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The generalized order-k Fibonacci–Pell sequence by matrix methods

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Abstract

In this paper, we consider the usual and generalized order-k Fibonacci and Pell recurrences, then we define a new recurrence, which we call *generalized order-k F–P sequence*. Also we present a systematic investigation of the generalized order-k F–P sequence. We give the generalized Binet formula, some identities and an explicit formula for sums of the generalized order-k F–P sequence by matrix methods. Further, we give the generating function and combinatorial representations of these numbers. Also we present an algorithm for computing the sums of the generalized order-k Pell numbers, as well as the Pell numbers themselves.

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1. Introduction

The Fibonacci and Pell sequences can be generalized as follows: let r be a nonnegative integer such that $r \geqslant 0$ and $4^{r-1}+1 \neq 0$. Define the generalized Fibonacci sequence as shown

$$x_{n+1} = 2^r x_n + x_{n-1}, \tag{1}$$

where $x_0 = 0$ and $x_1 = 1$. When $r = 0$, then $x_n = F_n$ (nth Fibonacci number) and when $r = 1$, then $x_n = P_n$ (nth Pell number).

Miles [19] define the generalized k-Fibonacci numbers as shown for $n > k \geqslant 2$

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k},$$

where $f_1 = f_2 = \cdots = f_{k-2} = 0$ and $f_{k-1} = f_k = 1$. Then the author have studied some properties of the sequence $\{f_n\}$.

Further, in [15], the authors consider the generalized k-Fibonacci numbers, then they give the Binet formula of the generalized Fibonacci sequence $\{f_n\}$.

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Er [8] give the definition of the k sequences of the generalized order-k Fibonacci numbers as follows: for $n>0$ and $1\leqslant i\leqslant k$

$$g_n^i=\sum_{j=1}^{k} g_{n-j}^i, \tag{2}$$

with initial conditions

$$g_n^i=\begin{cases}1 & \text{if } i=1-n\\ 0 & \text{otherwise}\end{cases} \quad \text{for } 1-k\leqslant n\leqslant 0,$$

where g_n^i is the nth term of the ith sequence. Also Er give the generating matrix for the generalized order-k Fibonacci sequence $\{g_n^i\}$.

Also in [12], the authors give the relationships between the generalized order-k Fibonacci numbers g_n^i and the generalized order-k Lucas numbers (see for more details about the generalized Lucas numbers [23]), and give the some useful identities and Binet formulas of the generalized order-k Fibonacci sequence $\{g_n^i\}$ and Lucas sequence $\{l_n^i\}$.

In [13], the authors define the k sequences of the generalized order- k Pell numbers as follows: for $n>0$ and $1\leqslant i\leqslant k$

$$P_n^i=2P_{n-1}^i+P_{n-2}^i+\cdots+P_{n-k}^i, \tag{3}$$

with initial conditions

$$P_n^i=\begin{cases}1 & \text{if } i=1-n\\ 0 & \text{otherwise}\end{cases} \quad \text{for } 1-k\leqslant n\leqslant 0,$$

where P_n^i is the nth term of the ith generalized order-k Pell sequence. Also authors give the generating matrix, Binet formula, sums and combinatorial representations of the terms of generalized order-k Pell sequence $\{P_n^i\}$.

The above sequences are a special case of a sequence which is defined recursively as a linear combination of the preceding k terms

$$a_{n+k}=c_1a_{n+k-1}+c_2a_{n+k-2}+\cdots+c_ka_n,$$

where $c_1,c_2,\ldots,c_k$ are real constants. In [11], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A=\begin{bmatrix} c_1 & c_2 & \ldots & c_{k-1} & c_k\\ 1 & 0 & \ldots & 0 & 0\\ 0 & 1 & \ldots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \ldots & 1 & 0\end{bmatrix}. \tag{4}$$

Then by an inductive argument, he obtains

$$A^n\begin{bmatrix} a_0\\ a_1\\ \vdots\\ a_{k-1}\end{bmatrix}=\begin{bmatrix} a_n\\ a_{n+1}\\ \vdots\\ a_{n+k-1}\end{bmatrix}.$$

Linear recurrence relations are of great interest and have been a central part of number theory. These recursions appear almost everywhere in mathematics and computer science (for more details see [17,5,14,20,9]). For example, many families of orthogonal polynomials, including the Tchebychev polynomials and the Dickson polynomials, satisfy recurrence relations. Linear recurrence relations are of importance in approximation theory and cryptography and they have arisen in computer graphics and time series analysis. Furthermore, higher order linear recurrences have leaderships to the new ideas for the generalization of orthogonal polynomials satisfying second order linear recurrences to the order-k. In a recently published article [1], the authors define the orthogonal polynomial satisfying a generalized order-k linear recurrence.

Also matrix methods are of importance in recurrence relations. For example, the generating matrices are useful tools for the number sequences satisfying a recurrence relation. Further, the combinatorial matrix theory is very important tool to obtain results for number theory [2]. In [15,12,13], the authors define certain generalizations of the usual Fibonacci, Pell and Lucas numbers by matrix methods and then obtain the Binet formulas and combinatorial representations of the generalizations of these number sequence. Furthermore, using matrix methods for computing of properties of recurrence relations are very convenient to parallel algorithm in computer science (see [4,6,7,18,21,22,25]).

Now we define and study properties common to Fibonacci and Pell numbers by investigating a number sequence that satisfy both Fibonacci and Pell recurrences of second order and kth order in matrix representation. Then extending the matrix representation, we give sums of the generalized Fibonacci and Pell numbers subscripted from 1 to n could be derived directly using this representation. Furthermore, using matrix methods, we obtain the generalized Binet formula and combinatorial representation of the new sequence.

2. A generalization of the Fibonacci and Pell numbers

In this section, we define a new order-k generalization of the Fibonacci and Pell numbers, we call *generalized order-k F–P numbers*. Then we obtain the generating matrix of the generalized order-k F–P sequence. Also using the generating matrix, we derive some interesting identities for the generalized order-k F–P numbers which is the well-known formula for the usual or generalized order-k Fibonacci and Pell numbers. We start with the definition of the generalized order-k F–P sequence.

Define k sequences of the generalized order-k F–P numbers as shown: for $n > 0$, $m \geqslant 0$ and $1 \leqslant i \leqslant k$

$$u_n^i = 2^m u_{n-1}^i + u_{n-2}^i + \cdots + u_{n-k}^i,$$

with initial conditions for $1 - k \leqslant n \leqslant 0$

$$u_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise}, \end{cases}$$

where u_n^i is the nth term of the ith generalized F–P sequence. For example, when $k = i = 2$ and $m = 1$, the sequence $\{u_n^i\}$ is reduced to the usual Pell sequence $\{P_n\}$. Also when $m = 0$, the sequence $\{u_n^i\}$ is reduced to the generalized order-k Fibonacci sequence $\{g_n^i\}$.

By the definition of generalized F–P numbers, we can write the following vector recurrence relation:

$$\begin{bmatrix} u_{n+1}^i \\ u_n^i \\ u_{n-1}^i \\ \vdots \\ u_{n-k+2}^i \end{bmatrix} = T \begin{bmatrix} u_n^i \\ u_{n-1}^i \\ u_{n-2}^i \\ \vdots \\ u_{n-k+1}^i \end{bmatrix}, \tag{5}$$

where T is the companion matrix of order k as follows:

$$T = \begin{bmatrix} 2^m & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{6}$$

The matrix T is said to be generalized order-k F–P matrix. To deal with the k sequences of the generalized order-k F–P numbers, we define an $(k \times k)$ matrix $D_n = [d_{ij}]$ as follows:

$$D_n = \begin{bmatrix} u_n^1 & u_n^2 & \dots & u_n^k \\ u_{n-1}^1 & u_{n-1}^2 & \dots & u_{n-1}^k \\ \vdots & \vdots & & \vdots \\ u_{n-k+1}^1 & u_{n-k+1}^2 & \dots & u_{n-k+1}^k \end{bmatrix}. \tag{7}$$

If we expand Eq. (5) to the k columns, then we can obtain the following matrix equation:

$$D_n = T D_{n-1}. \tag{8}$$

Then we have the following lemma.

Lemma 1. *Let the $k \times k$ matrices D_n and T have the forms* (7) *and* (6), *respectively. Then for $n \geqslant 1$*

$$D_n = T^n.$$

Proof. We know that $D_n = T D_{n-1}$. Then, by an inductive argument, we can write $D_n = T^{n-1} D_1$. Since the definition of generalized order-k F–P numbers, we obtain $D_1 = T$ and so $D_n = T^n$ which is desired. □

Then we can give the following theorem.

Theorem 2. *Let the matrix D_n have the form* (7). *Then for $n \geqslant 1$*

$$\det D_n = \begin{cases} 1 & \text{if } k \text{ is odd}, \\ (-1)^n & \text{if } k \text{ is even}. \end{cases}$$

Proof. From Lemma 1, we have $D_n = T^n$. Thus, $\det D_n = \det(T^n) = (\det T)^n$ where $\det T = (-1)^k$. Thus,

$$\det D_n = \begin{cases} 1 & \text{if } k \text{ is even}, \\ (-1)^n & \text{if } k \text{ is odd}. \end{cases}$$

So the proof is complete. □

Now we give some relations involving the generalized order-k F–P numbers.

Theorem 3. *Let u_n^i be the generalized order-k F–P number, for $1 \leqslant i \leqslant k$. Then, for $m \geqslant 0$ and $n \geqslant 1$*

$$u_{n+m}^i = \sum_{j=1}^{k} u_m^j u_{n-j+1}^i.$$

Proof. By Lemma 1, we know that $D_n = T^n$; so we rewrite it as $D_n = D_{n-1} D_1 = D_1 D_{n-1}$. In other words, D_1 is commutative under matrix multiplication. Hence, more generalizing, we can write

$$D_{n+m} = D_n D_m = D_m D_n. \tag{9}$$

Consequently, an element of D_{n+m} is the product of a row D_n and a column of D_m; that is

$$u_{n+m}^i = \sum_{j=1}^{k} u_m^j u_{n-j+1}^i.$$

Thus, the proof is complete. □

For example, if we take $k = 2$ and $m = 0$ in Theorem 3, the sequence $\{u_n^i\}$ is reduced to the usual Fibonacci sequence and we have

$$\begin{aligned} F_{n+m}^2 &= \sum_{j=1}^{2} F_m^j F_{n-j+1}^2 \\ &= F_m^1 F_n^2 + F_m^2 F_{n-1}^2 \end{aligned}$$

and, since $F_n^1 = F_{n+1}^2$ for all positive n and $k = 2$, we obtain

$$F_{n+m}^2 = F_{m+1}^2 F_n^2 + F_m^2 F_{n-1}^2,$$

where F_n^2 is the usual Fibonacci number. Indeed, we generalize the following relation involving the usual Fibonacci numbers [24]

$$F_{n+m} = F_{m+1} F_n + F_m F_{n-1}.$$

For later use, we give the following lemma.

Lemma 4. *Let u_n^i be the generalized order-k F–P number. Then*

$$u_{n+1}^i = u_n^1 + u_n^{i+1} \text{ for } 2 \leqslant i \leqslant k-1, \tag{10}$$

$$u_{n+1}^1 = 2^m u_n^1 + u_n^2, \tag{11}$$

$$u_{n+1}^k = u_n^1. \tag{12}$$

Proof. From (9), we have $D_{n+1} = D_n D_1$. Then by using a property of matrix multiplication, the proof is readily seen. □

Generalizing Theorem 3, we can give the following Corollary without proof since $D_n = T^n$ and so $D_{n+t} = D_{n+r} D_{t-r}$ for all positive integers t and r such that $t > r$.

Corollary 5. *Let u_n^i be the generalized order-k F–P number. Then for $n, t \geqslant 1$ such that $t > r$*

$$u_{n+t}^i = \sum_{j=1}^{k} u_{n+r}^j u_{t-r-j+1}^i.$$

3. Generalized Binet formula

In 1843, Binet derived the following formula for the Fibonacci numbers:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are given by $(1 \mp \sqrt{5})/2$. Also in [16], using the generating function, the author give the Binet formula for generalized Fibonacci numbers.

In this section, we derive the generalized Binet formula for generalized order-k F–P numbers by using matrix methods. Before this, we give some results.

From the companion matrices, it is well known that the characteristic equation of the matrix T given by (6) is $x^k - 2^m x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$ which is also the characteristic equation of generalized order-k F–P numbers.

Lemma 6. *The equation $x^{k+1} - (2^m + 1)x^k + (2^m - 1)x^{k-1} + 1 = 0$ does not have multiple roots for $k \geqslant 2$ and $m \geqslant 0$.*

Proof. Let $f(x) = x^k - 2^m x^{k-1} - x^{k-2} - \cdots - x - 1$ and $h(x) = (x-1) f(x) = x^{k+1} - (2^m + 1)x^k + (2^m - 1)x^{k-1} + 1$. Thus, 1 is a root but not a multiple root of $h(x)$, since $k \geqslant 2$ and $f(1) \neq 1$. Suppose that α is a multiple root of $h(x)$ for all integers k and m such that $k \geqslant 2$ and $m \geqslant 0$. Note that $\alpha \neq 0$ and $\alpha \neq 1$. Since α is a multiple root, $h(\alpha) = \alpha^{k+1} - (2^m + 1)\alpha^k + (2^m - 1)\alpha^{k-1} + 1 = 0$ and

$$\begin{aligned} h'(\alpha) &= (k+1)\alpha^k - (2^m + 1)k\alpha^{k-1} + (2^m - 1)(k-1)\alpha^{k-2} \\ &= \alpha^{k-2}[(k+1)\alpha^2 - (2^m + 1)k\alpha + (2^m - 1)(k-1)] = 0. \end{aligned}$$

Thus, $\alpha_{1,2} = ((2^m + 1)k \mp \sqrt{\Delta})/2(k+1)$ where $\Delta = (2^m + 1)^2 k^2 - 4(2^m - 1)\left(k^2 - 1\right)$ and hence, for α_1

$$0 = -h(\alpha_1) = \alpha_1^{k-1}(-\alpha_1^2 + (2^m + 1)\alpha_1 - (2^m - 1)) - 1.$$

Let $a_{k,m} = \alpha_1^{k-1}(-\alpha_1^2 + (2^m + 1)\alpha_1 - (2^m - 1))$. Then we write the above equation as follows: for $k \geqslant 2$ and $m \geqslant 0$

$$0 = a_{k,m} - 1.$$

However, if we choose $k = 2$ and $m = 2$, $a_{2,2} = 39$, $a_{2,2} \neq 1$, a contradiction. Similarly, hence, for α_2

$$0 = -h(\alpha_2) = \alpha_2^{k-1}(-\alpha_2^2 + (2^m + 1)\alpha_2 - (2^m - 1)) - 1.$$

Let $b_{k,m} = \alpha_2^{k-1}(-\alpha_2^2 + (2^m + 1)\alpha_2 - (2^m - 1))$. Then we write the above equation as follows: for $k \geqslant 2$ and $m \geqslant 0$

$$0 = b_{k,m} - 1.$$

For $k = 2$ and $m = 2$, $b_{2,2} = -\frac{13}{27}$, $b_{2,2} \neq 1$, a contradiction because we suppose that α is a multiple root for any integers k and m such that $k \geqslant 2$ and $m \geqslant 0$. Therefore, the equation $h(x) = 0$ does not have multiple roots. $\square$

Consequently, by Lemma 6, the equation $x^k - 2^m x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$ does not have multiple roots for $k \geqslant 2$ and $m \geqslant 0$.

Let $f(\lambda)$ be the characteristic polynomial of the generalized order-k F–P matrix T. Then $f(\lambda) = \lambda^k - 2^m \lambda^{k-1} - \lambda^{k-2} - \cdots - \lambda - 1$, which is mentioned above. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of matrix T. Then, by Lemma 6, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. Let V be a $k \times k$ Vandermonde matrix as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_1^{k-2} & \ldots & \lambda_1 & 1 \\ \lambda_2^{k-1} & \lambda_2^{k-2} & \ldots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_k^{k-1} & \lambda_k^{k-2} & \ldots & \lambda_k & 1 \end{bmatrix}.$$

Let e_k^i be a $k \times 1$ matrix as follows:

$$e_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and $V_j^{(i)}$ be a $k \times k$ matrix obtained from V by replacing the jth column of V by e_k^i. Denote V^T by Λ.

Then we obtain the generalized Binet formula for the generalized order-k F–P numbers with following theorem.

Theorem 7. *Let u_n^i be the nth term of ith F–P sequence, for $1 \leqslant i \leqslant k$. Then*

$$u_{n-i+1}^j = \frac{\det(V_j^{(i)})}{\det(V)}.$$

Proof. Since the eigenvalues of T are distinct, T is diagonalizable. Since Λ is invertible $\Lambda^{-1} T \Lambda = E$ where $E = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Hence, T is similar to E. So we obtain $T^n \Lambda = \Lambda E^n$. By Lemma 1, $T^n = D_n$. Then we have the following linear system of equations:

$$\begin{aligned}
&d_{i1}\lambda_1^{k-1} + d_{i2}\lambda_1^{k-2} + \cdots + d_{ik} = \lambda_1^{n+k-i},\\
&d_{i1}\lambda_2^{k-1} + d_{i2}\lambda_2^{k-2} + \cdots + d_{ik} = \lambda_2^{n+k-i},\\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
&d_{i1}\lambda_k^{k-1} + d_{i2}\lambda_k^{k-2} + \cdots + d_{ik} = \lambda_k^{n+k-i},
\end{aligned}$$

where $D_n = [d_{ij}]_{k\times k}$. Thus, for each $j = 1, 2, \ldots, k$, we obtain

$$d_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}.$$

Note that $d_{ij} = u_{n-i+1}^j$. So we have the conclusion. $\square$

4. Sums of the generalized F–P numbers

Now we extend the matrix representation of generalized order-k F–P numbers in Section 1 and derive a generating matrix for the sums of the generalized order-k F–P numbers subscripted from 1 to n.

To calculate the sums S_n of the generalized order-k F–P numbers, defined by

$$S_n = \sum_{i=0}^{n-1} u_i^1$$

since $u_{n+1}^k = u_n^1$ which given in Lemma 4, we rewrite it as

$$S_n = \sum_{i=1}^{n} u_i^k.$$

Let A and W_n be $(k+1)\times(k+1)$ square matrix, such that

$$A = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & & & & \\ 0 & & T & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \tag{13}$$

and

$$W_n = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ S_n & & & & \\ S_{n-1} & & D_n & & \\ \vdots & & & & \\ S_{n-k+1} & & & & \end{bmatrix}, \tag{14}$$

where T and D_n are the $k\times k$ matrices as in (6) and (7), respectively.

Using formula (12), we obtain the following equation:

$$S_n = u_{n-1}^1 + S_{n-1}$$

and so we derive the following matrix recurrence equation:

$$W_n = W_{n-1}A.$$

Inductively, we also have

$$W_n = W_1 A^n.$$

Since $S_{-i}=0$, $1\leqslant i\leqslant k$ and by the definition of the generalized order-k F–P numbers, we thus infer $W_1=A$, and in general, $W_n=A^n$. So we obtain the generating matrix for the sums of the generalized order-k F–P numbers S_n.

Since $W_n=A^n$, we write

$$W_{n+1}=W_nW_1=W_1W_n \tag{15}$$

which shows that W_1 is commutative as well under matrix multiplication. By an application of Eq. (15), the sums of generalized order-k F–P numbers satisfy the recurrence relation:

$$S_n=1+2^mS_{n-1}+\sum_{i=2}^{k}S_{n-i}. \tag{16}$$

Substituting $S_n=u_{n-1}^1+S_{n-1}$ into the Eq. (16) and by (12), we express u_n^k in terms of the sums of the generalized order-k F–P numbers

$$u_n^k=1+(2^m-1)S_{n-1}+\sum_{i=2}^{k}S_{n-i}. \tag{17}$$

For example, when $k=2$ and $m=0$, the sequence $\{u_n^i\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and so the above equation is reduced to

$$F_n=1+S_{n-2}.$$

So we derive the well-known result [24]

$$\sum_{i=1}^{n-2}F_i=F_n-1.$$

5. An explicit formula for the sums of generalized F–P numbers

In this section, we derive an explicit formula for the sums of generalized order-k F–P numbers subscripted from 1 to n by matrix methods. Recall that the $(k+1)\times(k+1)$ matrix A be as in (13). We consider the characteristic equation of matrix A, then the characteristic polynomial of A is

$$K_A(\lambda)=\begin{vmatrix} 1-\lambda & 0 & 0 & \dots & 0 \\ 1 & & & & \\ 0 & & T-\lambda I & & \\ \vdots & & & & \\ 0 & & & & \end{vmatrix},$$

where the $k\times k$ matrix T given by (6) and I is the identity matrix of order k. Computing the above determinant by the Laplace expansion of determinant with respect to the first row gives as

$$K_A(\lambda)=(1-\lambda)|T-\lambda I|. \tag{18}$$

From Section 2, we have the characteristic polynomial of matrix T, then we obtain the characteristic equation of matrix A as follows:

$$x^{k+1}-(2^m+1)x^k+(2^m-1)x^{k-1}+1=0.$$

From Lemma 6, we know that the above equation does not have multiple roots. Thus, we can diagonalize the matrix A. Recall that the eigenvalues of the matrix T are $\lambda_1,\lambda_2,\dots,\lambda_k$. By (18) and previous results, we have that the eigenvalues of matrix A are $1,\lambda_1,\lambda_2,\dots,\lambda_k$ and all of them are distinct.

We define an $(k+1)\times(k+1)$ matrix G as follows:

$$G = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \dfrac{-1}{(2^m+k-2)} & \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \dfrac{-1}{(2^m+k-2)} & \lambda_1^{k-2} & \lambda_2^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{-1}{(2^m+k-2)} & \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \dfrac{-1}{(2^m+k-2)} & 1 & 1 & \dots & 1 \end{bmatrix}, \tag{19}$$

where $1, \lambda_1, \lambda_2, \dots, \lambda_k$ are the eigenvalues of A.

Then we give an explicit formula for the sums of generalized order-k F–P numbers with the following theorem.

Theorem 8. *Let S_n denote the sums of the terms of the sequence $\{u_n^k\}$ subscripted from 1 to n. Then*

$$S_n = \left(\sum_{i=1}^{k} u_n^i\right) \Big/ (2^m+k-2),$$

where u_n^i is the nth term of ith generalized order-k F–P sequence.

Proof. It is easily verified that

$$AG = GM, \tag{20}$$

where $M = \mathrm{diag}(1, \lambda_1, \lambda_2, \dots, \lambda_k)$ and the matrices A and G given by (13) and (19), respectively. If we compute the $\det G$ by the Laplace expansion according to the first row, then, by the definition of $k\times k$ Vandermonde matrix V, we easily obtain $\det G = \det V$. We know that $\det V \neq 0$ since the all λ_i's are distinct for $1 \leqslant i \leqslant k$. Thus, $\det G \neq 0$ and so the matrix G is invertible. Therefore, we write (20) as $G^{-1}AG = M$. Then the matrix A is similar to the matrix M and so we obtain the matrix equation $G^{-1}A^nG = M^n$ or $A^nG = GM^n$. Since $A^n = W_n$ and $S_n = (W_n)_{2,1}$, $W_nG = GM^n$. Thus, by a matrix multiplication, the conclusion is easily obtained. □

Taking by $k=2$ and $m=1$, then the sequence $\{u_n^i\}$ is reduced to the usual Pell sequence and by Theorem 8, we obtain

$$\sum_1^n P_i = \frac{P_n^1 + P_n^2 - 1}{2}$$

and since $P_n^1 = P_{n+1}^2$,

$$\sum_1^n P_i = \frac{P_{n+1}^2 + P_n^2 - 1}{2}$$

which is well-known fact from [10].

6. Generating function and combinatorial representation

In this section we give the generating function and combinatorial representation of generalized order-k F–P sequence $\{u_n^k\}$. We start with the generating function of the sequence $\{u_n^k\}$.

Let

$$G_k(x)=u_1^k+u_2^kx+u_3^kx^2+\cdots u_k^kx^{k-1}+u_{k+1}^kx^k+\cdots+u_{n+1}^kx^n+\cdots .$$

Then

$$\begin{aligned}&G_k(x)-2^mxG_k(x)-x^2G_k(x)-\cdots-x^kG_k(x)\\&\quad=(1-2^mx-x^2-\cdots-x^k)G_k(x)\\&\quad=u_1^k+x(u_2^k-2^mu_1^k)+x^2(u_3^k-2^mu_2^k-u_1^k)\\&\qquad+\cdots+x^k(u_{k+1}^k-2^mu_k^k-u_{k-1}^k-\cdots-u_2^k-u_1^k)\\&\qquad+\cdots+x^n(u_{n+1}^k-2^mu_n^k-u_{n-1}^k-\cdots-u_{n-k+2}^k-u_{n-k+1}^k)+\cdots .\end{aligned}$$

By the definition of generalized order-k F–P numbers, we obtain

$$G_k(x)-2^mxG_k(x)-x^2G_k(x)-\cdots-x^kG_k(x)=u_1^k$$

and since $u_1^k=1$,

$$G_k(x)=(1-2^mx-x^2-\cdots-x^k)^{-1}$$

for $0\leqslant 2^mx+x^2+\cdots+x^k<1$.

Let $f_k(x)=2^mx+x^2+\cdots+x^k$. Then $0\leqslant f_k(x)<1$ and we give exponential representation for generalized order-k F–P numbers

$$\begin{aligned}\ln G_k(x)=&\ \ln[1-(2^mx+x^2+\cdots+x^k)]^{-1}\\=&\ -\ln[1-(2^mx+x^2+\cdots+x^k)]\\=&\ -[-(2^mx+x^2+\cdots+x^k)-\tfrac{1}{2}(2^mx+x^2+\cdots+x^k)^2\\&\ -\cdots-\tfrac{1}{n}(2^mx+x^2+\cdots+x^k)^n-\cdots]\\=&\ x[(2^m+x+x^2+\cdots+x^{k-1})+\tfrac{1}{2}(2^m+x+x^2+\cdots+x^{k-1})^2\\&\ +\cdots+\tfrac{1}{n}(2^m+x+x^2+\cdots+x^{k-1})^n+\cdots]\\=&\ x\sum_{n=0}^{\infty}\frac{1}{n}(2^m+x+x^2+\cdots+x^{k-1})^n.\end{aligned}$$

Thus,

$$G_k(x)=\exp\left(x\sum_{n=0}^{\infty}\frac{1}{n}(2^m+x+x^2+\cdots+x^{k-1})^n\right).$$

Now we give combinatorial representation of the generalized order-k F–P numbers. In [3], the authors obtained an explicit formula for the elements in the nth power of the companion matrix and gave some interesting applications. The companion matrix A be as in (4), then we find the following theorem in [3].

Theorem 9. *The* (i,j) *entry* $a_{ij}^{(n)}(c_1,c_2,\ldots,c_k)$ *in the matrix* $A^n(c_1,c_2,\ldots,c_k)$ *is given by the following formula*:

$$a_{ij}^{(n)}(c_1,c_2,\ldots,c_k)=\sum_{(t_1,t_2,\ldots,t_k)}\frac{t_j+t_{j+1}+\cdots+t_k}{t_1+t_2+\cdots+t_k}\times\binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k}c_1^{t_1}\ldots c_k^{t_k}, \tag{21}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+kt_k=n-i+j$, *and the coefficients in* (21) *is defined to be* 1 *if* $n=i-j$.

Then we have the following Corollary.

```
function f=sums(k,n,m)
S=0;I=0;
%create initial the last row
%result matrix
for  j=1:k
   I(j,j)=1;
end
for l=1:n-k+1
     for i=1:k
      Temp(i)=2^m*I(1,i);
          for j=2:k
         Temp(i)=Temp(i)+I(j,i);
          end
   end
     S=S+I(1,1);
     for i=k:-1:2
          for j=1:k
         I(i,j)=I(i-1,j);
          end
   end
   for j=1:k
      I(1,j)=Temp(j);
     end
     Temp=0;
end
R=[S,I(1,1:k)];
%fill the matrix
for   j=1:k+1
   C(k+1,j)=R(j);
end
C(1,1)=1;
for i=k:-1:2
   C(i,k+1)=C(i+1,2);
   C(i,1)=C(i+1,1)+C(i+1,2);
   C(i,2)=C(i+1,3)+2^m*C(i+1,2);
     for   j=3:k
      C(i,j)=C(i+1,j+1)+C(i+1,2);
     end

end
S
```

Fig. 1. Illustrates the MATLAB code concerning the algorithm.

Corollary 10. *Let u_n^i be the generalized order-k* F–P *number for* $1 \leqslant i \leqslant k$. *Then*

$$u_n^i = \sum_{(r_1, r_2, \ldots, r_k)} \frac{r_k}{r_1 + r_2 + \cdots + r_k} \times \binom{r_1 + r_2 + \cdots + r_k}{r_1, r_2, \ldots, r_k} 2^{mr_1},$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + \cdots + kr_k = n - i + k$.

Proof. In Theorem 9, if $j = k$ and $c_1 = 2^m$, then the proof is immediately seen from (6). □

7. An algorithm for the sum

In this section, we are going to give a computational algorithm for the sums of the generalized order-k F–P numbers. The sums of generalized order-k F–P numbers may be estimated via the equation $S_n = u_n^1 + S_{n-1}$ mentioned above. But this estimation based on the matrix multiplication is not useful for the large values of n, m and k. Consequently, to obtain the matrix W_n generating sums of the generalized order-k F–P numbers, we give an algorithm that is entirely based on the sums of the series rather than the one based on matrix multiplication. The following considerations have played important results in forming such an algorithm. First, whatever values n, m and k take, the first line of the resulting matrix is fixed as follows: for all i and j, $W_i(1, j) = 1$ for $j = i$ and $W_i(1, j) = 0$ for $j > i$. Thus, there is no need for further processes for the first line. Second, when $i = k$, the elements of the $(i - k + 1)$th line of D_i can be obtained by using the identity matrix I of order k. Following the second step, when $i > k$, the operation to do is to erase the kth line of I and for $j = 2, 3, \ldots, k - 1$ to move the jth row into $(j + 1)$th line and the $(i - k + 1)$th line of D_{i-1} into the first line of I. This operation is repeated by using the second step until $i = n$. Finally, after obtaining the $(n - k + 1)$th line of the matrix D_n, it is easy to compute the other element of W_n by using the equation $S_n = u_n^1 + S_{n-1}$ and the Eq. (12) (Fig. 1).

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