section 5, we rigorously prove the homogenization convergence for the two-scale multi-continuum system. The paper ends with the conclusions in Section 6.

Throughout the paper, by ∇, we denote the gradient with respect to x of a function that depends only on the variable x, or the variables x and t. By ∇_x, we denote the partial gradient with respect to x of a function that depends on x, t and also other variables. Repeated indices indicate summation. The notation # denotes spaces of periodic functions.

2. Problem formulation

2.1. Homogenization of multi-continuum systems

In multi-continuum approaches, equations for each continuum are written separately. We denote by u_i the solution for ith continuum. In the general case where each continuum interacts with every other continuum, we have the following system of equations introduced in [10]

$$\mathcal{C}_{ii}^{\epsilon}(x)\frac{\partial u_i^{\epsilon}(t,x)}{\partial t} = \mathrm{div}(\kappa_i^{\epsilon}(x)\nabla u_i^{\epsilon}(t,x)) + Q_i^{\epsilon}(u_1^{\epsilon}(t,x),\dots,u_N^{\epsilon}(t,x)) + q_i, \quad \text{in } \Omega$$

where $\Omega \subset \mathbb{R}^d$ is a domain ($d=2,3$), κ_i^{ϵ} are the multiscale permeability and $\mathcal{C}_{ii}^{\epsilon}$ are the multiscale porosities, q_i are the source terms, and the functions Q_i^{ϵ} of $(u_1,\dots,u_N)$ are exchange terms (see [1–5]) that describe the interaction of continua; ϵ represents the microscopic scale of the local variation.

In this paper, we consider a dual-continuum system. Let Y be the unit cube in $\mathbb{R}^d$. Let $\mathcal{C}_{ii}(x,y)$, $\kappa_i(x,y)$ $(i=1,2)$ be continuous functions on $\Omega\times Y$ which are Y-periodic with respect to y and q be a function in $L^2(\Omega)$. We assume further that there is a constant $c>0$ such that for all $x\in\Omega, y\in Y$

$$\mathcal{C}_{ii}(x,y)\ge c, \qquad \kappa_i(x,y)\ge c, \qquad Q(x,y)\ge c. \tag{2.1}$$

We define the two scale coefficients as

$$\mathcal{C}_{ii}^{\epsilon}(x)=\mathcal{C}_{ii}(x,\frac{x}{\epsilon}), \qquad \kappa_i^{\epsilon}(x)=\kappa_i(x,\frac{x}{\epsilon}), \qquad Q^{\epsilon}(x)=Q(x,\frac{x}{\epsilon}).$$

We consider in this paper the case where the interaction terms are scaled as $O(1/\epsilon^2)$; this case has the most interesting cell problems in the form of a coupled system. We consider the multiscale dual-continuum system

$$\begin{aligned}
\mathcal{C}_{11}^{\epsilon}(x)\frac{\partial u_1^{\epsilon}(t,x)}{\partial t} &= \mathrm{div}(\kappa_1^{\epsilon}(x)\nabla u_1^{\epsilon}(t,x)) + \frac{1}{\epsilon^2}Q^{\epsilon}(x)(u_2^{\epsilon}(t,x)-u_1^{\epsilon}(t,x)) + q,\\
\mathcal{C}_{22}^{\epsilon}(x)\frac{\partial u_2^{\epsilon}(t,x)}{\partial t} &= \mathrm{div}(\kappa_2^{\epsilon}(x)\nabla u_2^{\epsilon}(t,x)) + \frac{1}{\epsilon^2}Q^{\epsilon}(x)(u_1^{\epsilon}(t,x)-u_2^{\epsilon}(t,x)) + q,
\end{aligned} \tag{2.2}$$

with the Dirichlet boundary condition $u_1^{\epsilon}(t,x)=u_2^{\epsilon}(t,x)=0$ for $x\in\partial\Omega$, and with the initial condition $u_1^{\epsilon}(0,x)=g_1$, $u_2^{\epsilon}(0,x)=g_2$ where g_1 and g_2 are in $L^2(\Omega)$. We consider the following two-scale asymptotic expansion of u_1^{ϵ} and u_2^{ϵ}.

$$u_1^{\epsilon}(t,x)=u_{10}(t,x,\frac{x}{\epsilon})+\epsilon u_{11}(t,x,\frac{x}{\epsilon})+\cdots, \qquad u_2^{\epsilon}(t,x)=u_{20}(t,x,\frac{x}{\epsilon})+\epsilon u_{21}(t,x,\frac{x}{\epsilon})+\cdots,$$

where the functions $u_{1i}(t,x,y)$ and $u_{2i}(t,x,y)$ are periodic with respect to y. Performing the two-scale asymptotic expansion, from (2.2) we obtain

$$\begin{aligned}
&\mathcal{C}_{11}\frac{\partial(u_{10}+\epsilon u_{11}+\cdots)}{\partial t}\\
&\quad=(\mathrm{div}_x+\frac{1}{\epsilon}\mathrm{div}_y)(\kappa_1(\nabla_x+\frac{1}{\epsilon}\nabla_y)(u_{10}+\epsilon u_{11}+\cdots))+\frac{1}{\epsilon^2}Q(u_{20}+\epsilon u_{21}-u_{10}-\epsilon u_{11}+\cdots)+q,\\
&\mathcal{C}_{22}\frac{\partial(u_{20}+\epsilon u_{21}+\cdots)}{\partial t}\\
&\quad=(\mathrm{div}_x+\frac{1}{\epsilon}\mathrm{div}_y)(\kappa_2(\nabla_x+\frac{1}{\epsilon}\nabla_y)(u_{20}+\epsilon u_{21}+\cdots))+\frac{1}{\epsilon^2}Q(u_{10}+\epsilon u_{11}-u_{20}-\epsilon u_{21}+\cdots)+q,
\end{aligned} \tag{2.3}$$

For the $O(\epsilon^{-2})$ terms, we obtain,

$$\begin{aligned}
&\mathrm{div}_y(\kappa_1(x,y)\nabla_y u_{10}(t,x,y))+Q(x,y)(u_{20}(t,x,y)-u_{10}(t,x,y))=0\\
&\mathrm{div}_y(\kappa_2(x,y)\nabla_y u_{20}(t,x,y))+Q(x,y)(u_{10}(t,x,y)-u_{20}(t,x,y))=0.
\end{aligned}$$

From this, we have

$$\begin{aligned}
&-\int_Y \kappa_1\nabla_y u_{10}\cdot\nabla_y u_{10}dy+\int_Y Q(u_{20}-u_{10})u_{10}dy=0\\
&-\int_Y \kappa_2\nabla_y u_{20}\cdot\nabla_y u_{20}dy+\int_Y Q(u_{10}-u_{20})u_{20}dy=0
\end{aligned}$$

Adding these two equations, we obtain

$$\int_Y \kappa_1\nabla_y u_{10}\cdot\nabla_y u_{10}dy+\int_Y \kappa_2\nabla_y u_{20}\cdot\nabla_y u_{20}dy+\int_Y Q(u_{20}-u_{10})^2dy=0.$$

This implies $\nabla_y u_{10} = 0$, $\nabla_y u_{20} = 0$. i.e. u_{10} and u_{20} are independent of y, and $u_{10}(t,x) = u_{20}(t,x) = u_0(t,x)$ as $Q(x,y) > c > 0\ \forall x \in \Omega, y \in Y$. For the $O(\epsilon^{-1})$ terms in (2.3), we have,

$$\begin{aligned} &\mathrm{div}_x(\kappa_1 \nabla_y u_{10}) + \mathrm{div}_y(\kappa_1 \nabla u_{10}) + \mathrm{div}_y(\kappa_1 \nabla_y u_{11}) + Q(u_{21} - u_{11}) = 0 \\ &\mathrm{div}_x(\kappa_2 \nabla_y u_{20}) + \mathrm{div}_y(\kappa_2 \nabla u_{20}) + \mathrm{div}_y(\kappa_2 \nabla_y u_{21}) + Q(u_{11} - u_{21}) = 0. \end{aligned}$$

Since u_{10} and u_{20} are independent of y, we have

$$\begin{aligned} &\mathrm{div}_y(\kappa_1 \nabla_y u_{11}) + Q(u_{21} - u_{11}) = -\,\mathrm{div}_y(\kappa_1 \nabla u_0) \\ &\mathrm{div}_y(\kappa_2 \nabla_y u_{21}) + Q(u_{11} - u_{21}) = -\,\mathrm{div}_y(\kappa_2 \nabla u_0) \end{aligned}$$

Thus $u_{11} = \frac{\partial u_0}{\partial x_i} N_1^i$ and $u_{21} = \frac{\partial u_0}{\partial x_i} N_2^i$ where $N_1^i(x,\cdot) \in H^1_\#(Y)/\mathbb{R}$, and $N_2^i(x,\cdot) \in H^1_\#(Y)/\mathbb{R}$ are solutions of the cell problem

$$\begin{aligned} &\mathrm{div}_y(\kappa_1(x,y)(e^i + \nabla_y N_1^i)) + Q(x,y)(N_2^i - N_1^i) = 0 \\ &\mathrm{div}_y(\kappa_2(x,y)(e^i + \nabla_y N_2^i)) + Q(x,y)(N_1^i - N_2^i) = 0, \end{aligned} \tag{2.4}$$

where e^i is the ith unit vector in the standard basis of $\mathbb{R}^d$. For the $O(\epsilon^0)$ terms in (2.3), integrating over Y, one has

$$\begin{aligned} &\int_Y \mathcal{C}_{11} \frac{\partial u_0}{\partial t} dy = \int_Y \mathrm{div}_x(\kappa_1 \nabla u_0) dy + \int_Y \mathrm{div}_x(\kappa_1 \nabla_y u_{11}) dy + \int_Y Q(u_{22} - u_{12}) dy + \int_Y q dy \\ &\int_Y \mathcal{C}_{22} \frac{\partial u_0}{\partial t} dy = \int_Y \mathrm{div}_x(\kappa_2 \nabla u_0) dy + \int_Y \mathrm{div}_x(\kappa_2 \nabla_y u_{21}) dy + \int_Y Q(u_{12} - u_{22}) dy + \int_Y q dy. \end{aligned}$$

Adding these two equations, one obtains the homogenized equation

$$\left(\int_Y \mathcal{C}_{11} dy + \int_Y \mathcal{C}_{22} dy \right) \frac{\partial u_0}{\partial t} = \mathrm{div}(\kappa_1^* \nabla u_0) + \mathrm{div}(\kappa_2^* \nabla u_0) + \int_Y 2q dy \qquad \text{in } \Omega \tag{2.5}$$

where the x-dependent permeabilities are defined as

$$\kappa_{1ij}^*(x) = \int_Y \kappa_1(x,y)(\delta_{ij} + \frac{\partial N_1^j(x,y)}{\partial y_i}) dy, \qquad \kappa_{2ij}^*(x) = \int_Y \kappa_2(x,y)(\delta_{ij} + \frac{\partial N_2^j(x,y)}{\partial y_i}) dy \tag{2.6}$$

We will show later that the matrix $\kappa_{1ij}^*(x) + \kappa_{2ij}^*(x)$ is symmetric and positive definite. We will also show that the initial condition for u_0 is

$$u_0(0,x) = \frac{\langle C_{11} \rangle g_1(x) + \langle C_{22} \rangle g_2(x)}{\langle C_{11} \rangle + \langle C_{22} \rangle} \tag{2.7}$$

where $\langle C_{ii} \rangle = \int_Y C_{ii}(y) dy$ for $i = 1, 2$. Eq. (2.5) together with initial condition (2.7) has a unique solution (see, e.g., [20]).

2.2. Uniqueness of solution to the cell problem

We write the system (2.4) in the variational form as

$$\begin{aligned} &\int_Y \kappa_1(x,y) \nabla_y N_1^i(x,y) \cdot \nabla_y \phi_1(y) dy - \int_Y Q(x,y)(N_2^i - N_1^i)\phi_1(y) dy = -\int_Y \kappa_1(x,y) e^i \cdot \nabla_y \phi_1(y) dy \\ &\int_Y \kappa_2(x,y) \nabla_y N_2^i(x,y) \cdot \nabla_y \phi_2(y) dy - \int_Y Q(x,y)(N_1^i - N_2^i)\phi_2(y) dy = -\int_Y \kappa_2(x,y) e^i \cdot \nabla_y \phi_2(y) dy \end{aligned} \tag{2.8}$$

where $\phi_1, \phi_2 \in H^1_\#(Y)$. Let W be the space $H^1_\#(Y) \times H^1_\#(Y)/(c,c)$, $c \in \mathbb{R}$. The space W is equipped with the norm

$$|||(\phi_1, \phi_2)||| = \|\nabla_y \phi_1\|_{L^2(Y)} + \|\nabla_y \phi_2\|_{L^2(Y)} + \|\phi_1 - \phi_2\|_{L^2(Y)}.$$

For $x \in \Omega$, we define the bilinear form $B(x;\cdot,\cdot) : W \times W \to \mathbb{R}$ as

$$\begin{aligned} B(x;(\phi_1,\phi_2),(\psi_1,\psi_2)) = &\int_Y \kappa_1(x,y) \nabla_y \phi_1(y) \cdot \nabla_y \psi_1(y) dy + \int_Y \kappa_2(x,y) \nabla_y \phi_2(y) \cdot \nabla_y \psi_2(y) dy \\ &+ \int_Y Q(x,y)(\phi_1(x,y) - \phi_2(x,y))(\psi_1(x,y) - \psi_2(x,y)) dy \end{aligned}$$

for $(\phi_1,\phi_2) \in W$ and $(\psi_1,\psi_2) \in W$. From (2.1), we deduce that the bilinear form B is uniformly coercive and bounded with respect to $x \in \Omega$, i.e. there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$B(x;(\phi_1,\phi_2),(\phi_1,\phi_2)) \geq c_1 |||(\phi_1,\phi_2)|||^2, \qquad \text{and} \qquad B(x;(\phi_1,\phi_2),(\psi_1,\psi_2)) \leq c_2 |||(\phi_1,\phi_2)||| \cdot |||(\psi_1,\psi_2)|||$$

for all $(\phi_1,\phi_2) \in W$ and $(\psi_1,\psi_2) \in W$. Adding the two equations in (2.8), we obtain

$$B(x;(N_1^i,N_2^i),(\phi_1,\phi_2)) = -\int_Y \kappa_1(x,y) e^i \cdot \nabla_y \phi_1(y) dy - \int_Y \kappa_2(x,y) e^i \cdot \nabla_y \phi_2(y) dy.$$

Theorem 2.1. *Problem (2.8) has a unique solution* $(N_1^i, N_2^i) \in W$.

Proof. The conclusion follows from the boundedness and coerciveness of the bilinear form B and the Lax–Milgram lemma. □

3. Hierarchical finite element algorithm

Computing effective coefficients $\kappa_i^*(x)$ requires the solutions of the cell problems (2.4) at many macroscopic points which can be very expensive. We develop in this section the hierarchical FE method which computes the solution of the cell problems at a dense network of macroscopic points using only an essentially optimal number of degrees of freedom which is equal to that for solving one cell problem (apart from a multiplying logarithmic factor). We assume that the coefficients are sufficiently smooth with respect to the macroscopic variable x. We make the following assumption.

Assumption 3.1. There is a constant $C > 0$ such that for all $x, x' \in \Omega$,

$$\|\kappa_1(x,\cdot) - \kappa_1(x',\cdot)\|_{L^\infty(Y)} \le C|x - x'|, \qquad \|\kappa_2(x,\cdot) - \kappa_2(x',\cdot)\|_{L^\infty(Y)} \le C|x - x'|,$$
$$\text{and} \qquad \|Q(x,\cdot) - Q(x',\cdot)\|_{L^\infty(Y)} \le C|x - x'|.$$

Remark. The main necessary condition for our proposed method to work is that the two scale coefficients possess Lipschitz (or Holder) smoothness with respect to the macroscopic variable. However, this assumption is reasonable as the macroscopic properties of the media normally vary smoothly.

3.1. Overview of hierarchical algorithm

We develop an efficient hierarchical finite element algorithm to solve the coupled cell problem (2.4) numerically and to approximate the effective properties $\kappa_i^*(x)$ in (2.6) for a dense network of macroscopic points $x \in \Omega$. We follow the algorithm introduced in [18].

We outline the algorithm as follows.

Step 1 : Build nested finite element spaces. We employ Galerkin FE to obtain an approximation of the solution $(N_1^i, N_2^i) \in W$ of (2.4) for each macroscopic point $x \in \Omega$ using FE spaces of different levels of resolution. We assume that there exists a hierarchy of FE spaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_L \subset H^1_\#(Y)$ where the integer index L denotes the resolution level. We assume further the following approximation properties: for $w \in H^2_\#(Y)$,

$$\inf_{\phi \in \mathcal{V}_{L-l}} \|\nabla_y(w - \phi)\|_{L^2(Y)} + 2^{L-l}\|w - \phi\|_{L^2(Y)} \le C2^{-L+l}\|w\|_{H^2(Y)}, \tag{3.1}$$

where the constant C is independent of L and l.

Step 2 : Build a hierarchy of macrogrids. We solve the cell equations at different macroscopic points $x \in \Omega$ with different levels of accuracy. We use the solutions solved with a higher accuracy level to correct the solutions obtained with a lower accuracy level. We achieve this by solving the cell problems at different macroscopic points using different FE spaces in the hierarchy in Step 1. This can be done by constructing a hierarchy of macro-grid points. We construct a nested macro-grid, $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_L \subset \Omega$ as follows. First, we build an initial grid $\mathcal{T}_0$ with a proper grid spacing H, the maximal distance between neighboring nodes. We then inductively construct $\mathcal{T}_l$, a refinement of $\mathcal{T}_{l-1}$, with grid spacing $H2^{-l}$. Then, we define the hierarchy of macro-grids, $\{S_0, S_1, \dots, S_L\}$ as $S_0 = \mathcal{T}_0$, $S_1 = \mathcal{T}_1 \backslash S_0$, and for each $l > 1$, we have

$$S_l = \mathcal{T}_l \Big\backslash \left(\bigcup_{k<l} S_k \right).$$

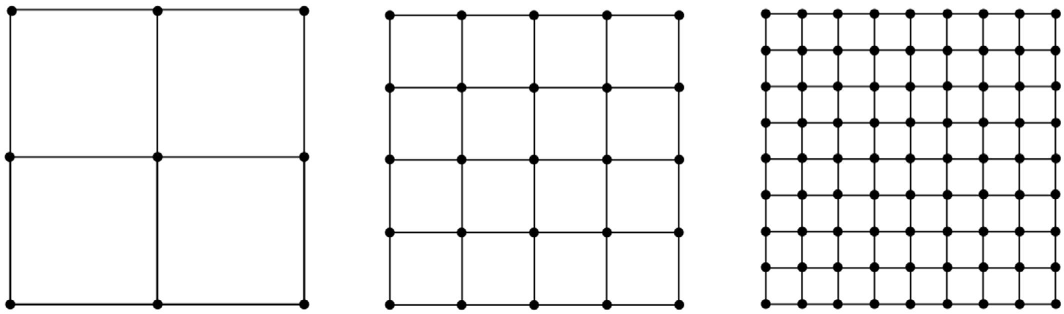
We call the nodes in the lowest level grid S_0 the anchor points. In this way, we obtain a dense hierarchy of the macro-grids. That is, each point $x \in S_l$ has at least one point from one of the previous levels, $x' \in \bigcup_{k<l} S_k$ such that $\operatorname{dist}(x, x') < O(H2^{-l})$. Figs. 1 and 2 show an example of 3-level hierarchy of macrogrids $\mathcal{T}_l$, S_l, $l = 1, 2, 3$, constructed in $\Omega = [0, 1]^2$.

Step 3 : Calculating the correction term. We relate the nested FE spaces and the hierarchy of macrogrids for our algorithm. We first solve the cell problems at anchor points using the standard Galerkin FE with FE space $\mathcal{V}_L$. That is, for the points in the coarsest macro-grid S_0, we solve the cell problems with the finest mesh. More precisely, we find $\bar{N}_1^i(x,\cdot), \bar{N}_2^i(x,\cdot) \in \mathcal{V}_L$, such that

$$B(x; (\bar{N}_1^i, \bar{N}_2^i), (\phi_1, \phi_2)) = -\int_Y \kappa_1(x,y)e^i \cdot \nabla_y \phi_1(y)dy - \int_Y \kappa_2(x,y)e^i \cdot \nabla_y \phi_2(y)dy$$

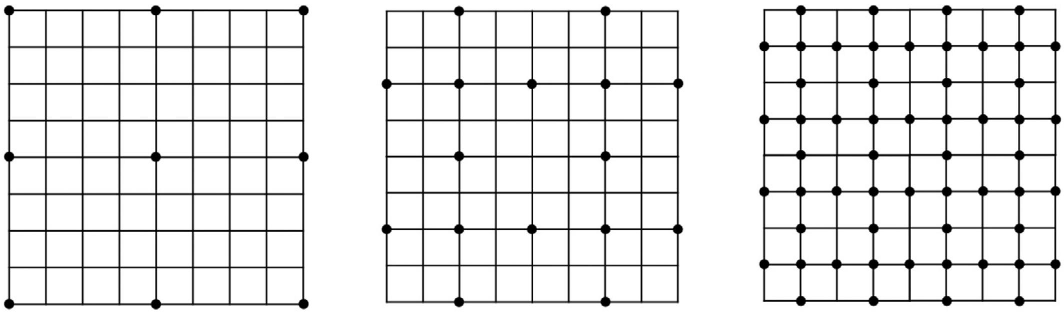
for all $\phi_1, \phi_2 \in \mathcal{V}_L$. Proceeding inductively, for $x \in S_l$ ($l = 1, \dots, L$), we choose the points $\{x_1, x_2, \dots, x_n\} \in (\bigcup_{l'<l} S_{l'})$ so that the distance between x and each point in $\{x_1, x_2, \dots, x_n\}$ is $O(H2^{-l})$. This is possible from the assumption for the hierarchy of macroscopic points above. We define the lth macro-grid interpolation by

$$I_l^x(N_k^i) = \sum_{j=1}^n c_j N_k^i(x_j, \cdot),$$



(a) $\mathcal{T}_0$ (b) $\mathcal{T}_1$ (c) $\mathcal{T}_2$

Fig. 1. 3-level nested macrogrids.



(a) S_0 (b) S_1 (c) S_2

Fig. 2. 3-level hierarchy of macrogrids.

where the coefficients c_j satisfy $\sum_{j=1}^{n} c_j = 1$ $(k = 1, 2)$. We refer to the lth macro-grid interpolation of Galerkin approximations as $I_l^x(\bar{N}_k^i) = \sum_{j=1}^{n} c_j \bar{N}_k^i(x_j, \cdot)$. We solve the following problem: Find $\bar{N}_1^{i}{}^{c}(x, \cdot), \bar{N}_2^{i}{}^{c}(x, \cdot) \in \mathcal{V}_{L-l}$ such as

$$
\begin{aligned}
&B(x; (\bar{N}_1^{i}{}^{c}, \bar{N}_2^{i}{}^{c}), (\phi_1, \phi_2)) \\
&= -\sum_{j=1}^{n} c_j \int_Y (\kappa_1(x, y) - \kappa_1(x_j, y))\nabla_y \bar{N}_1^i(x_j, y) \cdot \nabla_y \phi_1(y) dy - \sum_{j=1}^{n} c_j \int_Y (\kappa_1(x, y) - \kappa_1(x_j, y)) e^i \cdot \nabla_y \phi_1(y) dy \\
&\quad - \sum_{j=1}^{n} c_j \int_Y (\kappa_2(x, y) - \kappa_2(x_j, y))\nabla_y \bar{N}_2^i(x_j, y) \cdot \nabla_y \phi_2(y) dy - \sum_{j=1}^{n} c_j \int_Y (\kappa_2(x, y) - \kappa_2(x_j, y)) e^i \cdot \nabla_y \phi_2(y) dy \\
&\quad + \sum_{j=1}^{n} c_j \int_Y (Q(x_j, y) - Q(x, y))(\bar{N}_1^i(x_j, y) - \bar{N}_2^i(x_j, y))(\phi_1(y) - \phi_2(y)) dy,
\end{aligned}
\tag{3.2}
$$

for all $\phi_1, \phi_2 \in \mathcal{V}_{L-l}$. Note that right-hand side data is all known since we have already computed $\{\bar{N}_k^i(x_j, \cdot)\}_{j=1}^n$ inductively using finer mesh spaces at macro-grid points in $(\bigcup_{l'<l} S_{l'})$. We let

$$
\bar{N}_k^i(x, \cdot) = \bar{N}_k^{i}{}^{c}(x, \cdot) + I_l^x(\bar{N}_k^i), \tag{3.3}
$$

be the FE approximation for $N_k^i(x, \cdot)$ where $k = 1, 2$. A main goal of this paper is to prove that the approximation (3.3) for $N_k^i(x, \cdot)$ has the same order of accuracy compared to the approximation we obtain by solving (2.8) using the finest FE space $\mathcal{V}_L$ at all macroscopic points. We also prove that we reduce the computation cost with the approximation (3.3) to the optimal level.

Remark. In the following, for simplicity, we use a simple 1-point interpolation to compute the correction term $(\bar{N}_1^{i}{}^{c}, \bar{N}_2^{i}{}^{c})$. More precisely, for $x \in S_l$ we choose $x' \in (\bigcup_{l'<l} S_{l'})$ such that $\mathrm{dist}(x, x') < O(H2^{-l})$. We let

$$
I_l^x(\bar{N}_k^i) = \bar{N}_k^i(x', \cdot), \quad k = 1, 2
$$

be the macro-grid interpolation. The FE approximation is

$$\bar{N}_k^i(x,\cdot) = \bar{N_k^i}^c(x,\cdot) + \bar{N}_k^i(x',\cdot), \quad k = 1, 2.$$

Remark. Note that as the level l goes higher, we use coarser FE spaces for the corresponding finer macro grids. This balance guarantees that although we use coarser FE spaces, the FE error is still optimal, but with much less computation cost.

3.2. Error estimates

We require that the coefficients κ_i and Q satisfy Assumption 3.1 and (2.1). We prove that the hierarchical method achieves the same order of accuracy as the full solve. For simplicity, we consider 1-point interpolation for our proof; the proof for the general case is similar.

Lemma 3.1. *There exists a positive number C such that $|||(N_1^i(x,\cdot), N_2^i(x,\cdot))||| \le C$ for all $x \in \Omega$.*

Proof. From (2.8), we obtain

$$\begin{aligned}
&B(x; (N_1^i(x,\cdot), N_2^i(x,\cdot)), (N_1^i(x,\cdot), N_2^i(x,\cdot)))\\
&= -\int_Y \kappa_1(x,y)e^i\cdot\nabla_y N_1^i(x,y)dy - \int_Y \kappa_2(x,y)e^i\cdot\nabla_y N_2^i(x,y)dy.
\end{aligned}$$

Using the uniform coercivity of the bilinear form $B(x;\cdot,\cdot)$ with respect to x, we get

$$C|||(N_1^i(x,\cdot), N_2^i(x,\cdot))||| \le (\|\nabla_y N_1^i(x,\cdot)\|_{L^2(Y)} + \|\nabla_y N_2^i(x,\cdot)\|_{L^2(Y)})$$

for $C > 0$. From this we get the conclusion. □

Let $N_k^{ic}(x,\cdot) = N_k^i(x,\cdot) - N_k^i(x',\cdot)$. We have that $(N_1^{ic}(x,\cdot), N_2^{ic}(x,\cdot)) \in W$ satisfies

$$\begin{aligned}
&B(x; (N_1^{ic}, N_2^{ic}), (\phi_1,\phi_2))\\
&= -\int_Y (\kappa_1(x,y)-\kappa_1(x',y))\nabla_y N_1^i(x',y)\cdot\nabla_y\phi_1(y)dy - \int_Y (\kappa_1(x,y)-\kappa_1(x',y))e^i\cdot\nabla_y\phi_1(y)dy\\
&\quad - \int_Y (\kappa_2(x,y)-\kappa_2(x',y))\nabla_y N_2^i(x',y)\cdot\nabla_y\phi_2(y)dy - \int_Y (\kappa_2(x,y)-\kappa_2(x',y))e^i\cdot\nabla_y\phi_2(y)dy\\
&\quad + \int_Y (Q(x',y)-Q(x,y))(N_1^i(x',y)-N_2^i(x',y))(\phi_1(y)-\phi_2(y))dy
\end{aligned}\tag{3.4}$$

$\forall(\phi_1,\phi_2)\in W$.

Proposition 3.2. *There exists $C > 0$ such that*

$$|||(N_1^{ic}(x,\cdot), N_2^{ic}(x,\cdot))||| \le C|x-x'|$$

for $x \in \mathcal{T}_L$.

Proof. From (3.4), for $(\phi_1,\phi_2) = (N_1^{ic}(x,\cdot), N_2^{ic}(x,\cdot)) \in W$ we have

$$\begin{aligned}
&B(x; (N_1^{ic}, N_2^{ic}), (N_1^{ic}, N_2^{ic}))\\
&= -\int_Y (\kappa_1(x,y)-\kappa_1(x',y))\nabla_y N_1^i(x',y)\cdot\nabla_y N_1^{ic}(x,y)dy - \int_Y (\kappa_1(x,y)-\kappa_1(x',y))e^i\cdot\nabla_y N_1^{ic}(x,y)dy\\
&\quad - \int_Y (\kappa_2(x,y)-\kappa_2(x',y))\nabla_y N_2^i(x',y)\cdot\nabla_y N_2^{ic}(x,y)dy - \int_Y (\kappa_2(x,y)-\kappa_2(x',y))e^i\cdot\nabla_y N_2^{ic}(x,y)dy\\
&\quad + \int_Y (Q(x',y)-Q(x,y))(N_1^i(x',y)-N_2^i(x',y))(N_1^{ic}(x,y)-N_2^{ic}(x,y))dy.
\end{aligned}$$

As $\nabla_y N_1^i(x',\cdot)$ and $\nabla_y N_2^i(x',\cdot)$ are uniformly bounded in $L^2(Y)$ with respect to $x\in\Omega$ by Lemma 3.1. From Assumption 3.1 we have

$$\begin{aligned}
&|||(N_1^{ic}(x,\cdot), N_2^{ic}(x,\cdot))|||^2\\
&\qquad \le C|x-x'|(\|\nabla_y N_1^{ic}(x,\cdot)\|_{L^2(Y)} + \|\nabla_y N_2^{ic}(x,\cdot)\|_{L^2(Y)} + \|N_2^{ic}(x,\cdot)-N_1^{ic}(x,\cdot)\|_{L^2(Y)}).
\end{aligned}$$

Thus

$$|||(N_1^{ic}(x,\cdot), N_2^{ic}(x,\cdot))||| \le C|x-x'| \tag{3.5}$$

where the constant C is independent of x. □

Lemma 3.3. *There is a positive constant C such that $\|\Delta_y N_1^i(x,\cdot)\|_{L^2(Y)} + \|\Delta_y N_2^i(x,\cdot)\|_{L^2(Y)} \le C$ for all $x \in \Omega$.*

Proof. We rewrite cell problem (2.4) as

$$\kappa_1 \Delta_y N_1^i + \nabla_y \kappa_1 \cdot \nabla_y N_1^i + \mathrm{div}_y(\kappa_1 e^i) + Q(x,y)(N_2^i - N_1^i) = 0$$

$$\kappa_2 \Delta_y N_2^i + \nabla_y \kappa_2 \cdot \nabla_y N_2^i + \mathrm{div}_y(\kappa_2 e^i) + Q(x,y)(N_1^i - N_2^i) = 0.$$

Rearranging these equations, we have,

$$\Delta_y N_1^i = -\frac{1}{\kappa_1}(\nabla_y \kappa_1 \cdot \nabla_y N_1^i + \mathrm{div}_y(\kappa_1 e^i) + Q(x,y)(N_2^i - N_1^i))$$

$$\Delta_y N_2^i = -\frac{1}{\kappa_2}(\nabla_y \kappa_2 \cdot \nabla_y N_2^i + \mathrm{div}_y(\kappa_2 e^i) + Q(x,y)(N_1^i - N_2^i)).$$

By the uniform boundedness of $|||(N_1^i(x,\cdot), N_2^i(x,\cdot))|||$ with respect to x and Lemma 3.1, we deduce that $\|\Delta_y N_1^i(x,\cdot)\|_{L^2(Y)}$ and $\|\Delta_y N_2^i(x,\cdot)\|_{L^2(Y)}$ are uniformly bounded for all $x \in \Omega$. □

Lemma 3.4. *There exists a positive constant C such that*

$$\|\Delta_y N_1^{i\,c}(x,\cdot)\|_{L^2(Y)} \le C|x - x'|, \qquad \|\Delta_y N_2^{i\,c}(x,\cdot)\|_{L^2(Y)} < C|x - x'|$$

for all $x \in \mathcal{T}_L$.

Proof. From (3.4), we have

$$\begin{aligned}
\kappa_1(x,y)\Delta_y N_1^{i\,c}(x,y) + \nabla_y \kappa_1(x,y) \cdot \nabla_y N_1^{i\,c}(x,y) &= -Q(x,y)(N_2^{i\,c}(x,y) - N_1^{i\,c}(x,y)) \\
&\quad - \nabla_y(\kappa_1(x,y) - \kappa_1(x',y)) \cdot \nabla_y N_1^i(x',y) - (\kappa_1(x,y) - \kappa_1(x',y))\Delta_y N_1^i(x',y) \\
&\quad - \mathrm{div}_y(\kappa_1(x,y) - \kappa_1(x',y)e^i) + (Q(x',y) - Q(x,y))(N_2^i(x',y) - N_1^i(x',y)),
\end{aligned}$$

$$\begin{aligned}
\kappa_2(x,y)\Delta_y N_2^{i\,c}(x,y) + \nabla_y \kappa_2(x,y) \cdot \nabla_y N_2^{i\,c}(x,y) &= -Q(x,y)(N_1^{i\,c}(x,y) - N_2^{i\,c}(x,y)) \\
&\quad - \nabla_y(\kappa_2(x,y) - \kappa_2(x',y)) \cdot \nabla_y N_2^i(x',y) - (\kappa_2(x,y) - \kappa_2(x',y))\Delta_y N_2^i(x',y) \\
&\quad - \mathrm{div}_y(\kappa_2(x,y) - \kappa_2(x',y)e^i) + (Q(x',y) - Q(x,y))(N_1^i(x',y) - N_2^i(x',y)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta_y N_1^{i\,c}(x,y) = \frac{1}{\kappa_1}\{&-\nabla_y \kappa_1(x,y) \cdot \nabla_y N_1^{i\,c}(x,y) - Q(x,y)(N_2^{i\,c}(x,y) - N_1^{i\,c}(x,y)) \\
&- \nabla_y(\kappa_1(x,y) - \kappa_1(x',y)) \cdot \nabla_y N_1^i(x',y) - (\kappa_1(x,y) - \kappa_1(x',y))\Delta_y N_1^i(x',y) \\
&- \mathrm{div}_y(\kappa_1(x,y) - \kappa_1(x',y)e^i) + (Q(x',y) - Q(x,y))(N_2^i(x',y) - N_1^i(x',y))\},
\end{aligned}$$

$$\begin{aligned}
\Delta_y N_2^{i\,c}(x,y) = \frac{1}{\kappa_2}\{&-\nabla_y \kappa_2(x,y) \cdot \nabla_y N_2^{i\,c}(x,y) - Q(x,y)(N_1^{i\,c}(x,y) - N_2^{i\,c}(x,y)) \\
&- \nabla_y(\kappa_2(x,y) - \kappa_2(x',y)) \cdot \nabla_y N_2^i(x',y) - (\kappa_2(x,y) - \kappa_2(x',y))\Delta_y N_2^i(x',y) \\
&- \mathrm{div}_y(\kappa_2(x,y) - \kappa_2(x',y)e^i) + (Q(x',y) - Q(x,y))(N_1^i(x',y) - N_2^i(x',y))\}.
\end{aligned}$$

From Lemma 3.1 and Proposition 3.2, we have

$$\|\Delta_y N_1^{i\,c}(x,\cdot)\|_{L^2(Y)}, \|\Delta_y N_2^{i\,c}(x,\cdot)\|_{L^2(Y)} \le C|x - x'|.$$

for some constant $C > 0$. □

We choose $(N_1^{i\,c}, N_2^{i\,c}) \in W$ such that

$$\int_Y (N_1^{i\,c} + N_2^{i\,c})dy = 0.$$

We then have

Lemma 3.5. *There is a positive constant C such that $\|N_1^{i\,c}(x,\cdot)\|_{L^2(Y)} \le C|x - x'|$ and $\|N_2^{i\,c}(x,\cdot)\|_{L^2(Y)} \le C|x - x'|$ for all $x \in \mathcal{T}_L$.*

Proof. We note that

$$2(\|N_1^{i\,c}\|^2_{L^2(Y)} + \|N_2^{i\,c}\|^2_{L^2(Y)}) = \|N_1^{i\,c} + N_2^{i\,c}\|^2_{L^2(Y)} + \|N_1^{i\,c} - N_2^{i\,c}\|^2_{L^2(Y)}. \tag{3.6}$$

Since $\int_Y (N_1^{i\,c} + N_2^{i\,c})dy = 0$, by Poincare inequality, and (3.5), the following inequalities hold.

$$\|N_1^{i\,c} + N_2^{i\,c}\|_{L^2(Y)} \le C\|\nabla_y(N_1^{i\,c} + N_2^{i\,c})\|_{L^2(Y)} \le C(\|\nabla_y N_1^{i\,c}\|_{L^2(Y)} + \|\nabla_y N_2^{i\,c}\|_{L^2(Y)}) \le C|x-x'|$$

And then by (3.6),

$$2(\|N_1^{i\,c}\|^2_{L^2(Y)} + \|N_2^{i\,c}\|^2_{L^2(Y)}) \le C|x-x'|^2. \quad \square$$

Proposition 3.6. *There is a constant $C > 0$ such that $\|N_1^{i\,c}\|_{H^2(Y)} \le C|x-x'|$ and $\|N_2^{i\,c}\|_{H^2(Y)} \le C|x-x'|$ for all $x \in \mathcal{T}_L$.*

Proof. Let $\omega \subset \mathbb{R}^d$ be a domain such that $Y \subset \omega$. Let $\phi \in \mathcal{C}_0^\infty(\omega)$ be such that $\phi = 1$ in Y. We have

$$\Delta_y(\phi N_1^{i\,c}) = \Delta_y\phi N_1^{i\,c} + 2\nabla\phi\cdot\nabla N_1^{i\,c} + \phi\Delta_y N_1^{i\,c}.$$

Since $\phi N_1^{i\,c} = 0$ on $\partial\omega$, applying elliptic regularity, we have

$$\|N_1^{i\,c}\|_{H^2(Y)} \le \|\phi N_1^{i\,c}\|_{H^2(\omega)} \le \|\Delta_y\phi N_1^{i\,c} + 2\nabla_y\phi\cdot\nabla_y N_1^{i\,c} + \phi\Delta_y N_1^{i\,c}\|_{L^2(\omega)}. \tag{3.7}$$

By Proposition 3.2, Lemmas 3.4 and 3.5, and the Y-periodicity of $N_1^{i\,c}$,

$$\|N_1^{i\,c}(x,\cdot)\|_{L^2(\omega)} \le C|x-x'|,\ \|\nabla_y N_1^{i\,c}(x,\cdot)\|_{L^2(\omega)} \le C|x-x'|,\ \|\Delta_y N_1^{i\,c}(x,\cdot)\|_{L^2(\omega)} \le C|x-x'|$$

for all $x \in \mathcal{T}_L$. Then from (3.7), $\|N_1^{i\,c}\|_{H^2(Y)} \le C|x-x'|$. Similarly, $\|N_2^{i\,c}\|_{H^2(Y)} \le C|x-x'|$ for $C > 0$. $\square$

We consider the problem: Find $\bar{\bar{N}}_1^{i\,c}(x,y) \in \mathcal{V}_{L-l}$ and $\bar{\bar{N}}_2^{i\,c}(x,y) \in \mathcal{V}_{L-l}$ such that

$$\begin{aligned}
&B(x;(\bar{\bar{N}}_1^{i\,c}, \bar{\bar{N}}_2^{i\,c}), (\phi_1,\phi_2))\\
&= -\int_Y(\kappa_1(x,y)-\kappa_1(x',y))\nabla_y N_1^i(x',y)\cdot\nabla_y\phi_1(y)dy - \int_Y(\kappa_1(x,y)-\kappa_1(x',y))e^i\cdot\nabla_y\phi_1(y)dy\\
&\quad -\int_Y(\kappa_2(x,y)-\kappa_2(x',y))\nabla_y N_2^i(x',y)\cdot\nabla_y\phi_2(y)dy - \int_Y(\kappa_2(x,y)-\kappa_2(x',y))e^i\cdot\nabla_y\phi_2(y)dy\\
&\quad +\int_Y(Q(x',y)-Q(x,y))(N_1^i(x',y)-N_2^i(x',y))(\phi_1(y)-\phi_2(y))dy,
\end{aligned} \tag{3.8}$$

for all $\phi_1 \in \mathcal{V}_{L-l}$ and $\phi_2 \in \mathcal{V}_{L-l}$. This is the FE approximation of (3.4). We then have the following result.

Lemma 3.7. *There is a positive constant C^0 such that*

$$|||(N_1^{i\,c}(x,\cdot) - \bar{\bar{N}}_1^{i\,c}(x,\cdot), N_2^{i\,c}(x,\cdot) - \bar{\bar{N}}_2^{i\,c}(x,\cdot))||| \le C^0 2^{-L}.$$

Proof. It follows from Cea's Lemma, Proposition 3.6 and (3.1) that

$$|||(N_1^{i\,c} - \bar{\bar{N}}_1^{i\,c}, N_2^{i\,c} - \bar{\bar{N}}_2^{i\,c})||| \le C2^{-(L-l)}(\|N_1^{i\,c}\|_{H^2(Y)} + \|N_2^{i\,c}\|_{H^2(Y)}) \le C2^{-(L-l)}|x-x'| \le C^0 2^{-L}. \quad \square$$

Proposition 3.8. *There is a constant $c_l > 0$ which only depends on the level S_l of $x \in \mathcal{T}_L$ such that*

$$|||(\bar{N}_1^i(x,\cdot) - N_1^i(x,\cdot), \bar{N}_2^i(x,\cdot) - N_2^i(x,\cdot))||| \le c_l 2^{-L}.$$

Proof. We will prove the proposition by induction. The conclusion holds for $l = 0$. We assume that for all $x' \in S_{l'}$ where $l' \le l-1$.

$$|||(\bar{N}_1^i(x',\cdot) - N_1^i(x',\cdot), \bar{N}_2^i(x',\cdot) - N_2^i(x',\cdot))||| \le c_{l-1} 2^{-L}. \tag{3.9}$$

From (3.2) and (3.8), we have

$$\begin{aligned}
&B(x;(\bar{N}_1^{i\,c}(x,\cdot) - \bar{\bar{N}}_1^{i\,c}(x,\cdot), \bar{N}_2^{i\,c}(x,\cdot) - \bar{\bar{N}}_2^{i\,c}(x,\cdot)), (\phi_1,\phi_2))\\
&= -\int_Y(\kappa_1(x,y)-\kappa_1(x',y))\nabla_y(\bar{N}_1^i(x',y) - N_1^i(x',y))\cdot\nabla_y\phi_1(y)dy\\
&\quad -\int_Y(\kappa_2(x,y)-\kappa_2(x',y))\nabla_y(\bar{N}_2^i(x',y) - N_2^i(x',y))\cdot\nabla_y\phi_2(y)dy\\
&\quad +\int_Y(Q(x',y)-Q(x,y))((\bar{N}_1^i(x',y) - \bar{N}_2^i(x',y)) - (N_1^i(x',y) - N_2^i(x',y)))(\phi_1(y)-\phi_2(y))dy
\end{aligned}$$

for all $\phi_1 \in \mathcal{V}_{L-l}$ and $\phi_2 \in \mathcal{V}_{L-l}$. From Assumption 3.1 and the induction hypothesis, we have

$$\| \! | (\bar{N}_1^{i\,c}(x,\cdot) - \bar{\bar{N}}_1^{i\,c}(x,\cdot), \bar{N}_2^{i\,c}(x,\cdot) - \bar{\bar{N}}_2^{i\,c}(x,\cdot)) \| \! | \le \gamma c_{l-1} 2^{-L-l}. \tag{3.10}$$

where $\gamma > 0$ is independent of x and l. By Lemma 3.7 and (3.10),

$$\begin{aligned} \| \! | (N_1^{i\,c}(x,\cdot) - \bar{N}_1^{i\,c}(x,\cdot), N_2^{i\,c}(x,\cdot) - \bar{N}_2^{i\,c}(x,\cdot)) \| \! | &\le \| \! | (N_1^{i\,c}(x,\cdot) - \bar{\bar{N}}_1^{i\,c}(x,\cdot), N_2^{i\,c}(x,\cdot) - \bar{\bar{N}}_2^{i\,c}(x,\cdot)) \| \! | \\ &\quad + \| \! | (\bar{N}_1^{i\,c}(x,\cdot) - \bar{\bar{N}}_1^{i\,c}(x,\cdot), \bar{N}_2^{i\,c}(x,\cdot) - \bar{\bar{N}}_2^{i\,c}(x,\cdot)) \| \! | \le C^0 2^{-L} + \gamma c_{l-1} 2^{-L-l}. \end{aligned} \tag{3.11}$$

Using $\bar{N}_k^i(x,y) = \bar{N}_k^{i\,c}(x,y) + \bar{N}_k^i(x',y)$, We have

$$\| \! | (N_1^i(x,\cdot) - \bar{N}_1^i(x,\cdot), N_2^i(x,\cdot) - \bar{N}_2^i(x,\cdot)) \| \! | \le c_l 2^{-L},$$

where

$$c_l = \gamma c_{l-1} 2^{-l} + c_{l-1} + C^0. \quad \square \tag{3.12}$$

Theorem 3.9. *Under Assumption 3.1 and the uniform boundedness of $\kappa_i(x,y)$ and $Q(x,y)$, there is a positive constant C_* which depends only on the functions κ_1, κ_2 and Q so that,*

$$\| \! | (N_1^i(x,\cdot) - \bar{N}_1^i(x,\cdot), N_2^i(x,\cdot) - \bar{N}_2^i(x,\cdot)) \| \! | \le C_* l 2^{-L} \tag{3.13}$$

for $x \in S_l$.

Proof. We let $\bar{l}$ be an integer independent of L such that $l2^{-l} < \frac{1}{2\gamma}$ for $l > \bar{l}$. And let

$$C_* = \max\left\{ \max_{0 \le l \le \bar{l}} \left\{ \frac{c_l}{l} \right\}, 2C^0 \right\}, \tag{3.14}$$

where C^0 and c_l are the constants in Lemma 3.7 and Proposition 3.8. Now we prove

$$\| \! | (N_1^i(x,\cdot) - \bar{N}_1^i(x,\cdot), N_2^i(x,\cdot) - \bar{N}_2^i(x,\cdot)) \| \! | \le C_* l 2^{-L} \tag{3.15}$$

by induction. From (3.14), this holds for all $l \le \bar{l}$. Suppose that (3.15) holds for all $l' \le l$. Then from (3.12), we obtain

$$c_l \le ((l-1)C_* + \frac{1}{2\gamma}\gamma C_* + \frac{C_*}{2}) = C_* l. \qquad \square \tag{3.16}$$

Theorem 3.10. *The total number of degrees of freedom required to solve (2.8) for all points in $S_0, S_1, \ldots, S_L$ is $\mathcal{O}((L+1)2^{dL})$ for the hierarchical solve while it is $\mathcal{O}((2^{dL})^2)$ in the full solve where cell problems are solved with the finest mesh level at all macrogrid points.*

Proof. Since the number of macroscopic points in S_l is $\mathcal{O}(2^{dl})$, and the space $\mathcal{V}_{L-l}$ is of dimension $\mathcal{O}(2^{d(L-l)})$, the total number of degrees of freedom for solving (2.8) for all points in S_l is $\mathcal{O}(2^{dl})\mathcal{O}(2^{d(L-l)}) = \mathcal{O}(2^{dL})$. Therefore, the total number of degrees of freedom required to solve (2.8) for all points in $S_0, S_1, \ldots, S_L$ is $\mathcal{O}((L+1)2^{dL})$. $\square$

4. Numerical example

In this section, we apply the hierarchical finite element algorithm to a numerical example for computing the effective coefficients of a multiscale multi-continuum system at a dense network of macrogrid points. To show the accuracy of the algorithm, we compare the results to the approximations to the effective coefficients obtained from solving the cell problems using the finest meshes at all macroscopic points.

4.1. Numerical implementation

We let $\Omega = [0,1]^2$ be the macroscopic domain and $Y = [0,1]^2$ be the unit cell. We consider the locally periodic coefficients

$$\begin{aligned} \kappa_1(x_1, y_1, y_2) &= (2 - ax_1)\cos(2\pi y_1)\sin(2\pi y_2) + 3 \\ \kappa_2(x_1, y_1, y_2) &= (2 - ax_1)\sin(2\pi y_1)\cos(2\pi y_2) + 3 \\ Q(x_1, y_1, y_2) &= (1 + ax_1)\sin(2\pi y_1)\sin(2\pi y_2) + 3 \end{aligned}$$

where the constant a is chosen below. We use 4 square meshes in $[0,1]^2$ to construct a nested sequence of FE spaces, $\{\mathcal{V}_{3-l}\}_{l=0}^3$ so that the mesh size of each space is $h_l = 2^l \cdot 2^{-4}$ for $l = 0, 1, 2, 3$. Since κ_1, κ_2 and Q are independent of x_2, we only consider 1-dimensional macrogrids in $[0,1]$. The nested macrogrids $\{\mathcal{T}_l\}_{l=0}^L \subset [0,1]$ and the subsequent macrogrid

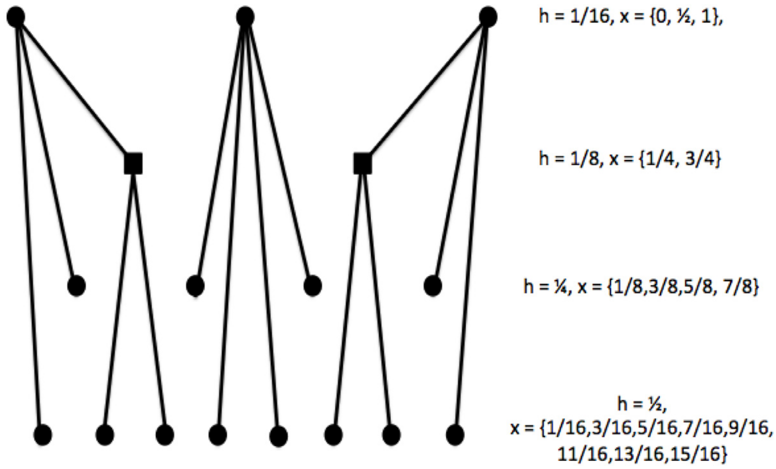


Fig. 3. The hierarchy of one dimensional macrogrids and corresponding mesh size of FE spaces for 1-pt interpolation method. The lines indicate correction relations. The squares indicate the points at which the solutions are corrected with the lower level solutions and used once more to correct upper level solutions.

hierarchy, $\{S_l\}_{l=0}^3$ are constructed as follows. We first let $\mathcal{T}_0 = S_0 = \{0, \frac{1}{2}, 1\}$. Considering that our macrogrids have grid spacing $H2^{-l}$ for $l = 0, 1, 2, 3$, where $H = \frac{1}{2}$ in this case, we have following hierarchy of macrogrids.

$$S_0 = \{0, \frac{1}{2}, 1\}, \qquad S_1 = \{\frac{1}{4}, \frac{3}{4}\}, \;\; S_2 = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}, \qquad S_3 = \{\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}\}$$

Fig. 3 indicates how these macrogrids and the approximation spaces are related in numerical implementation.

We implement the algorithm as follows. For $x' \in S_0 = \{0, \frac{1}{2}, 1\}$, we solve (2.8) for $\bar{N}_1^i(x', \cdot)$, $\bar{N}_2^i(x', \cdot) \in \mathcal{V}_3$, for all $\phi_1, \ \phi_2 \in \mathcal{V}_3$ by the standard Galerkin FEM. We then use a simple 1-point interpolation to compute the correction terms. That is, for $x \in S_l$ we choose $x' \in (\bigcup_{k<l} S_k)$ such that $|x' - x| \le 2^{-l}$. We let the lth macrogrid interpolation be

$$I_l^x(\bar{N}_k^i) = \bar{N}_k^i(x', \cdot), \quad (k = 1, 2).$$

We find $\bar{N}_1^{i,c}(x, y)$ and $\bar{N}_2^{i,c}(x, y)$ in $\mathcal{V}_{L-l}$ such that

$$\begin{aligned} &\int_Y \kappa_1(x, y)\nabla_y \bar{N}_1^{i,c}(x, y) \cdot \nabla_y \phi_1(y)dy - \int_Y Q(x, y)(\bar{N}_2^{i,c}(x, y) - \bar{N}_1^{i,c}(x, y))\phi_1(y)dy \\ &= -\int_Y \kappa_1(x, y)\nabla_y \bar{N}_1^i(x', y) \cdot \nabla_y \phi_1(y)dy - \int_Y \kappa_1(x, y)e^i \cdot \nabla_y \phi_1(y)dy \\ &\quad + \int_Y Q(x, y)(\bar{N}_2^i(x', y) - \bar{N}_1^i(x', y))\phi_1(y)dy, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\int_Y \kappa_2(x, y)\nabla_y \bar{N}_2^{i,c}(x, y) \cdot \nabla_y \phi_2(y)dy - \int_Y Q(x, y)(\bar{N}_1^{i,c}(x, y) - \bar{N}_2^{i,c}(x, y))\phi_2(y)dy \\ &= -\int_Y \kappa_2(x, y)\nabla_y \bar{N}_2^i(x', y) \cdot \nabla_y \phi_2(y)dy - \int_Y \kappa_2(x, y)e^i \cdot \nabla_y \phi_2(y)dy \\ &\quad + \int_Y Q(x, y)(\bar{N}_1^i(x', y) - \bar{N}_2^i(x', y))\phi_2(y)dy, \end{aligned} \tag{4.2}$$

for $\forall \phi_1, \phi_2 \in \mathcal{V}_{L-l}$. We let

$$\bar{N}_k^i(x, \cdot) = \bar{N}_k^i(x', \cdot) + \bar{N}_k^{i,c}(x, \cdot), \quad (k = 1, 2)$$

be the approximation to $N_k^i(x, \cdot)$. We continue inductively. For example, for $x = \frac{1}{2} \in S_0$, we compute $\bar{N}_1^i(\frac{1}{2}, \cdot)$, $\bar{N}_2^i(\frac{1}{2}, \cdot)$ using the standard Galerkin FEM. Then for $\frac{3}{8} \in S_1$, we find the correction terms $\bar{N}_1^{i,c}(\frac{3}{8}, \cdot)$, $\bar{N}_2^{i,c}(\frac{3}{8}, \cdot) \in \mathcal{V}_{L-1}$ that satisfy (4.1) and (4.2), where $x' = \frac{1}{2}$. And we let the solutions at $x = \frac{3}{8}$ be

$$\bar{N}_k^i(\frac{3}{8}, y) = \bar{N}_k^i(\frac{1}{2}, y) + \bar{N}_k^{i,c}(\frac{3}{8}, y), \quad (k = 1, 2).$$

We continue this procedure based on Fig. 3.

Table 1
a = 1, the effective coefficients $\kappa^*_{111}(x_1)$ and $\kappa^*_{211}(x_1)$ computed by full mesh reference and hierarchical solve along with percentage relative errors between those.

x_1	$\kappa^*_{111}(x_1)$			$\kappa^*_{211}(x_1)$		
	Full solve	Hierarchical solve	Errors (%)	Full solve	Hierarchical solve	Errors (%)
0	2.8211	2.8211	0.0000	2.8304	2.8304	0.0000
$\frac{1}{16}$	2.8333	2.8267	0.2312	2.8413	2.8397	0.0582
$\frac{1}{8}$	2.8448	2.8408	0.1414	2.8518	2.8491	0.0968
$\frac{3}{16}$	2.8559	2.8593	0.1184	2.8619	2.8624	0.0159
$\frac{1}{4}$	2.8664	2.8641	0.0803	2.8716	2.8707	0.0322
$\frac{5}{16}$	2.8765	2.8690	0.2605	2.8809	2.8787	0.0764
$\frac{3}{8}$	2.8860	2.8887	0.0933	2.8898	2.8919	0.0712
$\frac{7}{16}$	2.8952	2.8998	0.1608	2.8983	2.8995	0.0390
$\frac{1}{2}$	2.9038	2.9038	0.0000	2.9065	2.9065	0.0000
$\frac{9}{16}$	2.9120	2.9078	0.1450	2.9143	2.9133	0.0349
$\frac{5}{8}$	2.9199	2.9178	0.0706	2.9217	2.9201	0.0564
$\frac{11}{16}$	2.9273	2.9319	0.1572	2.9288	2.9303	0.0496
$\frac{3}{4}$	2.9343	2.9351	0.0288	2.9355	2.9361	0.0180
$\frac{13}{16}$	2.9409	2.9383	0.0857	2.9419	2.9416	0.0093
$\frac{7}{8}$	2.9471	2.9485	0.0476	2.9479	2.9492	0.0414
$\frac{15}{16}$	2.9530	2.9558	0.0979	2.9536	2.9543	0.0229
1	2.9584	2.9584	0.0000	2.9598	2.9590	0.0000

Tables 1 and 2 indicate κ^*_{111} and κ^*_{211} obtained by both the hierarchical solve and the full solve where the finest mesh is used for all cell problems, at each x_1 and the relative errors between them, where relative errors are calculated by $\frac{100|\kappa^*_{full}-\kappa^*_{hier}|}{\kappa^*_{full}}$ with obvious notations for $a = 1$ and $a = 0.1$ respectively. The results show clearly that the effective coefficients obtained from the hierarchical algorithm are very closed to the reference effective coefficients. We can see from the tables that relatively large errors occur at the highest level macroscopic points where more than one layer of corrections is performed, i.e. the corrector itself is corrected by the solution at a macroscopic point belonging to a lower level. We note that the error for the case $a = 0.1$ is much smaller as the change of κ_i in x is much smaller. That is, large Lipschitz constants in Assumption 3.1 tend to result in large errors. The results in Tables 1 and 2 are obtained when only one corrector point is employed. If we use more corrector points, the error can be reduced significantly. In Table 3 we show the relative errors, in comparison to the coefficients obtained from the full solve where the finest mesh is used for all the cell problems, for the effective coefficients obtained from the hierarchical solve for the two cases where one-point and two-point interpolations are used. The table shows that the result can be improved by employing two-point interpolation.

5. Proof of homogenization convergence

In this section, we prove rigorously the homogenization convergence, i.e. the convergence of the solution of the two-scale equation (2.2) to the solution of the homogenized equation (2.5). Throughout this section, we denote the spaces $L^2(\Omega)$ and $H^1_0(\Omega)$ as H and V respectively. We recall the two-scale multi-continuum system

$$C^\epsilon_{11}\frac{\partial u^\epsilon_1(t,x)}{\partial t} - \mathrm{div}(\kappa^\epsilon_1(x)\nabla u^\epsilon_1(t,x)) - \frac{1}{\epsilon^2}Q^\epsilon(x)(u^\epsilon_2(t,x) - u^\epsilon_1(t,x)) = q, \tag{5.1}$$

$$C^\epsilon_{22}\frac{\partial u^\epsilon_2(t,x)}{\partial t} - \mathrm{div}(\kappa^\epsilon_2(x)\nabla u^\epsilon_2(t,x)) - \frac{1}{\epsilon^2}Q^\epsilon(x)(u^\epsilon_1(t,x) - u^\epsilon_2(t,x)) = q. \tag{5.2}$$

We have the following theorem.

Lemma 5.1. *The solutions $(u^\epsilon_1, u^\epsilon_2)$ of (5.1) and (5.2) are uniformly bounded in $L^\infty(0,T;H)$ and $L^2(0,T;V)$.*

Proof. Multiplying ϕ_1 and $\phi_2 \in V$ to (5.1) and (5.2) respectively and integrating over Ω, one has

$$\begin{aligned}
&\int_\Omega C^\epsilon_{11}\frac{\partial u^\epsilon_1}{\partial t}\phi_1 dx + \int_\Omega \kappa^\epsilon_1\nabla u^\epsilon_1\cdot\nabla\phi_1 dx - \int_\Omega \frac{1}{\epsilon^2}Q^\epsilon(u^\epsilon_2 - u^\epsilon_1)\phi_1 dx = \int_\Omega q\phi_1 dx,\\
&\int_\Omega C^\epsilon_{22}\frac{\partial u^\epsilon_2}{\partial t}\phi_2 dx + \int_\Omega \kappa^\epsilon_2\nabla u^\epsilon_2\cdot\nabla\phi_2 dx - \int_\Omega \frac{1}{\epsilon^2}Q^\epsilon(u^\epsilon_1 - u^\epsilon_2)\phi_2 dx = \int_\Omega q\phi_2 dx.
\end{aligned} \tag{5.3}$$

Table 2
a = .1, the effective coefficients $\kappa^*_{111}(x_1)$ and $\kappa^*_{211}(x_1)$ computed by full mesh reference and hierarchical solve along with percentage relative errors between those.

x_1	$\kappa^*_{111}(x_1)$			$\kappa^*_{211}(x_1)$		
	Full solve	Hierarchical solve	Errors (%)	Full solve	Hierarchical solve	Errors (%)
0	2.8210	2.8211	0.0000	2.8304	2.8304	0.0000
$\frac{1}{16}$	2.8224	2.8217	0.0241	2.8315	2.8314	0.0061
$\frac{1}{8}$	2.8236	2.8231	0.0161	2.8326	2.8323	0.0107
$\frac{3}{16}$	2.8248	2.8252	0.0125	2.8337	2.8338	0.0020
$\frac{1}{4}$	2.8261	2.8257	0.0112	2.8348	2.8347	0.0040
$\frac{5}{16}$	2.8273	2.8263	0.0347	2.8359	2.8356	0.0099
$\frac{3}{8}$	2.8285	2.8289	0.0154	2.8370	2.8373	0.0103
$\frac{7}{16}$	2.8297	2.8303	0.0232	2.8381	2.8383	0.0059
$\frac{1}{2}$	2.8309	2.8309	0.0000	2.8392	2.8392	0.0000
$\frac{9}{16}$	2.8321	2.8314	0.0230	2.8403	2.8401	0.0058
$\frac{5}{8}$	2.8333	2.8328	0.0150	2.8413	2.8410	0.0101
$\frac{11}{16}$	2.8345	2.8354	0.0327	2.8424	2.8427	0.0095
$\frac{3}{4}$	2.8356	2.8359	0.0100	2.8435	2.8436	0.0037
$\frac{13}{16}$	2.8368	2.8364	0.0125	2.8445	2.8445	0.0019
$\frac{7}{8}$	2.8380	2.8384	0.0144	2.8456	2.8459	0.0098
$\frac{15}{16}$	2.8391	2.8398	0.0222	2.8466	2.8468	0.0056
1	2.8403	2.8403	0.0000	2.8477	2.8477	0.0000

Summing these equations, we get

$$\begin{aligned}
&\int_\Omega \mathcal{C}^\epsilon_{11}\frac{\partial u^\epsilon_1(t)}{\partial t}\phi_1 dx + \int_\Omega \kappa^\epsilon_1 \nabla u^\epsilon_1(t)\cdot\nabla\phi_1 dx - \int_\Omega \frac{1}{\epsilon^2}Q^\epsilon(u^\epsilon_2(t)-u^\epsilon_1(t))\phi_1 dx\\
&+\int_\Omega \mathcal{C}^\epsilon_{22}\frac{\partial u^\epsilon_2}{\partial t}(t)\phi_2 dx + \int_\Omega \kappa^\epsilon_2 \nabla u^\epsilon_2(t)\cdot\nabla\phi_2 dx - \int_\Omega \frac{1}{\epsilon^2}Q^\epsilon(u^\epsilon_1(t)-u^\epsilon_2(t))\phi_2 dx\\
&\qquad = \int_Q q(t)\phi_1 dx + \int_\Omega q(t)\phi_2 dx
\end{aligned} \tag{5.4}$$

$\forall \phi_1,\phi_2 \in V$. Substituting u^ϵ_1 and u^ϵ_2 into ϕ_1 and ϕ_2 in (5.4) respectively, we have

$$\begin{aligned}
&\int_\Omega \mathcal{C}^\epsilon_{11}\frac{\partial u^\epsilon_1(t)}{\partial t}u^\epsilon_1(t)dx + \int_\Omega \mathcal{C}^\epsilon_{22}\frac{\partial u^\epsilon_2(t)}{\partial t}u^\epsilon_2(t)dx + \int_\Omega \kappa^\epsilon_1\nabla u^\epsilon_1(t)\cdot\nabla u^\epsilon_1(t)dx\\
&+\int_\Omega \kappa^\epsilon_2\nabla u^\epsilon_2(t)\cdot\nabla u^\epsilon_2(t)dx + \frac{1}{\epsilon^2}\int_\Omega Q^\epsilon(u^\epsilon_2(t)-u^\epsilon_1(t))^2 dx = \int_\Omega q u^\epsilon_1(t) + \int_\Omega q u^\epsilon_2(t)dx.
\end{aligned}$$

Integrating this equation over $(0,\tau)$, we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{11}|u^\epsilon_1(\tau,x)|^2 dx + \frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{22}|u^\epsilon_2(\tau,x)|^2 dx + \int_0^\tau\int_\Omega \kappa^\epsilon_1\nabla u^\epsilon_1\cdot\nabla u^\epsilon_1 dxdt\\
&+\int_0^\tau\int_\Omega \kappa^\epsilon_2\nabla u^\epsilon_2\cdot\nabla u^\epsilon_2 dxdt + \frac{1}{\epsilon^2}\int_0^\tau\int_\Omega Q^\epsilon(u^\epsilon_2-u^\epsilon_1)^2 dxdt\\
&=\int_0^\tau\int_\Omega q u^\epsilon_1 dxdt + \int_0^\tau\int_\Omega q u^\epsilon_2 dxdt + \frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{11}|u^\epsilon_1(0,x)|^2 dx + \frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{22}|u^\epsilon_2(0,x)|^2 dx.
\end{aligned} \tag{5.5}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{11}|u^\epsilon_1(\tau,x)|^2 dx + \frac{1}{2}\int_\Omega \mathcal{C}^\epsilon_{22}|u^\epsilon_2(\tau,x)|^2 dx + \int_0^\tau\int_\Omega \kappa^\epsilon_1\nabla u^\epsilon_1\cdot\nabla u^\epsilon_1 dxdt + \int_0^\tau\int_\Omega \kappa^\epsilon_2\nabla u^\epsilon_2\cdot\nabla u^\epsilon_2 dxdt\\
&\le c\int_0^\tau\int_\Omega |q|^2 dxdt + \delta\int_0^\tau\int_\Omega |u^\epsilon_1|^2 dxdt + c\int_0^\tau\int_\Omega |q|^2 dxdt\\
&\quad+\delta\int_0^\tau\int_\Omega |u^\epsilon_2|^2 dxdt + \int_\Omega |\mathcal{C}^\epsilon_{11}||u^\epsilon_1(0,x)|^2 dx + \int_\Omega |\mathcal{C}^\epsilon_{22}||u^\epsilon_2(0,x)|^2 dx.
\end{aligned}$$

Table 3
Percentage relative errors between full mesh reference solve and hierarchical solve when a = 1.

1-pt interpolation

x_1	Relative errors (%)	
	κ^*_{111}	κ^*_{211}
$\frac{1}{16}$	0.2312	0.0582
$\frac{1}{8}$	0.1414	0.0968
$\frac{3}{16}$	0.1184	0.0159
$\frac{1}{4}$	0.0803	0.0322
$\frac{5}{16}$	0.2605	0.0764
$\frac{3}{8}$	0.0933	0.0712
$\frac{7}{16}$	0.1608	0.0390
$\frac{9}{16}$	0.1450	0.0349
$\frac{5}{8}$	0.0706	0.0564
$\frac{11}{16}$	0.1572	0.0496
$\frac{3}{4}$	0.0288	0.0180
$\frac{13}{16}$	0.0857	0.0093
$\frac{7}{8}$	0.0476	0.0414
$\frac{15}{16}$	0.0979	0.0229

2-pt interpolation

x_1	Relative errors (%)	
	κ^*_{111}	κ^*_{211}
$\frac{1}{16}$	0.0072	0.0022
$\frac{1}{8}$	0.0091	0.0030
$\frac{3}{16}$	0.099	0.0026
$\frac{1}{4}$	0.0068	0.0013
$\frac{5}{16}$	0.0080	0.0021
$\frac{3}{8}$	0.0061	0.0020
$\frac{7}{16}$	0.0042	0.0013
$\frac{9}{16}$	0.0026	0.0008
$\frac{5}{8}$	0.0031	0.0011
$\frac{11}{16}$	0.0033	0.0009
$\frac{3}{4}$	0.0021	0.0004
$\frac{13}{16}$	0.0026	0.0007
$\frac{7}{8}$	0.0020	0.0007
$\frac{15}{16}$	0.0014	0.0004

Using the uniform boundedness from below of C^ϵ_{11} and C^ϵ_{22}, we have

$$c\|u^\epsilon_1(\tau,\cdot)\|^2_H + c\|u^\epsilon_2(\tau,\cdot)\|^2_H + \int_0^\tau\int_\Omega \kappa^\epsilon_1\nabla u^\epsilon_1\cdot\nabla u^\epsilon_1 dxdt + \int_0^\tau\int_\Omega \kappa^\epsilon_2\nabla u^\epsilon_2\cdot\nabla u^\epsilon_2 dxdt$$
$$\le c + \delta\int_0^T \|u^\epsilon_1(t,\cdot)\|^2_H dt + \delta\int_0^T \|u^\epsilon_2(t,\cdot)\|^2_H dt.$$

Choosing δ sufficiently small, we deduce that u^ϵ_1 and u^ϵ_2 are uniformly bounded in $L^\infty(0,T;H)$ and $L^2(0,T;V)$. □

Note that because of the 5th term of Eq. (5.5), $\lim_{\epsilon\to 0} u^\epsilon_1 = \lim_{\epsilon\to 0} u^\epsilon_2$. Thus, there exist subsequences of u^ϵ_1 and u^ϵ_2, which we still denote by u^ϵ_1 and u^ϵ_2, and u_0 such that

$$u^\epsilon_1, u^\epsilon_2 \rightharpoonup u_0 \text{ in } L^2(0,T;V).$$

Recall that $(N^i_1, N^i_2) \in W$ is the solution of cell problem.

$$\begin{aligned}&\mathrm{div}_y(\kappa_1(x,y)(e^i + \nabla_y N^i_1(x,y))) + Q(x,y)(N^i_2(x,y) - N^i_1(x,y)) = 0\\&\mathrm{div}_y(\kappa_2(x,y)(e^i + \nabla_y N^i_2(x,y))) + Q(x,y)(N^i_1(x,y) - N^i_2(x,y)) = 0.\end{aligned} \tag{5.6}$$

We assume that N_1^i and N_2^i are sufficiently smooth with respect to both x and y. Let $\omega_1(x) = \frac{x_i}{\epsilon} + N_1^i(x, \frac{x}{\epsilon})$ and $\omega_2(x) = \frac{x_i}{\epsilon} + N_2^i(x, \frac{x}{\epsilon})$. We define ω_1^ϵ and ω_2^ϵ as

$$\omega_1^\epsilon(x) = \epsilon\omega_1(x, \frac{x}{\epsilon}), \qquad \omega_2^\epsilon(x) = \epsilon\omega_2(x, \frac{x}{\epsilon}).$$

Assuming that κ_1, κ_2, N_1^i and N_2^i are sufficiently smooth, for all $\psi_1, \psi_2 \in V$ we have

$$\begin{aligned}
&-\int_\Omega \mathrm{div}(\kappa_1^\epsilon(x)\nabla\omega_1^\epsilon(x))\psi_1(x)dx - \frac{1}{\epsilon^2}\int_\Omega Q^\epsilon(x)(\omega_2^\epsilon(x) - \omega_1^\epsilon(x))\psi_1(x)dx \\
&= -\frac{1}{\epsilon}\int_\Omega \mathrm{div}_y(\kappa_1(x, \frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x, \frac{x}{\epsilon})))\psi_1(x)dx - \frac{1}{\epsilon}\int_\Omega Q(x, \frac{x}{\epsilon})(N_2^i(x, \frac{x}{\epsilon}) - N_1^i(x, \frac{x}{\epsilon}))\psi_1(x)dx \\
&\quad -\epsilon\int_\Omega \mathrm{div}_x(\kappa_1(x, \frac{x}{\epsilon})\nabla_x(N_1^i(x, \frac{x}{\epsilon})))\psi_1(x)dx - \int_\Omega \mathrm{div}_x(\kappa_1(x, \frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x, \frac{x}{\epsilon})))\psi_1(x)dx \\
&\quad - \int_\Omega \mathrm{div}_y(\kappa_1(x, \frac{x}{\epsilon})\nabla_x N_1^i(x, \frac{x}{\epsilon}))\psi_1(x)dx \\
&= -\epsilon\int_\Omega \mathrm{div}_x(\kappa_1(x, \frac{x}{\epsilon})\nabla_x N_1^i(x, \frac{x}{\epsilon}))\psi_1(x)dx - \int_\Omega \mathrm{div}_x(\kappa_1(x, \frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x, \frac{x}{\epsilon})))\psi_1(x)dx \\
&\quad - \int_\Omega \mathrm{div}_y(\kappa_1(x, \frac{x}{\epsilon})\nabla_x N_1^i(x, \frac{x}{\epsilon}))\psi_1(x)dx
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
&-\int_\Omega \mathrm{div}(\kappa_2^\epsilon(x)\nabla\omega_2^\epsilon(x))\psi_2(x)dx - \frac{1}{\epsilon^2}\int_\Omega Q^\epsilon(x)(\omega_1^\epsilon(x) - \omega_2^\epsilon(x))\psi_2(x)dx \\
&= -\frac{1}{\epsilon}\int_\Omega \mathrm{div}_y(\kappa_2(x, \frac{x}{\epsilon})(e^i + \nabla_y N_2^i(x, \frac{x}{\epsilon})))\psi_2(x)dx - \frac{1}{\epsilon}\int_\Omega Q(x, y)(N_1^i(x, \frac{x}{\epsilon}) - N_2^i(x, \frac{x}{\epsilon}))\psi_2(x)dx \\
&\quad -\epsilon\int_\Omega \mathrm{div}_x(\kappa_2(x, \frac{x}{\epsilon})\nabla_x(N_2^i(x, \frac{x}{\epsilon})))\psi_2(x)dx - \int_\Omega \mathrm{div}_x(\kappa_2(x, \frac{x}{\epsilon})(e^i + \nabla_y N_2^i(x, \frac{x}{\epsilon})))\psi_2(x)dx \\
&\quad - \int_\Omega \mathrm{div}_y(\kappa_2(x, \frac{x}{\epsilon})\nabla_x N_2^i(x, \frac{x}{\epsilon}))\psi_2(x)dx \\
&= -\epsilon\int_\Omega \mathrm{div}_x(\kappa_2(x, \frac{x}{\epsilon})\nabla_x N_2^i(x, \frac{x}{\epsilon}))\psi_2(x)dx - \int_\Omega \mathrm{div}_x(\kappa_2(x, \frac{x}{\epsilon})(e^i + \nabla_y N_2^i(x, \frac{x}{\epsilon})))\psi_2(x)dx \\
&\quad - \int_\Omega \mathrm{div}_y(\kappa_2(x, \frac{x}{\epsilon})\nabla_x N_2^i(x, \frac{x}{\epsilon}))\psi_2(x)dx
\end{aligned} \tag{5.8}$$

due to (5.6). Let $\phi_1(x) = \phi(x)\omega_1^\epsilon(x)$, $\phi_2(x) = \phi(x)\omega_2^\epsilon(x)$ where $\phi \in \mathcal{C}_0^\infty(\Omega)$ in (5.3), we have

$$\begin{aligned}
&\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\omega_1^\epsilon dx + \int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\phi\omega_2^\epsilon dx + \int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon \cdot \nabla(\phi\omega_1^\epsilon)dx + \int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon \cdot \nabla(\phi\omega_2^\epsilon)dx \\
&+ \int_\Omega \frac{1}{\epsilon^2}Q^\epsilon(u_1^\epsilon - u_2^\epsilon)(\omega_1^\epsilon - \omega_2^\epsilon)\phi dx = \int_\Omega q\phi\omega_1^\epsilon dx + \int_\Omega q\phi\omega_2^\epsilon dx.
\end{aligned} \tag{5.9}$$

Let $\psi_1(x)$ and $\psi_2(x)$ in (5.7) and (5.8) be ϕu_1^ϵ and ϕu_2^ϵ respectively. We have

$$\begin{aligned}
&\int_\Omega \kappa_1^\epsilon\nabla\omega_1^\epsilon \cdot \nabla(\phi u_1^\epsilon)dx + \int_\Omega \kappa_2^\epsilon\nabla\omega_2^\epsilon \cdot \nabla(\phi u_2^\epsilon)dx + \int_\Omega \frac{1}{\epsilon^2}Q^\epsilon(\omega_1^\epsilon - \omega_2^\epsilon)(u_1^\epsilon - u_2^\epsilon)\phi dx \\
&= -\epsilon\int_\Omega \mathrm{div}_x(\kappa_1(x, \frac{x}{\epsilon})\nabla_x N_1^i(x, \frac{x}{\epsilon}))\phi u_1^\epsilon dx - \int_\Omega \mathrm{div}_x(\kappa_1(x, \frac{x}{\epsilon})(e^i + \nabla_y N_1^i(x, \frac{x}{\epsilon})))\phi u_1^\epsilon dx \\
&\quad - \int_\Omega \mathrm{div}_y(\kappa_1(x, \frac{x}{\epsilon})\nabla_x N_1^i(x, \frac{x}{\epsilon}))\phi u_1^\epsilon dx - \epsilon\int_\Omega \mathrm{div}_x(\kappa_2(x, \frac{x}{\epsilon})\nabla_x N_2^i(x, \frac{x}{\epsilon}))\phi u_2^\epsilon dx \\
&\quad - \int_\Omega \mathrm{div}_x(\kappa_2(x, \frac{x}{\epsilon})(e^i + \nabla_y N_2^i(x, \frac{x}{\epsilon})))\phi u_2^\epsilon dx - \int_\Omega \mathrm{div}_y(\kappa_2(x, \frac{x}{\epsilon})\nabla_x N_2^i(x, \frac{x}{\epsilon}))\phi u_2^\epsilon dx.
\end{aligned} \tag{5.10}$$

Let $\psi \in C_0^\infty(0,T)$. We multiply (5.9) and (5.10) by ψ and integrate over $(0,T)$ with respect to t. After subtracting the resulting equations by each other, we obtain

$$
\begin{aligned}
&\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt + \int_0^T\int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon\cdot\nabla\phi\omega_1^\epsilon\psi dxdt - \int_0^T\int_\Omega \kappa_1^\epsilon\nabla\omega_1^\epsilon\cdot\nabla\phi u_1^\epsilon\psi dxdt\\
&+\int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\phi\psi\omega_2^\epsilon dxdt + \int_0^T\int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon\cdot\nabla\phi\omega_2^\epsilon\psi dxdt - \int_0^T\int_\Omega \kappa_2^\epsilon\nabla\omega_2^\epsilon\cdot\nabla\phi u_2^\epsilon\psi dxdt\\
&=\int_0^T\int_\Omega q\phi\omega_1^\epsilon\psi dxdt + \int_0^T\int_\Omega q\phi\omega_2^\epsilon\psi dxdt\\
&\ \ +\epsilon\int_0^T\int_\Omega \mathrm{div}_x(\kappa_1(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_1^i(\cdot,\frac{\cdot}{\epsilon}))\phi u_1^\epsilon\psi dxdt + \int_0^T\int_\Omega \mathrm{div}_x(\kappa_1(\cdot,\frac{\cdot}{\epsilon})(e^i+\nabla_y N_1^i(\cdot,\cdot\epsilon)))\phi u_1^\epsilon\psi dxdt\\
&\ \ +\int_0^T\int_\Omega \mathrm{div}_y(\kappa_1(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_1^i(\cdot,\frac{\cdot}{\epsilon}))\phi u_1^\epsilon\psi dxdt + \epsilon\int_0^T\int_\Omega \mathrm{div}_x(\kappa_2(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_2^i(\cdot,\frac{\cdot}{\epsilon}))\phi u_2^\epsilon\psi dxdt\\
&\ \ +\int_0^T\int_\Omega \mathrm{div}_x(\kappa_2(\cdot,\frac{\cdot}{\epsilon})(e^i+\nabla_y N_2^i(\cdot,\frac{\cdot}{\epsilon})))\phi u_2^\epsilon\psi dxdt + \int_0^T\int_\Omega \mathrm{div}_y(\kappa_2(\cdot,\frac{\cdot}{\epsilon})\nabla_x N_2^i(\cdot,\frac{\cdot}{\epsilon}))\phi u_2^\epsilon\psi dxdt.
\end{aligned}
\tag{5.11}
$$

We have the following lemma.

Lemma 5.2. *The functions $\int_0^T\psi(t)u_1^\epsilon(x,t)dt$ and $\int_0^T\psi(t)u_2^\epsilon(x,t)dt$ converge strongly in H to $\int_0^T\psi(t)u_0(x,t)dt$.*

Proof. This is the standard result in Jikov et al. [17]. As u_1^ϵ is uniformly bounded in $L^2(0,T;V)$, $\int_0^T\psi(t)u_1^\epsilon(x,t)dt$ is uniformly bounded in V when $\epsilon\to 0$. Thus we can extract a subsequence which converges weakly in V and strongly in H. As for all $\phi\in C_0^\infty(\Omega)$,

$$\int_\Omega\int_0^T\psi(t)u_1^\epsilon(x,t)\phi(x)dtdx \to \int_\Omega\int_0^T\psi(t)u_0(x,t)\phi(x)dtdx,$$

the limit is $\int_0^T\psi(t)u_0(x,t)dt$. □

We have

$$\int_0^T\int_\Omega C_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt = -\int_\Omega C_{11}^\epsilon\left(\int_0^T u_1^\epsilon\frac{\partial\psi}{\partial t}dt\right)\phi\omega_1^\epsilon dx.$$

As C_{11}^ϵ converges weakly to $\int_Y C_{11}(x,y)dy$ in H, $\int_0^T u_1^\epsilon\frac{\partial\psi}{\partial t}dt$ converges weakly to $\int_0^T u_0\frac{\partial\psi}{\partial t}dt$ in V, we have

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_0^T\int_\Omega C_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}\phi\psi\omega_1^\epsilon dxdt &= -\int_0^T\int_\Omega\left(\int_Y C_{11}(x,y)dy\right)u_0\frac{\partial\psi}{\partial t}\phi x_i dxdt\\
&=\int_0^T\int_\Omega\left(\int_Y C_{11}(x,y)dy\right)\frac{\partial u_0}{\partial t}\psi\phi x_i dxdt.
\end{aligned}
$$

We note that

$$
\begin{aligned}
\kappa_1^\epsilon(x)\nabla\omega_1^\epsilon(x) &= \kappa_1(x,\frac{x}{\epsilon})\big((e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon}))+\epsilon\nabla_x N_1^i(x,\frac{x}{\epsilon})\big),\\
\kappa_2^\epsilon(x)\nabla\omega_2^\epsilon(x) &= \kappa_2(x,\frac{x}{\epsilon})\big((e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon}))+\epsilon\nabla_x N_2^i(x,\frac{x}{\epsilon})\big).
\end{aligned}
$$

Also, note that due to Y-periodicity of κ and N^i, we have

$$
\begin{aligned}
&\kappa_1(x,\frac{x}{\epsilon})(e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon}))\rightharpoonup\int_Y\kappa_1(x,y)(e^i+\nabla_y N_1^i(x,y))dy\\
&\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon}))\rightharpoonup\int_Y\kappa_2(x,y)(e^i+\nabla_y N_2^i(x,y))dy\ \text{ in } H.
\end{aligned}
$$

We observe that $x_i + \epsilon N_1^i \to x_i$ strongly in H and $\int_\Omega q\phi\omega_k^\epsilon dx \to \int_\Omega q\phi x_i dx$ since $\omega_k^\epsilon\phi \rightharpoonup x_i\phi$ in H. Passing to the limit in (5.11), we obtain from Lemma 5.2,

$$\begin{aligned}
&\int_0^T\int_\Omega\left(\int_Y C_{11}dy\right)\frac{\partial u_0}{\partial t}\phi\psi x_i dxdt + \lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot\nabla\phi\psi x_i dxdt\\
&-\int_0^T\int_\Omega\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\cdot\nabla\phi\psi u_0 dxdt + \int_0^T\int_\Omega\left(\int_Y C_{22}dy\right)\frac{\partial u_0}{\partial t}\phi\psi x_i dxdt\\
+\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot\nabla\phi\psi x_i dxdt &-\int_0^T\int_\Omega\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\cdot\nabla\phi\psi u_0 dxdt\\
&= 2\int_0^T\int_\Omega q\phi\psi x_i dxdt + \lim_{\epsilon\to 0}\int_0^T\int_\Omega \mathrm{div}_x(\kappa_1(\cdot,\frac{\cdot}{\epsilon})(e^i+\nabla_y N_1^i(\cdot,\frac{\cdot}{\epsilon})))\phi\psi u_1^\epsilon dxdt\\
&\quad+\lim_{\epsilon\to 0}\int_0^T\int_\Omega \mathrm{div}_x(\kappa_2(\cdot,\frac{\cdot}{\epsilon})(e^i+\nabla_y N_2^i(\cdot,\frac{\cdot}{\epsilon})))\phi\psi u_2^\epsilon dxdt.
\end{aligned} \tag{5.12}$$

Let ϕ_1 and ϕ_2 in (5.3) be ϕx_i where $\phi \in C_0^\infty(\Omega)$. Adding the two equations, we have

$$\begin{aligned}
&\int_0^T\int_\Omega C_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}\phi\psi x_i dxdt + \int_0^T\int_\Omega C_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t}\phi\psi x_i dxdt + \int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot\nabla(\phi x_i)\psi dxdt\\
&+\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot\nabla(\phi x_i)\psi dxdt = 2\int_0^T\int_\Omega q\phi\psi x_i dxdt.
\end{aligned}$$

Passing to the limit, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega\left(\int_Y C_{11}dy\right)\frac{\partial u_0}{\partial t}\phi\psi x_i dxdt + \int_0^T\int_\Omega\left(\int_Y C_{22}dy\right)\frac{\partial u_0}{\partial t}\phi\psi x_i dxdt\\
&+\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\nabla\cdot(\phi x_i)\psi dxdt + \lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot\nabla(\phi x_i)\psi dxdt = 2\int_0^T\int_\Omega q(\phi x_i)\psi dxdt.
\end{aligned} \tag{5.13}$$

Using (5.12) and (5.13), one obtains

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot e^i\phi\psi dxdt + \lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot e^i\phi\psi dxdt\\
&= -\int_0^T\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla\phi\psi u_0 dxdt - \int_0^T\int_\Omega\left(\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\right)\cdot\nabla\phi\psi u_0 dxdt\\
&\quad-\lim_{\epsilon\to 0}\int_0^T\int_\Omega \mathrm{div}_x(\kappa_1(x,\frac{x}{\epsilon})(e^i+\nabla_y N_1^i(x,\frac{x}{\epsilon})))\phi\psi u_1^\epsilon dxdt\\
&\quad-\lim_{\epsilon\to 0}\int_0^T\int_\Omega \mathrm{div}_x(\kappa_2(x,\frac{x}{\epsilon})(e^i+\nabla_y N_2^i(x,\frac{x}{\epsilon})))\phi\psi u_2^\epsilon dxdt.
\end{aligned}$$

Since κ_1, κ_2, N_1^i and N_2^i are independent of t, by Lemma 5.2, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot e^i\phi\psi dxdt + \lim_{\epsilon\to 0}\int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot e^i\phi\psi dxdt\\
&= -\int_0^T\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla\phi\psi u_0 dxdt - \int_0^T\int_\Omega\left(\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\right)\cdot\nabla\phi\psi u_0 dxdt\\
&\quad-\int_0^T\int_\Omega \mathrm{div}\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\phi\psi u_0 dxdt - \int_0^T\int_\Omega \mathrm{div}\left(\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\right)\phi\psi u_0 dxdt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^T\left(\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot e^i\phi dx\right)\psi dt + \lim_{\epsilon\to 0}\int_0^T\left(\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot e^i\phi dx\right)\psi dt\\
&= \int_0^T\left(\int_\Omega\left(\int_Y \kappa_1(e^i+\nabla_y N_1^i)dy\right)\cdot\nabla u_0\phi dx\right)\psi dt + \int_0^T\left(\int_\Omega\left(\int_Y \kappa_2(e^i+\nabla_y N_2^i)dy\right)\cdot\nabla u_0\phi dx\right)\psi dt
\end{aligned}$$

From this, we deduce

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\left[\int_0^T\int_\Omega \kappa_1^\epsilon(x)\nabla u_1^\epsilon(x)\cdot\nabla\phi\psi\, dxdt + \int_0^T\int_\Omega \kappa_2^\epsilon(x)\nabla u_2^\epsilon(x)\cdot\nabla\phi\psi\, dxdt\right]\\
&= \lim_{\epsilon\to 0}\left[\int_0^T\int_\Omega \kappa_1^\epsilon\nabla u_1^\epsilon\cdot e^i\frac{\partial\phi}{\partial x_i}\psi\, dxdt + \int_0^T\int_\Omega \kappa_2^\epsilon\nabla u_2^\epsilon\cdot e^i\frac{\partial\phi}{\partial x_i}\psi\, dxdt\right]\\
&= \int_0^T\int_\Omega\left(\int_Y \kappa_1(x,y)(\delta_{ij}+\frac{\partial N_1^i(x,y)}{\partial y_j})dy\right)\frac{\partial u_0}{\partial x_j}(x)\frac{\partial\phi}{\partial x_i}\psi\, dxdt\\
&\quad + \int_0^T\int_\Omega\left(\int_Y \kappa_2(x,y)(\delta_{ij}+\frac{\partial N_2^i(x,y)}{\partial y_j})dy\right)\frac{\partial u_0}{\partial x_j}(x)\frac{\partial\phi}{\partial x_i}\psi\, dxdt
\end{aligned}
\tag{5.14}
$$

For consistency with formula (2.6), we note the following result.

Lemma 5.3. $\int_Y \kappa_1\frac{\partial N_1^j(x,y)}{\partial y_i}dy + \int_Y \kappa_2\frac{\partial N_2^j(x,y)}{\partial y_i}dy = \int_Y \kappa_1\frac{\partial N_1^i(x,y)}{\partial y_j}dy + \int_Y \kappa_2\frac{\partial N_2^i(x,y)}{\partial y_j}dy$

Proof. From the cell problem, we have

$$
\begin{aligned}
&\int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_1^j dy + \int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_1^j dy\\
&\qquad + \int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_2^j dy + \int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_2^j dy = 0\\
&\int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_1^i dy + \int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_1^i dy\\
&\qquad + \int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_2^i dy + \int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_2^i dy = 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_Y \kappa_1\frac{\partial N_1^j}{\partial y_i}dy + \int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_1^j dy + \int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_2^j dy + \int_Y \kappa_2\frac{\partial N_2^j}{\partial y_i}dy\\
&= \int_Y \kappa_1\frac{\partial N_1^i}{\partial y_j}dy + \int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_1^i dy + \int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_2^i dy + \int_Y \kappa_2\frac{\partial N_2^i}{\partial y_j}dy.
\end{aligned}
\tag{5.15}
$$

Now we show

$$
\begin{aligned}
&\int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_1^j dy + \int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_2^j dy\\
&= \int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_1^i dy + \int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_2^i dy.
\end{aligned}
$$

From the cell problem, we know that

$$
\begin{aligned}
&\int_Y \kappa_2(e^i+\nabla_y N_2^i)\cdot\nabla_y N_1^j dy + \int_Y \kappa_1(e^i+\nabla_y N_1^i)\cdot\nabla_y N_2^j dy\\
&= \int_Y Q(N_1^i-N_2^i)N_1^j + Q(N_2^i-N_1^i)N_2^j dy\\
&= \int_Y Q(N_1^iN_1^j - N_2^iN_1^j + N_2^iN_2^j - N_1^iN_2^j)dy\\
&= \int_Y Q(N_1^j-N_2^j)N_1^i + Q(N_2^j-N_1^j)N_2^i dy\\
&= \int_Y \kappa_2(e^j+\nabla_y N_2^j)\cdot\nabla_y N_1^i dy + \int_Y \kappa_1(e^j+\nabla_y N_1^j)\cdot\nabla_y N_2^i dy.
\end{aligned}
\tag{5.16}
$$

Thus, by (5.15) and (5.16), we have the result. □

Theorem 5.4. *Assume that the solutions N_1^i and N_2^i of cell problem (2.8) belong to $C^2(\bar\Omega, C^2(\bar Y))$ and the coefficients κ_1 and κ_2 belong to $C^1(\bar\Omega, C^1(\bar Y))$. The limit function u_0 of the sequences u_1^ϵ, u_2^ϵ is the unique solution of the homogenized equation (2.5) with the initial condition (2.7).*

Proof. Note that from Eq. (5.1), we obtain

$$\int_0^T\int_\Omega \mathcal{C}_{11}^\epsilon \frac{\partial u_1^\epsilon}{\partial t}\phi dx\psi dt + \int_0^T\int_\Omega \kappa_1^\epsilon \nabla u_1^\epsilon\cdot\nabla\phi dx\psi dt + \int_0^T\int_\Omega \mathcal{C}_{22}^\epsilon \frac{\partial u_2^\epsilon}{\partial t}\phi dx\psi dt + \int_0^T\int_\Omega \kappa_2^\epsilon \nabla u_2^\epsilon\cdot\nabla\phi dx\psi dt$$
$$= 2\int_0^T\int_\Omega q\phi dx\psi dt.$$

for all $\phi\in C_0^\infty(\Omega)$ and $\psi\in C_0^\infty((0,T))$. Passing to the limit, from (5.14), Lemmas 5.2 and 5.3, we have

$$\int_0^T\int_\Omega \{(\int_Y \mathcal{C}_{11}dy) + (\int_Y \mathcal{C}_{22}dy)\}\frac{\partial u_0}{\partial t}\phi dx\psi dt$$
$$= \int_0^T\int_\Omega \operatorname{div}(\kappa_1^*\nabla u_0)\phi dx\psi dt + \int_0^T\int_\Omega \operatorname{div}(\kappa_2^*\nabla u_0)\phi dx\psi dt + \int_0^T\int_\Omega 2q\phi dx\psi dt,$$

where

$$\kappa_{1ij}^*(x) = \int_Y \kappa_1(x,y)(\delta_{ij} + \frac{\partial N_1^j(x,y)}{\partial y_i})dy$$
$$\kappa_{2ij}^*(x) = \int_Y \kappa_2(x,y)(\delta_{ij} + \frac{\partial N_2^j(x,y)}{\partial y_i})dy.$$

We now show the initial condition. Adding (5.1) and (5.2), we have

$$\mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t} - \operatorname{div}(\kappa_1^\epsilon\nabla u_1^\epsilon) - \operatorname{div}(\kappa_2^\epsilon\nabla u_2^\epsilon) = 2q.$$

As u_1^ϵ and u_2^ϵ are bounded in $L^2(0,T;V)$, we deduce that $\mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}+\mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t}$ is bounded in $L^2(0,T;V')$. Let $\psi(t,x)\in C_0^\infty(0,T;V)$, i.e. $\psi(0,x)=\psi(T,x)=0$. We have

$$\int_0^T\int_\Omega(\mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t})\psi dxdt = -\int_0^T\int_\Omega\big(\mathcal{C}_{11}^\epsilon u_1^\epsilon + \mathcal{C}_{22}^\epsilon u_2^\epsilon\big)\frac{\partial\psi}{\partial t}dxdt$$
$$\to -\int_0^T\int_\Omega\big(\langle\mathcal{C}_{11}\rangle + \langle\mathcal{C}_{22}\rangle\big)u_0\frac{\partial\psi}{\partial t}dxdt = \int_0^T\int_\Omega\big(\langle\mathcal{C}_{11}\rangle + \langle\mathcal{C}_{22}\rangle\big)\frac{\partial u_0}{\partial t}\psi dxdt.$$

This shows that the weak limit of $\mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t}+\mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t}$ in $L^2(0,T;V')$ is $\big(\langle\mathcal{C}_{11}\rangle+\langle\mathcal{C}_{22}\rangle\big)\frac{\partial u_0}{\partial t}$. Now we choose $\psi\in C^\infty(0,T;V)$ so that $\psi(T,x)=0$. Then

$$\int_0^T\int_\Omega(\mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t})\psi dxdt$$
$$= -\int_0^T\int_\Omega\big(\mathcal{C}_{11}^\epsilon u_1^\epsilon + \mathcal{C}_{22}^\epsilon u_2^\epsilon\big)\frac{\partial\psi}{\partial t}dxdt + \int_\Omega\big(\mathcal{C}_{11}^\epsilon u_1^\epsilon(0,x) + \mathcal{C}_{22}^\epsilon u_2^\epsilon(0,x)\big)\psi(0,x)dx$$
$$\to -\int_0^T\int_\Omega\big(\langle\mathcal{C}_{11}\rangle + \langle\mathcal{C}_{22}\rangle\big)u_0\frac{\partial\psi}{\partial t}dxdt + \int_\Omega\big(\langle\mathcal{C}_{11}\rangle g_1 + \langle\mathcal{C}_{22}\rangle g_2\big)\psi(0,x)dx.$$

On the other hand

$$\int_0^T\int_\Omega(\mathcal{C}_{11}^\epsilon\frac{\partial u_1^\epsilon}{\partial t} + \mathcal{C}_{22}^\epsilon\frac{\partial u_2^\epsilon}{\partial t})\psi dxdt \to \int_0^T\int_\Omega\big(\langle\mathcal{C}_{11}\rangle + \langle\mathcal{C}_{22}\rangle\big)\frac{\partial u_0}{\partial t}\psi dxdt.$$
$$= -\int_0^T\int_\Omega\big(\langle\mathcal{C}_{11}\rangle + \langle\mathcal{C}_{22}\rangle\big)u_0\frac{\partial\psi}{\partial t}dxdt + \int_\Omega\big(\langle\mathcal{C}_{11}\rangle + \langle\mathcal{C}_{22}\rangle\big)u_0(0,x)\psi(0,x)dx.$$

This shows that $\big(\langle\mathcal{C}_{11}\rangle+\langle\mathcal{C}_{22}\rangle\big)u_0(0,x) = \langle\mathcal{C}_{11}\rangle g_1(x) + \langle\mathcal{C}_{22}\rangle g_2(x)$. i.e. the initial condition of u_0 is

$$u_0(0,x) = \frac{\langle\mathcal{C}_{11}\rangle g_1(x) + \langle\mathcal{C}_{22}\rangle g_2(x)}{\langle\mathcal{C}_{11}\rangle + \langle\mathcal{C}_{22}\rangle} \quad \square \tag{5.17}$$

6. Conclusions

In this paper, we developed an efficient algorithm for computing the effective coefficients of a coupled multiscale multi-continuum system. We derived the coupled cell problems and the homogenized equation from two-scale asymptotic expansion. We solved the cell problems using hierarchical FE algorithm and used the solutions to compute the effective coefficients. To establish the hierarchical FE algorithm, we first constructed a dense hierarchy of macrogrids and the corresponding nested FE spaces. Based on the hierarchy, we solve the cell problems using different resolution FE spaces at different macroscopic points. We use solutions solved with a higher level of accuracy to correct solutions obtained with a lower level of accuracy at nearby macroscopic points. We rigorously showed that this hierarchical FE method achieves the same order of accuracy as the reference full solve where cell problems at every macroscopic point are solved with the highest level of accuracy, at a significantly reduced computation cost, using an essentially optimal number of degrees of freedom. For numerical examples, we applied this algorithm to a multi-continuum model in a two dimensional domain. The algorithm was implemented on macroscopic points in a one dimensional domain. The numerical results strongly support the error estimates we provided in Section 3.

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