JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 57 (1995) 309–317

Some new results for chain-sequence polynomials

Erik A. van Doorn*

Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands

Received 23 October 1992; revised 15 March 1993

Abstract

We study the class $\mathscr{C}$ of (generalized) orthogonal polynomial sequences $\{P_n(x)\}_{n=0}^{\infty}$ satisfying a recurrence relation of the type

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n > 1,$$

where $\lambda_n \neq 0$ and the sequence $\{\lambda_{n+1}/(c_n c_{n+1})\}_{n=1}^{\infty}$ constitutes a *chain sequence*. We obtain a new characterization of $\mathscr{C}$ in terms of the moment sequence associated with an orthogonal polynomial sequence, and contribute to the solution of the problem of determining a (signed) orthogonalizing measure for a member of $\mathscr{C}$.

Keywords: Orthogonal polynomials; Orthogonalizing measure; Quasi-definite moment functional; Kernel polynomials; Zeros; Separation property

1. Introduction

Our starting point will be the familiar three-terms recurrence relation for orthogonal polynomials. Thus consider a sequence of monic polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying

$$\begin{aligned} &P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n > 1, \\ &P_0(x) = 1, \qquad P_1(x) = x - c_1, \end{aligned} \tag{1}$$

where the coefficients are real.

It is well known that when $\lambda_n > 0$ for all $n > 1$ the zeros of $P_n(x)$ are real and distinct, and between each pair of consecutive zeros of $P_{n+1}(x)$ there is precisely one zero of $P_n(x)$. Moreover, there exists a positive Borel measure ψ on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)\psi(\mathrm{d}x) = k_n\delta_{nm} \tag{2}$$

*e-mail: doorn@math.utwente.nl.

SSDI 0377-0427(93)E0254-J

with $k_n > 0$. When $\lambda_n > 0$ for all $n > 1$ we shall refer to $\{P_n(x)\}$ as an *orthogonal polynomial sequence* (*OPS*).

In the more general framework $\lambda_n \neq 0$ for all $n > 1$ we shall refer to $\{P_n(x)\}$ as a *generalized orthogonal polynomial sequence* (*GOPS*). Little can be said in general about the polynomials of a GOPS but for the existence of a finite signed Borel measure ψ on $\mathbb{R}$ such that (2) holds with $k_n \neq 0$. However, as shown in [10], there exists a class of GOPSs which, in general, are not OPSs but have properties resembling those of OPSs as far as zeros are concerned. This class is denoted by $\mathscr{C}$ and defined as follows.

Definition 1. Let $\{P_n(x)\}$ be a GOPS satisfying (1). Then $\{P_n(x)\} \in \mathscr{C}$ if $c_n \neq 0$ for all $n \geqslant 1$ and the sequence $\{\alpha_n\}_{n=1}^{\infty}$ defined by

$$\alpha_n \equiv \lambda_{n+1}/(c_n c_{n+1}) \tag{3}$$

constitutes a *chain sequence*. (That is, there exists a *parameter sequence* $\{g_n\}_{n=0}^{\infty}$ satisfying $g_0 = 0$ and $0 < g_n < 1$, $n \geqslant 1$, such that $\alpha_n = (1 - g_{n-1})g_n$, $n \geqslant 1$.)

The elements of a GOPS in $\mathscr{C}$ will be called *chain-sequence polynomials*. Of course, if $c_n > 0$ for all $n \geqslant 1$ or $c_n < 0$ for all $n \geqslant 1$, and hence $\lambda_n > 0$ for all $n > 1$, then $\{P_n(x)\} \in \mathscr{C}$ constitutes an OPS and we are on familiar grounds. The interesting cases arise when the c_n, and hence the λ_n, differ in sign. The following was proved in [10], see also [5].

Theorem 2. *If $\{P_n(x)\} \in \mathscr{C}$ then the zeros of $P_n(x)$ are real, nonzero and simple, and between each pair of consecutive positive (negative) zeros of $P_{n+1}(x)$ there is precisely one zero of $P_n(x)$.*

The proof in [10] of the reality of the zeros of a chain-sequence polynomial hinges on the result that the sequence $\{P_n^*(x)\}_{n=0}^{\infty}$ of *kernel polynomials* associated with $\{P_n(x)\} \in \mathscr{C}$, defined by

$$P_n^*(x) \equiv x^{-1}(P_{n+1}(x) - P_{n+1}(0)P_n(x)/P_n(0)), \tag{4}$$

constitutes an OPS (see also Section 3). These kernel polynomials play a prominent role again in this paper, which is mainly concerned with orthogonalizing measures for chain-sequence polynomials. Indeed, it will be shown in Section 3 that a (signed) orthogonalizing measure for $\{P_n(x)\} \in \mathscr{C}$ can be constructed in terms of a (positive) orthogonalizing measure for the associated sequence of kernel polynomials $\{P_n^*(x)\}$ provided the latter measure has a finite moment of order -1.

The remainder of this paper is organized as follows. First, in Section 2, we present a new characterization of the class $\mathscr{C}$. Then, in Section 3, we obtain the result mentioned above and address related issues such as the status of the Hamburger moment problem for a sequence of kernel polynomials. Finally, in Section 4, we discuss a separation property of the zeros of a sequence of chain-sequence polynomials, in relation to the zeros of the associated kernel polynomials.

2. Characterizations

The known characterizations of $\mathscr{C}$ are collected in the next theorem, see [10].

Theorem 3. *Let $\{P_n(x)\}$ be a GOPS satisfying the recurrence relation* (1). *Then the following are equivalent:*

(i) $\{P_n(x)\} \in \mathscr{C}$;

(ii) *there exists a (unique) sequence of real numbers $\{\gamma_n\}_{n=2}^{\infty}$ such that, for all $n \geqslant 1$, $c_n = \gamma_{2n-1} + \gamma_{2n}$ ($\gamma_1 \equiv 0$), $\lambda_{n+1} = \gamma_{2n}\gamma_{2n+1}$ and $\gamma_{2n+1}\gamma_{2n+2} > 0$;*

(iii) *for all $n \geqslant 1$ one has $\lambda_{n+1}c_nc_{n+1} > 0$ and $(-1)^n c_1c_2 \cdots c_nP_n(0) > 0$.*

As an aside we observe from the third characterization that $\mathscr{C}$ is a subclass of the class of polynomial sequences studied by Sato [8], whose results may be invoked to obtain an alternative proof of Theorem 2.

Before establishing a fourth characterization of $\mathscr{C}$ we introduce some notation and results, see [2]. Let $\{P_n(x)\}$ then be any GOPS. With $\mathscr{L}$ denoting the corresponding moment functional we let $\mu_n \equiv \mathscr{L}[x^n]$, $n \geqslant 0$, and

$$\Delta_n \equiv \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \cdots & \cdots & \cdots \\ \mu_n & \cdots & \mu_{2n} \end{vmatrix}, \quad \Delta_n^{(1)} \equiv \begin{vmatrix} \mu_1 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots \\ \mu_{n+1} & \cdots & \mu_{2n+1} \end{vmatrix}, \qquad n \geqslant 0.$$

Then we have $\Delta_n \neq 0$ for all $n \geqslant 0$, and for the sake of definiteness we assume throughout $\mu_0 = \Delta_0 = 1$, which is no restriction of generality. The moment functional $\mathscr{L}^*$ is subsequently defined in terms of $\mathscr{L}$ by

$$\mathscr{L}^*[x^n] \equiv \mu_{n+1} \quad (\equiv \mathscr{L}[x^{n+1}]), \quad n \geqslant 0. \tag{5}$$

By [2, Theorem I.7.1] we know that if $P_n(0) \neq 0$ for all n, then $\mathscr{L}^*$ is quasi-definite and the polynomial sequence $\{P_n^*(x)\}$ defined by (4) constitutes the GOPS corresponding to $\mathscr{L}^*$.

When $\{P_n(x)\} \in \mathscr{C}$ we do know from Theorem 2 that $P_n(0) \neq 0$, while, in addition, $\{P_n^*(x)\}$ constitutes an OPS, as shown in [10] (see also Section 3). It follows that $\mathscr{L}^*$ is actually positive-definite in this case, provided $\mathscr{L}^*[1] = \mu_1 > 0$. We are now ready to prove the new characterization of $\mathscr{C}$.

Theorem 4. *Let $\{P_n(x)\}$ be a GOPS satisfying* (1). *Then*

$$\{P_n(x)\} \in \mathscr{C} \text{ and } c_1 > 0 \iff \Delta_n^{(1)} > 0 \text{ for all } n \geqslant 0.$$

Proof. First suppose $\{P_n(x)\} \in \mathscr{C}$ and $c_1 > 0$. Since $\mathscr{L}^*[1] = \mu_1 = \mathscr{L}[x] = c_1 > 0$, and hence $\mathscr{L}^*$ is positive-definite, it follows (with evident notation) that $\Delta_n^{(1)} = \Delta_n^* > 0$, $n \geqslant 0$.

Next let $\{P_n(x)\}$ be a GOPS satisfying (1) and $\Delta_n^{(1)} > 0$, $n \geqslant 0$. By [2, Theorem I.4.2] and [2, Example I.3.1] we have

$$\lambda_{n+1} = \Delta_{n-2}\Delta_n/\Delta_{n-1}^2, \quad n \geqslant 1,$$

and

$$P_n(0) = (-1)^n \Delta_{n-1}^{(1)}/\Delta_{n-1} \neq 0, \quad n \geq 0,$$

respectively, where $\Delta_{-1} = \Delta_{-1}^{(1)} \equiv 1$. Defining

$$\gamma_{2n} \equiv -P_n(0)/P_{n-1}(0), \quad \gamma_{2n+1} \equiv -\lambda_{n+1}P_{n-1}(0)/P_n(0), \qquad n \geq 1, \tag{6}$$

it follows that

$$\gamma_2 = c_1 = \mu_1 = \Delta_0^{(1)} > 0,$$

and, for $n > 0$,

$$\gamma_{2n}\gamma_{2n+1} = \lambda_{n+1}$$

and

$$\gamma_{2n+1} + \gamma_{2n+2} = -(\lambda_{n+1}P_{n-1}(0) + P_{n+1}(0))/P_n(0) = c_{n+1}.$$

Finally,

$$\gamma_{2n+1}\gamma_{2n+2} = \Delta_{n-2}^{(1)}\Delta_n^{(1)}/(\Delta_{n-1}^{(1)})^2 > 0, \quad n \geq 1,$$

and hence $\{P_n(x)\} \in \mathscr{C}$ by Theorem 3. $\square$

Concluding this section we note that a GOPS $\{P_n(x)\}$ satisfying (1) with $c_1 < 0$ can of course be renormalized to satisfy $c_1 > 0$. In fact, by considering the sequence $\{c_1^{-n}P_n(c_1x)\}$ instead of $\{P_n(x)\}$, one can get the normalization $c_1 = \Delta_0^{(1)} = 1$.

3. Orthogonalizing measures

Let $\{P_n(x)\} \in \mathscr{C}$ and let $\mathscr{L}$ be the corresponding moment functional. In what follows it will be convenient to use the second characterization in Theorem 3, that is, there exist real numbers $\{\gamma_n\}_{n=2}^{\infty}$ such that $\gamma_2 \neq 0$ and $\gamma_{2n+1}\gamma_{2n+2} > 0$ for all $n \geq 1$, while $\{P_n(x)\}$ satisfies the recurrence

$$\begin{aligned} &P_n(x) = (x - \gamma_{2n-1} - \gamma_{2n})P_{n-1}(x) - \gamma_{2n-2}\gamma_{2n-1}P_{n-2}(x), \quad n > 1, \\ &P_0(x) = 1, \qquad P_1(x) = x - \gamma_2. \end{aligned} \tag{7}$$

As shown in the previous section one can always normalize $\{P_n(x)\}$ such that

$$\gamma_2 = c_1 > 0, \tag{8}$$

and we shall tacitly assume the validity of (8). Evidently, the numbers γ_n, $n \geq 2$, can be obtained iteratively from the parameters in the recurrence relation (1) satisfied by $\{P_n(x)\}$.

It is shown in [10] that the kernel polynomials (4) associated with $\{P_n(x)\}$ satisfy the recurrence

$$\begin{aligned} &P_n^*(x) = (x - \gamma_{2n} - \gamma_{2n+1})P_{n-1}^*(x) - \gamma_{2n-1}\gamma_{2n}P_{n-2}^*(x), \quad n > 1, \\ &P_0^*(x) = 1, \quad P_1^*(x) = x - \gamma_2 - \gamma_3. \end{aligned} \tag{9}$$

Since $\gamma_{2n-1}\gamma_{2n} > 0$ for $n > 1$, it follows immediately that $\{P_n^*(x)\}$ constitutes an OPS, and hence there exists a positive orthogonalizing measure ψ^* for $\{P_n^*(x)\}$. Since the moment functional $\mathscr{L}^*$ corresponding with $\{P_n^*(x)\}$ can now be represented as

$$\mathscr{L}^*[x^n] = \int_{-\infty}^{\infty} x^n \psi^*(\mathrm{d}x), \quad n = 0, 1, \ldots, \tag{10}$$

the next theorem emerges.

Theorem 5. *If ψ^* is a (positive) orthogonalizing measure with a finite moment of order -1 (in the sense that $\psi^*(\{0\}) = 0$ and the integrals $\int_{(-\infty,0)} x^{-1}\psi^*(\mathrm{d}x)$ and $\int_{(0,\infty)} x^{-1}\psi^*(\mathrm{d}x)$ converge) for the kernel polynomials $\{P_n^*(x)\}$ associated with $\{P_n(x)\} \in \mathscr{C}$, then $\{P_n(x)\}$ constitutes a GOPS with respect to the (signed) measure ψ defined by*

$$\begin{aligned} &\psi(\mathrm{d}x) = x^{-1}\psi^*(\mathrm{d}x), \quad x \neq 0, \\ &\psi(\{0\}) = 1 - \int_{-\infty}^{\infty} x^{-1}\psi^*(\mathrm{d}x). \end{aligned} \tag{11}$$

Proof. Defining the moment functional $\mathscr{L}_\psi$ by

$$\mathscr{L}_\psi[x^n] \equiv \int_{-\infty}^{\infty} x^n \psi(\mathrm{d}x), \quad n = 0, 1, \ldots,$$

it follows from (5), (10) and (11) that $\mathscr{L}_\psi[x^n] = \mathscr{L}^*[x^{n-1}] = \mu_n = \mathscr{L}[x^n]$ for $n > 0$, while $\mathscr{L}_\psi[1] = 1 = \mu_0 = \mathscr{L}[1]$. Hence $\mathscr{L}_\psi = \mathscr{L}$, as required. □

Remark 6. S. Belmehdi and P. Maroni (personal communications) kindly demonstrated that the above result may be obtained in a constructive way by employing the theory developed in [6], see also [7]. In addition, Maroni showed that the representation (10) for $\mathscr{L}^*$ leads to a representation for $\mathscr{L}$ which incorporates the present one but is valid under milder conditions. Then, however, we go beyond the setting of finite (signed) Borel measures.

We observe that the measure ψ defined by (11) is positive (negative) on the positive (negative) real axis. Evidently, a finite orthogonalizing measure for $\{P_n(x)\}$ with this property can exist *only* if there exists a (positive) orthogonalizing measure for $\{P_n^*(x)\}$ with a finite moment of order -1. It may be shown that when $\gamma_n > 0$ for all $n > 1$, there always exists a positive measure for $\{P_n^*(x)\}$ with a finite moment of order -1. In general, however, this is not the case as the next example shows.

Example 7. Let $\gamma_2 = \frac{1}{2}\sqrt{2}$ and, for $n \geqslant 1$, $\gamma_{2n+1} = -\sqrt{2}$ and $\gamma_{2n+2} = \frac{1}{2}\sqrt{2}$, and $\{P_n(x)\}$ and $\{P_n^*(x)\}$ the polynomial sequences satisfying the recurrences (7) and (9), respectively. Defining $Q_n^*(x) = (-1)^n P_n^*(-2x - \frac{3}{2}\sqrt{2})$, it is easy to see that $\{Q_n^*(x)\}$ satisfies the recurrence

$$\begin{aligned} &Q_n^*(x) = 2xQ_{n-1}^*(x) - Q_{n-2}^*(x), \quad n > 1, \\ &Q_0^*(x) = 1, \qquad Q_1^*(x) = 2x + \sqrt{2}. \end{aligned}$$

According to [2, pp. 205–206], $\{Q_n^*(x)\}$ is orthogonal with respect to a (unique) positive measure which has zero mass outside the interval $(-1, 1)$ with the exception of a point mass $\frac{1}{2}$ at the point $-\frac{3}{4}\sqrt{2}$. It follows that $\{P_n^*(x)\}$ is orthogonal with respect to a (unique) positive measure which has zero mass outside the interval $(-2-\frac{3}{2}\sqrt{2}, 2-\frac{3}{2}\sqrt{2})$ with the exception of a point mass $\frac{1}{2}$ at 0. So we cannot use (11) to obtain an orthogonalizing measure for $\{P_n(x)\}$.

The problem thus arises of finding a criterion in terms of $\{\gamma_n\}$ for the existence of a positive measure with a finite moment of order -1 for $\{P_n^*(x)\}$. Before discussing this problem, however, we address the problem of finding a criterion for the existence of a *unique* positive measure for $\{P_n^*(x)\}$, that is, we will look into the status of the Hamburger moment problem (Hmp) for $\{P_n^*(x)\}$.

A criterion due to Hamburger, see [9, Theorem 2.17], tells us that the Hmp for $\{P_n^*(x)\}$ is determined if and only if

$$\sum_{n=0}^{\infty} ((p_n^*(0))^2 + (p_n^{*(1)}(0))^2) = \infty, \tag{12}$$

where $\{p_n^*(x)\}$ are the *orthonormal* polynomials and $\{p_n^{*(1)}(x)\}$ the *orthonormal numerator polynomials* associated with $\{P_n^*(x)\}$. We recall that the *monic numerator polynomials* $\{P_n^{*(1)}(x)\}$ associated with $\{P_n^*(x)\}$ satisfy the recurrence (9) with γ_n replaced by γ_{n+2}, see [2]. Obviously, whether (12) holds true or not does not depend on the normalization one chooses for the moment functionals $\mathscr{L}^*$ and $\mathscr{L}^{*(1)}$ associated with $\{P_n^*(x)\}$ and $\{P_n^{*(1)}(x)\}$, respectively. But for the sake of definiteness (and in concurrence with (5)) we let

$$\mathscr{L}^*[1] \equiv \gamma_2 \quad \text{and} \quad \mathscr{L}^{*(1)}[1] \equiv \gamma_2\gamma_4. \tag{13}$$

We also define

$$H_n \equiv \prod_{i=1}^{n} (\gamma_{2i+1}/\gamma_{2i+2}), \quad K_n \equiv \sum_{j=0}^{n} \prod_{i=1}^{j} (\gamma_{2i}/\gamma_{2i+1}), \qquad n \geq 0, \tag{14}$$

where an empty product denotes unity. With these conventions we are ready to compute the terms in (12).

We first observe with induction from (9) that

$$P_n^*(0) = (-1)^n K_n \prod_{i=1}^{n} \gamma_{2i+1}. \tag{15}$$

Exploiting the relation between monic orthogonal and orthonormal polynomials, see e.g. [2, Eq. (I.4.10)], it subsequently follows after some algebra that

$$(p_n^*(0))^2 = \gamma_2^{-1} H_n K_n^2, \quad n \geq 0. \tag{16}$$

Next proceeding in the same manner with respect to the numerator polynomials, we readily obtain

$$P_n^{*(1)}(0) = (-1)^n \gamma_2^{-1} (K_{n+1} - 1) \prod_{i=1}^{n+1} \gamma_{2i+1}, \quad n \geq 0, \tag{17}$$

and

$$(p_n^{*(1)}(0))^2 = \gamma_2^{-2} H_{n+1}(K_{n+1} - 1)^2, \quad n \geq 0, \tag{18}$$

so that we get the following theorem.

Theorem 8. *The Hmp for $\{P_n^*(x)\}$ is determined if and only if*

$$\sum_{n=0}^{\infty} H_n(K_n^2 + (K_n - 1)^2) = \infty .$$

Remark 9. When $\gamma_n > 0$ for all $n \geqslant 2$ (and hence $\{P_n(x)\}$ is an OPS), $\{K_n\}$ is increasing so that one has $K_n \geqslant K_0 = 1$ for all n. It follows from the above theorem that in this case the Hmp for $\{P_n^*(x)\}$ is determined if and only if $\Sigma H_n K_n^2 = \infty$, which is in accordance with [3, Theorem 3].

When the Hmp for $\{P_n^*(x)\}$ is indeterminate then, see [9, Theorem 2.13], there are infinitely many positive Borel measures ψ^* with discrete support and zero mass at 0 with respect to which $\{P_n^*(x)\}$ constitutes an OPS. Hence, in this case there are infinitely many finite signed Borel measures ψ of the type (11) with respect to which $\{P_n(x)\}$ is orthogonal. An interesting question is whether there exists a "best" measure for $\{P_n^*(x)\}$, that is, an (extremal) measure whose (discrete) support coincides with the set of accumulation points of the zeros of all $P_n^*(x)$. Defining

$$L_n(z) \equiv -\gamma_2 P_{n-1}^{*(1)}(z)/P_n^*(z), \tag{19}$$

Chihara [4] shows (recall the normalization (13)) that the answer to this question is positive if and only if either $\{L_n(0)\}_n$ converges or $\{|L_n(0)|\}_n$ tends to ∞. (Actually, the present statement involves a minor correction of Chihara's formulation.) From (15) and (17) we note that

$$L_n(0) = 1 - 1/K_n, \tag{20}$$

so that the condition is met if and only if either $\{K_n\}$ converges or $\{|K_n|\}$ tends to ∞. Obviously, this "best" measure for $\{P_n^*(x)\}$, if it exists, may have a point mass at 0, in which case it does not lead to a measure for $\{P_n(x)\}$ via (11). From [9, Theorem 2.13] we readily observe that this happens if and only if $K_n \to 0$ as $n \to \infty$. We summarize the preceding results in the next theorem.

Theorem 10. *If the Hmp for $\{P_n^*(x)\}$ is indeterminate and either $K_n \to K \neq 0$ or $|K_n| \to \infty$ as $n \to \infty$ then there exists a unique (extremal) measure ψ^* for $\{P_n^*(x)\}$ whose support is discrete and coincides with the set of accumulation points of the zeros of all $P_n^*(x)$, while $\{P_n(x)\}$ is orthogonal with respect to the measure ψ which is well defined in terms of ψ^* by* (11).

When the Hmp for $\{P_n^*(x)\}$ is determined, so that there is a unique positive measure ψ^* for $\{P_n^*(x)\}$, the problem of finding conditions on $\{\gamma_n\}$ for the measure ψ^* to have a finite moment of order -1 is unsolved, but we conjecture the following.

Conjecture 11. Let the Hmp for $\{P_n^*(x)\}$ be determined. The orthogonalizing measure ψ^* for $\{P_n^*(x)\}$ has a finite moment of order -1 if and only if either $K_n \to K \neq 0$ or $|K_n| \to \infty$ as $n \to \infty$, in which case

$$\int_{-\infty}^{\infty} x^{-1}\psi^*(\mathrm{d}x) = 1 - K^{-1}, \tag{21}$$

which should be interpreted as 1 if $|K_n| \to \infty$ as $n \to \infty$.

Circumstantial evidence for the conjecture is provided by Theorem 10, and by (20) together with a generalization of *Markov's Theorem* to the effect that

$$\lim_{n\to\infty} L_n(z) = \int_{-\infty}^{\infty} (x-z)^{-1}\psi^*(\mathrm{d}x), \quad z\in\mathbb{C}\backslash\Lambda, \tag{22}$$

where

$$\operatorname{supp}(\psi^*) \subseteq \Lambda \equiv \bigcap_{N=1}^{\infty} \overline{\bigcup_{n=N}^{\infty} \Lambda_n}$$

and Λ_n denotes the set of zeros of P_n^*, see Berg [1] and Wall [11]. Also recall that in Example 7, in which ψ^* has no finite moment of order -1, we have $K_n = (\frac{1}{2})^n \to 0$ as $n\to\infty$.

Finally, one might wonder whether the OPS associated with a positive measure ψ^* with a finite moment of order -1 constitutes a sequence of kernel polynomials associated with a sequence of chain-sequence polynomials. It can readily be verified that the answer to this question is positive if and only if the moment functional associated with the measure ψ defined by (11) is quasi-definite, which is certainly not always the case.

4. A separation property

Let $\{P_n(x)\}\in\mathscr{C}$ and $\{P_n^*(x)\}$ the associated sequence of kernel polynomials. We let x_{nk} and x_{nk}^*, $k=1,2,\dots,n$ denote the (real) zeros of $P_n(x)$ and $P_n^*(x)$, respectively, and assume that they are numbered in increasing order of magnitude. From [10] we recall the following refinement of Theorem 2, where $x_{n0}\equiv-\infty$ and $x_{n,n+1}\equiv\infty$.

Theorem 12. *The number of positive (negative) zeros of $P_n(x)$ equals the number of positive (negative) elements in the set $\{c_1, c_2, \dots, c_n\}$; moreover one has, for $k=1,2,\dots,n+1$,*

$$x_{n+1,k} < x_{nk} < x_{n+1,k+1} \quad \textit{if } x_{n+1,k} > 0$$

and

$$x_{n+1,k-1} < x_{nk} < x_{n+1,k} \quad \textit{if } x_{n+1,k} < 0.$$

Now using the second representation of Theorem 3, we see from (4) and (6) that

$$xP_n^*(x) = P_{n+1}(x) + \gamma_{2n+2}P_n(x), \tag{23}$$

from which it follows that

$$P_{n+1}(x_{nk}) = x_{nk}P_n^*(x_{nk}), \quad k=1,2,\dots,n, \tag{24}$$

and

$$\gamma_{2n+2}P_n(x_{n+1,k}) = x_{n+1,k}P_n^*(x_{n+1,k}), \quad k=1,2,\dots,n+1. \tag{25}$$

In view of Theorem 12 and recalling that $c_{n+1}=\gamma_{2n+1}+\gamma_{2n+2}$ while $\gamma_{2n+1}\gamma_{2n+2}>0$ we conclude from (24) that

$$c_{n+1}>0 \Rightarrow x_{nk} < x_{nk}^* < x_{n,k+1}, \quad k=1,2,\dots,n,$$

and

$$c_{n+1} < 0 \Rightarrow x_{n,k-1} < x_{nk}^* < x_{nk}, \quad k = 1,2, \dots, n,$$

and from (25) that

$$x_{n+1,k} < x_{nk}^* < x_{n+1,k+1}, \quad k = 1,2, \dots, n+1.$$

Letting $[a,b]^+ \equiv \max\{a,b\}$ and $[a,b]^- \equiv \min\{a,b\}$, we can summarize the preceding results as follows.

Theorem 13. *For all $n = 1,2, \dots$ and $k = 1,2, \dots, n$ one has*

$$[x_{nk}, x_{n+1,k}]^+ < x_{nk}^* < [x_{n,k+1}, x_{n+1,k+1}]^- \quad \text{if } c_{n+1} > 0$$

and

$$[x_{n,k-1}, x_{n+1,k}]^+ < x_{nk}^* < [x_{nk}, x_{n+1,k+1}]^- \quad \text{if } c_{n+1} < 0.$$

We finally remark that the maxima and minima in Theorem 13 depend on the signs of the zeros involved and can be determined from Theorem 12.

Acknowledgements

The author would like to thank S. Belmehdi, C. Berg, T.S. Chihara, A.A. Jagers and P. Maroni for their interest and comments.

References

[1] C. Berg, Markov's theorem revisited, *J. Approx. Theory* **78** (1994) 260–275.
[2] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).
[3] T.S. Chihara, Indeterminate symmetric moment problems, *J. Math. Anal. Appl.* **85** (1982) 331–346.
[4] T.S. Chihara, Orthogonal polynomials with discrete spectra on the real line, *J. Approx. Theory* **42** (1984) 97–105.
[5] E.K. Ifantis and P.N. Panagopoulos, On the zeros of a class of polynomials defined by a three term recurrence relation, *J Math. Anal. Appl.* **182** (1994) 361–370.
[6] P. Maroni, Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques, in: M. Alfaro et al., Eds., *Orthogonal Polynomials and Their Applications*, Lecture Notes in Math. **1329** (Springer, Berlin, 1988) 279–290.
[7] P. Maroni, Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x-c)^{-1}L$, *Period. Math. Hungar.* **21** (1990) 223–248.
[8] K.-I. Sato, On zeros of a system of polynomials and application to sojourn time distributions of birth-and-death processes, *Trans. Amer. Math. Soc.* **309** (1988) 375–390.
[9] J. Shohat and J.D. Tamarkin, *The Problem of Moments*, Math. Surveys No. 1, revised edition (Amer. Math. Soc., Providence, RI, 1963).
[10] E.A. van Doorn, On a class of generalized orthogonal polynomials with real zeros, in: C. Brezinski, L. Gori and A. Ronveaux, Eds., *Orthogonal Polynomials and Their Applications* (Baltzer, Basel, 1991) 231–236.
[11] H.S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, New York, 1948).