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Laguerre–Sobolev orthogonal polynomials

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Abstract

In this paper, polynomials that are orthogonal with respect to the inner product

$$(f, g)_s = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x} dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x} dx$$

where $\alpha > -1$ and $\lambda \geq 0$, are studied. For these nonstandard orthogonal polynomials algebraic and differential properties as well as the relation with the classical Laguerre polynomials are obtained. Finally, some properties concerning the localization and separation of the zeros of these polynomials are deduced.

Keywords: Laguerre polynomials; Sobolev orthogonal polynomials; Zeros

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1. Introduction

In the last decades, several authors have devoted considerable attention to some particular examples of orthogonal polynomials in Sobolev spaces [2, 4, 6, 9, 10, 12]. From a numerical point of view, the approximation by spectral methods of differential equations in bounded or unbounded domains is in general performed by orthogonal polynomials, Jacobi or Laguerre polynomials (basically) [7, 8, 14]. Convergence results are derived from approximation properties of projection operators in

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the discrete spaces and the convergence estimates for these operators are obtained in appropriate weighted Sobolev spaces. However spectral methods based on Sobolev orthogonal polynomials have not been developed yet. One of the reasons for this situation may be the fact that properties about zeros as well as estimates for such a kind of polynomials are unknown.

Not much is known about Sobolev orthogonal polynomials in the general case. Only some of the usual properties of orthogonal polynomials can be translated to the more general context of Sobolev spaces. In fact, most algebraic properties fail (for example, the three term recurrence relation is lost), and, more importantly, the zeros of the orthogonal polynomials need not be simple, real and contained in the support of the underlying measures.

The above-mentioned authors have investigated in some detail several particular cases. In the present work we follow the same direction and we are trying to make a wide study of polynomials that are orthogonal with respect to the inner product

$$(f, g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x} dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x} dx \quad (1.1)$$

where $\alpha > -1$ and $\lambda \geq 0$. Specifically, we are going to say that a n th degree monic polynomial $Q_n^{(\alpha)}(x)$ is the n th *monic Laguerre–Sobolev orthogonal polynomial* if

$$(Q_n^{(\alpha)}, x^m)_S = 0, \quad m = 0, 1, \dots, n-1. \quad (1.2)$$

These orthogonal polynomials have been considered in [4] in the particular case $\alpha = 0$, and therefore some of our results generalizes those obtained by this author. Furthermore, Laguerre–Sobolev orthogonal polynomials appear as an interesting example of the so called *coherent pairs* in [10]. In the present paper, we recover, with a different technique, the results concerning some algebraic properties of Laguerre–Sobolev orthogonal polynomials. These results have not been considered in [16], as we can notice with a detailed reading.

In order to make this paper self-contained, we include in Section 2 the properties of the classical Laguerre polynomials needed for the development of our presentation.

In Section 3, Laguerre–Sobolev orthogonal polynomials are introduced and some of their algebraic properties are obtained. In particular, we get a relation between two consecutive orthogonal polynomials and the corresponding classical Laguerre polynomials which can be used to compute the Laguerre–Sobolev orthogonal polynomials.

In Section 4, we introduce a differential operator acting on the linear space of real polynomials. This operator plays an essential role in the remainder of the paper. Firstly it allows us to connect the Sobolev inner product with the inner product defined from the Laguerre weight function.

Secondly, this linear operator is self-adjoint with respect to the Sobolev inner product. That characteristic is deeply related to the differential properties obtained in Section 5, where we deduce the existence of a difference-differential relation, a Christoffel–Darboux-type formula, and a Rodrigues-type formula for the Laguerre–Sobolev orthogonal polynomials.

Finally, Section 6 is devoted to the study of the zeros. We will show that Laguerre–Sobolev orthogonal polynomial $Q_n^{(\alpha)}(x)$ has exactly n different and real roots, separating the roots of the n th Laguerre polynomial. Also, for $\alpha \geq 0$, all the roots are positive, and for $-1 < \alpha < 0$, at most one of them is negative.

2. Classical Laguerre polynomials

We consider the Laguerre inner product

$$(f, g) = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x} dx, \quad (2.1)$$

where $\alpha > -1$.

We will denote by $\{\hat{L}_n^{(\alpha)}(x)\}_n$ the sequence of monic Laguerre–Sonine orthogonal polynomials, that is the MOPS with respect to the inner product (2.1). For the classical Laguerre–Sonine orthogonal polynomials the following properties are known (see [5, pp.144–156; 18, Ch. V]):

Norm

Denote by

$$k_n^{(\alpha)} = \int_0^{+\infty} (\hat{L}_n^{(\alpha)}(x))^2 x^\alpha e^{-x} dx.$$

Then

$$k_n^{(\alpha)} = \Gamma(n + \alpha + 1)n!. \quad (2.2)$$

Three-term recurrence relation

$$x\hat{L}_n^{(\alpha)}(x) = \hat{L}_{n+1}^{(\alpha)}(x) + \beta_n^{(\alpha)}\hat{L}_n^{(\alpha)}(x) + \gamma_n^{(\alpha)}\hat{L}_{n-1}^{(\alpha)}(x), \quad n \geq 1, \quad (2.3)$$

$$\hat{L}_0^{(\alpha)}(x) = 1, \quad \hat{L}_1^{(\alpha)}(x) = x - (\alpha + 1),$$

$$\beta_n^{(\alpha)} = 2n + \alpha + 1, \quad \gamma_n^{(\alpha)} = n(n + \alpha). \quad (2.4)$$

Second-order differential equation

$$xy'' + [\alpha + 1 - x]y' + ny = 0. \quad (2.5)$$

Differential relation

$$\frac{d}{dx}\hat{L}_n^{(\alpha)}(x) = n\hat{L}_{n-1}^{(\alpha+1)}(x). \quad (2.6)$$

Rodrigues formula

$$x^\alpha e^{-x}\hat{L}_n^{(\alpha)}(x) = (-1)^n \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x}]. \quad (2.7)$$

Structure relations

$$x \frac{d}{dx} \hat{L}_n^{(\alpha)}(x) = n\hat{L}_n^{(\alpha)}(x) + n(n + \alpha)\hat{L}_{n-1}^{(\alpha)}(x). \quad (2.8)$$

$$x \frac{d}{dx} \hat{L}_n^{(\alpha)}(x) = -\hat{L}_{n+1}^{(\alpha)}(x) + [x - (n + \alpha + 1)]\hat{L}_n^{(\alpha)}(x). \quad (2.9)$$

From the explicit representation for the monic Laguerre polynomials we can obtain the value of the polynomial and its derivative at the point 0.

Proposition 2.1.

$$\hat{L}_n^{(\alpha)}(0) = \binom{n+\alpha}{n} (-1)^n n! = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} (-1)^n, \quad (2.10)$$

$$(\hat{L}_n^{(\alpha)})'(0) = \binom{n+\alpha}{n-1} (-1)^{n-1} n! = n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+2)} (-1)^{n-1}. \quad (2.11)$$

Moreover, we can relate two families of Laguerre polynomials whose parameters α differ in one.

Proposition 2.2.

$$\hat{L}_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + n L_{n-1}^{(\alpha+1)}(x) = \frac{1}{n+1} (\hat{L}_{n+1}^{(\alpha)})'(x) + (\hat{L}_n^{(\alpha)})'(x). \quad (2.12)$$

As usual, we will denote by

$$K_n(x, y) = \sum_{j=0}^n \frac{\hat{L}_j^{(\alpha)}(x) \hat{L}_j^{(\alpha)}(y)}{k_j^{(\alpha)}},$$

the n th kernel associated to the MOPS $\{L_n^{(\alpha)}\}_n$, and by

$$K_n^{(r,s)}(x, y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} K_n(x, y)$$

the corresponding partial derivatives. For these kernels, it is well known that for every polynomial $p(x)$, with degree less or equal to n , the following equality holds:

$$\int_0^{+\infty} K_n^{(0,s)}(x, y) p(x) x^\alpha e^{-x} dx = p^{(s)}(y), \quad (2.13)$$

where α_j is the j th Fourier coefficient of the polynomial $p(x)$ with respect to the orthogonal family $\{L_n^{(\alpha)}\}_n$. This equality is called *reproducing property of the kernels*.

Moreover, we know (see [5, 18]), that the family of polynomials $\{K_n(x, 0)\}$ is an orthogonal polynomial sequence with respect to the weight function $x^{\alpha+1}e^{-x}$, i.e.,

$$K_n(x, 0) = \frac{\hat{L}_n^{(\alpha)}(0)}{k_n^{(\alpha)}} L_n^{(\alpha+1)}(x). \quad (2.14)$$

3. Laguerre–Sobolev orthogonal polynomials

Now, we consider the inner product defined by

$$(f, g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x} dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x} dx, \quad (3.1)$$

where $\alpha > -1$, $\lambda \geq 0$. We will denote by $\{Q_n^{(\alpha)}\}_n$ the MOPS with respect to the inner product $(\cdot, \cdot)_S$, and we will call such a sequence *Sobolev–Laguerre orthogonal polynomials*.

From the usual expression of orthogonal polynomials as a ratio of determinants, it is very easy to deduce that, for $n \geq 2$, every coefficient of $Q_n^{(\alpha)}(x)$ is a rational function in λ with numerator and denominator of degree $n - 1$. Thus, it makes sense to define the *limit polynomial* with respect to the parameter λ by

$$R_0^{(\alpha)}(x) = Q_0^{(\alpha)}(x) = \hat{L}_0^{(\alpha)}(x) = 1,$$

$$R_1^{(\alpha)}(x) = Q_1^{(\alpha)}(x) = \hat{L}_1^{(\alpha)}(x),$$

$$R_n^{(\alpha)}(x) = \lim_{\lambda \rightarrow \infty} Q_n^{(\alpha)}(x),$$

which is a monic polynomial of exact degree n . From the orthogonality properties of the polynomial $Q_n^{(\alpha)}(x)$, we can deduce for $R_n^{(\alpha)}(x)$, $n \geq 2$

Theorem 3.1. *Let $n \geq 2$. Then*

$$\int_0^{+\infty} R_n^{(\alpha)}(x) x^\alpha e^{-x} dx = 0, \quad (3.2)$$

$$\int_0^{+\infty} (R_n^{(\alpha)})'(x) x^m x^\alpha e^{-x} dx = 0, \quad 0 \leq m \leq n - 2. \quad (3.3)$$

Proof. From the definition of $R_n^{(\alpha)}(x)$ we have

$$\begin{aligned} \int_0^{+\infty} R_n^{(\alpha)}(x) x^\alpha e^{-x} dx &= \int_0^{+\infty} \lim_{\lambda \rightarrow \infty} Q_n^{(\alpha)}(x) x^\alpha e^{-x} dx = \lim_{\lambda \rightarrow \infty} (Q_n^{(\alpha)}, 1)_S = 0, \\ \int_0^{+\infty} (R_n^{(\alpha)})'(x) x^m x^\alpha e^{-x} dx &= \int_0^{+\infty} \lim_{\lambda \rightarrow \infty} (Q_n^{(\alpha)})'(x) x^m x^\alpha e^{-x} dx \\ &= \lim_{\lambda \rightarrow \infty} \left(-\frac{1}{\lambda(m+1)} \right) \int_0^{+\infty} Q_n^{(\alpha)}(x) x^{m+1} x^\alpha e^{-x} dx = 0, \quad 0 \leq m \leq n - 2. \quad \square \end{aligned}$$

In particular, (3.3) implies that

$$(R_n^{(\alpha)})'(x) = n \hat{L}_{n-1}^{(\alpha)}(x), \quad n \geq 2, \quad (3.4)$$

but, by relation (2.6) and Proposition 2.2,

$$R_n^{(\alpha)}(x) = \hat{L}_n^{(\alpha)}(x) + n \hat{L}_{n-1}^{(\alpha)}(x), \quad n \geq 2. \quad (3.5)$$

We must point out that, for $0 < \alpha$, (3.5) implies $R_n^{(\alpha)}(x) = L_n^{(\alpha-1)}(x)$, i.e., $R_n^{(\alpha)}(x)$ is the classical monic Laguerre polynomial of degree n , associated to the weight function $x^{\alpha-1} e^{-x}$. If $-1 < \alpha \leq 0$ then $R_n^{(\alpha)}(x)$ is a quasi-orthogonal polynomial of order one with respect to the MOPS $\{\hat{L}_n^{(\alpha)}\}_n$.

These remarks allow us to relate Laguerre–Sobolev orthogonal polynomials with classical Laguerre polynomials.

Proposition 3.2. *The following relation holds*

$$\hat{L}_n^{(\alpha)}(x) + n \hat{L}_{n-1}^{(\alpha)}(x) = Q_n^{(\alpha)}(x) + d_{n-1}(\lambda) Q_{n-1}^{(\alpha)}(x), \quad (3.6)$$

where

$$d_{n-1}(\lambda) = \frac{n(\hat{L}_{n-1}^{(\alpha)}, \hat{L}_{n-1}^{(\alpha)})}{(Q_{n-1}^{(\alpha)}, Q_{n-1}^{(\alpha)})_S} = \frac{nk_{n-1}^{(\alpha)}}{\tilde{k}_{n-1}^{(\alpha)}} \quad (3.7)$$

or equivalently,

$$R_n^{(\alpha)}(x) = Q_n^{(\alpha)}(x) + d_{n-1}(\lambda)Q_{n-1}^{(\alpha)}(x). \quad (3.8)$$

Proof. We can expand the polynomial $R_n^{(\alpha)}(x)$ in terms of the Laguerre–Sobolev polynomials in the following way:

$$R_n^{(\alpha)}(x) = Q_n^{(\alpha)}(x) + \sum_{i=0}^{n-1} d_i(\lambda)Q_i^{(\alpha)}(x).$$

The coefficients can be easily computed by the expressions

$$d_i(\lambda) = \frac{(R_n^{(\alpha)}, Q_i^{(\alpha)})_S}{(Q_i^{(\alpha)}, Q_i^{(\alpha)})_S} = \frac{\int_0^{+\infty} [\hat{L}_n^{(\alpha)}(x) + n\hat{L}_{n-1}^{(\alpha)}(x)]Q_i^{(\alpha)}(x)x^\alpha e^{-x} dx}{\tilde{k}_i^{(\alpha)}}.$$

Then, $d_i(\lambda) = 0$, $0 \leq i \leq n-2$. Finally,

$$d_{n-1}(\lambda) = \frac{\int_0^{+\infty} [\hat{L}_n^{(\alpha)}(x) + n\hat{L}_{n-1}^{(\alpha)}(x)]Q_{n-1}^{(\alpha)}(x)x^\alpha e^{-x} dx}{\tilde{k}_{n-1}^{(\alpha)}} = \frac{nk_{n-1}^{(\alpha)}}{\tilde{k}_{n-1}^{(\alpha)}}. \quad \square$$

Notice that, according (3.6) we show an example of the so-called coherent pair.

Now, we obtain a recurrence relation for the coefficients $d_{n-1}(\lambda)$ with the shape of a continued fraction

Proposition 3.3.

$$d_n(\lambda) = \frac{(n+1)(n+\alpha)}{n(2+\lambda) + \alpha - d_{n-1}(\lambda)}, \quad n \geq 1, \quad (3.9)$$

with the initial condition $d_1(\lambda) = 2(\alpha+1)/(\lambda+\alpha+1)$.

Proof. Since $(Q_{n+1}^{(\alpha)}, Q_n^{(\alpha)})_S = 0$, if we substitute (3.8), we get

$$0 = (Q_{n+1}^{(\alpha)}, Q_n^{(\alpha)})_S = (R_{n+1}^{(\alpha)}, R_n^{(\alpha)})_S - d_n(\lambda)(R_n^{(\alpha)}, R_n^{(\alpha)})_S + d_n(\lambda)d_{n-1}(\lambda)(R_{n-1}^{(\alpha)}, R_n^{(\alpha)})_S.$$

In this way, we deduce

$$d_n(\lambda) = \frac{(R_{n+1}^{(\alpha)}, R_n^{(\alpha)})_S}{(R_n^{(\alpha)}, R_n^{(\alpha)})_S - d_{n-1}(\lambda)(R_n^{(\alpha)}, R_{n-1}^{(\alpha)})_S},$$

and the only thing we must do is to compute the above inner products. Finally, writing expression (3.6) for $n=2$, we deduce the initial condition. \square

By induction it follows that $d_n(\lambda)$ is a rational function in λ with numerator of degree $n - 1$ and denominator of degree n .

Proposition 3.4. *The coefficient $d_n(\lambda)$ can be expressed as:*

$$d_n(\lambda) = (n+1)(n+\alpha)q_{n-1}(\lambda)/q_n(\lambda),$$

where the polynomials $\{q_n(\lambda)\}_n$ satisfy the three-term recurrence relation

$$q_0(\lambda) = 1,$$

$$q_1(\lambda) = \lambda + \alpha + 1,$$

$$q_n(\lambda) = [n\lambda + 2n + \alpha]q_{n-1}(\lambda) - n(n + \alpha - 1)q_{n-2}(\lambda), \quad n \geq 2.$$

Remark. If we denote $q_n^*(\lambda) = q_n(2\lambda - 2)$, in terms of monic polynomials the above expression becomes

$$\hat{q}_0^*(\lambda) = 1,$$

$$\hat{q}_1^*(\lambda) = \lambda + (\alpha - 1)/2,$$

$$\hat{q}_n^*(\lambda) = \left[\lambda + \frac{\alpha}{2n} \right] \hat{q}_{n-1}^*(\lambda) - \frac{n + \alpha - 1}{4(n-1)} \hat{q}_{n-2}^*(\lambda), \quad n \geq 3. \quad (3.10)$$

If we compare (3.10) with the recurrence formula for the monic *generalized Pollaczek* polynomials, see [5, p. 185], we deduce that the polynomials $\{\hat{q}_n^*(\lambda)\}_n$ are the co-recursive polynomials with parameter $\frac{1}{2}$ for the generalized Pollaczek polynomials (see also [10]).

On the other hand, we know that the sequence $\{d_n(\lambda)\}_n$ is always positive, and from Eq. (3.9), we can deduce the following bound:

$$\frac{(n+1)(n+\alpha)}{(2+\lambda)n+\alpha} < d_n(\lambda) < (2+\lambda)(n+1) + \alpha.$$

In this way $d_n(\lambda) = O(n)$.

If we study the behaviour of $d_n(\lambda)$ with respect to the parameter λ , an inductive reasoning shows that $d_n(\lambda)$ is a strictly decreasing function of λ ; and, since $d_n(\lambda)$ is a rational function in λ with degrees $n - 1$ in the numerator and n in the denominator, respectively, we have

$$\lim_{\lambda \rightarrow \infty} d_n(\lambda) = 0, \quad \forall n \in \mathbb{N}.$$

This last situation corresponds to the marginal case $\lambda \rightarrow +\infty$, where, the limit polynomial $R_n^{(\alpha)}(x)$ arises again.

The marginal value $\lambda = 0$ corresponds to the case where the inner product (3.1) coincides with the usual Laguerre inner product, and the two MOPS are equal, i.e., $Q_n^{(\alpha)}(x) = \hat{L}_n^{(\alpha)}(x)$, $\forall n \in \mathbb{N}$. Therefore

$$d_n(0) = n + 1, \quad \forall n \in \mathbb{N}.$$

In particular, because d_n is a decreasing function on $[0, +\infty)$, then

$$d_n(\lambda) < n + 1.$$

Next, we give an expression for the norm of the polynomials $Q_n^{(\alpha)}(x)$ in terms of the norm of the classical Laguerre polynomials

Proposition 3.5. *We have*

$$\tilde{k}_n^{(\alpha)} = (Q_n^{(\alpha)}, Q_n^{(\alpha)})_S = k_n^{(\alpha)}(1 - \lambda A_n),$$

where $\{A_n\}_n$ is the sequence given by $A_n = (Q_n^{(\alpha)})'(0)/L_n^{(\alpha)}(0)$.

Proof. Using the differential relation (2.6), and the expressions (2.2) and (2.10) for $k_{n-1}^{(\alpha)}$ and $\hat{L}_{n-1}^{(\alpha)}(0)$, we obtain

$$\begin{aligned} (Q_n^{(\alpha)}, Q_n^{(\alpha)})_S &= (Q_n^{(\alpha)}, \hat{L}_n^{(\alpha)})_S \\ &= k_n^{(\alpha)} + \lambda n \int_0^{+\infty} (Q_n^{(\alpha)})'(x) L_{n-1}^{(\alpha+1)}(x) x^\alpha e^{-x} dx \\ &= k_n^{(\alpha)} + \lambda n \frac{k_{n-1}^{(\alpha)}}{L_{n-1}^{(\alpha)}(0)} \int_0^{+\infty} (Q_n^{(\alpha)})'(x) K_{n-1}(x, 0) x^\alpha e^{-x} dx \\ &= k_n^{(\alpha)} + \lambda n \frac{k_{n-1}^{(\alpha)}}{L_{n-1}^{(\alpha)}(0)} (Q_n^{(\alpha)})'(0) = k_n^{(\alpha)} \left[1 - \lambda \frac{(Q_n^{(\alpha)})'(0)}{\hat{L}_n^{(\alpha)}(0)} \right]. \quad \square \end{aligned}$$

4. The linear operator \mathcal{F}

Now, we will use the classical character of the Laguerre's weight function. In fact, we know that $\omega(x) = x^\alpha e^{-x}$ satisfies, on the interval $[0, +\infty)$, the following first-order linear differential equation with polynomial coefficients:

$$x\omega'(x) + (x - \alpha)\omega(x) = 0. \quad (4.1)$$

We can define a linear operator \mathcal{F} on the linear space of the real polynomials \mathbb{P} by means of

$$\mathcal{F} = x\mathcal{I} - \lambda[\alpha - x]D - \lambda xD^2, \quad (4.2)$$

where \mathcal{I} stands for the identity operator, D denotes the differentiation operator and D^2 denotes the second derivative operator. Obviously, the operator \mathcal{F} maps polynomials of exact degree n in polynomials of exact degree $n + 1$.

The role of this operator is shown in the next proposition.

Proposition 4.1. *Let $p(x), q(x) \in \mathbb{P}$ be arbitrary polynomials. Then*

$$(xp(x), q(x))_S = \int_0^{+\infty} p(x) \mathcal{F}q(x) x^\alpha e^{-x} dx = (p(x), \mathcal{F}q(x)).$$

Proof. Integration by parts gives

$$\begin{aligned}(xp(x), q(x))_S &= \int_0^{+\infty} xp(x)q(x)x^\alpha e^{-x} dx + \lambda \int_0^{+\infty} [xp(x)]' q'(x)x^\alpha e^{-x} dx \\ &= \int_0^{+\infty} xp(x)q(x)x^\alpha e^{-x} dx - \lambda \int_0^{+\infty} xp(x)[q'(x)x^\alpha e^{-x}]' dx \\ &= \int_0^{+\infty} p(x)\mathcal{F}q(x)x^\alpha e^{-x} dx. \quad \square\end{aligned}$$

In this section, we will show that the linear operator \mathcal{F} is symmetric with respect to the Sobolev inner product (3.1).

Theorem 4.2. *The linear operator \mathcal{F} is symmetric with respect to the Sobolev inner product (3.1), i.e.*

$$(\mathcal{F}p, q)_S = (p, \mathcal{F}q)_S, \quad \forall p, q \in \mathbb{P}.$$

Proof. If $\alpha > 0$, integrating by parts in the second term, the constant of integration vanishes and we have

$$(\mathcal{F}p, q)_S = \int_0^{+\infty} \mathcal{F}p(x)q(x)x^\alpha e^{-x} dx - \lambda \int_0^{+\infty} \mathcal{F}p(x)[q'(x)x^\alpha e^{-x}]' dx = (p, \mathcal{F}q)_S.$$

For $-1 < \alpha \leq 0$, it suffices to show this result for the elements of the canonical basis. Thus, the same argument shows that, for $n \geq 2$,

$$(\mathcal{F}x^n, x^m)_S = (x^n, \mathcal{F}x^m)_S, \quad \forall m \in \mathbb{N}$$

holds.

Next, if $n = 0, m = 1$ we have $\mathcal{F}(1) = x$, $\mathcal{F}(x) = x^2 - \lambda(\alpha - x)$ and therefore

$$(\mathcal{F}(1), x)_S = (x, x)_S = \Gamma(\alpha + 3) + \lambda\Gamma(\alpha + 1),$$

$$(1, \mathcal{F}(x))_S = (1, x^2 - \lambda(\alpha - x))_S = \Gamma(\alpha + 3) + \lambda\Gamma(\alpha + 1).$$

Finally, the case $n = 1, m = 0$ follows by symmetry and the case $m = 0, n = 0$ is trivial. \square

From the above properties, we can show that the polynomial $x\hat{L}_n^{(\alpha)}(x)$ is a quasi-orthogonal polynomial of order two with respect to the inner product (3.1), and we shall compute the coefficients of this relation in the following way.

Proposition 4.3.

$$x\hat{L}_n^{(\alpha)}(x) = Q_{n+1}^{(\alpha)}(x) + a_n^{(\alpha)}Q_n^{(\alpha)}(x) + a_{n-1}^{(\alpha)}Q_{n-1}^{(\alpha)}(x), \quad (4.3)$$

where

$$a_n^{(n)} = (n + \alpha) + \frac{n + 1}{1 - \lambda A_n} = (n + \alpha) + d_n(\lambda),$$

$$a_{n-1}^{(n)} = \frac{n(n + \alpha)}{1 - \lambda A_{n-1}} = (n + \alpha)d_{n-1}(\lambda).$$

Proof. From (2.3), we get

$$\begin{aligned} x\hat{L}_n^{(\alpha)}(x) &= \hat{L}_{n+1}^{(\alpha)}(x) + \beta_n^{(\alpha)}\hat{L}_n^{(\alpha)}(x) + \gamma_n^{(\alpha)}\hat{L}_{n-1}^{(\alpha)}(x) \\ &= \hat{L}_{n+1}^{(\alpha)}(x) + (n + 1)\hat{L}_n^{(\alpha)}(x) + (n + \alpha)[\hat{L}_n^{(\alpha)}(x) + n\hat{L}_{n-1}^{(\alpha)}(x)], \end{aligned}$$

and, using (3.6), we conclude

$$x\hat{L}_n^{(\alpha)}(x) = Q_{n+1}^{(\alpha)}(x) + d_n(\lambda)Q_n^{(\alpha)}(x) + (n + \alpha)[Q_n^{(\alpha)}(x) + d_{n-1}(\lambda)Q_{n-1}^{(\alpha)}(x)]. \quad \square$$

From Propositions 3.2 and 4.3, we obtain a four-term recurrence relation for Laguerre–Sobolev orthogonal polynomials.

Corollary 4.4. (Four-term recurrence relation). *Laguerre–Sobolev polynomials $\{Q_n^{(\alpha)}\}$ satisfy the following recurrence relation:*

$$\begin{aligned} xQ_n^{(\alpha)}(x) + xd_{n-1}(\lambda)Q_{n-1}^{(\alpha)}(x) \\ = Q_{n+1}^{(\alpha)}(x) + [2n + \alpha + d_n(\lambda)]Q_n^{(\alpha)}(x) \\ + [n(n + \alpha - 1) + (2n + \alpha)d_{n-1}(\lambda)]Q_{n-1}^{(\alpha)}(x) + n(n + \alpha - 1)d_{n-2}(\lambda)Q_{n-2}^{(\alpha)}(x). \end{aligned}$$

On the other hand, we can show that $\mathcal{F}Q_n^{(\alpha)}(x)$ is a quasi-orthogonal polynomial of order two with respect to the classical Laguerre polynomials.

Proposition 4.5. *The following relation holds:*

$$\mathcal{F}Q_n^{(\alpha)}(x) = \hat{L}_{n+1}^{(\alpha)}(x) + b_n^{(n)}\hat{L}_n^{(\alpha)}(x) + b_{n-1}^{(n)}\hat{L}_{n-1}^{(\alpha)}(x),$$

where

$$b_n^{(n)} = (n + 1) + (n + \alpha)(1 - \lambda A_n) = (n + 1) \left[1 + \frac{n + \alpha}{d_n(\lambda)} \right],$$

$$b_{n-1}^{(n)} = n(n + \alpha)(1 - \lambda A_n) = \frac{n(n + 1)(n + \alpha)}{d_n(\lambda)}.$$

By using the three-term recurrence relation for Laguerre polynomials the above relation can be expressed as

$$\mathcal{F}Q_n^{(\alpha)}(x) = [x - \lambda(n + \alpha)A_n]\hat{L}_n^{(\alpha)}(x) - n(n + \alpha)\lambda A_n\hat{L}_{n-1}^{(\alpha)}(x). \quad (4.4)$$

Proof. We expand the polynomial $\mathcal{F}Q_n^{(\alpha)}$ in terms of Laguerre polynomials:

$$\mathcal{F}Q_n^{(\alpha)}(x) = \hat{L}_{n+1}^{(\alpha)}(x) + \sum_{i=0}^n b_i^{(n)} \hat{L}_i^{(\alpha)}(x),$$

and, using Proposition 4.1

$$b_i^{(n)} = \frac{(\mathcal{F}Q_n^{(\alpha)}(x), \hat{L}_i^{(\alpha)}(x))}{k_i^{(\alpha)}} = \frac{(Q_n^{(\alpha)}(x), x\hat{L}_i^{(\alpha)}(x))_S}{k_i^{(\alpha)}}$$

follows.

Therefore, by the orthogonality of the polynomial $Q_n^{(\alpha)}(x)$, we have $b_i^{(n)} = 0$, for $0 \leq i \leq n-2$. On the other hand, we know that

$$b_{n-1}^{(n)} = \frac{\tilde{k}_n^{(\alpha)}}{k_{n-1}^{(\alpha)}} = \frac{k_n^{(\alpha)}(1 - \lambda A_n)}{k_{n-1}^{(\alpha)}} = n(n + \alpha)(1 - \lambda A_n).$$

Finally, using (4.3), we compute the coefficient $b_n^{(n)}$:

$$\begin{aligned} b_n^{(n)} &= \frac{(Q_n^{(\alpha)}, x\hat{L}_n^{(\alpha)})_S}{k_n^{(\alpha)}} = \frac{(Q_n^{(\alpha)}, Q_{n+1}^{(\alpha)} + a_n^{(n)}Q_n^{(\alpha)} + a_{n-1}^{(n)}Q_{n-1}^{(\alpha)})_S}{k_n^{(\alpha)}} \\ &= a_n^{(n)} \frac{\tilde{k}_n}{k_n} = (n+1) + (n+\alpha)(1 - \lambda A_n). \quad \square \end{aligned}$$

Remark. By substitution of (2.8) in expression (4.4), we deduce

$$\begin{aligned} \mathcal{F}Q_n^{(\alpha)}(x) &= x\hat{L}_n^{(\alpha)}(x) - \lambda A_n [n\hat{L}_n^{(\alpha)}(x) + n(n+\alpha)\hat{L}_{n-1}^{(\alpha)}(x)] - \lambda\alpha A_n \hat{L}_n^{(\alpha)}(x) \\ &= x \left[\frac{1}{n+1} (\hat{L}_{n+1}^{(\alpha)})'(x) + (1 - \lambda A_n) (\hat{L}_n^{(\alpha)})'(x) \right] - \lambda\alpha A_n \hat{L}_n^{(\alpha)}(x) \\ &= (x - \lambda\alpha A_n) \frac{1}{n+1} (\hat{L}_{n+1}^{(\alpha)})'(x) + [(1 - \lambda A_n)x - \lambda\alpha A_n] (\hat{L}_n^{(\alpha)})'(x). \end{aligned}$$

In the particular case when $\alpha = 0$, the above relation can be written in the following way:

$$\begin{aligned} \mathcal{F}Q_n^{(0)}(x) &= x [Q_n^{(0)}(x) - \lambda(Q_n^{(0)})'(x) - \lambda(Q_n^{(0)})''(x)] \\ &= x \left[\frac{1}{n+1} (\hat{L}_{n+1}^{(0)})'(x) + (1 - \lambda A_n) (\hat{L}_n^{(0)})'(x) \right], \end{aligned}$$

and, after simplification we conclude

$$Q_n^{(0)}(x) + \lambda(Q_n^{(0)})'(x) - \lambda(Q_n^{(0)})''(x) = \frac{1}{n+1} (\hat{L}_{n+1}^{(0)})'(x) + (1 - \lambda A_n) (\hat{L}_n^{(0)})'(x).$$

This relation coincides with the expression obtained in [4, Theorem 3.3].

5. A difference-differential relation

At last, we will show a difference-differential relation satisfied by the polynomials, which coefficients are deduced from the previous proposition.

Proposition 5.1. (Difference-Differential Relation). *The following relation holds:*

$$\mathcal{F}(Q_n^{(\alpha)}) = Q_{n+1}^{(\alpha)}(x) + c_n^{(n)} Q_n^{(\alpha)}(x) + c_{n-1}^{(n)} Q_{n-1}^{(\alpha)}(x), \quad (5.1)$$

where

$$c_n^{(n)} = d_n(\lambda) + \frac{(n+1)(n+\alpha)}{d_n(\lambda)},$$

$$c_{n-1}^{(n)} = \frac{n(n+\alpha)(1-\lambda A_n)}{(1-\lambda A_{n-1})} = (n+1)(n+\alpha) \frac{d_{n-1}(\lambda)}{d_n(\lambda)}.$$

Proof. Expand $\mathcal{F}Q_n^{(\alpha)}$ in terms of the polynomials $\{Q_n^{(\alpha)}\}$:

$$\mathcal{F}(Q_n^{(\alpha)}) = Q_{n+1}^{(\alpha)}(x) + \sum_{i=0}^n c_i^{(n)} Q_i^{(\alpha)}(x)$$

where, from the symmetric character of \mathcal{F} , we obtain

$$c_i^{(n)} = \frac{(\mathcal{F}Q_n^{(\alpha)}, Q_i^{(\alpha)})_S}{(Q_i^{(\alpha)}, Q_i^{(\alpha)})_S} = \frac{(Q_n^{(\alpha)}, \mathcal{F}Q_i^{(\alpha)})_S}{\tilde{k}_i^{(\alpha)}}.$$

Consequently, we have $c_i^{(n)} = 0$, for $0 \leq i \leq n-2$. In this way, the previous expansion contains only three terms

$$\mathcal{F}(Q_n^{(\alpha)}) = Q_{n+1}^{(\alpha)}(x) + c_n^{(n)} Q_n^{(\alpha)}(x) + c_{n-1}^{(n)} Q_{n-1}^{(\alpha)}(x).$$

We point out that

$$c_{n-1}^{(n)} = \frac{\tilde{k}_n^{(\alpha)}}{\tilde{k}_{n-1}^{(\alpha)}} = \frac{n(n+\alpha)(1-\lambda A_n)}{(1-\lambda A_{n-1})}.$$

Finally, from Proposition 4.5 we get

$$c_n^{(n)} = \frac{(\mathcal{F}Q_n^{(\alpha)}, Q_n^{(\alpha)})_S}{\tilde{k}_n^{(\alpha)}} = \frac{(\hat{L}_{n+1}^{(\alpha)} + b_n^{(n)} \hat{L}_n^{(\alpha)} + b_{n-1}^{(n)} \hat{L}_{n-1}^{(\alpha)}, Q_n^{(\alpha)})_S}{\tilde{k}_n^{(\alpha)}}$$

$$= \frac{(n+1)}{1-\lambda A_n} + (n+\alpha)(1-\lambda A_n) = d_n(\lambda) + \frac{(n+1)(n+\alpha)}{d_n(\lambda)}. \quad \square$$

Using the expressions of the coefficients in the above relation, we can deduce an interesting result.

Proposition 5.2. *The polynomial $\mathcal{F}Q_n^{(\alpha)}(x)$ can be expressed in terms of the limit polynomials $R_n^{(\alpha)}(x)$ as follows:*

$$\mathcal{F}Q_n^{(\alpha)}(x) = R_{n+1}^{(\alpha)}(x) + (n+\alpha)(1-\lambda A_n)R_n^{(\alpha)}(x). \quad (5.2)$$

For $\alpha > 0$, the polynomial $\mathcal{F}Q_n^{(\alpha)}(x)$ is a quasi-orthogonal polynomial of order one with respect to the MOPS $\{L_n^{(\alpha-1)}\}_n$

$$\mathcal{F}(Q_n^{(\alpha)}(x)) = L_{n+1}^{(\alpha-1)}(x) + (n + \alpha)(1 - \lambda A_n)L_n^{(\alpha-1)}(x).$$

From the difference-differential relation, we can deduce a *Christoffel–Darboux-type formula*, as it is shown in the next proposition.

Proposition 5.3. (Christoffel–Darboux-type Formula). *Laguerre–Sobolev polynomials satisfy*

$$\mathcal{F}_x(L_n(x, y)) - \mathcal{F}_y(L_n(x, y)) = \frac{1}{\tilde{k}_n^{(\alpha)}} [Q_{n+1}^{(\alpha)}(x)Q_n^{(\alpha)}(y) - Q_n^{(\alpha)}(x)Q_{n+1}^{(\alpha)}(y)]$$

where $L_n(x, y)$ denotes the n th kernel associated to the SMOPS $\{Q_n^{(\alpha)}(x)\}_n$, i.e.,

$$L_n(x, y) = \sum_{i=0}^n \frac{Q_i^{(\alpha)}(x)Q_i^{(\alpha)}(y)}{\tilde{k}_i^{(\alpha)}}.$$

Proof. Write relation (5.1) for the variables x and y :

$$\mathcal{F}_x(Q_i^{(\alpha)}(x)) = Q_{i+1}^{(\alpha)}(x) + c_i^{(i)}Q_i^{(\alpha)}(x) + c_{i-1}^{(i)}Q_{i-1}^{(\alpha)}(x),$$

$$\mathcal{F}_y(Q_i^{(\alpha)}(y)) = Q_{i+1}^{(\alpha)}(y) + c_i^{(i)}Q_i^{(\alpha)}(y) + c_{i-1}^{(i)}Q_{i-1}^{(\alpha)}(y).$$

Multiply the first equation by $Q_i^{(\alpha)}(y)$, and by $Q_i^{(\alpha)}(x)$ the second one, subtract to get

$$\begin{aligned} Q_i^{(\alpha)}(y)\mathcal{F}_x(Q_i^{(\alpha)}(x)) - Q_i^{(\alpha)}(x)\mathcal{F}_y(Q_i^{(\alpha)}(y)) &= Q_{i+1}^{(\alpha)}(x)Q_i^{(\alpha)}(y) - Q_i^{(\alpha)}(x)Q_{i+1}^{(\alpha)}(y) \\ &\quad + c_{i-1}^{(i)}[Q_{i-1}^{(\alpha)}(x)Q_i^{(\alpha)}(y) - Q_i^{(\alpha)}(x)Q_{i-1}^{(\alpha)}(y)]. \end{aligned}$$

If we divide the above relation by $\tilde{k}_i^{(\alpha)}$, using the value of $c_{i-1}^{(i)}$, we deduce

$$\begin{aligned} &\frac{\mathcal{F}_x(Q_i^{(\alpha)}(x)Q_i^{(\alpha)}(y)) - \mathcal{F}_y(Q_i^{(\alpha)}(y)Q_i^{(\alpha)}(x))}{\tilde{k}_i^{(\alpha)}} \\ &= \frac{1}{\tilde{k}_i^{(\alpha)}} [Q_{i+1}^{(\alpha)}(x)Q_i^{(\alpha)}(y) - Q_i^{(\alpha)}(x)Q_{i+1}^{(\alpha)}(y)] \\ &\quad + \frac{1}{\tilde{k}_{i-1}^{(\alpha)}} [Q_{i-1}^{(\alpha)}(x)Q_i^{(\alpha)}(y) - Q_i^{(\alpha)}(x)Q_{i-1}^{(\alpha)}(y)]. \end{aligned}$$

We now sum these equalities for $i = 0, \dots, n$ and we get the desired result. \square

Next, we will obtain a *Rodrigues-type formula* for the Laguerre–Sobolev polynomials. This result generalizes the usual Rodrigues formula, since when $\lambda = 0$ both expressions are the same. In the same way, that formula generalizes the expression obtained in [4, Theorem 3.2] for $\alpha = 0$, using a variational technique.

Proposition 5.4. (Rodrigues-type formula). *Laguerre–Sobolev polynomials $Q_n^{(\alpha)}(x)$ satisfy the following Rodrigues-type formula:*

$$x^\alpha e^{-x} \mathcal{F} Q_n^{(\alpha)}(x) = (-1)^n x D^n [(x^n - \lambda(n + \alpha) A_n x^{n-1}) x^\alpha e^{-x}]. \quad (5.3)$$

Proof. The polynomial $\mathcal{F} Q_n^{(\alpha)}(x)$ can be expressed in terms of the classical Laguerre polynomials. In fact, by Proposition 4.5, we have

$$\mathcal{F} Q_n^{(\alpha)}(x) = x \hat{L}_n^{(\alpha)}(x) - \lambda(n + \alpha) A_n [\hat{L}_n^{(\alpha)}(x) + n \hat{L}_{n-1}^{(\alpha)}(x)].$$

Using the structure relation (2.9) for Laguerre polynomials we deduce

$$\mathcal{F} Q_n^{(\alpha)}(x) = x \hat{L}_n^{(\alpha)}(x) + \lambda(n + \alpha) A_n [(\alpha - x) \hat{L}_{n-1}^{(\alpha)}(x) + x (\hat{L}_{n-1}^{(\alpha)})'(x)].$$

Now, we can introduce the weight function $x^\alpha e^{-x}$, and obtain

$$\begin{aligned} \mathcal{F} Q_n^{(\alpha)}(x) &= x \hat{L}_n^{(\alpha)}(x) + \lambda(n + \alpha) A_n \frac{x}{x^\alpha e^{-x}} [x^{\alpha-1} e^{-x} ((\alpha - x) \hat{L}_{n-1}^{(\alpha)}(x) + x (\hat{L}_{n-1}^{(\alpha)})'(x))] \\ &= x \hat{L}_n^{(\alpha)}(x) + \lambda(n + \alpha) A_n \frac{x}{x^\alpha e^{-x}} D[x^\alpha e^{-x} \hat{L}_{n-1}^{(\alpha)}(x)] \end{aligned}$$

and using the Rodrigues formula for Laguerre polynomials (2.7) we conclude

$$\begin{aligned} \mathcal{F} Q_n^{(\alpha)}(x) &= (-1)^n \frac{x}{x^\alpha e^{-x}} D^n [x^{n+\alpha} e^{-x}] + \lambda(n + \alpha) A_n \frac{x}{x^\alpha e^{-x}} (-1)^{n-1} D^n [x^{n+\alpha-1} e^{-x}] \\ &= (-1)^n \frac{x}{x^\alpha e^{-x}} D^n [(x^n - \lambda(n + \alpha) A_n x^{n-1}) x^\alpha e^{-x}]. \quad \square \end{aligned}$$

The Rodrigues formula can be useful in order to obtain strong asymptotics estimates for the Laguerre–Sobolev orthogonal polynomials. When $\alpha = 0$, the following result has been obtained in [17].

Proposition 5.5. *For $\alpha = 0$*

$$Q_n^{(0)}(x) = \frac{\lambda + 2 + \sqrt{\lambda^2 + 4\lambda}}{\lambda + \sqrt{\lambda^2 + 4\lambda}} \frac{1}{2\sqrt{\pi}} (-x)^{1/4} n^{-3/4} e^{x/2} e^{2\sqrt{-n}x} \left[1 + O\left(\frac{1}{\sqrt{n}}\right) \right]$$

uniformly for λ on compact subsets of $\mathbb{C} \setminus [-4, 0]$ and uniformly for x on compact subsets of $\mathbb{C} \setminus [0, +\infty)$. Here $\sqrt{\lambda^2 + 4\lambda} > 0$ for $\lambda > 0$, and $-\pi < \arg(x) < \pi$.

This result for $\alpha > 0$ remains open.

6. The zeros

The behaviour of the zeros of $Q_n^{(\alpha)}(x)$, for λ sufficiently large, follows from the properties of the zeros of the limit polynomial $R_n^{(\alpha)}(x)$, as it has been pointed in [13].

Theorem 6.1. *For a sufficiently large λ , the Sobolev orthogonal polynomial $Q_n^{(\alpha)}(x)$ has exactly n different and real roots that separate the roots of $\hat{L}_n^{(\alpha)}(x)$. Also, for $\alpha \geq 0$, all the roots belong to the interval $[0, +\infty)$, and for $-1 < \alpha < 0$, at most one of them is negative.*

Proof. For $\alpha > 0$, $R_n^{(\alpha)}(x) = L_n^{(\alpha-1)}(x)$, has all its roots real, and simple. They are contained in $[0, +\infty)$ and are separated by the zeros of $\hat{L}_n^{(\alpha)}(x)$ as a consequence of relation (2.12).

When $-1 < \alpha \leq 0$, the polynomial $R_n^{(\alpha)}(x)$ is a quasi-orthogonal polynomial of order one, therefore it has $n-1$ real and different roots contained in $[0, +\infty)$ that separate those of $\hat{L}_n^{(\alpha)}(x)$. This implies that all the roots are real and different, and at most one root is negative (see [5, p. 65]).

In the case $\alpha = 0$, Brenner has shown that all the roots belong to $[0, +\infty)$ [4]. \square

In this section we will show that the previous properties hold for every value of λ , using the properties of the linear operator \mathcal{F} . First, we need some technical results.

Lemma 6.2. For $n \geq 2$ and $\alpha \geq 0$, we have

$$(-1)^n Q_n^{(\alpha)}(0) > 0. \quad (6.1)$$

Proof. The relation (3.6) for $x = 0$, provides a recurrence relation in order to compute $Q_n^{(\alpha)}(0)$, as follows. Write

$$Q_n^{(\alpha)}(0) = \hat{L}_n^{(\alpha)}(0) + nL_{n-1}^{(\alpha)}(0) - d_{n-1}(\lambda)Q_{n-1}^{(\alpha)}(0) = -\alpha\hat{L}_{n-1}^{(\alpha)}(0) - d_{n-1}Q_{n-1}^{(\alpha)}(0),$$

and divide in the previous relation by $\hat{L}_n^{(\alpha)}(0)$, and denote

$$B_n = Q_n^{(\alpha)}(0)/\hat{L}_n^{(\alpha)}(0).$$

Then, we deduce

$$B_n = \frac{\alpha}{n + \alpha} + \frac{d_{n-1}(\lambda)}{n + \alpha} B_{n-1}.$$

From this relation, we can deduce some interesting consequences. First, since $B_0 = B_1 = 1$, if $\alpha \geq 0$, we have $B_n > 0$, $\forall n \in \mathbb{N}$, and therefore $Q_n^{(\alpha)}(0) \neq 0, \forall n \in \mathbb{N}$. In fact, we have $(-1)^n Q_n^{(\alpha)}(0) > 0$, i.e.,

$$Q_{2n}^{(\alpha)}(0) > 0, \quad Q_{2n+1}^{(\alpha)}(0) < 0. \quad \square$$

Remark. If $-1 < \alpha < 0$, no global properties about the sign of B_n can be deduced, but if there exists $n_0 \in \mathbb{N}$ such that $B_{n_0} \leq 0$, then we have $B_n < 0$, $\forall n > n_0$. In this way, for $-1 < \alpha < 0$, if some polynomial $Q_n^{(\alpha)}(x)$ has a negative root, the next polynomials satisfy the same property. Analyzing the sign of $(Q_n^{(\alpha)})'(0)$ we immediately deduce that the negative root of $Q_n^{(\alpha)}(x)$ has odd multiplicity.

From the expression for B_2

$$B_2 = \frac{\alpha}{\alpha + 2} + \frac{d_1(\lambda)}{\alpha + 2},$$

we deduce that $B_2 < 0$, for λ sufficiently large and $-1 < \alpha < 0$, and therefore, we can always obtain inner products like (3.1) such that $B_n < 0, \forall n \geq 2$.

Lemma 6.3. Let $p(x)$ be a polynomial with degree k . Then

(i) If $\lambda\alpha = 0$, there exists a unique polynomial $p_1(x)$ with degree k such that

$$\mathcal{F}p_1 = xp(x).$$

(ii) If $\lambda\alpha \neq 0$, there exists an unique polynomial $p_1(x)$ with degree k and an unique constant $c(p)$, satisfying

$$\mathcal{F}p_1 = xp(x) + x^2c(p).$$

Proof. (i) If $\lambda = 0$, the linear operator \mathcal{F} is the shift operator, therefore $p_1(x) = p(x)$. In the case $\alpha = 0$, we have $\mathcal{F} = x[I + \lambda D - \lambda D^2]$ and it suffices to take

$$p_1(x) = [I + \lambda D - \lambda D^2]^{-1} p(x).$$

(ii) Let

$$xp(x) = \sum_{i=1}^{k+1} a_i x^i, \quad p_1(x) = \sum_{i=0}^k b_i x^i,$$

the power expansions for the polynomials $xp(x)$ and $p_1(x)$. We are trying to obtain the coefficients b_0, b_1, \dots, b_k , and the constant $c(p)$, from the coefficients a_1, a_2, \dots, a_{k+1} . First, we must remark that the image of x^i by means of the operator \mathcal{F} can be expressed by

$$\mathcal{F}(x^i) = x^{i+1} + \lambda i x^i - \lambda i(\alpha + i - 1)x^{i-1}.$$

In this way, applying the linear operator \mathcal{F} to the polynomial $p_1(x)$, we get

$$\begin{aligned} \mathcal{F}p_1 &= \sum_{i=0}^k b_i \mathcal{F}(x^i) = \sum_{i=0}^k b_i [x^{i+1} + \lambda i x^i - \lambda i(\alpha + i - 1)x^{i-1}] \\ &= b_k x^{k+1} + [b_{k-1} + \lambda k b_k] x^k + \sum_{i=1}^{k-1} [b_{i-1} + \lambda i b_i - \lambda(i+1)(\alpha + i)b_{i+1}] x^i - \lambda \alpha b_1. \end{aligned}$$

Then, if we assume $\mathcal{F}p_1 = xp(x) + x^2c(p)$, i.e.,

$$\mathcal{F}p_1 = \sum_{i=1}^{k+1} a_i x^i + x^2c(p),$$

we obtain the following linear system:

$$\begin{aligned} b_k &= a_{k+1}, \\ b_{k-1} + \lambda k b_k &= a_k, \\ \dots \quad \dots \quad \dots \\ b_{i-1} + \lambda i b_i - \lambda(i+1)(\alpha + i)b_{i+1} &= a_i, \quad 3 \leq i \leq k-1, \\ \dots \quad \dots \quad \dots \\ b_1 + 2\lambda b_2 - 3\lambda(\alpha + 2)b_3 &= a_2 + c(p), \\ b_0 + \lambda b_1 - 2\lambda(\alpha + 1)b_2 &= a_1, \\ -\lambda \alpha b_1 &= 0. \end{aligned} \tag{6.2}$$

The unique solution of the preceding system of linear equations can be obtained in a recursive way, extracting each coefficient b_i from the corresponding equation. Moreover, since $\lambda\alpha \neq 0$, the last equation gives $b_1 = 0$, and therefore $p'_1(0) = 0$. \square

Corollary 6.4. *In the conditions of the previous lemma, if $\lambda\alpha \neq 0$, for every polynomial $q(x)$*

$$(q, p_1)_S = \int_0^{+\infty} p(x)q(x)x^\alpha e^{-x} dx + c(p) \int_0^{+\infty} xq(x)x^\alpha e^{-x} dx$$

holds.

Proof. If we integrate by parts, the constant term vanishes since $p'_1(0) = 0$, and the integral terms converge by construction. Thus

$$\begin{aligned} (q, p_1)_S &= \int_0^{+\infty} [p_1(x)x^\alpha e^{-x} - \lambda p'_1(x)(\alpha - x)x^{\alpha-1} e^{-x} - \lambda p''_1(x)x^\alpha e^{-x}]q(x) dx \\ &= \int_0^{+\infty} \frac{\mathcal{F}p_1}{x} q(x)x^\alpha e^{-x} dx \\ &= \int_0^{+\infty} p(x)q(x)x^\alpha e^{-x} dx + c(p) \int_0^{+\infty} xq(x)x^\alpha e^{-x} dx. \quad \square \end{aligned}$$

Lemma 6.5. *Let $0 \leq x_1 < x_2 < \dots < x_k$, be k nonnegative real numbers, and consider the polynomial*

$$xp(x) = x \prod_{i=1}^k (x - x_i),$$

then, if $\lambda\alpha \neq 0$, we have

$$(-1)^k c(p) > 0.$$

Proof. If $\lambda\alpha \neq 0$, then $b_1 = 0$, and the solution of system of linear equations (6.2) can be computed in recursive way. Since $a_{k+1} = 1$, using Cardano–Vieta formulæ, we get

$$(-1)^{k+1-i} a_i > 0, \quad i = 2, 3, \dots, k+1,$$

$$a_1 = (-1)^k x_1 x_2 \dots x_k.$$

Thus, we have $(-1)^k c(p) > 0$. \square

Lemma 6.6. *For $n \geq 2$, we have*

$$\int_0^{+\infty} x Q_n^{(\alpha)}(x) x^\alpha e^{-x} dx = (-1)^n \frac{\lambda}{\alpha + 1} k_1^{(\alpha)} \prod_{i=1}^{n-1} d_{n-i}(\lambda).$$

Proof. Using repeatedly (3.6), we deduce

$$\int_0^{+\infty} x Q_n^{(\alpha)}(x) x^\alpha e^{-x} dx = (-1)^{n-2} \prod_{i=1}^{n-2} d_{n-i}(\lambda) \int_0^{+\infty} [2x \hat{L}_1^{(\alpha)}(x) - d_1(\lambda) x Q_1^{(\alpha)}(x)] x^\alpha e^{-x} dx,$$

and, since $Q_1^{(\alpha)}(x) = \hat{L}_1^{(\alpha)}(x)$, we obtain

$$\int_0^{+\infty} x Q_n^{(\alpha)}(x) x^\alpha e^{-x} dx = (-1)^{n-2} \prod_{i=1}^{n-2} d_{n-i}(\lambda) k_1^{(\alpha)} \frac{2\lambda}{\lambda + \alpha + 1} = (-1)^n \frac{\lambda}{\alpha + 1} k_1^{(\alpha)} \prod_{i=1}^{n-1} d_{n-i}(\lambda). \quad \square$$

Theorem 6.7. Let $\lambda > 0$, then the polynomial $Q_n^{(\alpha)}(x)$, $n \geq 2$, has exactly n different real roots, and at least $n - 1$ of them are positive. When $\alpha \geq 0$, all the roots are positive.

Moreover, the roots of $Q_n^{(\alpha)}(x)$ separate those of $\hat{L}_n^{(\alpha)}(x)$. If we denote by $x_{n,1} < x_{n,2} < \dots < x_{n,n}$ the roots of $\hat{L}_n^{(\alpha)}(x)$, and by $y_{n,1} < y_{n,2} < \dots < y_{n,n}$ those of $Q_n^{(\alpha)}(x)$, then

$$y_{n,1} < x_{n,1} < y_{n,2} < x_{n,2} < \dots < y_{n,n} < x_{n,n}.$$

Proof. Define the polynomial

$$\omega_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^n (x - x_{n,j}).$$

Then, if $\alpha \neq 0$, by Lemma 6.3, there exists a polynomial $p_i(x)$ with degree $n - 1$, and a constant $c(\omega_i)$ such that

$$\mathcal{F}(p_i) = x\omega_i(x) + x^2 c(\omega_i).$$

From Corollary 6.4, we get

$$0 = (Q_n^{(\alpha)}, p_i)_S = \int_0^{+\infty} \omega_i(x) Q_n^{(\alpha)}(x) x^\alpha e^{-x} dx + c(\omega_i) \int_0^{+\infty} x Q_n^{(\alpha)}(x) x^\alpha e^{-x} dx.$$

The Gaussian quadrature formula based in the zeros of $\hat{L}_n^{(\alpha)}(x)$, leads to

$$0 = (Q_n^{(\alpha)}, p_i)_S = \lambda_i \omega_i(x_{n,i}) Q_n^{(\alpha)}(x_{n,i}) + c(\omega_i) \int_0^{+\infty} x Q_n^{(\alpha)}(x) x^\alpha e^{-x} dx,$$

where $\lambda_i > 0$ denotes the corresponding Christoffel numbers, and from Lemmas 6.5 and 6.6, we deduce that the second term is negative. Thus

$$\lambda_i \omega_i(x_{n,i}) Q_n^{(\alpha)}(x_{n,i}) = \lambda_i (\hat{L}_n^{(\alpha)})'(x_{n,i}) Q_n^{(\alpha)}(x_{n,i}) > 0,$$

then $Q_n^{(\alpha)}(x_{n,i}) \neq 0$ and also $\text{sgn}(Q_n^{(\alpha)}(x_{n,i})) = (-1)^{n-i}$. Thus $Q_n^{(\alpha)}(x)$ changes sign between any two consecutive zeros of $\hat{L}_n^{(\alpha)}(x)$, and since $\text{sgn}(Q_n^{(\alpha)}(x_{n,1})) = (-1)^{n-1}$, $Q_n^{(\alpha)}(x)$ has n different real roots, which separate those of $\hat{L}_n^{(\alpha)}(x)$, in the announced position.

If $\alpha = 0$, taking integration by parts, we have

$$\begin{aligned} 0 &= (Q_n^{(\alpha)}, p_i)_S = \int_0^{+\infty} \omega_i(x) Q_n^{(\alpha)}(x) e^{-x} dx - \lambda Q_n^{(\alpha)}(0) p_i'(0) \\ &= \lambda_i \omega_i(x_{n,i}) Q_n^{(\alpha)}(x_{n,i}) - \lambda Q_n^{(\alpha)}(0) p_i'(0). \end{aligned}$$

Using the same reasoning as in Lemma 6.5, we obtain

$$(-1)^{n-2} p_i'(0) = (-1)^{n-2} b_1 > 0,$$

and since $(-1)^n Q_n^{(\alpha)}(0) > 0$, we get

$$\lambda_i \omega_i(x_{n,i}) Q_n^{(\alpha)}(x_{n,i}) > 0,$$

therefore $Q_n^{(\alpha)}(x_{n,i}) \neq 0$ and also $\text{sgn}(Q_n^{(\alpha)}(x_{n,i})) = (-1)^{n-i}$, and the conclusion follows in the same way.

Finally, we must remember that, from (6.1), for $\alpha \geq 0$, we have $\text{sgn}(Q_n^{(\alpha)}(0)) = (-1)^n$, and the position of the roots will be

$$0 < y_{n,1} < x_{n,1} < y_{n,2} < x_{n,2} < \dots < y_{n,n} < x_{n,n}. \quad \square$$

7. Open problems

1. Because $x_{n,1} \rightarrow 0$, when $n \rightarrow \infty$, when $\alpha \geq 0$, $y_{n,1} \rightarrow 0$. A first open question is to give some estimates for this convergence and to compare with the convergence to zero of $x_{n,1}$.
2. In the case $-1 < \alpha < 0$ the main interesting question concerns the characterization of the location for $y_{n,1}$ with respect to the support of the measure in terms of α and λ . Some numerical examples have been studied (see Figs. 1 and 2 and Tables 1 and 2).

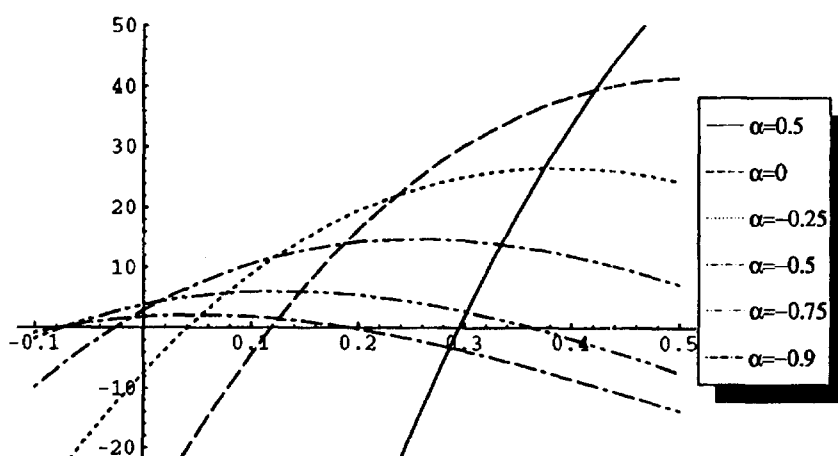


Fig. 1. Graphs of the polynomials $Q_5^{(\alpha)}(x)$ in the proximity of the origin, for $\lambda = 0.2$ and different values of α .

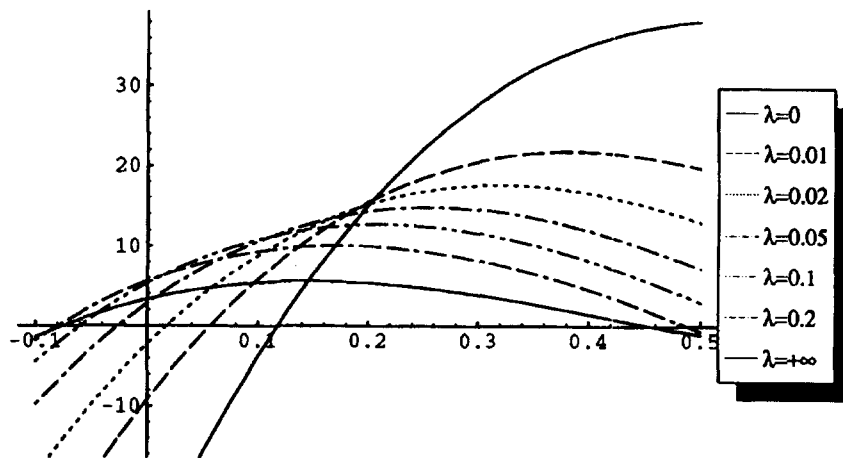


Fig. 2. Graphs of the polynomials $Q_s^{(x)}(x)$ in the proximity of the origin, for $\alpha = -\frac{1}{2}$ and different values of λ .

Table 1

Values for the least zero $y_{n,1}(\lambda)$ of the polynomial $Q_n^{(x)}(x)$, for $\alpha = -\frac{1}{2}$, $n = 1, 2, \dots, 10$ and different values of λ

	0	0.05	0.1	0.2	0.4	1	$\lambda \rightarrow \infty$
1	0.5	0.5	0.5	0.5	0.5	0.5	0.5
2	0.275255	0.257376	0.240427	0.209197	0.15631	0.051597	-0.1610893
3	0.190164	0.155367	0.124995	0.0758911	0.0108782	-0.070946	-0.1333454
4	0.145304	0.0958143	0.0579705	0.00708406	-0.0440012	-0.087491	-0.0989709
5	0.117581	0.0564115	0.0170997	-0.0262357	-0.060057	-0.07999	-0.0776557
6	0.098747	0.0291956	-0.0073488	-0.0406814	-0.0613204	-0.068983	-0.063823
7	0.0851154	0.0102887	-0.0213765	-0.0456133	-0.0574498	-0.059147	-0.0541788
8	0.0747919	-0.002706	-0.028935	-0.0459313	-0.0521892	-0.051200	-0.0470726
9	0.0667022	-0.011469	-0.0325601	-0.0441224	-0.0469525	-0.044918	-0.0416176
10	0.0601921	-0.017227	-0.0338344	-0.0414379	-0.0422254	-0.039929	-0.0372973

Table 2

Values for the least zero $y_{n,1}(\lambda)$ of the polynomial $Q_n^{(x)}(x)$, for $\lambda = 0.2$, $n = 1, 2, \dots, 10$ and different values of α

	0.5	0	-0.25	-0.5	-0.75	-0.9
1	1.5	1.	0.75	0.5	0.25	0.1
2	0.872867	0.531625	0.367608	0.209197	0.058132	-0.026135
3	0.579842	0.316644	0.192761	0.0758911	-0.0291145	-0.0750912
4	0.406634	0.193394	0.0956882	0.00708406	-0.064169	-0.083088
5	0.295022	0.118883	0.0409684	-0.0262357	-0.0736026	-0.0777522
6	0.220466	0.0734233	0.010759	-0.0406814	-0.0724749	-0.0694389
7	0.169629	0.0456072	-0.00546753	-0.0456133	-0.0675371	-0.0612413
8	0.134342	0.0285039	-0.013832	-0.0459313	-0.0615754	-0.0539764
9	0.10938	0.0179175	-0.01781	-0.0441224	-0.0556995	-0.047764
10	0.0913553	0.0113187	-0.0193594	-0.0414379	-0.05032	-0.0425173

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