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Explicit irregular sampling formulas

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Abstract

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An important new class of irregular sampling sequences is investigated. The sampling formulas can be expressed in terms of standard functions. For the first time estimates for the most common errors are given.

Keywords: Sampling, nonuniform, Lagrange interpolation, error estimates.

1. Introduction

Let B_β^p denote the class of all $L^p(\mathbb{R})$-functions that are band-limited to $[-\beta, \beta]$ (cf. Section 2). The classical Whittaker–Shannon–Kotel'nikov sampling theorem [6,14,17,19] states that

$$f(z)=\sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z\in\mathbb{C}, \tag{1.1}$$

i.e., the function f can be reconstructed from its values at the integers, provided that $f\in B_\beta^p$ for $1\leqslant p<\infty$, $\beta\leqslant\pi$ or $p=\infty$, $\beta<\pi$. Setting $t_n := n$, $G(z) := \pi^{-1}\sin\pi z$, formula (1.1) can be rewritten as

$$f(z)=\sum_{n=-\infty}^{\infty} f(t_n)\frac{G(z)}{G'(t_n)(z-t_n)}, \quad z\in\mathbb{C}. \tag{1.2}$$

The function G can be interpreted as a canonical product with respect to the integers (cf. Section 3), i.e.,

$$G(z)=z\prod_{k=1}^{\infty}\left(1-\frac{z}{t_n}\right)\left(1-\frac{z}{t_{-n}}\right). \tag{1.3}$$

Hence in view of (1.2), (1.3) it is justified to call the sampling series (1.1) a *Lagrange interpolation formula* with infinitely many knots (cf. [11]).

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One often speaks of the classical sampling theorem as the uniform sampling theorem because the underlying sequence of knots (the sequence of integers) is equidistantly spaced. Nonuniform, or irregular, sampling theory investigates which not necessarily equidistantly spaced sequences of sampling knots admit the reconstruction of band-limited functions; in particular it is asked under which assumptions on f and $\{t_n\}_{n\in\mathbb{Z}}$ do the formulas (1.2), (1.3) remain valid. The present paper deals with a class of sequences $\{t_n\}_{n\in\mathbb{Z}}$ of sampling points for which the product $G(z)$ in (1.3) can be expressed in terms of Gamma-, sinc- or other standard functions, i.e., for which the sampling series can be explicitly given. The sequences in question are essentially of the form

$$t_n = \begin{cases} n+D, & n>0, \\ n-D, & n<0, \end{cases} \quad |n| \text{ large}, \tag{1.4}$$

for some $D \in \mathbb{R}$. The parameter D can simulate the effect of adding or dropping finitely many sampling knots. Modifications, such as replacing finitely many knots or translating all knots by a fixed amount or multiplying all knots by a positive factor, are studied.

Section 3 contains the calculations needed to find a simple representation for $G(z)$. In Section 4 the corresponding sampling formulas are established. Section 5 is devoted to a study of the most important error types that occur in connection with nonuniform sampling series, namely truncation, amplitude, time-jitter and aliasing errors. The corresponding results are valid only for a rather small class of sequences ($-\frac{1}{2} \leqslant D \leqslant 0$ in the sense of (1.4)); however, this is the first time that error estimates for nonuniform sampling expansions of Lagrange type (1.2), (1.3) are given (Feichtinger's paper [8] deals with a different kind of nonuniform sampling series).

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural, integer, real and complex numbers, respectively. For $x \in \mathbb{R}$, the floor function $\lfloor x \rfloor$ is defined to be the largest integer $\leqslant x$.

Let $E \subset \mathbb{C}$ and let f_1, f_2 be real-valued, nonnegative functions on E. Then f_1 is *equivalent* to f_2 on E ($f_1 \sim f_2$ on E) if and only if there exist C_1, C_2 such that $0 < C_1 \leqslant C_2$ and $C_1 f_1(z) \leqslant f_2(z) \leqslant C_2 f_1(z)$, $z \in E$. Sometimes, the well-known o, $\mathcal{O}$-notation will be used as well.

Let $1 \leqslant p \leqslant \infty$. The number q is called the *conjugated index* of p, if $1/p + 1/q = 1$ (e.g., if $p=q=2$ or $p=1$, $q=\infty$ or $p=\infty$, $q=1$). The spaces $L^p(\mathbb{R})$ consist of all Lebesgue measurable functions f on $\mathbb{R}$ with $\|f\|_p := (\int_{-\infty}^{\infty} |f(x)|^p \, dx)^{1/p} < \infty$, and $L^\infty(\mathbb{R})$ denotes the space of all essentially bounded functions.

A sequence $\{t\} = \{t_n\}_{n\in\mathbb{Z}}$ is l^∞ if it is bounded, $\|\{t\}\|_{l^\infty} := \sup_{n\in\mathbb{Z}} |t_n|$. For $1 \leqslant p < \infty$, a sequence $\{t\}$ is element of the space l^p, iff $\|\{t\}\|_{l^p} := (\sum_{n\in\mathbb{Z}} |t_n|^p)^{1/p} < \infty$. Let $\{s\}$, $\{t\} \in l^p$, $1 \leqslant p \leqslant \infty$. Then there holds *Minkowski's inequality*, i.e., $\|\{s\}+\{t\}\|_{l^p} \leqslant \|\{s\}\|_{l^p} + \|\{t\}\|_{l^p}$. If p, q are conjugated indices and $\{s\} \in l^p$, $\{t\} \in l^q$, one has $\|\{st\}\|_{l^1} \leqslant \|\{s\}\|_{l^p} \|\{t\}\|_{l^q}$, where $\{st\}_n = s_n t_n$ (*Hölder's inequality*).

The spaces B^p_β, $1 \leqslant p \leqslant \infty$, $\beta \geqslant 0$, are made up of all entire functions f which are in $L^p(\mathbb{R})$ when restricted to $\mathbb{R}$ and fulfil $|f(x+\mathrm{i}y)| \leqslant \sup_{u\in\mathbb{R}} |f(u)| \mathrm{e}^{\beta|y|}$. All functions which belong to one of the spaces B_β^p are called *band-limited* to $[-\beta, \beta]$, since their (distributional) Fourier transform vanishes outside of $[-\beta, \beta]$. If $1 \leqslant p < \infty$ and $f \in B_\beta^p$, then one has firstly

$$|f(z)| \leqslant C \| f \|_p (1 + |y|)^{-1/p} \mathrm{e}^{\beta |y|}, \quad z = x + \mathrm{i}y \in \mathbb{C}, \tag{2.1}$$

(*Korevaar's inequality*; it can be obtained by simple modifications from the results presented in [13]), and secondly $\| f' \|_p \leqslant \beta \| f \|_p$ (*Bernstein's inequality*, cf. [16, p.115]). A third important estimate connected with band-limited functions is *Nikol'skiĭ's inequality*.

Lemma 2.1. *Let* $1 \leqslant p < \infty$, $\beta > 0$ *and let* $\{t\} = \{t_n\}_{n \in \mathbb{Z}}$ *be a sequence with*

$$\exists \delta,\ L > 0 \forall n \in \mathbb{Z} \quad \delta \leqslant t_{n+1} - t_n \leqslant L. \tag{2.2}$$

Then there exist constants C_1, $C_2 > 0$ (*dependent on* δ, L) *such that for all* $f \in B_\beta^p$,

$$C_1 \| f \|_p \leqslant \sup_{x \in \mathbb{R}} \left(\sum_{n=-\infty}^{\infty} |f(t_n - x)|^p \right)^{1/p} \leqslant C_2 \| f \|_p. \tag{2.3}$$

Proof. See [9, pp. 83–86] and [16, pp. 123–124]. □

In the calculations of this paper some formulas concerning the Gamma function are needed, namely the *functional equation* $\Gamma(z+1) = z\Gamma(z)$, $z \in \mathbb{C} \setminus \{-n;\ n \in \mathbb{N} \cup \{0\}\}$, the *reflection formula* $1/(\Gamma(z)\Gamma(1-z)) = \pi^{-1} \sin \pi z$, $z \in \mathbb{C}$, and the estimates given in the following lemma.

Lemma 2.2. (a) *Let* α, $\beta \in \mathbb{R}$ *and* $\eta > 0$. *Then there holds*

$$\left| \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \right| \sim |z|^{\alpha-\beta}, \quad on\ \{z = x + \mathrm{i}y \in \mathbb{C};\ |z| \geqslant \eta,\ x + \alpha \geqslant \eta,\ x + \beta \geqslant \eta\}.$$

(b) *Let* a_1, a_2, b_1, $b_2 \in \mathbb{C}$ *with* $a_1 + a_2 - b_1 - b_2 = 0$. *Then*

$$\lim_{M \to \infty} \frac{\Gamma(M + a_1)\Gamma(M + a_2)}{\Gamma(M + b_1)\Gamma(M + b_2)} = 1.$$

Proof. A partial proof of (a) and (b) can be found in [15], a detailed one is given in [9, pp. 11–14]. Estimate (a) is often stated as an asymptotic result (cf., e.g., [7]). □

The space UCB($\mathbb{R}$) consists of all uniformly continuous, bounded functions on $\mathbb{R}$ and is equipped with the norm $\| f \|_C := \sup_{x \in \mathbb{R}} |f(x)|$. In order to describe the smoothness of functions in UCB($\mathbb{R}$) one considers the Lipschitz classes ($\alpha > 0$, $L \geqslant 0$)

$$\mathrm{Lip}_L(\alpha, \mathrm{UCB}(\mathbb{R})) := \left\{ f \in \mathrm{UCB}(\mathbb{R});\ \sup_{|h| \leqslant \delta} \| f(\cdot) - f(\cdot + h) \|_C \leqslant L\delta^\alpha, \text{ for all } \delta > 0 \right\}.$$

For the spaces $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and UCB($\mathbb{R}$) the *means of de la Vallée-Poussin (delayed means)* are defined by

$$\mathrm{VP}_\rho f(x) := \frac{2\rho}{\pi} \int_{-\infty}^{\infty} f(x-u) \frac{\sin(\frac{1}{2}\rho u) \sin(\frac{3}{2}\rho u)}{(\frac{1}{2}\rho u)^2} \, \mathrm{d}u, \quad \rho > 0.$$

Lemma 2.3. (a) *Let $r \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 1]$ and $L \geqslant 0$. Then for all f with $f^{(r)} \in \mathrm{Lip}_L(\alpha, \mathrm{UCB}(\mathbb{R}))$ and all $\rho \geqslant 1$,*

$$\| \mathrm{VP}_\rho f - f \|_C \leqslant CL\rho^{-r-\alpha}.$$

(b) *Let $f \in \mathrm{UCB}(\mathbb{R})$ and assume that $\exists \gamma \in (0, 1]$, $M_f > 0$ such that $|f(x)| \leqslant M_f |x|^{-\gamma}$, $|x| \geqslant 1$. Then*

$$|\mathrm{VP}_\rho f(x)| \leqslant 3(M_f + \|f\|_C)|x|^{-\gamma}, \quad |x| \geqslant 1.$$

Proof. See [18] for these and further results on the delayed means. □

3. The sampling sequences $\{t_D^0\}$ and related sequences

Definition 3.1. (a) A strictly increasing sequence $\{t\} = \{t_n\}_{n \in \mathbb{Z}}$ of real numbers is called *equidistant*, if there exist constants $\tau \in \mathbb{R}$, $\sigma > 0$ such that $t_n = \tau + \sigma n$, $n \in \mathbb{Z}$.

(b) A sequence $\{t\}$ of real numbers will be called a *perturbed equidistant sequence* (with respect to the equidistant sequence $\{\tau + \sigma n\}_{n \in \mathbb{Z}}$), if there exists a constant $L \geqslant 0$ such that $|t_n - \tau - \sigma n| \leqslant L$, $n \in \mathbb{Z}$.

(c) Let $D \in \mathbb{R}$ and $N_0 := \max\{0, \lceil -D \rceil\}$. The symbol $\{t_D^0\}$ denotes the sequence defined by

$$t_{D,n}^0 := \begin{cases} n - D, & n \leqslant -N_0 - 1, \\ \dfrac{n}{N_0 + 1}, & -N_0 \leqslant n \leqslant N_0, \\ n + D, & n \geqslant N_0 + 1. \end{cases}$$

The members of each sequence $\{t_D^0\}$ are in strictly increasing order; in fact, the minimum distance between two consecutive members is not smaller than $1/(N_0 + 1)$. The sequences $\{t_D^0\}$ are symmetrical: $t_{D,0} = 0$ and $t_{D,-n} = -t_{D,n}$, $n \in \mathbb{N}$. The sequence $\{t_0^0\}$ turns out to be the sequence of integers, i.e., the $\{t_D^0\}$ sequences generalize the classical sampling sequence. It will be our main concern to see whether the sampling theorem extends to values of D different from zero.

Definition 3.2. (a) Let $\{t\}$ be a perturbed equidistant sequence. The *canonical product* with respect to $\{t\}$ is defined by

$$G(z) := G(\{t\}; z) := g(t_0; z) \prod_{k=1}^{\infty} g(t_k; z) g(t_{-k}; z), \quad z \in \mathbb{C},$$

where $g(s; z) := 1 - z/s$, if $s \in \mathbb{R} \backslash \{0\}$ and $g(0; z) = z$.

(b) Let $D \in \mathbb{R}$. For abbreviation, set $G_D(z) := G(\{t_D^0\}; z)$.

Since in every bounded subset of $\mathbb{C}$, $|g(t_k; z)g(t_{-k}; z) - 1| = \mathscr{O}(k^{-2})$ for large k, the product $G(z)$ is well-defined and represents an entire function with zeros at t_n (if a member of $\{t\}$ occurs more than once, the function G has a zero of corresponding multiplicity).

The canonical products G_D can be represented in terms of well-known functions and one can give a fine characterization of their growth behaviour.

Proposition 3.3. *Let $D \in \mathbb{R}$ and N_0, $\{t_D^0\}$, G_D be defined as above.*

(a) *With*

$$P_{2N_0+1}(z) := \prod_{k=-N_0}^{N_0} \left(z - \frac{k}{N_0+1}\right) \quad \text{and} \quad C_D := \frac{\Gamma^2(N_0+1+D)(N_0+1)^{2N_0}}{(N_0!)^2},$$

one has

$$G_D(z) = \frac{(-1)^{N_0} C_D P_{2N_0+1}(z)}{\Gamma(N_0+1+D-z)\Gamma(N_0+1+D+z)}, \qquad z \in \mathbb{C}, \tag{3.1}$$

$$G_D(z) = C_D P_{2N_0+1}(z) \frac{\Gamma(z-N_0-D)}{\Gamma(z+N_0+D+1)} \frac{\sin \pi(z-D)}{\pi}, \qquad z \in \mathbb{C}\setminus A_1, \tag{3.2}$$

$$G_D(z) = C_D P_{2N_0+1}(z) \frac{\Gamma(-z-N_0-D)}{\Gamma(-z+N_0+D+1)} \frac{\sin \pi(z+D)}{-\pi}, \qquad z \in \mathbb{C}\setminus A_2, \tag{3.3}$$

where $A_1 = \{n+D,\ n \leqslant N_0\}$, $A_2 = \{n-D,\ n \geqslant -N_0\}$.

(b) *For $z \in \mathbb{C}$ with $|\Im(z)| \leqslant \frac{1}{2}$ and $|\Re(z)| > N_0 + D + \frac{1}{2}$ set $N(z) := \lfloor \Re(z) - D + \frac{1}{2} \rfloor$, if $\Re(z) > 0$, and $N(z) := -N(-z)$, if $\Re(z) < 0$. Provided Ω is large enough, there holds for $z = x + \mathrm{i}y \in \mathbb{C}$, $|z| \geqslant \Omega$,*

$$|G_D(z)| \sim |z|^{-2D} \mathrm{e}^{\pi|y|} \begin{cases} 1, & |\Im(z)| > \frac{1}{2}, \\ |z - t_{D,N}^0|, & |\Im(z)| \leqslant \frac{1}{2}, \end{cases} \tag{3.4}$$

in particular,

$$|G_D(x)| = \mathcal{O}(|x|^{-2D}), \quad x \in \mathbb{R}, \ |x| \to \infty. \tag{3.5}$$

(c) *The sequence $\{G_D'(t_{D,n}^0)\}_{n \in \mathbb{Z}}$ is strictly alternating for large n and*

$$|G_D'(t_{D,n}^0)| \sim (|n|+1)^{-2D}, \quad n \in \mathbb{Z}. \tag{3.6}$$

Proof. (a) From the definition of $G_D(z)$, applying the functional equation of the Gamma function, one obtains

$$\begin{aligned}
G_D(z) &= z \prod_{k=1}^{N_0} \left(1 - \frac{(N_0+1)^2 z^2}{k^2}\right) \prod_{k=N_0+1}^{\infty} \left(1 - \frac{z^2}{(k+D)^2}\right) \\
&= \frac{(-1)^{N_0}(N_0+1)^{2N_0}}{(N_0!)^2} \prod_{k=-N_0}^{N_0} \left(z - \frac{k}{N_0+1}\right) \\
&\quad \times \lim_{M\to\infty} \prod_{k=N_0+1}^{M} \frac{(k+D-z)(k+D+z)}{(k+D)^2} \\
&= \frac{(N_0+1)^{2N_0}\Gamma^2(N_0+1+D)}{(N_0!)^2} \frac{(-1)^{N_0} P_{2N_0+1}(z)}{\Gamma(N_0+1+D-z)\Gamma(N_0+1+D+z)} \\
&\quad \times \lim_{M\to\infty} \frac{\Gamma(M+1+D-z)\Gamma(M+1+D+z)}{\Gamma^2(M+1+D)}.
\end{aligned}$$

For each $z \in \mathbb{C}$, the quotient $\Gamma(M+1+D-z)\Gamma(M+1+D+z)/\Gamma^2(M+1+D)$ tends to 1 as $M \to \infty$ in view of Lemma 2.2(b). Thus (3.1) holds. The other two representation formulas of part (a) can be derived by the reflection formula, e.g., if $z \in \mathbb{C} \setminus \{n+D;\ n \leqslant N_0\}$, one has

$$\frac{1}{\Gamma(N_0+1+D-z)\Gamma(N_0+1+D+z)} = \frac{1}{\Gamma(z-N_0-D)\Gamma(1-(z-N_0-D))} \times \frac{\Gamma(z-N_0-D)}{\Gamma(z+N_0+D+1)}$$
$$= (-1)^{N_0} \frac{\sin \pi(z-D)}{\pi} \frac{\Gamma(z-N_0-D)}{\Gamma(z+N_0+D+1)}.$$

Equation (3.3) could also be proved by substituting $-z$ for z in (3.2) and noting that $G_D(z)$ is an odd function of z.

(b) Both sides of (3.4) are even functions of z; so one can assume $\Re(z) \geqslant 0$ without loss of generality. Let $E_\Omega^+ := \{z \in \mathbb{C};\ \Re(z) \geqslant 0 \text{ and } |z| \geqslant \Omega\}$, and choose $\Omega := N_0 + D + 1$. Then (3.2) is valid on E_Ω^+. Making use of the functional equation of $\Gamma(z)$ again, one obtains

$$P_{2N_0+1}(z)\frac{\Gamma(z-N_0-D)}{\Gamma(z+N_0+D+1)} = \frac{P_{2N_0+1}(z)}{\prod_{k=\lfloor -D \rfloor}^{N_0}(z-k-D)} \frac{\Gamma(z-\lfloor -D \rfloor - D+1)}{\Gamma(z+N_0+D+1)}. \tag{3.7}$$

Since $P_{2N_0+1}(z)$ and $\prod_{k=\lfloor -D \rfloor}^{N_0}(z-k-D)$ are polynomials of degree $2N_0+1$ and $N_0 - \lfloor -D \rfloor + 1$, respectively, the zeros of which are real and no larger than $N_0/(N_0+1)$ and N_0+D, respectively (i.e., outside of E_Ω^+), one has $|P_{2N_0+1}(z)/\prod_{k=\lfloor -D \rfloor}^{N_0}(z-k-D)| \sim |z|^{N_0+\lfloor -D \rfloor}$ on E_Ω^+. The second factor on the right-hand side of (3.7) can be estimated with the help of Lemma 2.2(a). Indeed, noting that $x - \lfloor -D \rfloor - D + 1 = x + \lceil D \rceil - D + 1 \geqslant 1$, $x + N_0 + D + 1 \geqslant 1$ and $|z| \geqslant \Omega \geqslant 1$ on E_Ω^+,

$$|\Gamma(z-\lfloor -D \rfloor - D+1)/\Gamma(z+N_0+D+1)| \sim |z|^{-2D-N_0-\lfloor -D \rfloor}, \quad \text{on } E_\Omega^+,$$

and thus, keeping in mind that C_D is a nonvanishing constant,

$$\left| C_D \cdot P_{2N_0+1}(z) \frac{\Gamma(z-N_0-D)}{\Gamma(z+N_0+D+1)} \right| \sim |z|^{-2D}, \quad z \in E_\Omega^+.$$

In order to give an estimate of $\pi^{-1} \sin \pi(z-D)$, the remaining term of (3.2), assume firstly that $|\Im(z)| > \frac{1}{2}$ and recall the well-known equation $|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi y$. One clearly has

$$0 \leqslant \sin^2 \pi(x-D) \leqslant 1 \leqslant e^{2\pi y}$$

and

$$(1-e^{-\pi})\, e^{2\pi|y|} \leqslant 4 \sinh^2 \pi y = (1-e^{-2\pi|y|})^2\, e^{2\pi|y|} \leqslant e^{2\pi|y|},$$

and it readily follows that $|\pi^{-1} \sin \pi(z-D)| \sim e^{\pi|y|}$. Secondly, let $|\Im(z)| \leqslant \frac{1}{2}$. Then for z to be within E_Ω^+ it is necessary that $\Re(z) > N_0 + D + \frac{1}{2}$; hence $N = N(z) > N_0$ and $t_{D,N}^0 = N + D$. The definition of N and the restrictions on z imply that $\Re(z-N-D) = \Re(z) - N - D \in [-\frac{1}{2}, \frac{1}{2})$, and $\Im(z-N-D) = \Im(z) \in [-\frac{1}{2}, \frac{1}{2}]$. Since $\sin \pi z/(\pi z)$ is a continuous and nonvanishing function on the interval $\{z=x+\mathrm{i}y\in\mathbb{C};\ |x|\leqslant\frac{1}{2},\ |y|\leqslant\frac{1}{2}\}$, $\sin\pi(z-N-D)/\pi(z-N-D)$ is bounded from above and from below by positive constants which do not depend on N, and one has

$$\left|\frac{\sin\pi(z-D)}{\pi}\right|\sim\left|(z-t^0_{D,N})\frac{(-1)^N\sin\pi(z-N-D)}{\pi(z-N-D)}\right|$$
$$\sim|z-t^0_{D,N}|\sim\mathrm{e}^{\pi|y|}|z-t^0_{D,N}|,\quad z\in E^+_\Omega,\ |\Im(z)|\leqslant\tfrac{1}{2}.$$

The estimate (3.5) is obviously correct since one has $y=0$ and

$$|z-t^0_{D,N}|=|z-N-D|=\left(\Re^2(z-N-D)+\Im^2(z-N-D)\right)^{1/2}\leqslant\sqrt{\tfrac{1}{2}},$$

provided $z=x+\mathrm{i}y\in\mathbb{R}$ and x large.

(c) For nonnegative n (no loss of generality, since G'_D is odd and $\{t^0_D\}$ is symmetric) one obtains by a suitable application of the product law on (3.1) and (3.2), respectively,

$$G'_D(t^0_{D,n})=\begin{cases}1, & n=0,\\ \dfrac{(-1)^{N_0}C_D}{\Gamma(N_0+1+D-t^0_{D,n})\Gamma(N_0+1+D+t^0_{D,n})}P'_{2N_0+1}(t^0_{D,n}), & \\ & 1\leqslant n\leqslant N_0,\\ C_DP_{2N_0+1}(t^0_{D,n})\dfrac{\Gamma(n-N_0)}{\Gamma(n+N_0+2D+1)}(-1)^n, & n>N_0.\end{cases}\tag{3.8}$$

The sequence $\{G'_D(t^0_{D,n})\}_{n\in\mathbb{Z}}$ is strictly alternating for $n>N_0$, more precisely $\operatorname{sign}G'_D(t^0_{D,n})=(-1)^n$. In fact, both arguments of the Gamma function are positive $(n+N_0+2D+1>2(N_0+D)+1\geqslant1)$ and the polynomial $P_{2N_0+1}(x)$ does not change its (positive) sign for $x\geqslant1$, since all its zeros lie between -1 and 1.

For $n>N_0$ there holds $P_{2N_0+1}(t^0_{D,n})\sim(n+D)^{2N_0+1}\sim n^{2N_0+1}$, and by Lemma 2.2(a), $\Gamma(n-N_0)/\Gamma(n+N_0+2D+1)\sim n^{-2N_0-1-2D}$. This proves $|G'_D(t^0_{D,n})|\sim(|n|+1)^{-2D}$ for $n>N_0$ and (3.6) follows readily since all zeros of G_D are simple (i.e., $G'_D(t^0_{D,n})\neq0$) and $G'_D(t_{D,n})=-G'_D(t_{D,-n})$. $\square$

Example 3.4. For $D=0,\ \frac{1}{2},\ 1,\ -\frac{1}{2},\ -1$ one obtains by direct calculation or application of Proposition 3.3:

(1) $D=0$, $\{t^0_0\}=(\ldots,3,2,1,0,1,2,3,\ldots)$, i.e., the sequence of integers,

$$N_0=0,\qquad G_0(z)=z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)=\frac{\sin\pi z}{\pi},\qquad G'_0(t^0_{0,n})=(-1)^n,\quad n\in\mathbb{Z};$$

(2) $D=\frac{1}{2}$, $\{t^0_{1/2}\}=(\ldots,-\frac{7}{2},-\frac{5}{2},-\frac{3}{2},0,\frac{3}{2},\frac{5}{2},\frac{7}{2},\ldots)$,

$$N_0=0,\qquad G_{1/2}(z)=z\frac{\cos\pi z}{1-4z^2},\qquad G'_{1/2}(t^0_{1/2,n})=(-1)^n\frac{\pi(n+\frac{1}{2})}{4n(n+1)},\quad n\in\mathbb{N};$$

(3) $D=1$, $\{t_1^0\}=(\ldots,\ -4,\ -3,\ -2,\ 0,\ 2,\ 3,\ 4,\ldots)$,

$$N_0=0,\qquad G_1(z)=\frac{\sin \pi z}{\pi(1-z^2)},\qquad G_1'(t_{1,n}^0)=\frac{(-1)^n}{n(n+2)},\quad n\in\mathbb{N};$$

(4) $D=-\frac{1}{2}$, $\{t_{-1/2}^0\}=(\ldots,\ -\frac{5}{2},\ -\frac{3}{2},\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \ldots)$,

$$N_0=1,\qquad G_{-1/2}(z)=z\cos \pi z,\qquad G'_{-1/2}\left(t_{-1/2,n}^0\right)=(-1)^n\left(n-\tfrac{1}{2}\right)\pi,\quad n\in\mathbb{N};$$

(5) $D=-1$, $\{t_{-1}^0\}=(\ldots,\ -2,\ -1,\ -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ 2,\ \ldots)$,

$$N_0=1,\qquad G_{-1}(z)=(1-4z^2)\frac{\sin \pi z}{\pi},$$

$$G'_{-1}(t_{-1,n}^0)=(-1)^n\left(4(n-1)^2-1\right),\quad n\geqslant 2.$$

When a sampling sequence is modified, the corresponding canonical product changes accordingly. The next two lemmas describe the alterations caused by the most common modifications.

Lemma 3.5. *Let $\{t\}$ be a perturbed equidistant sequence. Let $\kappa\in\mathbb{Z}$, $\sigma>0$ and $\tau\in\mathbb{R}$. Then the sequences $\{t_{\cdot+\kappa}\}$, $\{\sigma\{t\}\}$ and $\{\tau+\{t\}\}$, defined by $\{t_{\cdot+\kappa}\}_n:=t_{n+\kappa}$ (index shift), $\{\sigma\{t\}\}_n:=\sigma t_n$ and $\{\tau+\{t\}\}_n:=\tau+t_n$ $(n\in\mathbb{Z})$, are perturbed equidistant sequences and*

$$G(\{t_{\cdot+\kappa}\};\ z)=G(\{t\};\ z),\qquad z\in\mathbb{C},\tag{3.9}$$

$$G(\{\sigma\{t\}\};\ z)=\sigma^{A}G\left(\{t\};\ \frac{z}{\sigma}\right),\qquad z\in\mathbb{C},\tag{3.10}$$

$$G(\{\tau+\{t\}\};\ z)=C_\tau G(\{t\};\ z-\tau),\qquad z\in\mathbb{C},\tag{3.11}$$

where A denotes the multiplicity of the number zero as a member of $\{t\}$ and C_τ is a suitable nonzero constant.

Proof. It is easy to check that the sequences under consideration are indeed perturbed equidistant sequences, hence the corresponding canonical products are well-defined. Equation (3.9) follows from a comparison of the respective partial sums, noting that $g(t_k;\ z)$ tends to 1 as $|k|\to\infty$, and (3.10) is obvious since $g(\sigma t_n;\ z)=1-z/(\sigma_{tn})=g(t_n;\ z/\sigma)$ if $t_n\neq 0$, and $g(\sigma t_n;\ z)=z=\sigma g(t_n;\ z/\sigma)$ if $t_n=0$. While $C_\tau=1$ in the trivial case $\tau=0$, one finds, for $\tau\neq 0$, that $g(\tau+t_n;\ z)=g_\tau(t_n)g(t_n;\ z-\tau)$, $n\in\mathbb{Z}$, where

$$g_\tau(s)=\begin{cases} g(\tau+s;\ \tau)=(g(s;\ -\tau))^{-1}, & s\in\mathbb{R}\setminus\{0,\ -\tau\},\\ -\tau^{-1}=(g(0;\ -\tau))^{-1}, & s=0,\\ g(0;\ \tau)=\tau, & s=-\tau.\end{cases}$$

Thus, (3.11) holds with

$$C_\tau=g_\tau(t_0)\prod_{k=1}^{\infty}g_\tau(t_k)g_\tau(t_{-k}),\quad \tau\neq 0.\tag{3.12}$$

Since each single factor of this product is different from zero and since the product (3.12) itself converges in view of $g_\tau(t_k)g_\tau(t_{-k})=1+\mathcal{O}(k^{-2})$ for $k\to\infty$, $C_\tau\neq 0$, as claimed. □

Lemma 3.6. *Let $\{t\}$ be a perturbed equidistant sequence. Let I_1, $I_2 \in \mathbb{N}$ and $I := I_1 - I_2$. Assume that the members $t_1, \ldots, t_{I_1}$ are replaced by $t'_1, \ldots, t'_{I_2}$ (all real-valued), and that the new sequence be $\{t'\}$ (formally: $t'_n := t_n$ for $n \leqslant 0$, $t'_n := t_{n+I}$ for $n > I_2$). The modified sequence $\{t'\}$ is a perturbed equidistant sequence, and*

$$G(\{t'\}; z) = \frac{\prod_{k=1}^{I_2} g(t'_k; z)}{\prod_{k=1}^{I_1} g(t_k; z)} G(\{t\}; z), \tag{3.13}$$

with removable singularities at t_k, $k = 1, \ldots, I_1$. Provided Ω is chosen large enough, there holds

$$|G(\{t'\}; z)| \sim |z|^{-I} |G(\{t'\}; z)|, \quad z \in \mathbb{C}, \ |z| \geqslant \Omega. \tag{3.14}$$

Proof. It is a trivial exercise to show that $\{t'\}$ is perturbed equidistantly; (3.13) may be proved by a comparison of the respective partial sums, and (3.14) holds since the modulus of a quotient of polynomials of degree I_1 and I_2 behaves like $|z|^{-1}$ provided z stays away from the zeros of both polynomials (as can be achieved by choosing, say, $\Omega := \max\{|t_k|, k = 1, \ldots, I_1, |t'_k|, k = 1, \ldots, I_2\} + 1$). □

Remark 3.7. Combining (3.12) with (3.9) (index shift), one can describe how the canonical product is changed by an arbitrary replacement of a finite number of knots (with arbitrary indices), as well as by dropping or adding finitely many numbers.

Remark 3.8. The sequences $\{t_D^0\}$, $2D \in \mathbb{Z}$, can be interpreted as modifications of $\{t_0^0\}$, the sequence of integers (in the sense of the preceding lemmas). Indeed, let $J \in \mathbb{N}$. Then $\{t_J^0\}$ can be obtained from $\{t_0^0\}$ by dropping $\pm 1, \ldots, \pm J$; $\{t_{(2J-1)/2}^0\}$ may be constructed by dropping $\pm \frac{1}{2}, \ldots, \pm \frac{1}{2}J$ from $\{\frac{1}{2} + \{t_0^0\}\}$ and adding 0 to it; $\{t_{-J}^0\}$ consists of $\{t_0^0\}$ plus the additional numbers $\pm 1/(J+1), \ldots, \pm J/(J+1)$; and $\{t_{-(2J-1)/2}^0\}$ is that modification of the sequence $\{\frac{1}{2} + \{t_0^0\}\}$ that results if the numbers $\pm \frac{1}{2}$ are discarded and $0, \pm 1/(J+1), \ldots, \pm J/(J+1)$ are added.

These observations can be noted as: "If $2D \in \mathbb{Z}$, the sequence $\{t_D^0\}$ has $2D$ members less (i.e., $-2D$ members more) than the sequence of integers". Thus, the introduction of a real-valued parameter D more or less generalizes the notion of dropping or adding points. However, one has to be very careful when using this characterization, since of course all the sequences $\{t_D^0\}$ (seen as sets) have the same cardinality.

4. The sampling formulas

Theorem 4.1. *Let $D \in \mathbb{R}$, and $\{t_D^0\}$ be as defined in Definition 3.1; I_1, $I_2 \in \mathbb{N}$ and $I := I_1 - I_2$. By $\{t\}$ denote any strictly increasing sequence that is a modification of $\{t_D^0\}$ in the sense that the points $t_{D,1}^0, \ldots, t_{D,I_1}^0$ are replaced by $t_1, \ldots, t_{I_2}$ (cf. Lemma 3.6), and by G denote the canonical product with respect to $\{t\}$. Let $1 \leqslant p \leqslant \infty$.*

If $2D+I<1/p$ (*i.e.*, $2D+I<0$, *if* $p=\infty$) *and* $f\in B_{\pi}^{p}$, *or if* $2D+I<1$ *and* $f\in B_{\beta}^{p}$ *for some* $\beta<\pi$, *then*

$$f(z)=\sum_{n=-\infty}^{\infty} f(t_n)\frac{G(z)}{G'(t_n)(z-t_n)}, \tag{4.1}$$

uniformly on each bounded subset of $\mathbb{C}$.

Proof. Let $R_m := m+D+\frac{1}{2}$, $m\in\mathbb{N}$. By (3.4) and (3.14) one obtains for $-\pi<\theta\leqslant\pi$, provided m is large enough,

$$C_1 R_m^{-2D-I}\,\mathrm{e}^{\pi R_m|\sin\theta|}\leqslant|G(\{t\};\,R_m\,\mathrm{e}^{\mathrm{i}\theta})|\leqslant C_2 R_m^{-2D-I}\,\mathrm{e}^{\pi R_m|\sin\theta|}, \tag{4.2}$$

where $C_2\geqslant C_1>0$ are constants that do not depend on m. Indeed, the factor $|R_m\,\mathrm{e}^{\mathrm{i}\theta}-t_{D,N}^{0}|$, with $N=N(R_m\,\mathrm{e}^{\mathrm{i}\theta})$, which occurs in formula (3.4) when $\Im(R_m\,\mathrm{e}^{\mathrm{i}\theta})\leqslant\frac{1}{2}$, is bounded from above and from below. To verify this, assume that $\Re(R_m\,\mathrm{e}^{\mathrm{i}\theta})>0$ (no loss of generality). Whenever $\Re(z)>0$, there holds $|z-t_{D,N}^{0}|\leqslant|\Re(z-N-D)|+|\Im(z-N-D)|\leqslant 1$ (noting the definition of N and the restrictions imposed on z, cf. the proof of Proposition 3.3(b)). The condition $|\Im(R_m\,\mathrm{e}^{\mathrm{i}\theta})|\leqslant\frac{1}{2}$ implies that $|\sin\theta|\leqslant 1/(2R_m)$, hence $\cos\theta\geqslant 1-\sin^2\theta\geqslant 1-1/(4R_m^2)$ and thus $|\Re(R_m\,\mathrm{e}^{\mathrm{i}\theta}-t_{D,N}^{0})|\geqslant|R_m-t_{D,N}^{0}|-(1-\cos\theta)R_m>\frac{1}{2}-1/(4R_m)$ for large m.

Now Korevaar's inequality (2.1) gives an estimate of band-limited functions from above. If $f\in B_{\beta}^{p}$, $\beta\geqslant 0$, one has for $R>0$, $-\pi<\theta\leqslant\pi$,

$$|f(R\,\mathrm{e}^{\mathrm{i}\theta})|\leqslant C\,\|f\|_p\begin{cases}(1+|R\sin\theta|)^{-1/p}\,\mathrm{e}^{\beta|R\sin\theta|}, & 1\leqslant p<\infty,\\ \mathrm{e}^{\beta|R\sin\theta|}, & p=\infty,\end{cases} \tag{4.3}$$

and in particular

$$|f(y\mathrm{i})|\leqslant C\,\|f\|_p\begin{cases}(1+|y|)^{-1/p}\,\mathrm{e}^{\pi|y|}, & 1\leqslant p<\infty,\\ \mathrm{e}^{\beta|y|}, & p=\infty,\end{cases}\quad y\in\mathbb{R}. \tag{4.4}$$

Let $S_{l,m}$ denote the positively oriented contour that consists of the two semicircles $\{R_m\,\mathrm{e}^{\mathrm{i}\theta};\ -\frac{1}{2}\pi<\theta<\frac{1}{2}\pi\}$ and $\{R_l\,\mathrm{e}^{\mathrm{i}\theta};\ \frac{1}{2}\pi<\theta<\frac{3}{2}\pi\}$ as well as the parts of the imaginary axis that connect the endpoints of these semicircles, i.e., $[R_m\mathrm{i},\,R_l\mathrm{i}]$ and $[-R_l\mathrm{i},\,-R_m\mathrm{i}]$. Let B be an arbitrary bounded subset of the complex plane, and let C_B be a constant such that $B\subset\{z\in\mathbb{C};\ |z|\leqslant C_B\}$.

Now one can define

$$\begin{aligned}\epsilon_{l,m}(z) &:= \frac{G(z)}{2\pi\mathrm{i}}\int_{S_{l,m}}\frac{f(\zeta)}{G(\zeta)(\zeta-z)}\,\mathrm{d}\zeta\\ &= \frac{G(z)}{2\pi\mathrm{i}}\left\{\int_{-\pi/2}^{\pi/2}\frac{f(R_m\,\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}R_m\,\mathrm{e}^{\mathrm{i}\theta}}{G(R_m\,\mathrm{e}^{\mathrm{i}\theta})(R_m\,\mathrm{e}^{\mathrm{i}\theta}-z)}\,\mathrm{d}\theta+\int_{R_m}^{R_l}\frac{f(y\mathrm{i})\mathrm{i}}{G(y\mathrm{i})(y\mathrm{i}-z)}\,\mathrm{d}y\right.\\ &\qquad\left.+\int_{\pi/2}^{3\pi/2}\frac{f(R_l\,\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}R_l\,\mathrm{e}^{\mathrm{i}\theta}}{G(R_l\,\mathrm{e}^{\mathrm{i}\theta})(R_l\,\mathrm{e}^{\mathrm{i}\theta}-z)}\,\mathrm{d}\theta+\int_{-R_l}^{-R_m}\frac{f(y\mathrm{i})\mathrm{i}}{G(y\mathrm{i})(y\mathrm{i}-z)}\,\mathrm{d}y\right\}.\end{aligned}$$

The inequalities (3.4), (3.14), (4.2)–(4.4) can be used to estimate $|\epsilon_{l,m}(z)|$ from above (for m, l large enough). Setting $r:=1/p$ for $1\leqslant p<\infty$ and $r:=0$ for $p=\infty$, and noting that, e.g.,

$|R_m - z| \geqslant R_m - C_B \geqslant CR_m$ on B provided m is large enough, one obtains for all $f \in B^p_\beta$, $\beta \leqslant \pi$, $1 \leqslant p \leqslant \infty$,

$$|\epsilon_{l,m}(z)| \leqslant C \| f \|_p \left\{ R_m^{2D+I} \int_{-\pi/2}^{\pi/2} \frac{e^{(\beta-\pi)R_m|\sin\theta|}}{(1+R_m|\sin\theta|)^r} \, d\theta \right.$$
$$+ R_l^{2D+I} \int_{\pi/2}^{3\pi/2} \frac{e^{(\beta-\pi)R_l|\sin\theta|}}{(1+R_l|\sin\theta|)^r} \, d\theta$$
$$\left. + \left| \left(\int_{R_m}^{R_l} + \int_{-R_l}^{-R_m} \right) \frac{e^{(\beta-\pi)|y|}(1+|y|)^{-r}}{|y|^{-2D-I}(|y|-C_B)} \, dy \right| \right\}.$$

If $\beta = \pi$ and $2D + I < r$, the last estimate implies that

$$|\epsilon_{l,m}(z)| \leqslant C \| f \|_p \begin{cases} R_m^{2D+I-r} + R_l^{2D+I-r}, & 1 < p \leqslant \infty, \\ R_m^{2D+I-1} \log R_m + R_l^{2D+I-r} \log R_l, & p = 1, \end{cases}$$
$$+ C \| f \|_p (\min\{R_l, R_m\})^{2D+I-r},$$

and $\epsilon_{l,m}(z)$ vanishes uniformly on B as $m, l \to \infty$. Now let $0 \leqslant \beta < \pi$. If $R > 0$, then

$$\int_0^{\pi/2} \frac{e^{(\beta-\pi)R\sin\theta}}{(1+R\sin\theta)^r} \, d\theta \leqslant \int_0^{\pi/2} \frac{e^{(\beta-\pi)2R\theta/\pi}}{(1+2R\theta/\pi)^r} \, d\theta = \frac{\pi}{2R} \int_0^R \frac{e^{(\beta-\pi)u}}{(1+u)^r} \, du \leqslant \frac{C}{R}.$$

Since $\int_R^\infty e^{(\beta-\pi)|y|}/|y|^{1+r-2D-I} \, dy = \mathscr{O}(e^{(\beta-\pi)R})$, $R \to \infty$, $|\epsilon_{l,m}(z)|$ vanishes uniformly on B whenever $2D + I < 1$.

The value of the contour integral $\epsilon_{l,m}(z)$ can be explicitly calculated. Indeed, it is easy to see that $f(\zeta)/(G(\zeta)(\zeta - z))$ is a meromorphic function with simple poles at z and at the knots $\{t_n\}_{n\in\mathbb{Z}}$; applying the residue theorem one obtains, provided $z \neq t_n$, $n \in \mathbb{Z}$ and l, m are so large that $z = t_n$, $n \in \mathbb{Z}$,

$$\epsilon_{l,m}(z) = G(z)\left(\operatorname{Res}\left(\frac{f(\cdot)}{G(\cdot)(\cdot - z)}; z \right) + \sum_{\substack{n \text{ with} \\ -R_{-l} < t_n < R_m}} \operatorname{Res}\left(\frac{f(\cdot)}{G(\cdot)(\cdot - z)}; t_n \right) \right) \tag{4.5}$$
$$= f(z) - \sum_{n=-l}^{m} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)}.$$

Since $\epsilon_{l,m}(z)$ tends to zero uniformly on B, the series $\sum_{n=-l}^m f(t_n)/(G'(t_n)(z - t_n))$ tends to $f(z)$ uniformly on $B \setminus \{t_n;\, n \in \mathbb{Z}\}$. This completes the proof since for $z = t_n$, $n \in \mathbb{Z}$, the result is obvious in view of the interpolatory property of the reconstruction functions, i.e.,

$$\lim_{z \to t_n} \frac{G(z)}{G'(t_k)(z - t_k)} = \delta_{nk}, \quad n, k \in \mathbb{Z}, \; n \neq k. \qquad \square$$

Remark 4.2. The assumption that $\{t\}$ is strictly increasing can be weakened. Equation (4.5) remains valid if finitely many of the knots coincide; then one only has to calculate the new values of the residues in order to find the appropriate sampling theorem. For a demonstration, see Example 4.4.

Corollary 4.3. *Let $\sigma > 0$ and $\tau \in \mathbb{R}$. Assume that $\{t\}$ is given as in Theorem 4.1 and that $G(z) := G(\{t\}; z)$. Let $1 \leqslant p \leqslant \infty$.*

If $2D + I < 1/p$ and $f \in B^p_{\pi/\sigma}$, or if $2D + I < 1$ and $f \in B^p_\beta$ for some $\beta < \pi$, then

$$f(z) = \sum_{n=-\infty}^{\infty} f(\tau + \sigma t_n) \frac{G((z-\tau)/\sigma)}{G'(t_n)((z-\tau)/\sigma - t_n)} \tag{4.6}$$

$$= \sum_{n=-\infty}^{\infty} f(\tau + \sigma t_n) \frac{H(z)}{H'(\tau + \sigma t_n)(z - (\tau + \sigma t_n))}, \quad z \in \mathbb{C}, \tag{4.7}$$

where $H(z)$ is the canonical product corresponding to the sequence $\{\tau + \sigma t_n\}_{n \in \mathbb{Z}}$.

Proof. Let $f \in B^p_{\beta/\sigma}$, $\beta \leqslant \pi$. Setting $g(z) := f(\tau + \sigma z)$, the function g fulfils $g|_{\mathbb{R}} \in L^p(\mathbb{R})$ and $|g(z)| \leqslant C \exp((\beta/\sigma)|\sigma y|)$ for $z = x + iy \in \mathbb{C}$, i.e., $g \in B^p_\beta$. Hence Theorem 4.1 can be applied and one obtains

$$g(z) = \sum_{n=-\infty}^{\infty} g(t_n) \frac{G(z)}{G'(t_n)(z - t_n)}, \quad z \in \mathbb{C}.$$

Substituting $(z - \tau)/\sigma$ for z yields (4.6), and the representation formula (4.7) holds since, by Lemma 3.5, $H(z) = C_\tau \sigma^\Lambda G((z-\tau)/\sigma)$ and $H'(\tau + \sigma t_n) = C_\tau \sigma^{\Lambda - 1} G'(t_n)$. □

Example 4.4. Let $\{t^V\}$ be the sequence obtained by adding a second occurrence of 0 to $\{t^0_0\}$, i.e., $\{t^V\} = (\ldots, -3, -2, -1, 0, 0, 1, 2, 3, \ldots)$. An alternative construction of $\{t^V\}$ is to modify $\{-\frac{1}{2} + \{t^0_{-1/2}\}\}$ by replacing $-\frac{1}{2}$ by 0. According to which construction is chosen, $\{t^V\}$ corresponds to the parameter set $D = 0$, $I = -1$ or $D = -\frac{1}{2}$, $I = 0$ (cf. Remark 3.8). Assume that $f \in B^\infty_\pi$. Theorem 4.1 may then be applied since $2D + I = -1 < 0$; however, since $\{t^V\}$ is not strictly increasing, the residues in (4.5) have to be recalculated. Noting that $G(z) := G(\{t^V\}; z) = (z/\pi) \sin \pi z$, one has (provided $z \neq 0$)

$$\begin{aligned}
\operatorname{Res}\left(\frac{f(\cdot)}{G(\cdot)(\cdot - z)}; 0\right) &= \left(\frac{f(u)u^2}{(u/\pi)(\sin \pi u)(u - z)}\right)'\Bigg|_{u=0} \\
&= \lim_{h \to 0}\left(\frac{f(0)}{(h-z)z} + \frac{f(h) - f(0)}{(h-z)(1/\pi)\sin \pi h}\right. \\
&\qquad \left. + \frac{f(0)}{h - z}\,\frac{\pi h - \sin \pi h}{h \sin \pi h}\right) \\
&= -\left(\frac{f(0)}{z^2} + \frac{f'(0)}{z}\right),
\end{aligned}$$

and one obtains for $z \in \mathbb{C} \setminus \mathbb{Z}$, uniformly on bounded subsets,

$$\begin{aligned}
f(z) &= -\sum_{n \in \mathbb{Z}} G(z) \operatorname{Res}\left(\frac{f(\cdot)}{G(\cdot)(\cdot - z)}; n\right) \\
&= (f(0) + zf'(0)) \frac{\sin \pi z}{\pi z} + \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{z}{n} \frac{\sin \pi(z - n)}{\pi(z - n)}.
\end{aligned} \tag{4.8}$$

The last formula, which obviously holds for $z \in \mathbb{Z}$ as well, is usually attributed to Valiron [6, p.12]. While the classical sampling series does not reconstruct all functions in B_π^∞ (e.g., it fails to reproduce $\sin \pi z \in B_\pi^\infty$), Valiron's interpolation series holds for all band-limited functions that are bounded on the real line, although only one additional term $(f'(0))$ is needed. According to Theorem 4.1 and Remark 4.2, all sequences with $2D + I < 0$ give rise to a sampling formula valid throughout B_π^∞. Since $2D + I = -1$ for $\{t^V\}$ and $2D + I = 0$ for $\{t_0^0\}$, Theorem 4.1 is a stronger result than Valiron's formula. By the way, it also allows one to calculate sampling series "of Valiron type" with derivative values, e.g., if one knot a, $a \notin \mathbb{Z}$, is added, one has for all $f \in B_\pi^\infty$,

$$f(z) = f(a)\frac{\sin \pi z}{\sin \pi a} + \sum_{n\in\mathbb{Z}} f(n)\frac{z-a}{n-a}\,\frac{\sin \pi(z-n)}{\pi(z-n)},$$

or if the knot 0 is replaced by the two knots $-\frac{1}{3}$ and $\frac{1}{3}$, the corresponding reconstruction formula, again valid for all $f \in B_\pi^\infty$, reads

$$f(z) = f(\tfrac{1}{3})\frac{(z+\frac{1}{3})\sin \pi z}{\sqrt{3}\,z} - f(-\tfrac{1}{3})\frac{(z-\frac{1}{3})\sin \pi z}{\sqrt{3}\,z}$$
$$+ \sum_{n\in\mathbb{Z}\setminus\{0\}} f(n)\frac{1-9z^2}{1-9n^2}\,\frac{n}{z}\,\frac{\sin \pi(z-n)}{\pi(z-n)}.$$

Theorem 4.5. *As in Theorem* 4.1, *let* $\{t\}$ *be a sequence characterized by the parameters* D, I_1, I_2 *and* I. *Let* $G(z) := G(\{t\}; z)$ *and* $1 \leqslant p \leqslant \infty$.

(a) *Let* $\{u\} = \{u_n\}_{n\in\mathbb{Z}}$ *be a sequence with* (2.2). *If* $f \in B_\beta^p$ *for some* $\beta \geqslant 0$ *and if*

$$2D + I \begin{cases} \leqslant 1, & p = 1, \\ < 1/p, & 1 < p \leqslant \infty, \end{cases}$$

then the series

$$\sum_{n=-\infty}^{\infty} \left| f(u_n)\frac{G(z)}{G'(t_n)(z-t_n)} \right| \tag{4.9}$$

converges uniformly on each bounded subset of $\mathbb{C}$.

(b) *If* $f \in B_\beta^p$ *for some* $\beta < \pi$ *and* $2D + I < 1/p$, *the sampling series* (4.1) *converges absolutely and uniformly, and its sum is* f.

Proof. (a) Let B denote an arbitrary bounded subset of $\mathbb{C}$. The product G is bounded on B since it is an entire function. From the definition of $\{t\}$ it is clear that $t_n \sim n$ and $|z - t_n| \geqslant C \cdot n$, provided $|n|$ is large enough, say $|n| \geqslant n_0 \in \mathbb{N}$. By (3.13) and the product law, one obtains for $n \leqslant 0$ or $n > I_2$,

$$G'(t_n) = G_D'(t_n)\prod_{k=1}^{I_2} g(t_k; t_n) \Big/ \prod_{k=1}^{I_1} g(t_{D,k}^0; t_n).$$

Since $t_n = t_{D,n}$, $n \leqslant 0$, and $t_n = t_{D,n+I}$, $n > I_2$, Proposition 3.3 implies that $|G'(t_n)| \geqslant C \cdot (|n|+1)^{-2D-I}$, and even $|G'(t_n)| \geqslant C \cdot |n|$, $n \in \mathbb{Z}$, as all zeros of G are simple.

Let $m > l \geqslant n_0$. By Hölder's inequality (with q as the conjugate index of p), the estimates just discussed and Nikol'skiĭ's inequality (2.3), one finds

$$\sum_{l\leqslant|n|\leqslant m}\left|f(u_n)\frac{G(z)}{G'(t_n)(z-t_n)}\right|$$

$$\leqslant\begin{cases}\displaystyle\sum_{l\leqslant|n|\leqslant m}|f(u_n)|\sup_{l\leqslant|n|\leqslant m}\left|\frac{G(z)}{G'(t_n)(z-t_n)}\right|, & p=1,\\ \displaystyle\Big(\sum_{l\leqslant|n|\leqslant m}|f(u_n)|^p\Big)^{1/p}\Big(\sum_{l\leqslant|n|\leqslant m}\left|\frac{G(z)}{G'(t_n)(z-t_n)}\right|^q\Big)^{1/q}, & 1<p<\infty,\\ \displaystyle\sup_{l\leqslant|n|\leqslant m}|f(u_n)|\sum_{l\leqslant|n|\leqslant m}\left|\frac{G(z)}{G'(t_n)(z-t_n)}\right|, & p=\infty,\end{cases}$$

$$\leqslant\begin{cases}\displaystyle C\cdot\sum_{l\leqslant|n|\leqslant m}|f(u_n)|\sup_{|n|\geqslant n_0}\{|n|^{2D+I-1}\}, & p=1,\\ \displaystyle C\cdot\|f\|_p\Big(\sum_{l\leqslant|n|\leqslant m}|n|^{(2D+I-1)q}\Big)^{1/q}, & 1<p\leqslant\infty.\end{cases}\tag{4.10}$$

The expressions thus obtained vanish under the assumptions of the theorem. Indeed, if $p=1$ and $2D+I\leqslant 1$, then the upper line of (4.10) tends to zero as $l\to\infty$ since $\{f(u_n)\}_{n\in\mathbb{Z}}\in l^1$ (Nikol'skiĭ's inequality); and if $1<p<\infty$ and $2D+I<1/p$, or if $p=\infty$ and $2D+I<0$, the exponent $(2D+I-1)q$ in the second line of (4.10) is less than -1, i.e., this expression vanishes as $l\to\infty$. Now Cauchy's convergence criterion implies the uniform convergence of (4.9).

(b) The sequence $\{t\}$ fulfils (2.2). Thus the assertion follows from part (a) and Theorem 4.1. □

Remark 4.6. The sampling formula (4.1) does not hold for all values of the parameter D. There are two main reasons for this fact. Firstly, for some sequences the sampled values at these knots may not uniquely determine f, e.g., $\sin\pi z$ and the null function both have $f(n)=0$ for all $n\in\mathbb{Z}$, thus there cannot be a sampling formula that reproduces all functions in B_π^∞ from their values at the integers.

Secondly, the series (4.5) may fail to converge. The following two examples illustrate this case. Let $\{t\}=\{t_D^0\}$ and $G=G_D$.

(i) Assume that $D>\frac{1}{2}$, $D\notin\mathbb{N}$. If $f(z)=(\sin\pi z)/(\pi z)\in B_\pi^2$ and $z=1$ ($\neq t_{D,n}^0$ for all $n\in\mathbb{Z}$), one has in view of Proposition 3.3(c)

$$\frac{f(t_n)}{G'(t_n)(z-t_n)}=\frac{\sin\pi(n+D)}{\pi(n+D)(1-n-D)C_{D,n}(|n|+1)^{-2D}}$$

$$=\frac{-(|n|+1)^{2D}}{(n+D)(n+D-1)}\frac{\sin\pi D}{\pi C_{D,n}},\quad n>N_0,$$

where $0 < C_1 \leqslant C_{D,n} \leqslant C_2$, $n \in \mathbb{Z}$, for suitable constants C_1, C_2. Since $(|n|+1)^{2D}/(n+D)(n+D-1) \sim n^{2D-2}$, $n > N_0$, the series (4.5) does not converge at $z=1$.

(ii) Assume that $D > 1$, $D \notin \mathbb{N}$, and let $f_L(z) := z^{-1}(\sin(\pi/L)z)^{L-1}$, $L \in \mathbb{N}$, $L > 1$. It is easy to show that $f \in B^2_{\pi(L-1)/L}$. Noting that for $n = 2kL$ ($k \in \mathbb{N}$ large)

$$f_L(t^0_{D,n}) = \left(\sin\frac{\pi}{L}(2kL+D)\right)^{L-1} = \left(\sin\frac{\pi D}{L}\right)^{L-1}$$

and estimating as in (i), one sees that the sequence

$$\left\{\frac{f_L(t^0_{D,n})G_D(1)}{G'(t^0_{D,n})(1-t^0_{D,n})}\right\}_{n\in\mathbb{N}}$$

contains a nonvanishing subsequence and thus cannot converge.

In [20,21] Yao and Thomas state that (4.1) holds whenever $\{t\}$ is a perturbed equidistant sequence with (2.2) (L arbitrary) and $f \in B^2_\beta$ with $\beta < \pi$. The calculations in (ii) imply that this statement cannot be true. However, it can be shown that under the assumptions made by Yao and Thomas the function f is determined by the values $f(t_n)$, $n \in \mathbb{Z}$ (cf. [9, p.81]).

Remark 4.7. The first author to study nonuniform sampling was Yen [22]. His paper contains a couple of very good ideas and several of his examples are particular cases of Theorem 4.1. Unfortunately his investigations lack some rigour, because he does not check whether his irregular sampling series converge or not.

5. Error estimates

The errors that may occur in uniform sampling have been intensively studied (see [6, pp. 15–23], [12, pp. 1583–1589], [1, pp. 82–89] and the references cited there). However, there are almost no papers on error estimates if the underlying sampling sequence is nonuniform. Only very recently Feichtinger presented a paper [8] which deals with error analysis in irregular sampling theory, but he generalizes the uniform sampling theorem in another direction than that used here (roughly speaking, he interprets the sampling series as a sum of translates, while we regard it as a generalized Lagrange interpolation formula). In this sense, the results of this section on error estimates of irregular samplings expansions are the first of their kind.

Assume that $\{t\}$ is a perturbed equidistant sequence with separated members (i.e., $\exists\delta > 0$ $\forall n \in \mathbb{Z}\; t_{n+1} - t_n > \delta$), and let $G(z) := G(\{t\}; z)$. The four most common error types are the following.

(1) The *truncation error*

$$T_N f(z) := f(z) - \sum_{n=-N}^{N} f(t_n)\frac{G(z)}{G'(t_n)(z-t_n)}. \tag{5.1}$$

(2) The *amplitude* (or *round-off*) *error*

$$A_\epsilon f(z) := f(z) - \sum_{n=-\infty}^{\infty} \tilde{f}(t_n)\frac{G(z)}{G'(t_n)(z-t_n)}. \tag{5.2}$$

This error occurs when the actual samples $f(t_n)$ are rounded; more precisely, when $f(t_n)$ is replaced by $\tilde{f}(t_n)$, the nearest member of the sequence $\{2\epsilon n\}_{n\in\mathbb{Z}}$, $\epsilon > 0$.

(3) The *time-jitter error*

$$J_\delta f(z) := f(z) - \sum_{n=-\infty}^{\infty} f(\tilde{t}_n) \frac{G(z)}{G'(t_n)(z-t_n)}, \tag{5.3}$$

due to sampling f not at the right knots t_n but at $\tilde{t}_n$ where $|\tilde{t}_n - t_n| \leqslant \delta$, all $n \in \mathbb{Z}$.

(4) The *aliasing error*

$$R_W f(z) := f(z) - \sum_{n=-\infty}^{\infty} f\left(\frac{t_n}{W}\right) \frac{G(Wz)}{G'(t_n)(Wz-t_n)}, \tag{5.4}$$

which is of particular importance if f is not band-limited and hence cannot be exactly reconstructed by a single sampling sum.

We only deal with the case that $\{t\} = \{t_D^0\}$ for some $D \in [-\frac{1}{2}, 0]$. Our results are generalizations of the results obtained for the regular case ($D = 0$). The most important element of proof will be the next result.

Proposition 5.1. *Let* $1 < p < \infty$ *and* $1/p + 1/q = 1$.

(a) *There holds* ($D = 0$)

$$\left(\sum_{n=-\infty}^{\infty} \left| \frac{\sin \pi(x-n)}{\pi(x-n)} \right|^q \right)^{1/q} \leqslant p, \quad x \in \mathbb{R}. \tag{5.5}$$

(b) *Let* $-\frac{1}{2} \leqslant D < 0$. *Then*

$$\left(\sum_{n=-\infty}^{\infty} \left| \frac{G_D(x)}{G_D'(t_{D,n}^0)(x - t_{D,n}^0)} \right|^q \right)^{1/q} \leqslant Cp, \quad x \in \mathbb{R}, \tag{5.6}$$

for a suitable constant C (*independent of* p).

Proof. (a) See [6, p.18] and [18, p.50].

(b) For abbreviation set

$$\psi_{D,n}(x) := \frac{G_D(x)}{G_D'(t_{D,n}^0)(x - t_{D,n}^0)}.$$

The symmetry properties of G_D and $\{t_D^0\}$ imply that $(\sum_{n=-\infty}^{\infty} |\psi_{D,n}(x)|^q)^{1/q} = (\sum_{n=-\infty}^{\infty} |\psi_{D,n}(-x)|^q)^{1/q}$. Hence it suffices to verify (5.6) for $x \geqslant 0$. Let $N_0 := \max\{0, \lceil -D \rceil\} = 1$, and choose $\Omega = N_0 + D + 1 = 2 + D$ as in Proposition 3.3(b). On the bounded set $[0, \Omega]$ one can proceed as in the proof of Theorem 4.5; one obtains (n_0 suitably chosen)

$$\left(\sum_{n=-\infty}^{\infty} |\psi_{D,n}(x)|^q \right)^{1/q} \leqslant C \left(\sum_{|n|<n_0} 1 + \sum_{|n| \geqslant n_0} |n|^{(2D-1)q} \right)^{1/q} \leqslant C \leqslant C \cdot p, \quad x \in [0, \Omega].$$

Now let $x > \Omega$. Defining N as in Proposition 3.3(b), one has $N = N(x) = \lfloor x - D + \frac{1}{2} \rfloor$, i.e., $x \in [N + D - \frac{1}{2},\ N + D + \frac{1}{2})$ and

$$\begin{aligned} |G_D(x)| &\leqslant C \cdot |x|^{-2D} |x - (N+D)| \\ &\leqslant C \cdot |x|^{-2D} \leqslant C \cdot \left(N + D + \tfrac{1}{2}\right)^{-2D}, \quad x > \Omega. \end{aligned} \tag{5.7}$$

Recall that by Proposition 3.3(c)

$$\left|G_D'(t_{D,n}^0)\right| \geqslant C \cdot (|n|+1)^{-2D}, \quad n \in \mathbb{Z}.$$

An application of these estimates yields (noting that $N \geqslant 2$, $-2D > 0$, $1 + 2D \geqslant 0$)

$$\begin{aligned} \sum_{n=-\infty}^{-1} |\psi_{D,n}(x)|^q &\leqslant C^q \Bigg\{ \left| \frac{\left(N + D + \frac{1}{2}\right)^{-2D}}{2^{-2D}(N+D)} \right|^q \\ &\quad + \sum_{n=-\infty}^{-2} \left| \frac{\left(N + D + \frac{1}{2}\right)^{-2D}}{(-n+1)^{-2D}\left(N + D - \frac{1}{2} - (n - D)\right)} \right|^q \Bigg\} \\ &\leqslant C^q \sum_{n=1}^{\infty} \left| \left(\frac{N + D + \frac{1}{2}}{N + n + 2D - \frac{1}{2}} \right)^{-2D} \frac{(n+1)^{2D}}{\left(N + n + 2D - \frac{1}{2}\right)^{1+2D}} \right|^q \\ &\leqslant C^q \sum_{n=1}^{\infty} n^{-q}, \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\lfloor N/2 \rfloor} |\psi_{D,n}(x)|^q &\leqslant C^q \Bigg\{ \left| \frac{\left(N + D + \frac{1}{2}\right)^{-2D}}{N + D - \frac{1}{2}} \right|^q + \left| \frac{\left(N + D + \frac{1}{2}\right)^{-2D}}{2^{-2D}(N + D - 1)} \right|^q \\ &\quad + \sum_{n=2}^{\lfloor N/2 \rfloor} \left| \frac{\left(N + D + \frac{1}{2}\right)^{-2D}}{(n+1)^{-2D}\left(N + D - \frac{1}{2} - (n + D)\right)} \right|^q \Bigg\} \\ &\leqslant C^q \sum_{n=0}^{\lfloor N/2 \rfloor} \left| \left(\frac{N + D + \frac{1}{2}}{\frac{1}{2}(N-1)} \right)^{-2D} \left(\frac{n+1}{\frac{1}{2}(N-1)} \right)^{1+2D} \frac{1}{n+1} \right|^q \\ &\leqslant C^q \sum_{n=1}^{\infty} n^{-q}, \end{aligned}$$

$$\begin{aligned} \sum_{n=\lfloor N/2 \rfloor + 1}^{N-1} |\psi_{D,n}(x)|^q &\leqslant C^q \sum_{n=\lfloor N/2 \rfloor + 1}^{N-1} \left| \frac{\left(N + D + \frac{1}{2}\right)^{-2D}}{(n+1)^{-2D}\left(N + D - \frac{1}{2} - (n + D)\right)} \right|^q \\ &\leqslant C^q \sum_{n=\lfloor N/2 \rfloor + 1}^{N-1} \left| \left(\frac{N + D + \frac{1}{2}}{\lfloor \frac{1}{2} N \rfloor + 2} \right)^{-2D} \frac{1}{N - n - \frac{1}{2}} \right|^q \\ &\leqslant C^q \sum_{n=1}^{\infty} n^{-q}, \end{aligned}$$

$$\sum_{n=N+1}^{\infty} |\psi_{D,n}(x)|^q \leqslant C^q \sum_{n=N+1}^{\infty} \left| \frac{(N+D+\frac{1}{2})^{-2D}}{(n+1)^{-2D}(n+D-(N+D+\frac{1}{2}))} \right|^q$$

$$\leqslant C^q \sum_{n=N+1}^{\infty} \left| \frac{1}{n-N-\frac{1}{2}} \right|^q \leqslant C^q \sum_{n=1}^{\infty} n^{-q},$$

and, in view of inequality (5.7),

$$|\psi_{D,N}(x)| = \left| \frac{G_D(x)}{x-(N+D)} \right| \frac{1}{|G_D'(t_{D,N}^0)|} \leqslant \left(\frac{N+D+\frac{1}{2}}{N+1} \right)^{-2D}$$

$$\leqslant C \leqslant C\left(\sum_{n=1}^{\infty} n^{-q} \right)^{1/q}.$$

Hence by Minkowski's inequality,

$$\left(\sum_{n=-\infty}^{\infty} |\psi_{D,n}(x)|^q \right)^{1/q} \leqslant \left(\sum_{n=-\infty}^{-1} |\psi_{D,n}(x)|^q \right)^{1/q} + \left(\sum_{n=0}^{\lfloor N/2 \rfloor} |\psi_{D,n}(x)|^q \right)^{1/q}$$

$$+ \left(\sum_{n=\lfloor N/2 \rfloor+1}^{N-1} |\psi_{D,n}(x)|^q \right)^{1/q} + |\psi_{D,N}(x)|$$

$$+ \left(\sum_{n=N+1}^{\infty} |\psi_{D,n}(x)|^q \right)^{1/q}$$

$$\leqslant C \cdot \left(\sum_{n=1}^{\infty} n^{-q} \right)^{1/q} \leqslant C \cdot \left(1 + \int_1^{\infty} x^{-q} \, dx \right)^{1/q}$$

$$= C \cdot p^{1/q} \leqslant C \cdot p.$$

The constant C does not depend on p. □

Remark 5.2. The analogue of Proposition 5.1 for $p=1$ holds, too. Indeed, one has $\sup_{n \in \mathbb{Z}} |\psi_{D,n}(x)| \leqslant C$, $x \in \mathbb{R}$, provided $-\frac{1}{2} \leqslant D \leqslant 0$ (cf. [9]).

Proposition 5.3 (Truncation error). *Let* $-\frac{1}{2} \leqslant D \leqslant 0$, $\{t\} = \{t_D^0\}$ *and* $G = G_D$.

(a) *Let* $1 \leqslant p < \infty$ *and* $f \in B_\pi^p$. *Then the series* (4.1) *is uniformly convergent on* $\mathbb{R}$, *i.e.*, $\|T_N f\|_C = o(1)$, $N \to \infty$.

(b) *If* $f \in B_\pi^\infty$ *and*

$$|f(x)| \leqslant M_f |x|^{-\gamma}, \quad |x| \geqslant 1, \text{ for some } \gamma > 0, \tag{5.8}$$

one has

$$\|T_N f\|_C \leqslant C M_f N^{-\gamma} \log N, \quad N \text{ large}. \tag{5.9}$$

Proof. A function $f \in B_\pi^\infty$ that satisfies (5.8) belongs to B_π^p for $p > \max\{1, 1/\gamma\}$. Thus under the assumptions of (a) and (b) the sampling theorem (Theorem 4.1) holds, and by Hölder's inequality and Proposition 5.1 one has for $1<p<\infty$ (case $p=1$ is similar),

$$\begin{aligned}\|T_N f\|_C &= \Big\| \sum_{|n|>N} f(t_{D,n}^0)\psi_{D,n}(x) \Big\|_C \\ &\leqslant \Big(\sum_{|n|>N} |f(t_{D,n}^0)|^p \Big)^{1/p} \sup_{x\in\mathbb{R}} \Big(\sum_{|n|>N} |\psi_{D,n}(x)|^q \Big)^{1/q} \\ &\leqslant Cp \Big(\sum_{|n|>N} |f(t_{D,n}^0)|^p \Big)^{1/p}.\end{aligned}$$

Assertion (a) follows since by Nikol'skiĭ's inequality $\{f(t_{D,n}^0)\}_{n\in\mathbb{Z}} \in l^p$. As to (b), assume that (5.8) is fulfilled, $p>1/\gamma$ and N large. Then

$$\begin{aligned}\sum_{|n|>N} |f(t_{D,n}^0)|^p &\leqslant 2M_f^p \sum_{n=N+1}^{\infty} |t_{D,n}^0|^{-\gamma p} \\ &\leqslant 2C^pM_f^p \sum_{n=N+1}^{\infty} n^{-\gamma p} \leqslant \frac{2C^pM_f^p}{\gamma p-1} N^{1-\gamma p},\end{aligned}$$

$$\|T_N f\|_C \leqslant CM_f(\gamma p-1)^{-1/p} N^{(1-\gamma p)/p} \leqslant CM_f N^{-\gamma}\, \mathrm{e}^{(1/p)\log N + \log p}.$$

Choosing $p=\log N$ the right-hand side of the last inequality becomes minimal and one obtains (5.9). □

Proposition 5.4 (Aliasing error). *Let* $-\frac{1}{2} \leqslant D \leqslant 0$, $\{t\} := \{t_D^0\}$ *and* $G := G_D$. *For all* $f \in \mathrm{UCB}(\mathbb{R})$ *with*

(i) $|f(x)| \leqslant M_f |x|^{-\gamma}$, $|x| \geqslant 1$, *for some* $\gamma \in (0,1]$, (5.10)

(ii) $f^{(r)} \in \mathrm{Lip}_L(\alpha; \mathrm{UCB}(\mathbb{R}))$, *for some* $\alpha \in (0,1]$, $r \in \mathbb{N}_0$,

there holds

$$\|R_W\|_C \leqslant C(f,r,\alpha,\gamma) W^{-r-\alpha} \log W, \quad W \textit{ large}.$$

Proof. This assertion for $D=0$ is proved in [6, Theorem 3.9, pp. 18–19]; (see also [2, pp. 262–264] and [3, pp. 190–192]). The estimates for $-\frac{1}{2} \leqslant D < 0$ can be derived in a similar way with some modifications due to the special form of the sampling points as will be sketched below.

For all $f \in \mathrm{UCB}(\mathbb{R})$ with (5.10) one has $\mathrm{VP}_{\pi W/2} f \in B_{\pi W}^p$ for all $p > 1/\gamma$; hence by Theorem 4.1 and Corollary 4.3,

$$\mathrm{VP}_{\pi W/2} f(x) = \sum_{n=-\infty}^{\infty} \mathrm{VP}_{\pi W/2} f\Big(\frac{t_{D,n}^0}{W}\Big) \frac{G_D(Wx)}{G_D'(t_{D,n}^0)(Wx - t_{D,n}^0)}.$$

Thus the aliasing error can be estimated as follows:

$$\begin{aligned}\|R_W f\|_C &\leqslant \|f - \mathrm{VP}_{\pi W/2} f\|_C \\ &\quad + \Big\| \sum_{n=-\infty}^{\infty} \Big(\mathrm{VP}_{\pi W/2} f\Big(\frac{t_{D,n}^0}{W}\Big) - f\Big(\frac{t_{D,n}^0}{W}\Big)\Big) \frac{G_D(Wx)}{G_D'(t_{D,n}^0)(Wx - t_{D,n}^0)} \Big\|_C.\end{aligned}$$

One has $\| f - \mathrm{VP}_{\pi W/2} f \|_C \leqslant C W^{-r-\alpha}$. By Hölder's inequality and Proposition 5.1(b) one obtains, provided $1/\gamma < p < \infty$,

$$\left\| \sum_{n=-\infty}^{\infty} \left(\mathrm{VP}_{\pi W/2} f\left(\frac{t_{D,n}^0}{W}\right) - f\left(\frac{t_{D,n}^0}{W}\right)\right) \frac{G_D(W_x)}{G_D'(t_{D,n}^0)(Wx - t_{D,n}^0)} \right\|_C$$

$$\leqslant Cp \left(\sum_{n=-\infty}^{\infty} \left| \mathrm{VP}_{\pi W/2} f\left(\frac{t_{D,n}^0}{W}\right) - f\left(\frac{t_{D,n}^0}{W}\right)\right|^p \right)^{1/p} \leqslant Cp(S_1 + S_2), \tag{5.11}$$

where

$$S_1 := \left(\sum_{n=-N}^{N} \left| \mathrm{VP}_{\pi W/2} f\left(\frac{t_{D,n}^0}{W}\right) - f\left(\frac{t_{D,n}^0}{W}\right)\right|^p \right)^{1/p} \leqslant C(2N+1)^{1/p} W^{-r-\alpha}, \tag{5.12}$$

$$S_2 := \left(\sum_{|n|>N} \left| \mathrm{VP}_{\pi W/2} f\left(\frac{t_{D,n}^0}{W}\right) - f\left(\frac{t_{D,n}^0}{W}\right)\right|^p \right)^{1/p}$$

$$\leqslant C \left(\sum_{|n|>N} \left| \frac{t_{D,n}^0}{W} \right|^{-\gamma p} \right)^{1/p} \leqslant C W^{\gamma} N^{(1-p\gamma)/p}, \tag{5.13}$$

noting Lemma 2.3. The desired result follows now by choosing $N = \lfloor W^{1+(r+\alpha)/\gamma} + 1 \rfloor$ and $p = (1 + (r+\alpha)/\gamma) \log W$ in (5.11)–(5.13). □

Proposition 5.5 (Round-off error, time-jitter error). *Let* $-\frac{1}{2} \leqslant D \leqslant 0$, $\{t\} := \{t_D^0\}$ *and* $G := G_D$. *For all* $f \in B_\pi^\infty$ *with* (5.10) *there holds*

$$\| A_\epsilon f \|_C \leqslant C(f, \gamma, D) \epsilon \log\left(\frac{1}{\epsilon}\right), \qquad \| J_\delta f \|_C \leqslant C(f, \gamma, D) \delta \log\left(\frac{1}{\delta}\right),$$

provided $\epsilon > 0$ *or* $\delta > 0$ *are sufficiently small.*

Proof. The case $D = 0$ is already known (see [2, pp. 265–269], [3, pp. 198–202], [5, pp. 104–106], [6, pp. 20–22]). In view of (5.10), $f \in B_\pi^p$ for all $p > 1/\gamma$, Theorem 4.1 may be applied and one has (cf. (5.2), (5.3))

$$(A_\epsilon f)(x) =: (E^{(1)} f)(x) = \sum_{n=-\infty}^{\infty} \left(f(t_{D,n}^0) - \tilde{f}(t_{D,n}^0) \right) \frac{G_D(x)}{G_D'(t_{D,n}^0)(x - t_{D,n}^0)},$$

$$(J_\delta f)(x) =: (E^{(2)} f)(x) = \sum_{n=-\infty}^{\infty} \left(f(t_{D,n}^0) - f(\tilde{t}_{D,n}^0) \right) \frac{G_D(x)}{G_D'(t_{D,n}^0)(x - t_{D,n}^0)}.$$

Let $\epsilon_n^{(1)} := f(t_{D,n}^0) - \tilde{f}(t_{D,n}^0)$. By definition of the round-off error, $|\epsilon_n^{(1)}| \leqslant \epsilon := \epsilon^{(1)}$, and (5.10) implies that $|\epsilon_n^{(1)}| \leqslant |f(t_{D,n}^0)| \leqslant C|n|^{-\gamma}$, $n \in \mathbb{Z} \setminus \{0\}$. Let $\delta_n := t_{D,n}^0 - \tilde{t}_{D,n}^0$, $\epsilon_n^{(2)} := f(t_{D,n}^0) - f(\tilde{t}_{D,n}^0)$. Since $f \in B_\pi^\infty$, one may apply the mean value theorem and Bernstein's inequality, obtaining $|\epsilon_n^{(2)}| = |\delta_n f'(\xi_n)| \leqslant \delta\pi \| f \|_\infty =: \epsilon^{(2)}$ for a suitable point ξ_n between $t_{D,n}^0$ and $\tilde{t}_{D,n}^0$.

By condition (5.10), $|\epsilon_n^{(2)}| \leqslant C(|t_{D,n}^0|^{-\gamma} + |\tilde{t}_{D,n}^0|^{-\gamma}) \leqslant C|n|^{-\gamma}$, $n \in \mathbb{Z}\setminus\{0\}$. Using Hölder's inequality, Proposition 5.1 and the estimates just discussed, one finds for $j = 1, 2$,

$$\| E^{(j)}f \|_C = \left\| \sum_{n=-\infty}^{\infty} \epsilon_n^{(j)} \frac{G_D(x)}{G_D'(t_{D,n}^0)(x - t_{D,n}^0)} \right\|_C \leqslant C_p \left(\sum_{n=-\infty}^{\infty} |\epsilon_n^{(j)}|^p \right)^{1/p}$$

$$\leqslant C_p \left(\left(\sum_{n=-N}^{N} |\epsilon_n^{(j)}|^p \right)^{1/p} + \left(\sum_{|n|>N} |\epsilon_n^{(j)}|^p \right)^{1/p} \right)$$

$$\leqslant C_p \left((2N+1)^{1/p} \epsilon^{(j)} + \left(\sum_{|n|>N} |n|^{-\gamma p} \right)^{1/p} \right)$$

$$\leqslant C_p \left((2N+1)^{1/p} \epsilon^{(j)} + N^{(1-\gamma p)/p} \right).$$

If one sets $N = (\epsilon^{(j)})^{-1/\gamma}$ and $p = (4/\gamma)\log(1/\epsilon^{(j)})$, the inequality reads

$$\| E^{(j)}f \|_C \leqslant C\epsilon^{(j)} \log\left(\frac{1}{\epsilon^{(j)}} \right), \quad j = 1, 2.$$

This completes the proof. □

Remark 5.6. The estimates of $|\epsilon_{l,m}(z)|$ given in the proof of Theorem 4.1 can be readily restated as results concerning the truncation error on bounded subsets of $\mathbb{C}$. Indeed, if the assumptions of Theorem 4.1 are fulfilled, B is a bounded subset of $\mathbb{C}$ and N is large, one has

$$\max_{z\in B} |T_N f(z)| = \max_{x \in B} |\epsilon_{N,N}(z)|$$

$$\leqslant C \| f \|_p \begin{cases} N^{2D+I-1/p}, & 1 < p \leqslant \infty \quad \text{and} \quad \beta = \pi, \\ N^{2D+I-1} \log N, & p = 1 \quad \text{and} \quad \beta = \pi, \\ N^{2D+I-1}, & 1 \leqslant p \leqslant \infty \quad \text{and} \quad \beta < \pi, \end{cases} \tag{5.14}$$

(note that $R_N = N + D + \frac{1}{2} \sim N$).

The classical sampling series is often considered inadequate for numerical purposes since its truncation error is only slowly decreasing (cf., e.g., [6, p. 23]); indeed, in that case ($p = 2$, $\beta = \pi$, $D = I = 0$) the estimate (5.14) reads $\max_{z\in B} |T_N f(z)| \leqslant C \| f \|_2 \sqrt{N}$. However, as can also be seen from (5.14), there is always one very simple way to speed up the reconstruction process: adding I sampling knots to the given sampling sequence decreases the truncation error by a factor $C \cdot N^{-I}$.

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