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Monotone method for a system of nonlinear mixed type implicit impulsive integro-differential equations in Banach spaces[☆]

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Abstract

In this paper, by using a monotone iterative technique in the presence of lower and upper solutions, we discuss the existence of solutions for a new system of nonlinear mixed type implicit impulsive integro-differential equations in Banach spaces. Under wide monotonicity conditions and the noncompactness measure conditions, we also obtain the existence of extremal solutions and a unique solution between lower and upper solutions.

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Keywords: Nonlinear mixed type implicit impulsive integro-differential equation system; Monotone iterative technique; Initial problem; Noncompactness measure; Existence and uniqueness

1. Introduction

It is well known that the theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, chemical, biological, engineering background and realistic mathematical model, and hence has been emerging as an important area of investigation in the last few decades, see [1–9]. Correspondingly, applications of the theory of impulsive differential equations to different areas were considered by many authors and some basic results on impulsive differential equations have been obtained (see, for example, [10–22], and the references therein). Furthermore, the existence of solutions to impulsive differential equations or impulsive integro-differential equations in Banach spaces has also been studied by many authors, see [1, 7,23–40,50,51].

Recently, He and He [51] investigated the existence of minimal and maximal solutions of impulsive integro-differential equations with periodic boundary conditions by establishing a comparison result and using the method of upper and lower solutions and the monotone iterative technique. Ahmad and Sivasundaram [7] developed the

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monotone method for impulsive hybrid set integro-differential equations in all its generality. Very recently, Li and Liu [27] pointed out "the monotone iterative technique in the presence of lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces". Further, Li and Liu used a monotone iterative technique in the presence of lower and upper solutions to discuss the existence of solutions for the initial value problem of the impulsive integro-differential equation of Volterra type in a Banach space E:

$$\begin{cases} u'(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad (i = 1, 2, \ldots, m), \\ u(0) = x_0, \end{cases}$$

where $f \in C(J \times E \times E, E)$, $J = [0, a]$, $0 < t_1 < t_2 < \cdots < t_m < a$ and $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. Under wide monotonicity conditions and the noncompactness measure condition of nonlinearity f, the authors also obtained the existence of extremal solutions and a unique solution between lower and upper solutions. On the other hand, Sun and Ma [34] used a monotone iterative technique in the presence of lower and upper solutions to discuss the existence of solutions for the following initial value problem of the impulsive integro-differential equation of Volterra type in a Banach space:

$$\begin{cases} u''(t) - f(x, u, u) = \theta, \quad x \in J,\ x \neq x_i, \\ \Delta u|_{x=x_i} = I_i(u(x_i)), \quad (i = 1, 2, \ldots, m), \\ \Delta u'|_{x=x_i} = \bar{I}_i(u(x_i)), \quad (i = 1, 2, \ldots, m), \\ u(0) = w_0, \quad u'(0) = w_1. \end{cases}$$

For more details of the monotone iterative methods, the readers can refer to [7,33,34,43–51] and the references therein.

In this paper, we study the following system of nonlinear mixed type implicit impulsive integro-differential equation problem in Banach spaces E_1 and E_2: Find $(x, y) : J \times J \to E_1 \times E_2$ such that

$$\begin{cases} x'(t) = f(t, x(t), y(t), \lambda Sx(t)), \quad t \neq t_k, \\ y'(t) = g(t, y(t), x(t), \mu Ty(t)), \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad (k = 1, 2, \ldots, m), \\ \Delta y|_{t=t_k} = \hat{I}_k(y(t_k)), \quad (k = 1, 2, \ldots, m), \\ x(t_0) = x_0, \quad y(t_0) = y_0, \end{cases} \tag{1.1}$$

where $J = [t_0, t_0 + a] \subset R = (-\infty, +\infty)$ is a compact interval, $t_0 < t_1 < \cdots < t_m < t_0 + a < +\infty$, $f : J \times E_1 \times E_2 \times E_1 \to E_1$ and $g : J \times E_2 \times E_1 \times E_2 \to E_2$ are continuous, $\lambda, \mu \geq 0$ are two constants, $x_0 \in E_1$, $y_0 \in E_2$,

$$Sx(t) = \int_{t_0}^{t} h(t, s)x(s)\mathrm{d}s$$

is a Volterra integral operator with integral kernel $h(t, s) \in C(D, \mathbb{R}^+)$, $D = \{(t, s)|s, t \in J, t \geq s\}$, $\mathbb{R}^+ = [0, +\infty)$,

$$Ty(t) = \int_{t_0}^{t} \kappa(t, s)y(s)\mathrm{d}s$$

is a Fredholm integral operator with integral kernel $\kappa(t, s) \in C(D_0, \mathbb{R}^+)$, $D_0 = \{(t, s)|s, t \in J\}$, and for $k = 1, 2, \ldots, m$, $I_k \in C[E_1, E_1]$, $\hat{I}_k \in C[E_2, E_2]$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$, respectively.

If $\lambda = 0$ and $\mu = 0$, then problem (1.1) reduces to finding $(x, y) : J \times J \to E_1 \times E_2$ such that

$$\begin{cases} x'(t) = f(t, x(t), y(t)), \quad t \neq t_k, \\ y'(t) = g(t, y(t), x(t)), \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad (k = 1, 2, \ldots, m), \\ \Delta y|_{t=t_k} = \hat{I}_k(y(t_k)), \quad (k = 1, 2, \ldots, m), \\ x(t_0) = x_0, \quad y(t_0) = y_0. \end{cases} \tag{1.2}$$

If $f = g, x = y, E_1 = E_2 = E$ and for $k = 1, 2, \ldots, m$, $I_k = \hat{I}_k$, then problem (1.2) further simplifies to finding $x : J \to E$ such that

$$\begin{cases} x'(t) = f(t, x(t), x(t)), & t \neq t_k, \\ \triangle x|_{t=t_k} = I_k(x(t_k)), & (k = 1, 2, \ldots, m), \\ x(t_0) = x_0. \end{cases} \tag{1.3}$$

Problem (1.3) was studied by some authors when $f(t, x, y) \equiv p(t, x)$ for all $t \in J$ and $x, y \in E$, see, for example, [1, 22,28].

Remark 1.1. For appropriate and suitable choices of f, g, λ, μ, S, T, I_k, $\hat{I}_k$ and E_i for $i = 1, 2$, it is easy to see that problem (1.1) includes a number (systems) of differential equations, impulsive differential equations, (impulsive) integro-differential equations studied by many authors as special cases, see, for example, [1–40,43,44,48–50] and the references therein.

The purpose of this paper is to discuss the existence of solutions for the new system of nonlinear mixed type implicit impulsive integro-differential equation (1.1) in Banach spaces by using a monotone iterative technique in the presence of lower and upper solutions. Further, under wide monotonicity conditions and the noncompactness measure conditions, we obtain the existence of extremal solutions and a unique solution between lower and upper solutions. The new and useful results obtained in this paper improve and extend some relevant results in abstract differential equations.

2. Preliminaries

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{x \in E | x \geq 0\}$ is normal with normal constant N. Let $J = [t_0, t_0 + a]$ (where $a > 0$), $t_0 < t_1 < \cdots < t_m < t_0 + a < +\infty$, $J_0 = [t_0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_k = (t_k, t_{k+1}], \ldots, J_m = (t_m, t_0 + a]$ and

$$PC(J, E) = \{x : J \to E |\, x(t) \text{ is continuous at } t \neq t_k, \text{ and left continuous at } t = t_k, \text{ and } x(t_k^+) \text{ exists}, \ k = 1, 2, \ldots, m\}.$$

Evidently, $PC(J, E)$ is a Banach space with norm $\|x\|_{PC} = \sup_{t\in J} x(t)$. Let $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$. An abstract function $(x, y) \in PC(J, E_1) \cap C^1(J', E_1) \cap PC(J, E_2) \cap C^1(J', E_2)$ is called a solution of problem (1.1) if $(x(t), y(t))$ satisfies all the equalities of (1.1).

Let

$$PC^1(J, E) = \{x \in PC(J, E) \cap C^1(J', E) |\, x'(t_k^+), x'(t_k^-) \text{ exist}, \ k = 1, 2, \ldots, m\},$$

where $x'(t_k^+)$ and $x'(t_k^-)$ represent the right and left derivatives of $x(t)$ at $t = t_k$, respectively. For $x \in PC^1(J, E)$, by virtue of the mean value theorem

$$x(t_k) - x(t_k - h) \in h\, \overline{co}\{x'(t) : t_k - h < t < t_k\} \quad (h > 0),$$

it is easy to see that the left derivative $x'_-(t_k)$ exists and

$$x'_-(t_k) = \lim_{h\to 0^+} h^{-1}[x(t_k) - x(t_k - h)] = x'(t_k^-),$$

where $\overline{co}\{x'(t) : t_k - h < t < t_k\}$ denotes the smallest closed convex subset containing $\{x'(t) : t_k - h < t < t_k\}$ in $PC^1(J, E)$, and $co(K) = \{x | x = \sum_{y\in K} \lambda_y y, \lambda_y \in [0, 1], \text{ there exist finite numbers } \lambda_y \neq 0 \text{ and } \sum_{y\in K} \lambda_y = 1\}$ for $K \subset PC^1(J, E)$. In what follows, $x'(t_k)$ is understood as $x'_-(t_k)$, hence $x' \in PC(J, E)$. Evidently, $PC^1(J, E)$ is a Banach space with norm $\|x\|_{PC^1} = \max\{\sup_{t\in J} \|x(t)\|, \sup_{t\in J} \|x'(t)\|\}$.

If $(x, y) \in PC(J, E_1) \cap C^1(J', E_1) \cap PC(J, E_2) \cap C^1(J', E_2)$ is a solution of problem (1.1), then by the continuity of f, g, $(x, y) \in PC^1(J, E_1) \cap PC^1(J, E_2)$.

A mapping $F : J \to E$ is differentiable at $t \in J$ if there exists a $F'(t) \in E$ such that the limits

$$\lim_{h\to 0^+} \frac{F(t + h) - F(t)}{h}$$

and

$$\lim_{h\to 0^+} \frac{F(t)-F(t-h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in E. At the endpoints of J, we consider the one-sided derivatives.

Let $C(J,E)$ denote the Banach space of all continuous E-value functions on interval J with norm $\|x\|_C = \max_{t\in J}\|x(t)\|$. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [38]. For any $B \subset C(J,E)$ and $t \in J$, set $B(t) = \{x(t)|x \in B\} \subset E$. If B is bounded in $C(J,E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \le \alpha(B)$.

Now, we first give the following lemmas in order to prove our main results.

Lemma 2.1 *([39]). Let $B \subset C(J,E)$ be bounded and equicontinuous. $\alpha(B(t))$ is continuous on J, and*

$$\alpha\left(\left\{\int_J x(t)\mathrm{d}t \,|\, x \in B\right\}\right) \le \int_J \alpha(B(t))\mathrm{d}t.$$

Lemma 2.2 *([40]). Let $B = \{x_n\} \subset PC(J,E)$ be a bounded and countable set. $\alpha(B(t))$ is a Lebesque integral on J, and*

$$\alpha\left(\left\{\int_J x_n(t)\mathrm{d}t\right\}\right) \le 2\int_J \alpha(B(t))\mathrm{d}t.$$

Lemma 2.3 *([27]). For any $p \in PC^1(J,\mathbb{B})$, $\nu \in \mathbb{B}$ and $\omega_k \in \mathbb{B}$, $k = 1,2,\ldots,m$, the line initial value problem*

$$\begin{cases} u'(t) + Mu(t) = p(t), \quad t \ne t_k, \\ \triangle u|_{t=t_k} = \omega_k, \quad (k = 1,2,\ldots,m), \\ u(t_0) = \nu, \end{cases} \tag{2.1}$$

has a unique solution $u \in PC^1(J,\mathbb{B})$ given by

$$u(t) = \nu \mathrm{e}^{-M(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{-M(t-s)} p(s)\mathrm{d}s + \sum_{t_0<t_k<t} \mathrm{e}^{-M(t-t_k)}\omega_k,$$

where $M \ge 0$ is a constant.

3. Main results

In this section, we are in a position to prove our main results concerning the solutions of the nonlinear mixed type implicit impulsive integro-differential equation system (1.1) in Banach spaces.

If a function $(\upsilon,\omega) \in PC^1(J,E_1) \times PC^1(J,E_2)$ satisfies

$$\begin{cases} \upsilon'(t) \le f(t,x(t),y(t),\lambda Sx(t)), \quad t \ne t_k, \\ \omega'(t) \le g(t,y(t),x(t),\mu Ty(t)), \quad t \ne t_k, \\ \triangle\upsilon|_{t=t_k} \le I_k(x(t_k)), \quad (k = 1,2,\ldots,m), \\ \triangle\omega|_{t=t_k} \le \hat{I}_k(y(t_k)), \quad (k = 1,2,\ldots,m), \\ \upsilon(t_0) \le x_0, \quad \omega(t_0) \le y_0, \end{cases} \tag{3.1}$$

we call it a lower solution of problem (1.1); if all the inequalities of (3.1) are inverse, we call it an upper solution of problem (1.1).

Lemma 3.1. *$(x,y) \in PC^1(J,E_1) \times PC^1(J,E_2)$ is a solution of problem (1.1) if and only if $x \in PC^1(J,E_1)$ and $y \in PC^1(J,E_2)$ satisfy the following impulsive integral equations*

$$\begin{cases} x(t) = x_0\mathrm{e}^{-M_1(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{-M_1(t-s)}[f(s,x(s),y(s),\lambda Sx(s)) + M_1x(s)]\mathrm{d}s + \sum_{t_0<t_k<t} \mathrm{e}^{-M_1(t-t_k)} I_k(x(t_k)), \\ y(t) = y_0\mathrm{e}^{-M_2(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{-M_2(t-s)}[g(s,y(s),x(s),\mu Ty(s)) + M_2y(s)]\mathrm{d}s + \sum_{t_0<t_k<t} \mathrm{e}^{-M_2(t-t_k)} \hat{I}_k(y(t_k)), \end{cases}$$

where $M_i > 0\,(i = 1, 2)$ *is a constant.*

Proof. The proof directly follows from Lemma 2.1 in [26] and it is omitted. □

Now, let us first list the following assumptions for convenience:

(H_1) There exist $u_0, v_0 \in PC^1[J, E_1]$, $\nu_0, \omega_0 \in PC^1[J, E_2]$ and constants $M_1, M_2 > 0$ such that for all $t \in J$, $v_0(t) \le u_0(t), \omega_0(t) \le \nu_0(t)$, $(v_0, \omega_0) \in PC^1(J, E_1) \times PC^1(J, E_2)$ and $(u_0, \nu_0) \in PC^1(J, E_1) \times PC^1(J, E_2)$ are lower and upper solutions of problem (1.1), respectively, and

$$f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1) \ge -M_1(x_2 - x_1),$$

for all $t \in J$ and $v_0(t) \le x_1 \le x_2 \le u_0(t)$, $\omega_0(t) \le y_1 \le y_2 \le \nu_0(t)$ and $\lambda S v_0(t) \le z_1 \le z_2 \le \lambda S u_0(t)$, and

$$g(t, y_2, x_2, \xi_2) - g(t, y_1, x_1, \xi_1) \ge -M_2(y_2 - y_1)$$

for all $t \in J$ and $v_0(t) \le x_1 \le x_2 \le u_0(t)$, $\omega_0(t) \le y_1 \le y_2 \le \nu_0(t)$ and $\mu T \omega_0(t) \le \xi_1 \le \xi_2 \le \mu T \nu_0(t)$.

(H_2) $I_k(x)$ and $\hat{I}_k(y)$ are increasing on intervals $[v_0(t), u_0(t)]$ and $[\omega_0(t), \nu_0(t)]$ for $t \in J$, $k = 1, 2, \ldots, m$, respectively, where $[v_0(t), u_0(t)] = \{x \in PC^1[J, E_1] | v_0(t) \le x(t) \le u_0(t), t \in J\}$ and $[\omega_0(t), \nu_0(t)] = \{x \in PC^1[J, E_2] | \omega_0(t) \le x(t) \le \nu_0(t), t \in J\}$.

(H_3) There exists $L_i > 0 (i = 1, 2)$ such that

$$\alpha(\{f(t, x_n(t), y_n(t), z_n(t))\}) \le L_1[\alpha(\{x_n(t)\}) + \alpha(\{z_n(t)\})],$$
$$\alpha(\{g(t, y_n(t), x_n(t), \xi_n(t))\}) \le L_2[\alpha(\{y_n(t)\}) + \alpha(\{\xi_n(t)\})]$$

for all $t \in J$ and increasing or decreasing monotonic sequences $\{x_n\} \subset [v_0(t), u_0(t)]$, $\{y_n\} \subset [\omega_0(t), \nu_0(t)]$, $\{z_n\} \subset [\lambda S v_0(t), \lambda S u_0(t)]$ and $\{\xi_n\} \subset [\mu T \omega_0(t), \mu T \nu_0(t)]$.

In what follows, we prove the following main result of this paper.

Theorem 3.1. *Let E_1 and E_2 be two ordered Banach spaces, whose positive cone $P_i\,(i = 1, 2)$ is normal, $f \in C(J \times E_1 \times E_2 \times E_1, E_1)$, $g \in C(J \times E_2 \times E_1 \times E_2, E_2)$, and $I_k \in C(E_1, E_1)$, $\hat{I}_k \in C(E_2, E_2)$, $k = 1, 2, \ldots, m$. Suppose that the conditions* (H_1)–(H_3) *hold. Then problem* (1.1) *has minimal and maximal solutions between (v_0, ω_0) and (u_0, ν_0), which can be obtained by a monotone iterative procedure starting from (v_0, ω_0) and (u_0, ν_0), respectively.*

Proof. For any $(x, y) \in PC^1(J, E_1) \times PC^1(J, E_2)$, define (Px, Qy) on $J \times J$ by the equation

$$\begin{cases} (Px)(t) = x_0 e^{-M_1(t-t_0)} + \displaystyle\int_{t_0}^{t} e^{-M_1(t-s)}[f(s, x(s), y(s), \lambda Sx(s)) + M_1 x(s)]ds \\ \quad + \displaystyle\sum_{t_0 < t_k < t} e^{-M_1(t-t_k)} I_k(x(t_k)), \\ (Qy)(t) = y_0 e^{-M_2(t-t_0)} + \displaystyle\int_{t_0}^{t} e^{-M_2(t-s)}[g(s, y(s), x(s), \mu T y(s)) + M_2 y(s)]ds \\ \quad + \displaystyle\sum_{t_0 < t_k < t} e^{-M_2(t-t_k)} \hat{I}_k(y(t_k)). \end{cases} \tag{3.2}$$

Now define $\|\cdot\|_*$ on $PC^1(J, E_1) \times PC^1(J, E_2)$ by

$$\|(x, y)\|_* = \|x\| + \|y\|, \quad \forall (x, y) \in PC^1(J, E_1) \times PC^1(J, E_2).$$

It is easy to see that $(PC^1(J, E_1) \times PC^1(J, E_2), \|\cdot\|_*)$ is a Banach space (see [41]). Thus, for any given $(x, y) \in PC^1(J, E_1) \times PC^1(J, E_2)$, it follows from (3.2) that

$$\begin{cases} (Px)'(t) = -M_1 P(x(t)) + M_1 x(t) + f(t, x(t), y(t), \lambda Sx(t)), \\ (Qy)'(t) = -M_2 Q(x(t)) + M_2 y(t) + g(t, y(t), x(t), \mu T y(t)), \end{cases}$$

and so $F(x, y) := (Px, Qy) \in PC^1(J, E_1) \times PC^1(J, E_2)$ is a continuous mapping from $PC^1(J, E_1) \times PC^1(J, E_2)$ into $PC^1(J, E_1) \times PC^1(J, E_2)$. By Lemma 3.1, the solution of problem (1.1) is equivalent to the fixed point of F. By assumptions (H_1) and (H_2), F is increasing in $[v_0, u_0] \times [\omega_0, \nu_0]$, and maps any bounded set in $[v_0, u_0] \times [\omega_0, \nu_0]$ into a bounded set.

Firstly, we show that $v_0 \le Pv_0$, $Pu_0 \le u_0$, $\omega_0 \le Q\omega_0$ and $Q\nu_0 \le \nu_0$. In fact, let $p(t) = v_0'(t) + M_1 v_0(t)$, by the definition of lower solution, $p \in PC^1(J, E_1)$ and $p(t) \le f(t, v_0(t), \omega_0(t), \lambda S v_0(t)) + M_1 v_0(t)$ for $t \in J'$. Because $v_0(t)$ is a solution of problem (2.1) for $\nu = v_0(t_0)$ and $\omega_k = \triangle v_0|_{t=t_k}$ $(k = 1, 2, \ldots, m)$, it follows from Lemma 2.3 that for all $t \in J$,

$$\begin{aligned} v_0(t) &= e^{-M_1(t-t_0)} v_0(t_0) + \int_{t_0}^t e^{-M_1(t-s)} p(s) ds + \sum_{t_0 < t_k < t} e^{-M_1(t-t_k)} \triangle v_0|_{t=t_k} \\ &\le e^{-M_1(t-t_0)} v_0 + \int_{t_0}^t e^{-M_1(t-s)} p(s) ds + \sum_{t_0 < t_k < t} e^{-M_1(t-t_k)} \triangle I_k(v(t_k)) \\ &\le Pv_0(t), \end{aligned}$$

i.e., $v_0 \le Pv_0$. Similarly, it can be shown that $Pu_0 \le u_0$, $\omega_0 \le Q\omega_0$ and $Q\nu_0 \le \nu_0$. Combining these facts and the increasing property of F in $[v_0, u_0] \times [\omega_0, \nu_0]$, we see that F maps $[v_0, u_0] \times [\omega_0, \nu_0]$ into itself and F is a continuously increasing operator.

Next, we define two sequences $\{(v_n, \omega_n)\}$ and $\{(u_n, \nu_n)\}$ in $[v_0, u_0] \times [\omega_0, \nu_0]$ by the iterative scheme

$$v_n = Pv_{n-1}, \qquad u_n = Pu_{n-1}, \qquad \omega_n = Q\omega_{n-1}, \qquad \nu_n = Q\nu_{n-1}, \quad n = 1, 2, \ldots. \tag{3.3}$$

Then by the monotonicity of F, we obtain

$$\begin{aligned} &v_0 \le v_1 \le \cdots \le v_n \le \cdots \le u_n \le \cdots \le u_1 \le u_0, \\ &\omega_0 \le \omega_1 \le \cdots \le \omega_n \le \cdots \le \nu_n \le \cdots \le \nu_1 \le \nu_0. \end{aligned} \tag{3.4}$$

We shall prove that $\{v_n\}$ and $\{u_n\}$ are uniformly convergent in J, and $\{\omega_n\}$ and $\{\nu_n\}$ are uniformly convergent in J.

For convenience, let $B = \{v_n | n \in \mathbb{N}\}$, $V = \{\omega_n | n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} | n \in \mathbb{N}\}$, $V_0 = \{\omega_{n-1} | n \in \mathbb{N}\}$. Since $B = P(B_0)$, $V = Q(V_0)$, by (3.2) and the boundedness of B_0 and V_0, we easily see that B and V are equicontinuous in every interval J_k', where $J_1' = [t_0, t_1]$ and $J_k' = (t_{k-1}, t_k]$, $k = 2, 3, \ldots, m$. From $B_0 = B \cup \{v_0\}$ and $V_0 = V \cup \{\omega_0\}$, it follows that $\alpha(B_0(t)) = \alpha(B(t))$ and $\alpha(V_0(t)) = \alpha(V(t))$ for $t \in J$. Letting

$$\phi(t) = (\alpha(B(t)), \alpha(V(t))) = (\alpha(B_0(t)), \alpha(V_0(t))), \quad t \in J,$$

by Lemma 2.1, we know that $\phi \in PC(J, \mathbb{R}^+) \times PC(J, \mathbb{R}^+)$. Going from J_1' to J_{m+1}' interval-by-interval, we show that $\phi(t) \equiv 0$ in J.

Indeed, for $t \in J$, there exists a J_k' such that $t \in J_k'$. By Lemma 2.1, we have that

$$\begin{aligned} \alpha(S(B_0)(t)) &= \alpha\left(\left\{\int_{t_0}^t h(t,s) v_{n-1}(s) ds | n \in \mathbb{N}\right\}\right) \\ &\le \sum_{j=1}^{k-1} \alpha\left(\left\{\int_{t_{j-1}}^{t_j} h(t,s) v_{n-1}(s) ds | n \in \mathbb{N}\right\}\right) + \alpha\left(\left\{\int_{t_{k-1}}^t h(t,s) v_{n-1}(s) ds | n \in \mathbb{N}\right\}\right) \\ &\le h_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \alpha(B_0(s)) ds + h_0 \int_{t_{k-1}}^t \alpha(B_0(s)) ds \\ &= h_0 \int_{t_0}^t \alpha(B_0(s)) ds \end{aligned}$$

and

$$\begin{aligned} \alpha(T(V_0)(t)) &= \alpha\left(\left\{\int_{t_0}^t \kappa(t,s) \omega_{n-1}(s) ds | n \in \mathbb{N}\right\}\right) \\ &\le \sum_{j=1}^{k-1} \alpha\left(\left\{\int_{t_{j-1}}^{t_j} \kappa(t,s) \omega_{n-1}(s) ds | n \in \mathbb{N}\right\}\right) + \alpha\left(\left\{\int_{t_{k-1}}^t \kappa(t,s) \omega_{n-1}(s) ds | n \in \mathbb{N}\right\}\right) \end{aligned}$$

$$\le \kappa_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \alpha(V_0(s))\mathrm{d}s + \kappa_0 \int_{t_{k-1}}^{t} \alpha(V_0(s))\mathrm{d}s$$
$$= \kappa_0 \int_{t_0}^{t} \alpha(V_0(s))\mathrm{d}s,$$

where $h_0 = \max\{|h(t,s)| : (t,s) \in D\}$ and $\kappa_0 = \max\{|\kappa(t,s)| : (t,s) \in D_0\}$. Thus,

$$\int_{t_0}^{t} \alpha(S(B_0)(s))\mathrm{d}s \le a h_0 \int_{t_0}^{t} \alpha(B_0(s))\mathrm{d}s, \qquad \int_{t_0}^{t} \alpha(T(V_0)(s))\mathrm{d}s \le a\kappa_0 \int_{t_0}^{t} \alpha(V_0(s))\mathrm{d}s. \tag{3.5}$$

For $t \in J_1'$, from (3.2), using Lemma 2.2, assumption (H_3) and (3.5), we have

$$\begin{aligned}
\alpha(B(t)) &= \alpha(P(B_0)(t)) \\
&= \alpha\left(\left\{\int_{t_0}^{t} \mathrm{e}^{-M_1(t-s)}(f(s, v_{n-1}(s), \omega_{n-1}(s), \lambda S v_{n-1}(s)) + M_1 v_{n-1}(s))\mathrm{d}s\right\}\right) \\
&\le 2\int_{t_0}^{t} \mathrm{e}^{-M_1(t-s)}\alpha\left(\{(f(s, v_{n-1}(s), \omega_{n-1}(s), \lambda S v_{n-1}(s)) + M_1 v_{n-1}(s))\}\right)\mathrm{d}s \\
&\le 2\int_{t_0}^{t} (L_1(\alpha(B_0(s)) + \lambda\alpha(S(B_0)(s))) + M_1\alpha(B_0(s)))\mathrm{d}s \\
&\le 2(L_1 + M_1)\int_{t_0}^{t} \alpha(B_0(s))\mathrm{d}s + 2L_1\lambda \int_{t_0}^{t} \alpha(S(B_0)(s))\mathrm{d}s \\
&\le 2(L_1 + M_1 + a h_0 L_1 \lambda)\int_{t_0}^{t} \alpha(B_0(s))\mathrm{d}s, \\
\alpha(V(t)) &= \alpha(Q(V_0)(t)) \\
&= \alpha\left(\left\{\int_{t_0}^{t} \mathrm{e}^{-M_2(t-s)}(f(s, \omega_{n-1}(s), v_{n-1}(s), \mu T \omega_{n-1}(s)) + M_2 \omega_{n-1}(s))\mathrm{d}s\right\}\right) \\
&\le 2\int_{t_0}^{t} \mathrm{e}^{-M_2(t-s)}\alpha\left(\{(f(s, \omega_{n-1}(s), v_{n-1}(s), \mu T \omega_{n-1}(s)) + M_2 \omega_{n-1}(s))\}\right)\mathrm{d}s \\
&\le 2\int_{t_0}^{t} (L_2(\alpha(V_0(s)) + \mu\alpha(T(V_0)(s))) + M_2\alpha(V_0(s)))\mathrm{d}s \\
&\le 2(L_2 + M_2)\int_{t_0}^{t} \alpha(V_0(s))\mathrm{d}s + 2L_2\mu \int_{t_0}^{t} \alpha(T(V_0)(s))\mathrm{d}s \\
&\le 2(L_2 + M_2 + a\kappa_0 L_2 \mu)\int_{t_0}^{t} \alpha(V_0(s))\mathrm{d}s,
\end{aligned}$$

and so

$$\begin{aligned}
\phi(t) &= (\alpha(B(t)), \alpha(V(t))) \\
&\le \left(2(L_1 + M_1 + a h_0 L_1\lambda)\int_{t_0}^{t} \alpha(B_0(s))\mathrm{d}s, 2(L_2 + M_2 + a\kappa_0 L_2\mu)\int_{t_0}^{t} \alpha(V_0(s))\mathrm{d}s\right) \\
&= \Gamma \int_{t_0}^{t} (\alpha(B_0(s)), \alpha(V_0(s)))\mathrm{d}s \\
&= \Gamma \int_{t_0}^{t} \phi(s)\mathrm{d}s,
\end{aligned}$$

where $\Gamma = \max\{2(L_1 + M_1 + a h_0 L_1 \lambda), 2(L_2 + M_2 + a\kappa_0 L_2 \mu)\}$. Hence, by the Bellman inequality, we know that $\phi(t) \equiv 0$ in J_1'. In particular, $(\alpha(B(t_1)), \alpha(V(t_1))) = (\alpha(B_0(t_1)), \alpha(V_0(t_1))) = \phi(t_1) = 0$, this means that $B(t_1)$, $B_0(t_1)$ and $V(t_1)$, $V_0(t_1)$ are precompact in E_1 and E_2, respectively. Therefore, $I_1(B_0(t_1))$ and $\hat{I}_1(V_0(t_1))$ are precompact in E_1 and E_2, respectively. This implies that

$$\alpha(I_1(B_0(t_1))) = 0 \quad \text{and} \quad \alpha(\hat{I}_1(V_0(t_1))) = 0.$$

Now, for $t \in J_2'$, by (3.2) and the above argument for J_1', we have

$$\begin{aligned}\phi(t) &= (\alpha(B(t)), \alpha(V(t)))\\ &\leq \left(2(L_1 + M_1 + ah_0 L_1\lambda)\int_{t_0}^{t}\alpha(B_0(s))\mathrm{d}s + \alpha(I_1(v_{n-1}(t_1))),\right.\\ &\qquad\left. 2(L_2 + M_2 + a\kappa_0 L_2\mu)\int_{t_0}^{t}\alpha(V_0(s))\mathrm{d}s + \alpha(\hat{I}_1(\omega_{n-1}(t_1)))\right)\\ &= \Gamma\int_{t_0}^{t}(\alpha(B_0(s)), \alpha(V_0(s)))\mathrm{d}s\\ &= \Gamma\int_{t_0}^{t}\phi(s)\mathrm{d}s.\end{aligned}$$

Again by the Bellman inequality, we know that $\phi(t) \equiv 0$ in J_2', from which we obtain that $\alpha(B_0(t_2)) = \alpha(V_0(t_2)) = 0$ and $\alpha(I_2(B_0(t_2))) = \alpha(\hat{I}_2(V_0(t_2))) = 0$.

Continuing such a process interval-by-interval up to J_{m+1}', we can prove that $\phi(t) \equiv 0$ in every J_k', $k = 1, 2, \ldots, m+1$.

For any J_k, if for all $n \in \mathbb{N}$, we modify the value of v_n and ω_n at $t = t_{k-1}$ via $v_n(t_{k-1}) = v_n(t_{k-1}^+)$ and $\omega_n(t_{k-1}) = \omega_n(t_{k-1}^+)$, respectively, then $\{v_n\} \subset C(J_k, E_1)$, $\{\omega_n\} \subset C(J_k, E_2)$ and they are equicontinuous. Since $\alpha(\{v_n(t)\}) \equiv 0$ and $\alpha(\{\omega_n(t)\}) \equiv 0$, $\{v_n(t)\}$ and $\{\omega_n(t)\}$ are precompact in E_1 and E_2 for every $t \in J_k$, respectively. By the Arzela–Ascoli theorem, we know that $\{v_n\}$ and $\{\omega_n(t)\}$ are precompact in $C(J_k, E_1)$ and $C(J_k, E_2)$, respectively. Hence, $\{v_n\}$ and $\{\omega_n\}$ have convergent subsequences in $C(J_k, E_1)$ and $C(J_k, E_2)$, respectively. Combining this with the monotonicity (3.4), we easily prove that $\{v_n\}$ itself is convergent in $C(J_k, E_1)$ and $\{\omega_n\}$ itself is convergent in $C(J_k, E_2)$. In particular, $\{v_n(t)\}$ and $\{\omega_n(t)\}$ are uniformly convergent in J_k'. Consequently, $\{v_n(t)\}$ and $\{\omega_n(t)\}$ are uniformly convergent over the whole of J.

Using an argument similar to that for $\{v_n(t)\}$ and $\{\omega_n(t)\}$, we can prove that $\{u_n(t)\}$ and $\{\nu_n(t)\}$ are also uniformly convergent in J. Hence, $\{v_n(t)\}$ and $\{u_n(t)\}$ are convergent in $PC^1(J, E_1)$, and $\{\omega_n(t)\}$ and $\{\nu_n(t)\}$ are convergent in $PC^1(J, E_2)$. Set

$$\underline{x} = \lim_{n\to\infty} v_n, \qquad \overline{x} = \lim_{n\to\infty} u_n \quad \text{in } PC^1(J, E_1), \tag{3.6}$$

$$\underline{y} = \lim_{n\to\infty} \omega_n, \qquad \overline{y} = \lim_{n\to\infty} \nu_n \quad \text{in } PC^1(J, E_2). \tag{3.7}$$

Letting $n \to \infty$ in (3.3) and (3.4), we see that $v_0 \leq \underline{x} \leq \overline{x} \leq u_0$, $\omega_0 \leq \underline{y} \leq \overline{y} \leq \nu_0$ and

$$\underline{x} = P\underline{x}, \qquad \underline{y} = Q\underline{y} \quad \text{and} \quad \overline{x} = P\overline{x}, \qquad \overline{y} = Q\overline{y},$$

i.e.,

$$(\underline{x}, \underline{y}) = F(\underline{x}, \underline{y}), \qquad (\overline{x}, \overline{y}) = F(\overline{x}, \overline{y}). \tag{3.8}$$

By the monotonicity of F, it is easy to see that $(\underline{x}, \underline{y})$ and $(\overline{x}, \overline{y})$ are the minimal and maximal fixed points of F in $[v_0, u_0] \times [\omega_0, \nu_0]$. That is, they are the minimal and maximal solutions of problem (1.1) between (v_0, ω_0) and (u_0, ν_0), respectively. This completes the proof. □

Remark 3.1. The conditions for an impulsive argument are dropped in Theorem 3.1, i.e., we do not need the following restrictions:

$$\alpha(I_k(x_k)) \leq M_k\alpha(x_k), \qquad \alpha(\bar{I}_k(y_k)) \leq N_k\alpha(y_k), \quad k = 1, 2, \ldots, m.$$

Further, the results do not rely on the Hausdorff measure of noncompactness, but use the Kuratowski measure of noncompactness. Therefore, Theorem 3.1 greatly improves the corresponding results in [39].

In Theorem 3.1, if E_1 and E_2 are weakly sequentially complete, the condition (H_3) holds automatically. In fact, by Theorem 2.2 of [42], any monotonic and order-bounded sequence is precompact. Let $\{x_n\}$ and $\{z_n\}$, $\{y_n\}$ and $\{\xi_n\}$ be two increasing or decreasing sequences obeying condition (H_3), respectively, then by condition (H_1),

$\{f(t, x_n, y_n, z_n) + M_1 x_n\}$ and $\{g(t, x_n, y_n, \xi_n) + M_2 y_n\}$ are monotonic and order-bounded sequences. By the property of measure of noncompactness, we have

$$\alpha(\{f(t, x_n, y_n, z_n) + M_1 x_n\}) \le \alpha(\{f(t, x_n, y_n, z_n) + M_1 x_n\}) + M_1 \alpha(\{x_n\}) = 0.$$
$$\alpha(\{g(t, x_n, y_n, \xi_n) + M_2 y_n\}) \le \alpha(\{g(t, x_n, y_n, \xi_n) + M_2 y_n\}) + M_2 \alpha(\{y_n\}) = 0.$$

Hence, condition (H_3) holds. From Theorem 3.1, we obtain the following result.

Corollary 3.1. *Let E_1 and E_2 be ordered and weakly sequentially complete Banach spaces, whose positive cone P_1 and P_2 are normal, respectively, $f \in C(J \times E_1 \times E_2 \times E_1, E_1)$, $g \in C(J \times E_2 \times E_1 \times E_2, E_2)$ and $I_k \in C(E_1, E_1)$, $\hat{I}_k \in C(E_2, E_2)$, $k = 1, 2, \ldots, m$. If conditions (H_1) and (H_2) are satisfied, then problem (1.1) has minimal and maximal solutions between (v_0, ω_0) and (u_0, ν_0), which can be obtained by a monotone iterative procedure starting from (v_0, ω_0) and (u_0, ν_0), respectively.*

Moreover, we shall discuss the uniqueness of the solution to problem (1.1) in $[v_0, u_0] \times [\omega_0, \nu_0]$. If we replace assumption (H_3) by the following assumption:

(H_4) There exist positive constants C_i $(i = 1, 2, 3, 4)$ such that

$$f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1) \le C_1(x_2 - x_1) + C_2(z_2 - z_1),$$
$$g(t, y_2, x_2, \xi_2) - g(t, y_1, x_1, \xi_1) \le C_3(y_2 - y_1) + C_4(\xi_2 - \xi_1)$$

for all $t \in J$, $v_0(t) \le x_1 \le x_2 \le u_0(t)$, $\omega_0(t) \le y_1 \le y_2 \le \nu_0(t)$, $\lambda S v_0(t) \le z_1 \le z_2 \le \lambda S u_0(t)$, $\mu T \omega_0(t) \le \xi_1 \le \xi_2 \le \mu T \nu_0(t)$, then we have the following unique existence result.

Theorem 3.2. *Let E_i be an ordered Banach space, whose positive cone P_i is normal for $i = 1, 2$, $f \in C(J \times E_1 \times E_2 \times E_1, E_1)$, $g \in C(J \times E_2 \times E_1 \times E_2, E_2)$ and $I_k \in C(E_1, E_1)$, $\hat{I}_k \in C(E_2, E_2)$, $k = 1, 2, \ldots, m$. If conditions (H_1), (H_2) and (H_4) hold, then problem (1.1) has a unique solution between (v_0, ω_0) and (u_0, ν_0), which can be obtained by a monotone iterative procedure starting from (v_0, ω_0) or (u_0, ν_0).*

Proof. We first prove that (H_1) and (H_4) imply (H_3). In fact, for $t \in J$, let $\{x_n\} \subset [v_0, u_0]$, $\{y_n\} \subset [\omega_0, \nu_0]$, $\{z_n\} \subset [\lambda S v_0(t), \lambda S u_0(t)]$ and $\{\xi_n\} \subset [\mu T \omega_0(t), \mu T \nu_0(t)]$ be increasing sequences. For $m, n \in \mathbb{N}$ with $m > n$, by (H_1) and (H_4),

$$\begin{aligned} \theta &\le (f(t, x_m, y_m, z_m) - f(t, x_n, y_n, z_n)) + M_1(x_m - x_n) \\ &\le (C_1 + M_1)(x_m - x_n) + C_2(z_m - z_n), \end{aligned}$$
$$\begin{aligned} \theta &\le (g(t, y_m, x_m, \xi_m) - g(t, y_n, x_n, \xi_n)) + M_2(y_m - y_n) \\ &\le (C_3 + M_1)(y_m - y_n) + C_4(\xi_m - \xi_n). \end{aligned}$$

By these and the normality of cone P_i $(i = 1, 2)$, we have

$$\begin{aligned} &\|f(t, x_m, y_m, z_m) - f(t, x_n, y_n, z_n)\| \\ &\quad \le N_1 \|(C_1 + M_1)(x_m - x_n) + C_2(z_m - z_n)\| + M_1 \|x_m - x_n\| \\ &\quad \le (M_1 + M_1 N_1 + N_1 C_1)\|x_m - x_n\| + N_1 C_2 \|z_m - z_n\| \end{aligned}$$

and

$$\begin{aligned} &\|g(t, y_m, x_m, \xi_m) - g(t, y_n, x_n, \xi_n)\| \\ &\quad \le N_2 \|(C_3 + M_2)(y_m - y_n) + C_4(\xi_m - \xi_n)\| + M_2 \|y_m - y_n\| \\ &\quad \le (M_2 + M_2 N_2 + N_2 C_3)\|y_m - y_n\| + N_2 C_4 \|\xi_m - \xi_n\|. \end{aligned}$$

From these inequalities and the definition of the measure of noncompactness, it follows that

$$\begin{aligned} \alpha(\{f(t, x_n, y_n, z_n)\}) &\le (M_1 + M_1 N_1 + N_1 C_1)\alpha(\{x_n\}) + N_1 C_2 \alpha(\{z_n\}) \\ &\le L_3(\alpha(\{x_n\}) + \alpha(\{z_n\})), \end{aligned}$$
$$\begin{aligned} \alpha(\{g(t, y_n, x_n, \xi_n)\}) &\le (M_2 + M_2 N_2 + N_2 C_3)\alpha(\{y_n\}) + N_2 C_4 \alpha(\{\xi_n\}) \\ &\le L_4(\alpha(\{y_n\}) + \alpha(\{\xi_n\})), \end{aligned}$$

where $L_3 = \max\{M_1 + M_1N_1 + N_1C_1, N_1C_2\}$ and $L_4 = \max\{M_2 + M_2N_2 + N_2C_3, N_2C_4\}$. If $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{\xi_n\}$ are two decreasing sequences, the above inequalities are also valid. Hence (H_3) holds.

Therefore, by Theorem 3.1, problem (1.1) has minimal solution $(\underline{x}, \underline{y})$ and maximal solution $(\overline{x}, \overline{y})$ in $[v_0, u_0] \times [\omega_0, \nu_0]$. By the proof of Theorem 3.1, (3.3), (3.4), (3.6) and (3.7) are valid. Going from J_1' to J_{m+1}' interval-by-interval, we show that $(\underline{x}, \underline{y}) \equiv (\overline{x}, \overline{y})$ in every J_k', $k = 1, 2, \ldots, m+1$.

Indeed, for $t \in J_1'$, by (3.6), (3.7) and (3.2) and assumption (H_4), we have

$$\begin{aligned}
\theta \le \overline{x}(t) - \underline{x}(t) &= P\overline{x}(t) - P\underline{x}(t) \\
&= \int_{t_0}^{t} e^{M_1(t-s)}(f(s, \overline{x}(s), \overline{y}(s), \lambda S\overline{x}(s)) - f(s, \underline{x}(s), \underline{y}(s), \lambda S\underline{x}(s)) + M_1(\overline{x}(s) - \underline{x}(s)))\mathrm{d}s \\
&\le \int_{t_0}^{t} e^{M_1(t-s)}((M_1 + C_1)(\overline{x}(s) - \underline{x}(s)) + \lambda C_2(S\overline{x}(s) - S\underline{x}(s)))\mathrm{d}s \\
&\le \int_{t_0}^{t} ((M_1 + C_1)(\overline{x}(s) - \underline{x}(s)) + \lambda C_2(S\overline{x}(s) - S\underline{x}(s)))\mathrm{d}s \\
&\le (M_1 + C_1)\int_{t_0}^{t} (\overline{x}(s) - \underline{x}(s))\mathrm{d}s + \lambda C_2 h_0 \int_{t_0}^{t}\int_{t_0}^{s} (\overline{x}(t) - \underline{x}(t))\mathrm{d}t\mathrm{d}s \\
&\le (M_1 + C_1 + a\lambda C_2 h_0)\int_{t_0}^{t} (\overline{x}(s) - \underline{x}(s))\mathrm{d}s
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\theta &\le \overline{y}(t) - \underline{y}(t) \\
&\le (M_2 + C_3 + a\lambda C_4 \kappa_0)\int_{t_0}^{t} (\overline{y}(s) - \underline{y}(s))\mathrm{d}s.
\end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) and the normality of cone P_i $(i = 1, 2)$ that

$$\|\overline{x}(t) - \underline{x}(t)\| \le N_1(M_1 + C_1 + a\lambda C_2 h_0)\int_{t_0}^{t} \|\overline{x}(s) - \underline{x}(s)\|\mathrm{d}s,$$

$$\|\overline{y}(t) - \underline{y}(t)\| \le N_2(M_2 + C_3 + a\lambda C_4 \kappa_0)\int_{t_0}^{t} \|\overline{y}(s) - \underline{y}(s)\|\mathrm{d}s.$$

By the Bellman inequality, these imply that $(\underline{x}(t), \underline{y}(t)) \equiv (\overline{x}(t), \overline{y}(t))$ in J_1'.

For $t \in J_2'$, since $I_1(\overline{x}(t_1)) = I_1(\underline{x}(t_1))$ and $\hat{I}_1(\overline{y}(t_1)) = \hat{I}_1(\underline{y}(t_1))$, using (3.2) and completely the same argument as above for $t \in J_1'$, we can prove that

$$\begin{aligned}
\|\overline{x}(t) - \underline{x}(t)\| &\le N_1(M_1 + C_1 + a\lambda C_2 h_0)\int_{t_0}^{t} \|\overline{x}(s) - \underline{x}(s)\|\mathrm{d}s \\
&= N_1(M_1 + C_1 + a\lambda C_2 h_0)\int_{t_1}^{t} \|\overline{x}(s) - \underline{x}(s)\|\mathrm{d}s, \\
\|\overline{y}(t) - \underline{y}(t)\| &\le N_2(M_2 + C_3 + a\lambda C_4 \kappa_0)\int_{t_0}^{t} \|\overline{y}(s) - \underline{y}(s)\|\mathrm{d}s \\
&= N_2(M_2 + C_3 + a\lambda C_4 \kappa_0)\int_{t_1}^{t} \|\overline{y}(s) - \underline{y}(s)\|\mathrm{d}s.
\end{aligned}$$

Again, by the Bellman inequality, we obtain that $(\underline{x}(t), \underline{y}(t)) \equiv (\overline{x}(t), \overline{y}(t))$ in J_2'.

Continuing such a process interval-by-interval up to J_{m+1}', we see that $(\underline{x}(t), \underline{y}(t)) \equiv (\overline{x}(t), \overline{y}(t))$ over the whole of J. Hence, $(x^*, y^*) := (\underline{x}(t), \underline{y}(t)) = (\overline{x}(t), \overline{y}(t))$ is the unique solution of problem (1.1) in $[v_0, u_0] \times [\omega_0, \nu_0]$, which can be obtained by the monotone iterative procedure (3.3) starting from (v_0, ω_0) or (ω_0, ν_0). This completes the proof. □

Remark 3.2. Using the same approach as in Theorems 3.1 and 3.2, we can consider initial value problems (1.2) and (1.3) and obtain analogous conclusions, respectively.

Remark 3.3. Using the above argument method interval-by-interval from J_1' to J_{m+1}', we can also improve the main results in [29] and [34], and delete some restrictive conditions there.

4. An example

Example 1. Consider the following system of nonlinear mixed type implicit impulsive integro-differential equations in Banach spaces E_1 and E_2: Find $(x, y) : J \times J \to E_1 \times E_2$ such that

$$\begin{cases} x_n'(t) = \dfrac{1}{20}\left\{\dfrac{e^{-6t}}{3n}[x_{n+1}^4 + (t-y_n)^3] + \lambda\displaystyle\int_0^t e^{-(6t+s)}x_n(s)ds\right\}, & \forall 0 \le t \le 1, t \ne \dfrac{1}{2}, \\ y_n'(t) = \dfrac{1}{9n}[y_{n+1}^4 + (t-x_n)^3] + \dfrac{\mu}{2n}\left[\displaystyle\int_0^1 e^{ts}y_{n+2}(s)ds\right]^3, & \forall 0 \le t \le 1, t \ne \dfrac{1}{2}, \\ \triangle x_n|_{t=1/2} = -\dfrac{2}{5}x_n\left(\dfrac{1}{2}\right), \\ \triangle y_n|_{t=1/2} = 4y_n\left(\dfrac{1}{2}\right), \\ x_n(0) = y_n(0) = 0 \quad (n = 1, 2, \ldots,). \end{cases} \tag{4.1}$$

Evidently, $(x_n(t), y_n(t)) \equiv (0, 0)$ $(n = 1, 2, \ldots)$ is a trivial solution of problem (4.1).

Theorem 4.1. *Problem (4.1) admits minimal and maximal solutions $(v(t), \omega(t))$ and $(u(t), \nu(t))$ which are continuously differentiable on $J \times J$ and satisfy*

$$0 \le v(t), u(t) \le \begin{cases} \dfrac{t}{n}, & \forall 0 \le t \le \dfrac{1}{2} \\ \dfrac{t}{n} - \dfrac{1}{5n}, & \forall \dfrac{1}{2} < t \le 1, \end{cases} \quad (n = 1, 2, \ldots),$$

$$0 \le \omega(t), \nu(t) \le \begin{cases} \dfrac{t}{n}, & \forall 0 \le t \le \dfrac{1}{2} \\ \dfrac{t}{n} + \dfrac{1}{8n}, & \forall \dfrac{1}{2} < t \le 1, \end{cases} \quad (n = 1, 2, \ldots),$$

where $J = [0, \frac{1}{2}] \cup (\frac{1}{2}, 1]$.

Proof. Let $t_0 = 0$, $a = 1$, $E_1 = E_2 = C_0 = \{x = (x_1, x_2, \ldots, x_n, \ldots) : x_n \to 0\}$ with norm $\|x\| = \sup_n |x_n|$ and $P_1 = P_2 = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in C_0 : x_n \ge 0, n = 1, 2, \ldots\}$. Then P_1 and P_2 are normal cones in E_1 and E_2, respectively, and problem (4.1) can be regarded to be of the form (1.1) in $E_1 \times E_2$. In this situation, $x_0 = y_0 = (0, 0, \ldots, 0, \ldots) = \theta$, $J = [0, 1]$, $h(t, s) = e^{-(6t+s)}$, $\kappa(t, s) = e^{ts}$, $x = (x_1, x_2, \ldots, x_n, \ldots)$, $y = (y_1, y_2, \ldots, y_n, \ldots)$, $z = (z_1, z_2, \ldots, z_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$ and $g = (g_1, g_2, \ldots, g_n, \ldots)$ in which

$$f_n(t, x, y, z) = \frac{1}{20}\left\{\frac{e^{-6t}}{3n}[(t-y_n)^3 + x_{n+1}^4] + \lambda z_n\right\},$$

$$g_n(t, x, y, z) = \frac{1}{9n}[(t-x_n)^3 + y_{n+1}^4] + \frac{\mu}{2n}z_n^3,$$

$m = 1$, $t_1 = \frac{1}{2}$ and

$$I_1(x) = -\frac{2}{5}x, \quad \forall x \in E_1 = C_0,$$

$$\hat{I}_1(y) = 4y, \quad \forall y \in E_2 = C_0.$$

Obviously, $f \in C[J \times E_1 \times E_2 \times E_1, E_1]$, $g \in C[J \times E_2 \times E_1 \times E_2, E_2]$, $I_1 \in C[E_1, E_1]$ and $\hat{I}_1 \in C[E_2, E_2]$. Let

$$\upsilon_0(t) = \omega_0(t) = (0, 0, \ldots, 0, \ldots), \quad \forall 0 \le t \le 1$$

$$u_0(t) = \begin{cases} \left(t, \frac{t}{2} \ldots, \frac{t}{n}, \ldots\right), & \forall 0 \le t \le \frac{1}{2} \\ \left(t - \frac{1}{5}, t - \frac{1}{10}, \ldots, \frac{t}{n} - \frac{1}{5n}, \ldots\right), & \forall \frac{1}{2} < t \le 1, \end{cases}$$

$$\nu_0(t) = \begin{cases} \left(t, \frac{t}{2}, \ldots, \frac{t}{n}, \ldots\right), & \forall 0 \le t \le \frac{1}{2} \\ \left(t + \frac{1}{8}, t + \frac{1}{16}, \ldots, \frac{t}{n} + \frac{1}{8n}, \ldots\right), & \forall \frac{1}{2} < t \le 1. \end{cases}$$

It is not difficult to verify that conditions (H_1)–(H_3) hold. Hence, our conclusion follows from Theorem 3.1. □

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