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Block incomplete factorization preconditioners for a symmetric block-tridiagonal M-matrix

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Abstract

We propose new block incomplete factorization preconditioners for a symmetric block-tridiagonal M-matrix which can be computed in parallel, and then theoretical properties for these block preconditioners are studied. Spectral properties of the transformed coefficient matrices with the block incomplete factorization preconditioners are also examined to see the convergence rate of the preconditioned CG(PCG) method. Lastly, numerical results of the PCG using the block incomplete factorization preconditioners are compared with those of the PCG using a standard incomplete factorization preconditioner to see how effective the block incomplete factorization preconditioners are. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The discretization of partial differential equations in 2D or 3D, by finite difference or finite element approximation, leads often to large sparse block-tridiagonal linear systems. In this paper, we consider the linear system of equations

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

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where A is a large sparse symmetric block-tridiagonal M-matrix blocked in the form

$$A = \begin{pmatrix} B_1 & -C_1 & 0 & \cdots & 0 \\ -C_1^{\mathrm{T}} & B_2 & -C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -C_{m-2}^{\mathrm{T}} & B_{m-1} & -C_{m-1} \\ 0 & \cdots & 0 & -C_{m-1}^{\mathrm{T}} & B_m \end{pmatrix}. \tag{2}$$

It is assumed that the diagonal blocks B_i of A are symmetric matrices and C_i's are nonnegative matrices. Since A is a large sparse matrix, direct solvers become prohibitively expensive because of the large amount of work and storage required. As an alternative, the conjugate gradient(CG) iterative method [13] is widely used for a symmetric M-matrix A which guarantees the positive-definiteness of A. Given an initial guess x_0, CG algorithm computes iteratively new approximations x_k to the true solution $x^* = A^{-1}b$. The iterate x_k is accepted as a solution if the residual $r_k = b - Ax_k$ satisfies $\| r_k \| / \| b \| \leqslant \mathrm{tol}$. In some cases, the convergence may be extremely slow. Hence, the original problem (1) must be transformed into a more tractable form. To do so, we consider a symmetric positive definite matrix K called the preconditioning matrix or preconditioner and apply the CG iterative solver to the preconditioned linear system $K^{-1}Ax = K^{-1}b$. Here, K should be chosen so that $K^{-1}A$ is a good approximation to the identity matrix. The CG method applied to the linear system $K^{-1}Ax = K^{-1}b$ is called the preconditioned CG(PCG) method with a preconditioner K.

Since the ultimate goal of the PCG method is to reduce the total execution time, the computation of preconditioner K should be done in parallel. One of the powerful preconditioning methods in terms of reducing the number of iterations is the incomplete Cholesky(IC) factorization method studied by Meijerink and van der Vorst [16]. A detailed review for the IC factorization method can be found in [3, 6, 11, 18]. However, it is very difficult to parallelize the IC factorization algorithm because of the recursive nature of the computation. On the other hand, polynomial preconditioners defined by $K^{-1} = p(A)$, where p is a polynomial, are easy to parallelize since they only involve the computation of matrix-vector operations, but they are not as powerful as the IC factorization preconditioners. In order to make the IC factorization method more suitable for vector computers and parallel architectures, incomplete block Cholesky factorizations using matrix blocks as basic entities were proposed [1, 2, 7, 9, 17].

The purpose of this paper is to propose *new block IC factorization preconditioners* for a symmetric block-tridiagonal M-matrix which can be computed *in parallel*. The block IC factorization preconditioners to be proposed in this paper are quite different from the incomplete block Cholesky factorization preconditioners introduced by Concus, Golub, and Meurant [9] which need the approximate inverses of pivot blocks. More specifically, let D be the block-diagonal matrix consisting of the diagonal blocks B_i of A and L the block strictly-lower triangular matrix consisting of the sub-diagonal blocks $-C_i^{\mathrm{T}}$ of A. Then, the coefficient matrix A can be expressed as

$$A = L + D + L^{\mathrm{T}}.$$

Incomplete block Cholesky factorization preconditioner K presented in [9] is of the form

$$K = (L + \Delta)\Delta^{-1}(L^{\mathrm{T}} + \Delta),$$

where Δ is a block-diagonal matrix whose diagonal blocks Δ_i satisfy the following block recurrence:

$$\Delta_1 = B_1,$$
$$\Delta_i = B_i - C_{i-1}^{\mathrm{T}}\Lambda_{i-1}C_{i-1}, \qquad 2 \leqslant i \leqslant m,$$

in which Λ_{i-1} is some sparse approximation to Δ_{i-1}^{-1}. Thus, to obtain incomplete block Cholesky factorization preconditioner K, sparse approximate inverses of the pivot blocks Δ_i must be formed. Many techniques for finding these approximate inverses were discussed in [4, 5, 9, 10]. In addition, some techniques designed for finding a sparse approximate inverse preconditioner of a matrix, which have recently been developed in [8, 12, 14] may be applied to approximate the inverses of pivot blocks. Most of incomplete block factorization preconditioners introduced up to date in the literature require sparse approximate inverses for pivot blocks. However, the block IC factorization preconditioners to be proposed in this paper are obtained by performing the standard IC factorization on each matrix block independently, so that they have no block recurrence which requires sparse approximate inverses for pivot blocks and thus they can be computed in parallel based on matrix blocks.

In Section 2, we consider some properties of the incomplete LU(or Cholesky) factorization on M-matrices. In Section 3, we propose new block IC factorization preconditioners for a symmetric block-tridiagonal M-matrix and their theoretical properties are studied. Spectral properties of the transformed coefficient matrices with block IC factorization preconditioners are also examined to see the convergence rate of the PCG method. In Section 4, we describe how to construct the effective block preconditioners for a special type of matrix which arises from five-point discretization of the second-order selfadjoint elliptic partial differential equation. In Section 5, we present numerical results of the PCG with block IC factorization preconditioners developed in this paper, and their results are compared with those of the PCG with a standard IC factorization preconditioner. Lastly, some conclusions are drawn.

2. Incomplete Lu factorizations

A general algorithm for building incomplete LU(ILU) factorizations for M-matrices can be derived by performing Gaussian elimination and dropping some elements in predetermined nondiagonal positions. To better understand the ILU factorization process for an M-matrix, we provide some important results in this section. Let P_n denote the set of all pairs of indices of off-diagonal matrix entries, that is,

$$P_n = \{(i,j) | i \neq j, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}.$$

A matrix $A = (a_{ij})$ is called a *Z-matrix* if $a_{ij} \leqslant 0$ for $i \neq j$. For two matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \leqslant B$ denotes $a_{ij} \leqslant b_{ij}$ for all i and j, and $A \geqslant B$ denotes $a_{ij} \geqslant b_{ij}$ for all i and j. A splitting $A = K - N$ is called a *regular splitting* of A if K is nonsingular, $K^{-1} \geqslant 0$, and $N \geqslant 0$. A matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leqslant 0$ for $i \neq j, A$ is nonsingular, and $A^{-1} \geqslant 0$.

Lemma 2.1. *Let A and B be M-matrices. If $A \leqslant B$, then $B^{-1} \leqslant A^{-1}$.*

Proof. A can be splitted into

$$A = A - 0 = B - (B - A),$$

where 0 denotes the zero matrix. Since A and B are M-matrices and $B - A \geqslant 0$, from Theorem 6.22 in [3] $B^{-1} \leqslant A^{-1}$. □

Theorem 2.2. (Meijerink and van der Vorst [16, p.150]). *Let A be an M-matrix. Then, for each zero pattern set $P \subset P_n$, there exist a unit lower triangular matrix $L = (l_{ij})$, an upper triangular matrix $U = (u_{ij})$, and a matrix $R = (r_{ij})$, with $l_{ij} = u_{ij} = 0$ if $(i,j) \in P$ and $r_{ij} = 0$ if $(i,j) \notin P$, such that $A = LU - R$ is a regular splitting of A. Moreover, L and U are also M-matrices.*

In Theorem 2.2, $A = \mathrm{LU} - R$ is called an incomplete LU(ILU) factorization of A corresponding to a zero pattern set $P \subset P_n$. In particular, if P is an empty set, then $R = 0$ and thus a complete LU factorization of A such that $A = \mathrm{LU}$ is obtained. The following theorem which is a little variant of Theorem 2.4 in [16] states the existence of an IC factorization for a symmetric M-matrix.

Theorem 2.3. *Let A be a symmetric M-matrix. Then, for each zero pattern set $P \subset P_n$ having the property that $(i,j) \in P$ implies $(j,i) \in P$, there exist an upper triangular matrix $U = (u_{ij})$, a diagonal matrix D whose ith diagonal element is u_{ii}^{-1}, and a matrix $R = (r_{ij})$, with $u_{ij} = 0$ if $(i,j) \in P$ and $r_{ij} = 0$ if $(i,j) \notin P$, such that $A = U^{\mathrm{T}}DU - R$ is a regular splitting of A. Moreover, U is also an M-matrix.*

In Theorem 2.3, $U^{\mathrm{T}}D$ is a unit lower triangular matrix and $A = U^{\mathrm{T}}DU - R$ is called an incomplete Cholesky(IC) factorization corresponding to a symmetric zero pattern set $P \subset P_n$. In particular, if P is an empty set, then a complete Cholesky factorization of A such that $A = U^{\mathrm{T}}DU$ is obtained.

Theorem 2.4. (Meijerink and van der Vorst [16, p.152]). *Let A and B be M-matrices, and let $A^{(1)}$ and $B^{(1)}$ be matrices obtained from A and B, respectively, by performing the first step of Gaussian elimination. If $A \leqslant B$, then $A^{(1)}$ and $B^{(1)}$ are M-matrices and $A^{(1)} \leqslant B^{(1)}$.*

Theorem 2.5. *Let A and B be $n \times n$ M-matrices, and let $A = L_1U_1 - R_1$ and $B = L_2U_2 - R_2$ be ILU factorizations corresponding to the same zero pattern set $P \subset P_n$. If $A \leqslant B$, then $L_2^{-1} \leqslant L_1^{-1}$ and $U_2^{-1} \leqslant U_1^{-1}$.*

Proof. The first step of ILU factorization process consists of dropping some off-diagonal elements in the first row and column of an M-matrix corresponding to a zero pattern set. Let $A^{(0)} = A$ and $B^{(0)} = B$. Let $\tilde{A}^{(0)}$ and $\tilde{B}^{(0)}$ be the matrices that are obtained by setting off-diagonal elements in the first rows and columns of $A^{(0)}$ and $B^{(0)}$ corresponding to the same zero pattern set P to zero, respectively. It follows that $A^{(0)} \leqslant \tilde{A}^{(0)}$ and $B^{(0)} \leqslant \tilde{B}^{(0)}$. Since $\tilde{A}^{(0)}$ and $\tilde{B}^{(0)}$ are Z-matrices, $\tilde{A}^{(0)}$ and $\tilde{B}^{(0)}$ are M-matrices. Since $A^{(0)}$ and $B^{(0)}$ use the same zero pattern set P, $A^{(0)} \leqslant B^{(0)}$ implies $\tilde{A}^{(0)} \leqslant \tilde{B}^{(0)}$. Let $L_1^{(1)}$ and $L_2^{(1)}$ be the elementary lower triangular matrices for the first elimination steps on $\tilde{A}^{(0)}$ and $\tilde{B}^{(0)}$, respectively. Since $\tilde{A}^{(0)}$ and $\tilde{B}^{(0)}$ are M-matrices and $\tilde{A}^{(0)} \leqslant \tilde{B}^{(0)}$, it can be easily shown that

$L_1^{(1)} \geqslant L_2^{(1)} \geqslant 0$. From Theorem 2.4, $A^{(1)} = L_1^{(1)}\tilde{A}^{(0)}$ and $B^{(1)} = L_2^{(1)}\tilde{B}^{(0)}$ are M-matrices and $A^{(1)} \leqslant B^{(1)}$. The second step of ILU factorization process consists of first dropping some off-diagonal elements in the second rows and columns of $A^{(1)}$ and $B^{(1)}$ and then eliminating the second columns using the second rows. Hence, $A^{(2)} = L_1^{(2)}\tilde{A}^{(1)}$ and $B^{(2)} = L_2^{(2)}\tilde{B}^{(1)}$ are obtained, where $\tilde{A}^{(1)}$, $\tilde{B}^{(1)}$, $L_1^{(2)}$, and $L_2^{(2)}$ are defined in the similar way as was done for the first elimination step. Repeating the above process until an upper triangular matrix is obtained, one has the following relation:

$$L_1^{(n-1)}L_1^{(n-2)}\cdots L_1^{(1)}\left(A^{(0)} + R_1\right) = A^{(n-1)} = U_1,$$

$$L_2^{(n-1)}L_2^{(n-2)}\cdots L_2^{(1)}\left(B^{(0)} + R_2\right) = B^{(n-1)} = U_2.$$

Since $A^{(n-1)}$ and $B^{(n-1)}$ are M-matrices and $A^{(n-1)} \leqslant B^{(n-1)}$, from Lemma 2.1,

$$\left(B^{(n-1)}\right)^{-1} \leqslant \left(A^{(n-1)}\right)^{-1}, \quad \text{i.e.,}\ U_2^{-1} \leqslant U_1^{-1}$$

Notice that $L_1^{-1} = L_1^{(n-1)}L_1^{(n-2)}\cdots L_1^{(1)}$ and $L_2^{-1} = L_2^{(n-1)}L_2^{(n-2)}\cdots L_2^{(1)}$. Since $0 \leqslant L_2^{(i)} \leqslant L_1^{(i)}$ for all $1 \leqslant i \leqslant n-1$, $L_2^{-1} \leqslant L_1^{-1}$. This completes the proof. $\square$

For symmetric M-matrices, a result similar to Theorem 2.5 is given in the following theorem.

Theorem 2.6. *Let A and B be $n \times n$ symmetric M-matrices, and let $A = U_1^{\mathrm{T}}D_1U_1 - R_1$ and $B = U_2^{\mathrm{T}}D_2U_2 - R_2$ be IC factorizations corresponding to the same symmetric zero pattern set $P \subset P_n$. If $A \leqslant B$, then $U_2^{-1} \leqslant U_1^{-1}$ and $D_2 \leqslant D_1$.*

Proof. Since $U_i^{\mathrm{T}}D_i$'s are unit lower triangular matrices for $i = 1, 2$, by Theorem 2.5 $U_2^{-1} \leqslant U_1^{-1}$. Notice that D_i is a diagonal matrix whose jth diagonal element is the reciprocal of the element in the jth row and jth column of U_i for each $i = 1, 2$. Hence, $U_2^{-1} \leqslant U_1^{-1}$ implies $D_2 \leqslant D_1$. $\square$

A comparison theorem for regular splittings which will be used for the proof of main results in Section 3 is presented below.

Theorem 2.7. (Axelsson [3, p. 219]). *Let $A = K_1 - N_1 = K_2 - N_2$ be regular splittings of A. If $K_2^{-1} \leqslant K_1^{-1}$, then*

$$\rho\left(K_1^{-1}N_1\right) \leqslant \rho\left(K_2^{-1}N_2\right)$$

where $\rho(K_i^{-1}N_i)$ denotes the spectral radius of $K_i^{-1}N_i$ for each $i = 1,2$.

3. Block IC factorization preconditioners

We first consider block IC factorization preconditioners for a symmetric block-tridiagonal M-matrix of the simplest form

$$A = \begin{pmatrix} B_1 & -C_1 \\ -C_1^{\mathrm{T}} & B_2 \end{pmatrix}. \tag{3}$$

Since A is a symmetric M-matrix, B_1 and B_2 are symmetric M-matrices. From the IC factorization process, we can find an upper triangular matrix U_i, a diagonal matrix D_i, and a matrix R_i such that $B_i = U_i^{\mathrm{T}} D_i U_i - R_i$ is a regular splitting of B_i for each $i = 1, 2$, see Theorem 2.3. If $A = K - N$ is a splitting of A and K is a symmetric positive-definite matrix which is easily invertible, then K can be used as a preconditioner for the PCG method. The effectiveness of the preconditioner K depends on how well K approximates A.

Theorem 3.1. *Let A be a symmetric M-matrix of the form* (3), *and let $B_i = U_i^{\mathrm{T}} D_i U_i - R_i$ be a regular splitting of B_i which can be obtained by the IC factorization process for each i = 1, 2. Let*

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \qquad \overline{U} = \begin{pmatrix} U_1 & -C_1 \\ 0 & U_2 \end{pmatrix},$$

$$\tilde{U} = \begin{pmatrix} U_1 & -(U_1^{\mathrm{T}} D_1)^{-1} C_1 \\ 0 & U_2 \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

If we let $M = U^{\mathrm{T}} D U$, $\overline{M} = \overline{U}^{\mathrm{T}} D \overline{U}$, and $\tilde{M} = \tilde{U}^{\mathrm{T}} D \tilde{U}$, then the following holds:
(a) $R = M - A \geqslant 0, \overline{R} = \overline{M} - A \geqslant 0$, *and* $\tilde{R} = \tilde{M} - A \geqslant 0$,
(b) $0 \leqslant U^{-1} \leqslant \overline{U}^{-1} \leqslant \tilde{U}^{-1}$,
(c) $0 \leqslant M^{-1} \leqslant \overline{M}^{-1} \leqslant \tilde{M}^{-1}$,
(d) $A = M - R = \overline{M} - \overline{R} = \tilde{M} - \tilde{R}$ *are regular splittings of A*,
(e) $\rho(\tilde{M}^{-1}\tilde{R}) \leqslant \rho(\overline{M}^{-1}\overline{R}) \leqslant \rho(M^{-1}R) < 1$.

Proof. For the proof of part (a), simple calculation yields

$$R = \begin{pmatrix} U_1^{\mathrm{T}} D_1 U_1 & 0 \\ 0 & U_2^{\mathrm{T}} D_2 U_2 \end{pmatrix} - \begin{pmatrix} B_1 & -C_1 \\ -C_1^{\mathrm{T}} & B_2 \end{pmatrix} = \begin{pmatrix} R_1 & C_1 \\ C_1^{\mathrm{T}} & R_2 \end{pmatrix},$$

$$\overline{R} = \begin{pmatrix} U_1^{\mathrm{T}} D_1 U_1 & -U_1^{\mathrm{T}} D_1 C_1 \\ -C_1^{\mathrm{T}} D_1 U_1 & U_2^{\mathrm{T}} D_2 U_2 + C_1^{\mathrm{T}} D_1 C_1 \end{pmatrix} - \begin{pmatrix} B_1 & -C_1 \\ -C_1^{\mathrm{T}} & B_2 \end{pmatrix}$$

$$= \begin{pmatrix} R_1 & C_1 - U_1^{\mathrm{T}} D_1 C_1 \\ C_1^{\mathrm{T}} - C_1^{\mathrm{T}} D_1 U_1 & R_2 + C_1^{\mathrm{T}} D_1 C_1 \end{pmatrix},$$

$$\tilde{R} = \begin{pmatrix} U_1^{\mathrm{T}} D_1 U_1 & -C_1 \\ -C_1^{\mathrm{T}} & U_2^{\mathrm{T}} D_2 U_2 + C_1^{\mathrm{T}} U_1^{-1} D_1^{-1} U_1^{-\mathrm{T}} C_1 \end{pmatrix} - \begin{pmatrix} B_1 & -C_1 \\ -C_1^{\mathrm{T}} & B_2 \end{pmatrix}$$

$$= \begin{pmatrix} R_1 & 0 \\ 0 & R_2 + C_1^{\mathrm{T}} U_1^{-1} D_1^{-1} U_1^{-\mathrm{T}} C_1 \end{pmatrix}.$$

Since $R_1 \geqslant 0$ and $R_2 \geqslant 0$, $R \geqslant 0$. From Theorem 2.2, U_1 is an M-matrix and so $U_1^{-1} \geqslant 0$ and $D_1^{-1} \geqslant 0$. It follows that $\tilde{R} \geqslant 0$. Since $U_1^{\mathrm{T}} D_1$ is a unit lower triangular matrix and its off-diagonal elements are nonpositive, $I - U_1^{\mathrm{T}} D_1 \geqslant 0$. It follows that $(I - U_1^{\mathrm{T}} D_1)C_1 \geqslant 0$. Hence, $\overline{R} \geqslant 0$.

For the proof of part (b), if we compute inverse matrices of U, $\overline{U}$, and $\tilde{U}$, then

$$U^{-1} = \begin{pmatrix} U_1^{-1} & 0 \\ 0 & U_2^{-1} \end{pmatrix}, \overline{U}^{-1} = \begin{pmatrix} U_1^{-1} & U_1^{-1} C_1 U_2^{-1} \\ 0 & U_2^{-1} \end{pmatrix},$$

$$\tilde{U}^{-1} = \begin{pmatrix} U_1^{-1} & U_1^{-1} (U_1^{\mathrm{T}} D_1)^{-1} C_1 U_2^{-1} \\ 0 & U_2^{-1} \end{pmatrix}.$$

Since $(U_1^{\mathrm{T}} D_1)^{-1}$ is a unit lower triangular nonnegative matrix and $C_1 \geqslant 0$, it is clear that $(U_1^{\mathrm{T}} D_1)^{-1} C_1 \geqslant C_1$. Hence, part (b) is proved. Since $M^{-1} = U^{-1} D^{-1} U^{-\mathrm{T}}$, $\overline{M}^{-1} = \overline{U}^{-1} D^{-1} \overline{U}^{-\mathrm{T}}$, and $\tilde{M}^{-1} = \tilde{U}^{-1} D^{-1} (\tilde{U})^{-\mathrm{T}}$, part (b) implies part (c). From (a) and (c), part (d) is proved. Since A is an M-matrix and $A = M - R$ is a regular splitting of A, it is easy to show that $\rho(M^{-1}R) < 1$. Hence, from Theorem 2.7 part (e) is proved. □

Theorem 3.2. *Let A be a symmetric M-matrix of the form* (3), *and let $U_i, D_i, D, \overline{U}, \tilde{U}, \overline{M}, \tilde{M}, \overline{R}$, and $\tilde{R}$ be defined as in Theorem* 3.1. *For a matrix E_1 such that $E_1 \geqslant C_1$, let*

$$\widehat{U} = \begin{pmatrix} U_1 & -E_1 \\ 0 & U_2 \end{pmatrix} \text{ and } \widehat{M} = \widehat{U}^{\mathrm{T}} D \widehat{U}.$$

If $U_1^{\mathrm{T}} D_1 E_1 \leqslant C_1$, then the following holds:
(a) $\widehat{R} = \widehat{M} - A \geqslant 0$,
(b) $0 \leqslant \overline{U}^{-1} \leqslant \widehat{U}^{-1} \leqslant \tilde{U}^{-1}$,
(c) $0 \leqslant \overline{M}^{-1} \leqslant \widehat{M}^{-1} \leqslant \tilde{M}^{-1}$,
(d) $A = \widehat{M} - \widehat{R}$ *is a regular splitting of* A,
(e) $\rho(\tilde{M}^{-1}\tilde{R}) \leqslant \rho(\widehat{M}^{-1}\widehat{R}) \leqslant \rho(\overline{M}^{-1}\overline{R}) < 1$.

Proof. By simple calculation, one obtains

$$\widehat{R} = \begin{pmatrix} R_1 & C_1 - U_1^{\mathrm{T}} D_1 E_1 \\ C_1^{\mathrm{T}} - E_1^{\mathrm{T}} D_1 U_1 & R_2 + E_1^{\mathrm{T}} D_1 E_1 \end{pmatrix} \tag{4}$$

$$\widehat{U}^{-1} = \begin{pmatrix} U_1^{-1} & U_1^{-1} E_1 U_2^{-1} \\ 0 & U_2^{-1} \end{pmatrix}.$$

Since $U_1^{\mathrm{T}} D_1 E_1 \leqslant C_1$, from Eq. (4) $\widehat{R} \geqslant 0$ which shows part (a). Since $(U_1^{\mathrm{T}} D_1)^{-1} \geqslant 0$, $U_1^{\mathrm{T}} D_1 E_1 \leqslant C_1 \leqslant E_1$ implies $C_1 \leqslant E_1 \leqslant (U_1^{\mathrm{T}} D_1)^{-1} C_1$. From this fact and Eq. (4), part (b) is proved. Proofs for the remaining parts can be done as in Theorem 3.1. □

The assumption $U_1^{\mathrm{T}} D_1 E_1 \leqslant C_1$ in Theorem 3.2 implies $E_1 \leqslant (U_1^{\mathrm{T}} D_1)^{-1} C_1$. Since C_1 is usually a sparse matrix, $(U_1^{\mathrm{T}} D_1)^{-1} C_1$ becomes less sparse than C_1 because of fill-in elements. If we drop some of fill-in elements of $(U_1^{\mathrm{T}}D_1)^{-1}C_1$, then we have a matrix E_1 such that $C_1 \leqslant E_1 \leqslant (U_1^{\mathrm{T}}D_1)^{-1}C_1$. It is easy to show that $0 \leqslant C_1 \leqslant E_1 \leqslant (U_1^{\mathrm{T}}D_1)^{-1}C_1$ does not imply $U_1^{\mathrm{T}}D_1E_1 \leqslant C_1$. Next example shows that there exist nonnegative matrices C_1 and E_1 such that $U_1^{\mathrm{T}}D_1E_1 \leqslant C_1 \leqslant E_1$, where E_1 is obtained by dropping some of fill-in elements of $(U_1^{\mathrm{T}}D_1)^{-1}C_1$.

Example 3.3. Let

$$U_1^{\mathrm{T}}D_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{pmatrix} \quad \text{and } C_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}.$$

Since $U_1^{\mathrm{T}}D_1$ is a unit lower triangular Z-matrix, $U_1^{\mathrm{T}}D_1$ is clearly an M-matrix. From simple calculation,

$$(U_1^{\mathrm{T}}D_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix} \quad \text{and} \quad (U_1^{\mathrm{T}}D_1)^{-1}C_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 3 \\ 8 & 12 & 12 \end{pmatrix}.$$

Let E_1 be a matrix obtained by dropping the entry 8 of $(U_1^{\mathrm{T}}D_1)^{-1}C_1$. That is,

$$E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 3 \\ 0 & 12 & 12 \end{pmatrix}.$$

Then, $0 \leqslant C_1 \leqslant E_1 \leqslant (U_1^{\mathrm{T}}D_1)^{-1}C_1$. On the other hand,

$$U_1^{\mathrm{T}}D_1E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ -8 & 1 & 3 \end{pmatrix} \leqslant C_1.$$

Next, we consider block IC factorization preconditioners for a symmetric block-tridiagonal M-matrix of the general form (2). Generalization of Theorems 3.1 and 3.2 to an M-matrix of the form (2) is complicated but easy, so that the following theorem is described without proof.

Theorem 3.4. *Let A be a symmetric block-tridiagonal M-matrix of the form* (2) *and let $B_i = U_i^{\mathrm{T}}D_iU_i - R_i$ be a regular splitting of B_i which can be obtained by the IC factorization process for each $i = 1,2,\ldots,m$. Suppose that for each $i = 1,2,\ldots,m-1$ E_i is a matrix which satisfies $U_i^{\mathrm{T}}D_iE_i \leqslant C_i \leqslant E_i$. Let*

$$D = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_m \end{pmatrix}, \quad U = \begin{pmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_m \end{pmatrix},$$

$$\overline{U} = \begin{pmatrix} U_1 & -C_1 & 0 & \cdots & 0 \\ 0 & U_2 & -C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{m-1} & -C_{m-1} \\ 0 & 0 & \cdots & 0 & U_m \end{pmatrix},$$

$$\widehat{U} = \begin{pmatrix} U_1 & -E_1 & 0 & \cdots & 0 \\ 0 & U_2 & -E_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{m-1} & -E_{m-1} \\ 0 & 0 & \cdots & 0 & U_m \end{pmatrix},$$

$$\tilde{U} = \begin{pmatrix} U_1 & -(U_1^{\mathrm{T}}D_1)^{-1}C_1 & 0 & \cdots & 0 \\ 0 & U_2 & -(U_2^{\mathrm{T}}D_2)^{-1}C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{m-1} & -(U_{m-1}^{\mathrm{T}}D_{m-1})^{-1}C_{m-1} \\ 0 & 0 & \cdots & 0 & U_m \end{pmatrix},$$

$M = U^{\mathrm{T}}DU, \overline{M} = \overline{U}^{\mathrm{T}}D\overline{U}, \widehat{M} = \widehat{U}^{\mathrm{T}}D\widehat{U}$, and $\tilde{M} = \tilde{U}^{\mathrm{T}}D\tilde{U}$. Then, the following holds:
(a) $R = M - A \geqslant 0, \overline{R} = \overline{M} - A \geqslant 0, \widehat{R} = \widehat{M} - A \geqslant 0$, *and* $\tilde{R} = \tilde{M} - A \geqslant 0$,
(b) $0 \leqslant U^{-1} \leqslant \overline{U}^{-1} \leqslant \widehat{U}^{-1} \leqslant \tilde{U}^{-1}$,
(c) $0 \leqslant M^{-1} \leqslant \overline{M}^{-1} \leqslant \widehat{M}^{-1} \leqslant \tilde{M}^{-1}$,
(d) $A = M - R = \overline{M} - \overline{R} = \widehat{M} - \widehat{R} = \tilde{M} - \tilde{R}$ *are regular splittings of* A,
(e) $\rho(\tilde{M}^{-1}\tilde{R}) \leqslant \rho(\widehat{M}^{-1}\widehat{R}) \leqslant \rho(\overline{M}^{-1}\overline{R}) \leqslant \rho(M^{-1}R) < 1$.

If $B_i = U_i^{\mathrm{T}}D_iU_i - R_i$ is an IC factorization of B_i, then

$$\begin{pmatrix} B_i + R_i & -C_i \\ -C_i^{\mathrm{T}} & 0 \end{pmatrix} = \begin{pmatrix} U_i^{\mathrm{T}} & 0 \\ -((U_i^{\mathrm{T}}D_i)^{-1}C_i)^{\mathrm{T}} & I \end{pmatrix} \begin{pmatrix} D_i & 0 \\ 0 & I \end{pmatrix}$$
$$\begin{pmatrix} U_i & -(U_i^{\mathrm{T}}D_i)^{-1}C_i \\ 0 & -C_i^{\mathrm{T}}(U_i^{\mathrm{T}}D_iU_i)^{-1}C_i \end{pmatrix}.$$

This equation shows that $(U_i^{\mathrm{T}}D_i)^{-1}C_i$ required for the construction of the upper triangular matrix $\tilde{U}$ in Theorem 3.4 can be computed at the time when the IC factorization of B_i is executed. In other words, since $(U_i^{\mathrm{T}}D_i)^{-1}$ is a product of elementary lower triangular matrices which are generated during the IC factorization process of B_i, $(U_i^{\mathrm{T}}D_i)^{-1}C_i$ is not computed explicitly using matrix-solve operations, but computed implicitly using elementary lower triangular matrices.

Since U_i's can be computed independently of one another, four types of the block IC factorization preconditioners M, $\overline{M}$, $\widehat{M}$, and $\tilde{M}$ presented in Theorem 3.4 can be computed *in parallel*. This inherent parallelism is a big advantage of four types of the block IC factorization preconditioners. The PCG method is used to test the effectiveness of the block preconditioners in Theorem 3.4, so the PCG algorithm with a preconditioner K is described below. Here, K is assumed to be a symmetric positive-definite matrix.

Algorithm : PCG

Choose x_0 and compute $r_0 = b - Ax_0$

Solve $Kw_0 = r_0$ and set $p_0 = w_0$

for $i = 0, 1, \ldots$

$\alpha_i = (r_i, w_i)/(p_i, Ap_i)$

$x_{i+1} = x_i + \alpha_i p_i$

$r_{i+1} = r_i - \alpha_i A p_i$

if $\| r_{i+1} \|_2 <$ tol, stop

Solve $Kw_{i+1} = r_{i+1}$

$\beta_i = (r_{i+1}, w_{i+1})/(r_i, w_i)$

$p_{i+1} = w_{i+1} + \beta_i p_i$

If x^* is the exact solution of $Ax = b$, then the well-known convergence property [15, p.187] of the PCG is

$$\| x_i - x^* \|_A \leqslant 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^i \| x_0 - x^* \|_A \tag{5}$$

where $\kappa = \lambda_{\max}(K^{-1}A)/\lambda_{\min}(K^{-1}A)$, and $\lambda_{\max}(K^{-1}A)$ and $\lambda_{\min}(K^{-1}A)$ denote the largest and smallest eigenvalues of $K^{-1}A$, respectively. From Eq. (5), we can see that κ needs to be close to 1 to ensure fast convergence of the PCG. In other words, we need to choose a preconditioner K such that eigenvalues of $K^{-1}A$ are clustered about 1. If $A = K - N$ is a splitting of A, then $K^{-1}A = I - K^{-1}N$. Hence, we want to make $\rho(K^{-1}N)$ as small as possible in order to make eigenvalues of $K^{-1}A$ clustered about 1. From this point of view, the PCG with the block preconditioner of type $\tilde{M}$ will converge faster than the PCG with any other type of the block preconditioner. When the block preconditioner of type M is used for the PCG, the preconditioner solve step for finding the vector w_i can also be computed in parallel. However, numerical experiments in Section 5 show that the effectiveness of the block preconditioner of type M is much worse than that of any other type of the block preconditioner. Notice that in this paper the effectiveness of the preconditioners is measured as the number of iterations of the PCG satisfying a termination criterion. That is, the smaller the number of iterations is, the more effective the preconditioner is.

4. Applications of block IC factorization preconditioners

The construction of four types of the block IC factorization preconditioners presented in Theorem 3.4 will be considered in this section for a special type of matrix described below. The matrix arises from five-point discretization of the second-order self-adjoint elliptic partial differential equation:

$$-\left(a(x,y)u_x(x,y)\right)_x-\left(b(x,y)u_y(x,y)\right)_y+c(x,y)u(x,y)=f(x,y) \tag{6}$$

with $a(x,y)>0$, $b(x,y)>0$, $c(x,y)\geqslant 0$, and $(x,y)\in\Omega$, where Ω is a square region, and with suitable boundary conditions on $\partial\Omega$ which denotes the boundary of Ω. The resulting matrix A is a symmetric M-matrix and thus positive definite, and the structure of A is of the form (2) with B_i's symmetric tridiagonal matrices and C_i's nonnegative diagonal matrices.

Since B_i is a tridiagonal matrix, the complete Cholesky factorization of B_i has no fill-in elements. More specifically, if $B_i=U_i^{\mathrm{T}}D_iU_i$ is the complete Cholesky factorization of B_i, then U_i is an upper bidiagonal matrix. Hence, four types of block IC factorization preconditioners are constructed using the complete Cholesky factorizations of B_i's rather than using the IC factorizations of B_i's. The block preconditioners defined in Theorem 3.4 which are constructed based on the Cholesky factorizations of 1×1 block matrices B_i are from now on called *1-block preconditioners.* In particular, for the construction of 1-block preconditioner of type $\widehat{M}$ E_i should be chosen so that $U_i^{\mathrm{T}}D_iE_i\leqslant C_i\leqslant E_i\,(1\leqslant i\leqslant m-1)$. We now describe how to choose such a matrix E_i. Suppose that $B_i=U_i^{\mathrm{T}}D_iU_i$ is the complete Cholesky factorization of B_i. Then, $(U_i^{\mathrm{T}}D_i)^{-1}C_i$ becomes a lower triangular matrix which is much less sparse than C_i. For each fixed i, let E_{ij}'s be matrices obtained by dropping some of fill-in elements of $(U_i^{\mathrm{T}}D_i)^{-1}C_i$, where $0\leqslant j\leqslant d-1$ and d is the order of matrix C_i, and the nonzero structures of E_{ij}'s for $d=9$ are illustrated in Fig. 1. Let Q_j be a zero pattern set corresponding to the matrix E_{ij}. Then, for each $0\leqslant j\leqslant d-1$

$$Q_j=\{(r,s)\,|\,r-s\neq 0,1,\ldots,j,1\leqslant r\leqslant d,1\leqslant s\leqslant d\}.$$

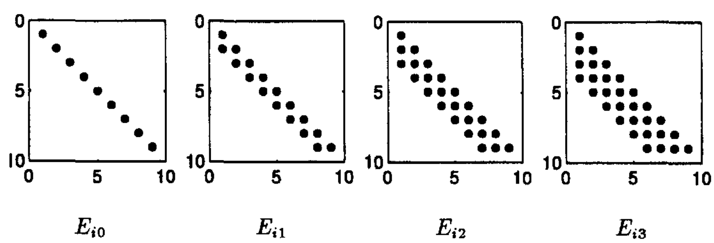

Fig. 1. Nonzero structures of E_{ij}'s.

For $j = d-1$, no dropping of fill-in elements is used, so it can be easily seen that $E_{i,d-1} = (U_i^{\mathrm{T}}D_i)^{-1}C_i$.

Since $(U_i^{\mathrm{T}}D_i)^{-1}$ is a unit lower triangular matrix and C_i is a diagonal matrix, diagonal entries of $(U_i^{\mathrm{T}}D_i)^{-1}C_i$ are equal to those of C_i and thus $E_{i0} = C_i$. For these matrices C_i and E_{ij}, the following theorem is obtained.

Theorem 4.1. *Let U_i's D_i's, and E_{ij}'s be defined as above. Then, for each $0 \leqslant j \leqslant d-1$*

$$U_i^{\mathrm{T}}D_iE_{ij} \leqslant C_i \leqslant E_{ij}.$$

Proof. Since $(U_i^{\mathrm{T}}D_i)^{-1}$ is a unit lower triangular nonnegative matrix and $C_i \geqslant 0$,

$$0 \leqslant C_i \leqslant (U_i^{\mathrm{T}}D_i)^{-1}C_i.$$

Since E_{ij} is obtained by dropping all fill-in elements except $j+1$ diagonals of $(U_i^{\mathrm{T}}D_i)^{-1}C_i$ (see Fig. 1),

$$C_i \leqslant E_{ij} \leqslant (U_i^{\mathrm{T}}D_i)^{-1}C_i.$$

Observe that $U_i^{\mathrm{T}}D_i$ is a unit lower triangular matrix and its off-diagonal elements are nonpositive. At the places (r,s) not belonging to the zero pattern set Q_j corresponding to E_{ij}, E_{ij} has the same elements as $(U_i^{\mathrm{T}}D_i)^{-1}C_i$ and hence $U_i^{\mathrm{T}}D_iE_{ij}$ has the same elments as C_i. Whereas, at the places (r,s) belonging to Q_j, $U_i^{\mathrm{T}}D_iE_{ij}$ has the nonpositive elements. Since $C_i \geqslant 0$, $U_i^{\mathrm{T}}D_iE_{ij} \leqslant C_i$. Hence, the proof is complete. □

Theorem 4.1 showed that the matrices E_{ij} satisfy the assumption in Theorem 3.4. So, if we let for each $0 \leqslant j \leqslant d-1$

$$\widehat{U}_j = \begin{pmatrix} U_1 & -E_{1j} & 0 & \cdots & 0 \\ 0 & U_2 & -E_{2j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{m-1} & -E_{m-1,j} \\ 0 & 0 & \cdots & 0 & U_m \end{pmatrix},$$

then $\widehat{M}_j = \widehat{U}_j^{\mathrm{T}}D\widehat{U}_j$ is a 1-block IC factorization preconditioner of type $\widehat{M}$ which is more effective than both M and $\overline{M}$ (see Theorem 3.4), where D, M, and $\overline{M}$ are defined the same as in Theorem 3.4. Since $E_{i,d-1} = (U_i^{\mathrm{T}}D_i)^{-1}C_i$ for each i, $\widehat{U}_{d-1} = \tilde{U}$ and thus $\widehat{M}_{d-1} = \tilde{M}$. It was already mentioned in Section 3 that $(U_i^{\mathrm{T}}D_i)^{-1} = L_{i,d-1}\cdots L_{i2}L_{i1}$, where $L_{ik} (1 \leqslant k \leqslant d-1)$ is an elementary lower triangular matrix which is equal to an identity matrix with some additional nonzero elements in the

k-th column below the diagonal. Then, efficient computation of E_{ij} can be done as follows:

$$E_{ij}^1 = C_i$$

for $k = 1, 2, \ldots, d-1$

$$\widehat{E}_{ij}^k = L_{ik} E_{ij}^k$$

$$E_{ij}^{k+1} = \widehat{E}_{ij}^k - \widehat{R}_{ij}^k$$

$$E_{ij} = E_{ij}^d$$

Here $\widehat{R}_{ij}^k$ has the same elements as $\widehat{E}_{ij}^k$ at the places (r,s) belonging to Q_j, and $\widehat{R}_{ij}^k$ has zero elements at the places (r,s) not belonging to Q_j.

For the purpose of getting more effective block IC factorization preconditioners than 1-block preconditioners mentioned in the above, we now consider *2-block preconditioners* which are constructed based on the IC factorizations of 2×2 block matrices rather than 1×1 block matrices B_i. For simplicity of exposition, suppose that A is a 4×4 symmetric block-tridiagonal M-matrix of the form (2), i.e., $m = 4$ is assumed in the form (2). First, A is partitioned into

$$A = \begin{pmatrix} \mathscr{B}_1 & -\mathscr{C}_1 \\ -\mathscr{C}_1^{\mathrm{T}} & \mathscr{B}_2 \end{pmatrix},$$

where

$$\mathscr{B}_1 = \begin{pmatrix} B_1 & -C_1 \\ -C_1^{\mathrm{T}} & B_2 \end{pmatrix}, \mathscr{B}_2 = \begin{pmatrix} B_3 & -C_3 \\ -C_3^{\mathrm{T}} & B_4 \end{pmatrix}, \quad \text{and} \quad \mathscr{C}_1 = \begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix}$$

are 2×2 block submatrices of A. Since A is assumed to be a symmetric M-matrix, $\mathscr{B}_i$ $(i = 1, 2)$ is also a symmetric M-matrix. It follows that the IC factorization of $\mathscr{B}_i$ exists, see Theorem 2.3. If $\mathscr{B}_i = U_i^{\mathrm{T}} D_i U_i$ is the complete Cholesky factorization of $\mathscr{B}_i$, then the nonzero structure of $\mathscr{B}_i$ and U_i for $d = 7$ are illustrated in Fig. 2.

As can be seen in Fig. 2, the complete Cholesky factorization of $\mathscr{B}_i$ now has a lot of fill-in elements, so that the IC factorization of $\mathscr{B}_i$ with some fill-ins needs to be considered for the construction of 2-block IC factorization preconditioners. For each fixed i, let $\mathscr{B}_i = U_{ij}^{\mathrm{T}} D_{ij} U_{ij} - R_{ij}$ be the IC factorization of $\mathscr{B}_i$, where $0 \leqslant j \leqslant d-1$, and the nonzero structures of U_{ij}'s for $d = 7$ are illustrated in Fig. 3. Notice that if $\mathscr{B}_i = U_i^{\mathrm{T}} D_i U_i$ is the complete Cholesky factorization of $\mathscr{B}_i$, then $U_{i,d-1} = U_i$ and $R_{i,d-1} = 0$.

If we let for each $0 \leqslant j \leqslant d-1$

$$\mathscr{D}_j^2 = \begin{pmatrix} D_{1j} & 0 \\ 0 & D_{2j} \end{pmatrix}, \qquad \mathscr{U}_j^2 = \begin{pmatrix} U_{1j} & 0 \\ 0 & U_{2j} \end{pmatrix},$$

$$\overline{\mathscr{U}}_j^2 = \begin{pmatrix} U_{1j} & -\mathscr{C}_1 \\ 0 & U_{2j} \end{pmatrix}, \qquad \tilde{\mathscr{U}}_j^2 = \begin{pmatrix} U_{1j} & -(U_{1j}^{\mathrm{T}} D_{1j})^{-1} \mathscr{C}_1 \\ 0 & U_{2j} \end{pmatrix},$$

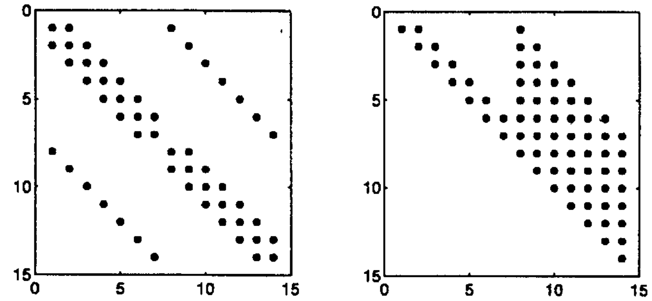

Fig. 2. Nonzero structures of $\mathcal{B}_i$ and U_i.

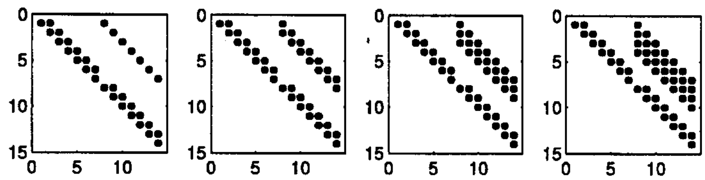

Fig. 3. Nonzero structures of U_{ij}'s.

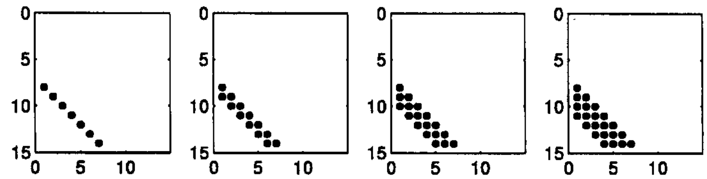

Fig. 4. Nonzero structures of $\mathscr{E}_{1j}$'s.

then $M_j^2 = (\mathcal{U}_j^2)^{\mathrm{T}}\mathcal{D}_j^2\mathcal{U}_j^2, \overline{M}_j^2 = (\overline{\mathcal{U}}_j^2)^{\mathrm{T}}\mathcal{D}_j^2\overline{\mathcal{U}}_j^2$, and $\tilde{M}_j^2 = (\tilde{\mathcal{U}}_j^2)^{\mathrm{T}}\mathcal{D}_j^2\tilde{\mathcal{U}}_j^2$ are 2-block IC factorization preconditioners, where the superscript 2 is used to represent 2-block preconditioners.

We now consider the construction of 2-block preconditioner of type $\widehat{M}$. For each $0 \leqslant j \leqslant d-1$, let $\mathscr{E}_{1j}$ be a matrix obtained by dropping some of fill-in elements of $(U_{1j}^{\mathrm{T}}D_{1j})^{-1}\mathscr{C}_1$, and the nonzero structures of $\mathscr{E}_{1j}$'s for $d = 7$ are illustrated in Fig. 4.

Notice that $\mathscr{E}_{1j}$ has nonzero elements on $j+1$ diagonals, see Fig. 4. Since $(U_{1j}^{\mathrm{T}}D_{1j})^{-1}$ is a unit dense lower triangular matrix for each $0 \leqslant j \leqslant d-1$, $\mathscr{E}_{10} = \mathscr{C}_1$ and the nonzero structure of $\mathscr{E}_{1,d-1}$ is the same as that of $(U_{1j}^{\mathrm{T}}D_{1j})^{-1}\mathscr{C}_1$ for each $0 \leqslant j \leqslant d-1$. Moreover, it is easy to see that $\mathscr{E}_{1,d-1} = (U_{1,d-1}^{\mathrm{T}}D_{1,d-1})^{-1}\mathscr{C}_1$. For these matrices $\mathscr{C}_1$ and $\mathscr{E}_{1j}$, $U_{1j}^{\mathrm{T}}D_{1j}\mathscr{E}_{1j} \leqslant \mathscr{C}_1 \leqslant \mathscr{E}_{1j}$ can be shown as

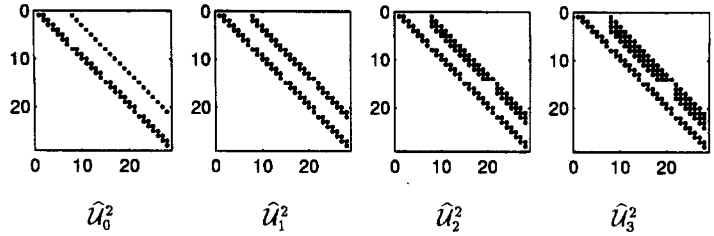

Fig. 5. Nonzero structures of $\widehat{\mathscr{U}}_j^2$'s.

in the proof of Theorem 4.1, where $0 \leqslant j \leqslant d-1$. If we let for each $0 \leqslant j \leqslant d-1$

$$\widehat{\mathscr{U}}_j^2 = \begin{pmatrix} U_{1j} & -\mathscr{E}_{1j} \\ 0 & U_{2j} \end{pmatrix},$$

then $\widehat{M}_j^2 = (\widehat{\mathscr{U}}_j^2)^{\mathrm{T}} \mathscr{D}_j^2 \widehat{\mathscr{U}}_j^2$ is a 2-block IC factorization preconditioner which is more effective than both M_j^2 and $\overline{M}_j^2$, where $\mathscr{D}_j^2$ is defined the same as above. Since $\mathscr{E}_{1,d-1} = (U_{1,d-1}^{\mathrm{T}} D_{1,d-1})^{-1} \mathscr{C}_1$, $\widehat{\mathscr{U}}_{d-1}^2 = \tilde{\mathscr{U}}_{d-1}^2$ and so $\widehat{M}_{d-1}^2 = \tilde{M}_{d-1}^2$. Since $\mathscr{E}_{10} = \mathscr{C}_1$, $\widehat{\mathscr{U}}_0^2 = \overline{\mathscr{U}}_0^2$ and so $\widehat{M}_0^2 = \overline{M}_0^2$. From Figs. 3 and 4, the nonzero structures of $\widehat{\mathscr{U}}_j^2$'s for $d = 7$ are illustrated in Fig. 5.

In the similar way as was done for 2-block preconditioners, *k-block preconditioners* M_j^k, $\overline{M}_j^k$, $\widehat{M}_j^k$, and $\tilde{M}_j^k$ which are based on the IC factorizations of $k \times k$ block matrices can be easily constructed. That is, the $m \times m$ block matrix A of the form (2) is first partitioned so that each submatrix of A is a $k \times k$ block matrix (it is assumed that m is divisible by k), and then the IC factorizations of $k \times k$ block matrices are carried out to construct k-block preconditioners. Then, it can be seen that $\widehat{M}_{d-1}^k = \tilde{M}_{d-1}^k$ and $\widehat{M}_0^k = \overline{M}_0^k$ for all k. Since the complete Cholesky factorizations of tridiagonal matrices B_i are used for construction of 1-block preconditioners, for each $0 \leqslant j \leqslant d-1$ $M_j^1 = M$, $\overline{M}_j^1 = \overline{M}$, $\tilde{M}_j^1 = \tilde{M}$, and $\widehat{M}_j^1 = \widehat{M}_j$. Notice that the construction of $(k+1)$-block preconditioners requires more storage and arithmetic than that of k-block preconditioners.

5. Numerical results

In this section, we provide numerical results of the PCG method using three different types of the k-block IC factorization preconditioners M_j^k, $\overline{M}_j^k$, and $\widehat{M}_j^k$ for linear systems $Ax = b$ with the special type of matrix A described in Section 4. For each type of preconditioner, numerical experiments are carried out for $0 \leqslant j \leqslant 3$ and $1 \leqslant k \leqslant 4$. However, numerical experiments for the k-block IC factorization preconditioner $\tilde{M}_j^k$ are not provided here since it requires a lot of fill-in elements causing too much storage and arithmetic. To evaluate the effectiveness of the k-block IC factorization preconditioners,we also provide numerical results of the PCG method using the standard IC factorization preconditioner with 0 extra diagonals which is called the ICCG(0) method in [16]. In all cases, the CG and PCG methods were started with $x_0 = 0$, and they were stopped when $\| r_i \| / \| b \| < 10^{-8}$. All numerical experiments have been carried out in double precision floating

Table 1
Number of iterations for Example 5.1 using M_j^k and $\overline{M}_j^k$

		PCG									
n	j	M_j^1	M_j^2	M_j^3	M_j^4	$\overline{M}_j^1$	$\overline{M}_j^2$	$\overline{M}_j^3$	$\overline{M}_j^4$	ICCG(0)	CG
48×48	0	81	71	64	61	47	45	45	45	42	92
	1		69	59	53		41	38	36		
	2		69	58	52		40	36	33		
	3		69	58	51		40	35	33		
60×60	0	101	86	79	74	57	55	54	54	50	115
	1		85	71	64		50	46	43		
	2		85	70	63		49	44	41		
	3		85	70	61		48	43	40		

Table 2
Number of iterations for Example 5.1 using $\widehat{M}_j^k$

		PCG					
n	j	$\widehat{M}_j^1$	$\widehat{M}_j^2$	$\widehat{M}_j^3$	$\widehat{M}_j^4$	ICCG(0)	CG
48×48	0	47	45	45	45	42	92
	1	40	36	34	33		
	2	38	34	31	30		
	3	38	33	30	29		
60×60	0	57	55	54	54	50	115
	1	49	44	42	40		
	2	47	41	38	36		
	3	46	40	37	35		

point arithmetic, and all data presented in Tables 1 – 6 represent *the number of iterations* satisfying the stopping criterion mentioned above.

Example 5.1. We consider Eq. (6) over the square region $\Omega = (0,1)\times(0,1)$ with $a(x,y) = b(x,y) = 1$, $c(x,y) = 0$, and Dirichlet condition $u = 0$ on $\partial\Omega$. That is, the following PDE problem is considered:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We have used two uniform meshes of $\Delta x = \Delta y = \frac{1}{49}$ and $\Delta x = \Delta y = \frac{1}{61}$, which leads to two matrices of order $n = 48 \times 48$ and $n = 60 \times 60$, where Δx and Δy refer to the mesh sizes in the x-direction and y-direction, respectively. Once the matrix A is constructed from five-point finite difference discretization of the PDE, the right-hand side vector b is chosen so that $b = A[1,1,\ldots,1]^{\mathrm{T}}$. Numerical results for this problem are listed in Tables 1 and 2.

Table 3
Number of iterations for Example 5.2 using M_j^k and $\overline{M}_j^k$

n	j	PCG M_j^1	M_j^2	M_j^3	M_j^4	$\overline{M}_j^1$	$\overline{M}_j^2$	$\overline{M}_j^3$	$\overline{M}_j^4$	ICCG(0)	CG
48×48	0	98	84	76	72	56	54	54	53	51	153
	1		82	70	63		48	44	41		
	2		82	69	61		46	41	38		
	3		82	69	61		46	40	37		
60×60	0	122	103	93	89	69	67	66	65	62	192
	1		102	86	78		59	54	51		
	2		102	85	75		57	51	47		
	3		102	85	75		56	50	46		

Table 4
Number of iterations for Example 5.2 using $\widehat{M}_j^k$

n	j	PCG $\widehat{M}_j^1$	$\widehat{M}_j^2$	$\widehat{M}_j^3$	$\widehat{M}_j^4$	ICCG(0)	CG
48×48	0	56	54	54	53	51	153
	1	46	41	39	38		
	2	43	38	35	34		
	3	42	37	34	32		
60×60	0	69	67	66	65	62	192
	1	57	51	48	47		
	2	53	46	43	41		
	3	52	45	41	39		

Example 5.2. We consider Eq. (6) over the square region $\Omega=(0,1)\times(0,1)$ with $a(x,y)=b(x,y)=\cos x, c(x,y)=0$, and Dirichlet condition $u=0$ on $\partial\Omega$. That is, the following PDE problem is considered:

$$\begin{cases} -\nabla\cdot(\cos x\nabla u)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

We have used the same uniform meshes as Example 5.1. Once the matrix A is constructed from five-point discretization of the PDE, the right-hand side vector b is chosen so that the exact solution is the discretization of $10xy(1-x)(1-y)\exp(x^{4.5})$. Numerical results for this problem are listed in Tables 3 and 4.

Example 5.3. We consider Eq. (6) over the square region $\Omega=(0,1)\times(0,1)$ with $a(x,y)=b(x,y)$, $c(x,y)=0$, and the boundary conditions $u=0$ for $y=0, u_x=0$ for $x=0$ and $x=1, u_y=0$ for

Table 5
Number of iterations for Example 5.3 using M_j^k and $\overline{M}_j^k$

n	j	PCG M_j^1	M_j^2	M_j^3	M_j^4	$\overline{M}_j^1$	$\overline{M}_j^2$	$\overline{M}_j^3$	$\overline{M}_j^4$	ICCG(0)	CG
49×48	0	168	120	108	102	83	80	78	78	75	NC
	1		118	101	89		69	63	59		
	2		120	98	88		66	58	55		
	3		119	98	87		66	58	54		
61×60	0	213	152	136	127	104	99	97	97	94	NC
	1		151	125	112		86	78	73		
	2		151	124	108		84	73	67		
	3		150	123	106		83	73	66		

Table 6
Number of iterations for Example 5.3 using $\widehat{M}_j^k$

n	j	PCG $\widehat{M}_j^1$	$\widehat{M}_j^2$	$\widehat{M}_j^3$	$\widehat{M}_j^4$	ICCG(0)	CG
49×48	0	83	80	78	78	75	NC
	1	68	60	56	54		
	2	65	55	51	48		
	3	64	54	49	46		
61×60	0	104	99	97	97	94	NC
	1	86	74	70	67		
	2	81	69	63	59		
	3	80	67	61	57		

$y = 1$, where

$$a(x,y) = \begin{cases} 1000, & 0.1 \leqslant x, y \leqslant 0.9, \\ 1 & \text{otherwise.} \end{cases}$$

We have used two uniform meshes of $\Delta x = \Delta y = 1/48$ and $\Delta x = \Delta y = 1/60$, which leads to two matrices of order $n = 49 \times 48$ and $n = 61 \times 60$. Once the matrix A is constructed from five-point discretization of the PDE, the right-hand side vector b is chosen so that the exact solution is the discretization of $10x^2y(1-x)^2(1-y)^2 \exp(x^{4.5})$. Numerical results for this problem are listed in Tables 5 and 6. NC in Tables 5 and 6 indicates that the CG method does not converge within 1000 iterations

As can be seen in Tables 1–6, the numerical results presented are in good agreement with the theoretical results presented in Theorem 3.4. That is, the block preconditioner of type $\widehat{M}$ is more effective than the block preconditioners of types M and $\overline{M}$. It can be also seen that the PCG with $(k+1)$-block preconditioners converges faster than the PCG with k-block preconditioners. As compared with the standard IC factorization preconditioner, $\widehat{M}_j^k$ is relatively effective when $j \geqslant 1$ and $k \geqslant 1$, and $\overline{M}_j^k$ is relatively effective when $j \geqslant 1$ and $k \geqslant 2$.

6. Conclusions

We presented in this paper four types of block IC factorization preconditioners which can be computed *in parallel*. Block IC factorization preconditioner of type $\tilde{M}$ may not be used in practical situations since it requires a lot of fill-in elements causing too much storage and arithmetic. Block preconditioner of type M has rich parallelism since both the computation of preconditioner and preconditioner solve step of the PCG can be done in parallel, but its effectiveness is much worse than other types of block preconditioners.

When using 1-block preconditioners, $\widehat{M}_j^1$ with $j \geqslant 1$ is strongly recommended as a preconditioner of the PCG. When using k-block preconditioners with $k \geqslant 2$, both $\overline{M}_j^k$ and $\widehat{M}_j^k$ with $j \geqslant 1$ are recommended as a preconditioner of the PCG. Notice that the construction of $\widehat{M}_j^k$ requires more storage and arithmetic than that of $\overline{M}_j^k$ and the number of arithmetic operations grows as j becomes large. From our experiments, it is not recommended to use large value of j and the optimal value of j usually ranges from 1 to 5. Future work will include applications of the block IC factorization preconditioners to more general type of problems and will include block incomplete LU factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix.

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