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Schur complements on Hilbert spaces and saddle point systems

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ABSTRACT

For any continuous bilinear form defined on a pair of Hilbert spaces satisfying the compatibility Ladyshenskaya–Babušca–Brezzi condition, symmetric Schur complement operators can be defined on each of the two Hilbert spaces. In this paper, we find bounds for the spectrum of the Schur operators only in terms of the compatibility and continuity constants. In light of the new spectral results for the Schur complements, we review the classical Babušca–Brezzi theory, find sharp stability estimates, and improve a convergence result for the inexact Uzawa algorithm. We prove that for any symmetric saddle point problem, the inexact Uzawa algorithm converges, provided that the inexact process for inverting the residual at each step has the relative error smaller than 1/3. As a consequence, we provide a new type of algorithm for discretizing saddle point problems, which combines the inexact Uzawa iterations with standard a posteriori error analysis and does not require the discrete stability conditions.

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1. Introduction

In the present literature, the abstract formulation and analysis for saddle point systems is based on the main properties of the operators B and B^* associated with a continuous bilinear form $b(\cdot,\cdot)$ defined on a pair of Hilbert spaces and satisfying the compatibility Ladyshenskaya–Babušca–Brezzi (LBB) condition. The properties of the two operators are described in terms of subspaces of dual spaces and polar sets, making the analysis less transparent than that in the finite dimensional case, see e.g., [9,10,15,17,19,24]. In this paper, we compose B^* and B with corresponding Riesz-canonical isometries $\mathcal{A}^{-1}$ and $\mathcal{C}^{-1}$ to get natural bounded operators $\mathcal{A}^{-1}B^*$ and $\mathcal{C}^{-1}B$ acting between the original Hilbert spaces, (see Section 2). The operators $\mathcal{A}^{-1}B^*$ and $\mathcal{C}^{-1}B$ are dual to each other as operators between Hilbert spaces, and the compositions $(\mathcal{C}^{-1}B)(\mathcal{A}^{-1}B^*)$ and $(\mathcal{A}^{-1}B^*)(\mathcal{C}^{-1}B)$ define symmetric and non-negative operators on Hilbert spaces.

In the particular case when $\mathcal{A}$ and B are matrices with $\mathcal{A}$ an invertible matrix, the standard saddle point system,

$$\begin{aligned} \mathcal{A}\mathbf{u} + B^*p &= \mathbf{f}, \\ B\mathbf{u} &= g, \end{aligned}$$

can be reduced to solving a problem in the p variable, by eliminating $\mathbf{u}$ from the first equation. Solving for p reduces to inverting the Schur complement $B\mathcal{A}^{-1}B^*$. In the infinite dimensional case, we have in general that $\mathbf{f}$ and g belong to dual spaces. Using the representation operators $\mathcal{A}^{-1}$ and $\mathcal{C}^{-1}$, the above system can be rewritten

$$\begin{aligned} \mathbf{u} + \mathcal{A}^{-1}B^*p &= \mathcal{A}^{-1}\mathbf{f}, \\ \mathcal{C}^{-1}B\mathbf{u} &= \mathcal{C}^{-1}g, \end{aligned}$$

where now $\mathcal{A}^{-1}B^*$ and $\mathcal{C}^{-1}B$ are operators between standard Hilbert spaces. The Schur complement of the above system is exactly $(\mathcal{C}^{-1}B)(\mathcal{A}^{-1}B^*)$. In light of the spectral properties emphasized in the next section, we will refer to *both* $(\mathcal{C}^{-1}B)(\mathcal{A}^{-1}B^*)$ and $(\mathcal{A}^{-1}B^*)(\mathcal{C}^{-1}B)$ as *Schur complement* type operators.

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We bound the spectrum of the Schur complements only in terms of the compatibility and continuity constants of the form b. Our approach for analyzing saddle point problems is based on the Schur complements and the properties of the two operators $\mathcal{A}^{-1}B^*$ and $\mathcal{C}^{-1}B$. By using these tools, new stability estimates for the solutions of saddle point problems can be found, and convergence results for Uzawa and Arrow–Hurwicz type algorithms can be improved. Convergence results for such algorithms at the continuous level, combined with standard techniques of discretization and a posteriori error estimates lead to adaptive algorithms for solving saddle point systems, see [3,4,12]. The main advantage of the new algorithms is that the LBB discrete condition is not needed.

The paper is organized as follows. In Section 2, we introduce the notation and the natural Schur operators for the general abstract case, and prove the main properties of the Schur complements and the connecting operators $\mathcal{A}^{-1}B^*$ and $\mathcal{C}^{-1}B$. In Section 3 and Appendix A, we reconsider the classical LBB theory in the light of Section 2 and find sharp stability estimates for the solution of a general saddle point system. In Section 4, we analyze the Inexact Uzawa Method (IUM) as introduced in [8, 13]. We consider the symmetric saddle point systems on abstract Hilbert spaces and generalize a finite dimensional result of Cheng, Hu and Zou from [11,16]. We prove that for any symmetric saddle point problem, the inexact Uzawa algorithm converges provided that the inexact process for inverting the residual at each step has the relative error smaller than a threshold $\delta_0 = \frac{2-\alpha M^2}{2+\alpha M^2}$, where α is the relaxation parameter of the algorithm and M is the continuity constant of the form b. In particular, for the choice $\alpha = \frac{1}{M^2}$, the threshold δ_0 becomes the universal constant $1/3$. As a consequence, in Section 4.1, we indicate a way that the inexact Uzawa algorithm can be combined with standard a posteriori error theory to discretize saddle point problems, without requiring discrete stability conditions.

2. Schur complements on Hilbert spaces

In this section, we start with a review of the notation of the classical LBB theory and introduce natural operators and norms for the general abstract case.

We let $\mathbf{V}$ and Q be two Hilbert spaces with inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively, with the corresponding induced norms $|\cdot|_{\mathbf{V}} = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_Q = \|\cdot\| = (\cdot,\cdot)^{1/2}$. The dual parings on $\mathbf{V}^* \times \mathbf{V}$ and $Q^* \times Q$ are denoted by $\langle\cdot,\cdot\rangle$. Here, $\mathbf{V}^*$ and Q^* denote the duals of $\mathbf{V}$ and Q, respectively. With the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$, we associate operators $\mathcal{A} : V \to V^*$ and $\mathcal{C} : Q \to Q^*$ defined by

$$\langle A\mathbf{u}, \mathbf{v}\rangle = a_0(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

and

$$\langle \mathcal{C}p, q\rangle = (p, q) \quad \text{for all } \ p, q \in Q.$$

The operators $\mathcal{A}^{-1} : V^* \to V$ and $\mathcal{C}^{-1} : Q^* \to Q$ are called the Riesz-canonical isometries and satisfy

$$a_0(\mathcal{A}^{-1}\mathbf{u}^*, \mathbf{v}) = \langle \mathbf{u}^*, \mathbf{v}\rangle, \quad |\mathcal{A}^{-1}\mathbf{u}^*|_{\mathbf{V}} = \|\mathbf{u}^*\|_{\mathbf{V}^*}, \mathbf{u}^* \in \mathbf{V}^*, \mathbf{v} \in \mathbf{V}, \tag{2.1}$$

$$(\mathcal{C}^{-1}p^*, q) = \langle p^*, q\rangle, \quad \|\mathcal{C}^{-1}p^*\| = \|p^*\|_{Q^*}, p^* \in Q^*, q \in Q. \tag{2.2}$$

Next, we consider that $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{V} \times Q$, satisfying the inf-sup condition. More precisely, we assume that

$$\inf_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\||\mathbf{v}|} = m > 0 \tag{2.3}$$

and

$$\sup_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\|\ |\mathbf{v}|} = M < \infty. \tag{2.4}$$

Here, and throughout this paper, the "inf" and "sup" are taken over nonzero vectors. With the form b, we associate the linear operators $B : V \to Q^*$ and $B^* : Q \to V^*$ defined by

$$\langle B\mathbf{v}, q\rangle = b(\mathbf{v}, q) = \langle B^*q, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V},\ q \in Q.$$

Let $\mathbf{V}_0$ be the kernel of B or $\mathcal{C}^{-1}B$, i.e.,

$$\mathbf{V}_0 = \mathrm{Ker}(B) = \{\mathbf{v} \in \mathbf{V} |\ B\mathbf{v} = 0\} = \{\mathbf{v} \in \mathbf{V} |\ \mathcal{C}^{-1}B\mathbf{v} = 0\}.$$

Due to (2.4), $\mathbf{V}_0$ is a closed subspace of $\mathbf{V}$. Before we present the main result of this section, we review a few useful functional analysis results.

For a bounded linear operator $T : X \to Y$ between two Hilbert spaces X and Y, we denote by T^t the Hilbert transpose of T. If $X = Y$, we say that T is symmetric if $T = T^t$. For a bounded linear operator $T : X \to X$, we denote the spectrum of the operator T by $\sigma(T)$. The following Proposition can be found in [18], Section 31.1.

Proposition 2.1. *The spectrum $\sigma(T)$ of a bounded symmetric operator T on a Hilbert space H, lies in the closed interval $[a, b]$ on the real axis, where*

$$a = \inf \frac{(Tx, x)}{(x, x)}, \qquad b = \sup \frac{(Tx, x)}{(x, x)}.$$

Moreover, $a, b \in \sigma(T)$ and consequently,

$$\rho(T) := \max\{|\lambda|, \lambda \in \sigma(T)\} = \|T\| = \max\{|a|, |b|\}.$$

Here, $(\cdot, \cdot)$ is the inner product on H.

The next result (see [21], Theorem 12.12), gives a characterization of the spectrum of normal operators in general and will be used in Section 4 for describing the spectrum of symmetric Schur operators.

Proposition 2.2. *Let T be a normal operator on a Hilbert space H. Then, T is invertible if and only if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in H$. Consequently, $\lambda \in \sigma(T)$ if and only if there exists a sequence $\{x_n\} \subset H$, such that $\|x_n\| = 1$ for all n, and $\|(T - \lambda)x_n\| \to 0$ as $n \to \infty$.*

The following lemma provides important properties of norms and operators to be used in this paper.

Lemma 2.3 (*Schur Complements*). *Let $\mathcal{A}$, $\mathcal{C}$, B, and B^* be the operators associated with the spaces $\mathbf{V}$, Q and the connecting form $b(\cdot, \cdot)$. Assume that (2.3) and (2.4) are satisfied.*

(i) ***The operators** $\mathcal{C}^{-1}B : \mathbf{V} \to Q$ **and** $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}$ **are symmetric to each other**, i.e.,*

$$(\mathcal{C}^{-1}B\mathbf{v}, q) = a_0(\mathbf{v}, \mathcal{A}^{-1}B^*q), \quad \mathbf{v} \in \mathbf{V}, q \in Q, \tag{2.5}$$

consequently,

$$(\mathcal{C}^{-1}B)^t = \mathcal{A}^{-1}B^* \quad \textit{and} \quad (\mathcal{A}^{-1}B^*)^t = \mathcal{C}^{-1}B.$$

(ii) ***The Schur complement on** Q is the operator $S_0 := \mathcal{C}^{-1}B\mathcal{A}^{-1}B^* : Q \to Q$. The operator S_0 is symmetric and positive definite on Q, satisfying*

$$(S_0p, p) = \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)^2}{|\mathbf{v}|^2}. \tag{2.6}$$

Consequently, $m^2, M^2 \in \sigma(S_0)$ and

$$\sigma(S_0) \subset [m^2, M^2]. \tag{2.7}$$

(iii) ***An orthogonal decomposition of** $\mathbf{V}$. The following estimate holds*

$$\|p\|_{S_0} := (S_0p, p)^{1/2} = |\mathcal{A}^{-1}B^*p|_{\mathbf{V}} \geq m\|p\| \quad \textit{for all } p \in Q. \tag{2.8}$$

Consequently, $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}$ has closed range, $\mathbf{V}_1 := \mathcal{A}^{-1}B^*(Q)$ is a closed subspace of $\mathbf{V}$ and $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}_1$ is an isomorphism. Moreover, $\mathbf{V}_0 = \mathrm{Ker}(\mathcal{C}^{-1}B) = \mathcal{A}^{-1}B^*(Q)^\perp$ and

$$\mathbf{V} = \mathrm{Ker}(\mathcal{C}^{-1}B) \oplus \mathcal{A}^{-1}B^*(Q) = \mathbf{V}_0 \oplus \mathbf{V}_1.$$

(iv) ***The Schur complement on** $\mathbf{V}$ is defined as the operator $S := \mathcal{A}^{-1}B^*\mathcal{C}^{-1}B : \mathbf{V} \to \mathbf{V}$. The operator S is symmetric and non-negative definite on $\mathbf{V}$, with $\mathrm{Ker}(S) = \mathbf{V}_0$, $S(\mathbf{V}) = \mathbf{V}_1$, and satisfies*

$$a_0(S\mathbf{u}, \mathbf{v}) = (\mathcal{C}^{-1}B\mathbf{u}, \mathcal{C}^{-1}B\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}. \tag{2.9}$$

(v) ***The Schur complement on** $\mathbf{V}_1 = \mathbf{V}_0^\perp$ is the restriction of S to $\mathbf{V}_1$, i.e., $S_1 := \mathcal{A}^{-1}B^*\mathcal{C}^{-1}B : \mathbf{V}_1 \to \mathbf{V}_1$. The operator S_1 is symmetric and positive definite on $\mathbf{V}_1$, satisfying*

$$\sigma(S_1) = \sigma(S_0) \subset [m^2, M^2]. \tag{2.10}$$

(vi) *$\mathcal{C}^{-1}B$ **is a double isometry.** The following statements hold*

$$\|\mathcal{C}^{-1}B\mathbf{u}_1\| = a_0(S_1\mathbf{u}_1, \mathbf{u}_1)^{1/2} := |\mathbf{u}_1|_{S_1} \geq m\,|\mathbf{u}_1|, \quad \mathbf{u}_1 \in \mathbf{V}_1, \tag{2.11}$$

and

$$\|\mathcal{C}^{-1}B\mathbf{u}_1\|_{S_0^{-1}} := (S_0^{-1}\mathcal{C}^{-1}B\mathbf{u}_1, \mathcal{C}^{-1}B\mathbf{u}_1)^{1/2} = |\mathbf{u}_1|, \quad \mathbf{u}_1 \in \mathbf{V}_1. \tag{2.12}$$

Consequently, $\mathcal{C}^{-1}B$ is an isometry from $(\mathbf{V}_1, |\cdot|_{S_1})$ to Q, and from $\mathbf{V}_1$ to $(Q, \|\cdot\|_{S_0^{-1}})$.

(vii) $\mathcal{A}^{-1}B^*$ ***is a double isometry.*** *The following identity holds*

$$|\mathcal{A}^{-1}B^*p|_{S_1^{-1}} := a_0(S_1^{-1}\mathcal{A}^{-1}B^*p, \mathcal{A}^{-1}B^*p)^{1/2} = \|p\|, \quad p \in Q. \tag{2.13}$$

Consequently, $\mathcal{A}^{-1}B^*$ *is an isometry from* Q *to* $(\mathbf{V}_1, |\cdot|_{S_1^{-1}})$*, and from* $(Q, \|\cdot\|_{S_0})$ *to* $\mathbf{V}_1$.

Proof. The proof follows by using standard functional analysis tools. For completeness, we include the main steps.
(i) For any $\mathbf{v} \in \mathbf{V}$, $q \in Q$, we have

$$(\mathcal{C}^{-1}B\mathbf{v}, q) = \langle B\mathbf{v}, q\rangle = \langle B^*q, \mathbf{v}\rangle = a_0(\mathcal{A}^{-1}B^*q, \mathbf{v}) = a_0(\mathbf{v}, \mathcal{A}^{-1}B^*q),$$

which proves (2.5).
(ii) The symmetry of S follows by using (i). Indeed,

$$S^t = \left((\mathcal{C}^{-1}B)(\mathcal{A}^{-1}B^*)\right)^t = (\mathcal{A}^{-1}B^*)^t(\mathcal{C}^{-1}B)^t = (\mathcal{C}^{-1}B)(\mathcal{A}^{-1}B^*) = S.$$

To prove (2.6), we let $p \in Q$ be fixed and consider the following problem:
Find $\mathbf{u} \in \mathbf{V}$ such that

$$a_0(\mathbf{u}, \mathbf{v}) = b(\mathbf{v}, p) \quad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{2.14}$$

Since the functional $\mathbf{v} \to b(\mathbf{v}, p)$ is continuous on $\mathbf{V}$, by the Riesz representation theorem (see e.g., [25]), we have that the unique solution $\mathbf{u}$ of (2.14) satisfies

$$a_0(\mathbf{u}, \mathbf{u}) = \|\mathbf{v} \to b(\mathbf{v}, p)\|^2_{\mathbf{V}^*} = \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)^2}{|\mathbf{v}|^2}. \tag{2.15}$$

On the other hand, from (2.14), we have

$$\mathcal{A}\mathbf{u} = B^*p \quad \text{or} \quad \mathbf{u} = \mathcal{A}^{-1}B^*p,$$

and

$$\begin{aligned} a_0(\mathbf{u}, \mathbf{u}) &= a_0(\mathcal{A}^{-1}B^*p, \mathcal{A}^{-1}B^*p) = ((\mathcal{A}^{-1}B^*)^t\mathcal{A}^{-1}B^*p, p) \\ &= (\mathcal{C}^{-1}B\mathcal{A}^{-1}B^*p, p) = (S_0p, p). \end{aligned} \tag{2.16}$$

Thus, (2.6) follows from (2.15) and (2.16). The inclusion (2.7) follows immediately from (2.6), (2.3), and (2.4).
(iii) From (2.16) and (2.7), we get

$$(S_0p, p) = a_0(\mathbf{u}, \mathbf{u}) = |\mathcal{A}^{-1}B^*p|^2_{\mathbf{V}} \geq m^2\|p\|^2.$$

For part (iv), we notice that

$$a_0(S\mathbf{u}, \mathbf{v}) = a_0((\mathcal{A}^{-1}B^*)(\mathcal{C}^{-1}B)\mathbf{u}, \mathbf{v}) = (\mathcal{C}^{-1}B\mathbf{u}, \mathcal{C}^{-1}B\mathbf{v}) \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}.$$

Thus, (2.9) holds and S is symmetric and non-negative definite. The relations $\mathrm{Ker}(S) = \mathbf{V}_0$ and $S(\mathbf{V}) = \mathbf{V}_1$ follow from (iii).
(v) By using (ii) and (iv), part (v) reduces to proving that $\sigma(S_1) = \sigma(S_0)$. This holds, because $S_0 = (\mathcal{C}^{-1}B)(\mathcal{A}^{-1}B^*)$ and $S_1 = (\mathcal{A}^{-1}B^*)(\mathcal{C}^{-1}B)$, where $(\mathcal{A}^{-1}B^*)$ and $(\mathcal{C}^{-1}B)$ are isomorphisms.
(vi) First, we have that (2.11) is a direct consequence of (2.9) and (2.10). Next, the identity

$$\begin{aligned} (S_0^{-1}\mathcal{C}^{-1}B\mathbf{u}_1, \mathcal{C}^{-1}B\mathbf{u}_1) &= \left((\mathcal{C}^{-1}B\mathcal{A}^{-1}B^*)^{-1}\mathcal{C}^{-1}B\mathbf{u}_1, \mathcal{C}^{-1}B\mathbf{u}_1\right) \\ &= \left((\mathcal{A}^{-1}B^*)^{-1}(\mathcal{C}^{-1}B)^{-1}(\mathcal{C}^{-1}B)\mathbf{u}_1, \mathcal{C}^{-1}B\mathbf{u}_1\right) \\ &= \left((\mathcal{A}^{-1}B^*)^{-1}\mathbf{u}_1, \mathcal{C}^{-1}B\mathbf{u}_1\right) \\ &= a_0((\mathcal{A}^{-1}B^*)(\mathcal{A}^{-1}B^*)^{-1}\mathbf{u}_1, \mathbf{u}_1) = |\mathbf{u}_1|^2, \end{aligned}$$

proves (2.12). □
(vii) The identity (2.13) follows in a similar way with (2.12).

The splitting $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1$ of Lemma 2.3 (iii) can be viewed as an abstract Helmholtz decomposition.

Lemma 2.4 (*Abstract Helmholtz Decomposition*). *Let* $b : \mathbf{V} \times Q \to \mathbb{R}$ *be a bilinear form satisfying* (2.3) *and* (2.4). *Then, any* $\mathbf{u} \in \mathbf{V}$ *has a unique orthogonal decomposition*

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1,$$

where $B\mathbf{u}_0 = 0$, *and* $\mathbf{u}_1 = \mathcal{A}^{-1}B^*p$, *for some* $p \in Q$. *In addition, we have*

$$|\mathbf{u}_1|_{S_1} = \|\mathcal{C}^{-1}B\mathbf{u}\| = \|B\mathbf{u}\|_{Q^*} \quad \text{and} \quad |\mathbf{u}_1|_{\mathbf{V}} = \|\mathcal{C}^{-1}B\mathbf{u}\|_{S_0^{-1}}. \tag{2.17}$$

Proof. According to Lemma 2.3, we only have to justify (2.17). Let $\mathbf{u} \in \mathbf{V}$ be fixed and let $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$ with $B\mathbf{u}_0 = 0$, and $\mathbf{u}_1 = \mathcal{A}^{-1}B^*p$ for some $p \in Q$. Then, $B\mathbf{u} = B\mathbf{u}_1$, and by using (2.11) and (2.12), we obtain (2.17).

3. Schur complements and stability estimates

In this section, we present the notation and some of the classical theory for saddle point systems in light of the spectral results of the Schur complements (the results of Lemma 2.3). We recover standard estimates and find sharp new stability estimates for the solutions of general case of a saddle point problem.

Next, we consider the general abstract saddle point problem. Assume that a bilinear form $a(\cdot,\cdot)$ is defined on $\mathbf{V}\times\mathbf{V}$ and satisfies

$$a(\mathbf{u},\mathbf{u}) \geq m_0\, a_0(\mathbf{u},\mathbf{u}), \quad \text{for all } \mathbf{u}\in\mathbf{V}_0, \quad \text{and} \tag{3.1}$$

$$\sup_{\mathbf{u}\in\mathbf{V}}\ \sup_{\mathbf{v}\in\mathbf{V}} \frac{a(\mathbf{u},\mathbf{v})}{|\mathbf{u}|\,|\mathbf{v}|} = M_0 < \infty. \tag{3.2}$$

With the form a, we associate the linear operator $A : V \to V^*$ defined by

$$\langle A\mathbf{u},\mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}) \quad \text{for all } \mathbf{u},\mathbf{v}\in\mathbf{V}.$$

Let $b : \mathbf{V}\times Q \to \mathbb{R}$ be a bilinear form satisfying (2.3) and (2.4). For $f\in\mathbf{V}^*$, $g\in Q^*$, we consider the following variational problem:

Find $(\mathbf{u},p)\in\mathbf{V}\times Q$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= \langle \mathbf{f},\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in\mathbf{V},\\ b(\mathbf{u},q) &= \langle g,q\rangle \quad \text{for all } q\in Q. \end{aligned} \tag{3.3}$$

The problem (3.3) is equivalent to the following reformulation:

Find $(\mathbf{u},p)\in\mathbf{V}\times Q$ such that

$$\begin{aligned} A\mathbf{u} + B^*p &= \mathbf{f},\\ B\mathbf{u} &= g. \end{aligned} \tag{3.4}$$

It is known that the above variational problem or system has a unique solution for any $f\in\mathbf{V}^*$, $g\in Q^*$ (see [9,10,15,17]). Next we present stability estimates for the isomorphism $(\mathbf{f},g)\to(\mathbf{u},p)$.

Theorem 3.1. *Assume that the bilinear form b satisfies (2.3) and (2.4), and the bilinear form a satisfies (3.1) and (3.2). Then, for any $(f,g)\in(\mathbf{V}^*,Q^*)$, the problem (3.3) has a unique solution $(\mathbf{u},p)\in(\mathbf{V},Q)$. Let $\mathbf{u}=\mathbf{u}_0+\mathbf{u}_1$ with $\mathbf{u}_0\in\mathbf{V}_0$ and $\mathbf{u}_1\in\mathbf{V}_1$ be the unique decomposition of $\mathbf{u}$. Then, the following estimates hold:*

$$|\mathbf{u}_1| = \|\mathcal{C}^{-1}g\|_{S_0^{-1}}, \quad \text{and} \quad |\mathbf{u}_1|_{S_1} = \|g\|_{Q^*}, \tag{3.5}$$

$$|\mathbf{u}_0| \leq \frac{1}{m_0}\|\mathbf{f}-A\mathbf{u}_1\|_{\mathbf{V}_0^*}, \tag{3.6}$$

$$\|p\|_{S_0} = \|\mathbf{f}-A\mathbf{u}\|_{\mathbf{V}_1^*} \leq \|\mathbf{f}\|_{\mathbf{V}_1^*} + M_0|\mathbf{u}|, \tag{3.7}$$

and

$$\begin{aligned} |\mathbf{u}| &\leq \frac{1}{m_0}\|\mathbf{f}\|_{\mathbf{V}_0^*} + \left(1+\frac{M_0}{m_0}\right)\frac{1}{m}\|g\|_{Q^*},\\ \|p\| &\leq \frac{1}{m}\frac{M_0}{m_0}\|\mathbf{f}\|_{\mathbf{V}_0^*} + \frac{1}{m}\|\mathbf{f}\|_{\mathbf{V}_1^*} + \frac{M_0}{m}\left(1+\frac{M_0}{m_0}\right)\|g\|_{Q^*}. \end{aligned} \tag{3.8}$$

To the best of the author's knowledge, the estimates (3.5) and (3.7) are new and interesting for theoretical analysis. More precise estimates are deduced for the symmetric case in Appendix. The estimate (3.8) can be found in e.g., [9,10,15,17]. For completeness, we include in Appendix a proof of the above theorem together with other sharp results and a more complete version of the Babušca lemma. The proofs avoid working with subspaces of dual spaces.

4. Schur complements and the inexact Uzawa algorithm the exact amount of inexactness we can afford

Besides the pure theoretical contribution to the classical stability theory for saddle point problems, the use of Schur complements turns out to be of practical interest in designing and analyzing Arrow–Hurwicz–Uzawa type algorithms for saddle point systems. In this section, we further motivate the efficiency of using Schur complements in finding the convergence factors of two algorithms.

We assume that the form $a(\cdot,\cdot)$ coincides with the form $a_0(\cdot,\cdot)$ which gives the inner product on $\mathbf{V}$. Consequently, we have $m_0=M_0=1$, and $A=\mathcal{A}$. The Uzawa algorithm for solving the Stokes system was introduced in [1]. It can be easily generalized to solve the general problem (3.3), provided that the form a is coercive on the whole space $\mathbf{V}$, see e.g., [7,13,3].

First, we review the Uzawa algorithm for the symmetric saddle point problem, and present a sharp convergence result for the inexact Uzawa algorithm.

Given a parameter $\alpha > 0$, called the relaxation parameter, the Uzawa algorithm for approximating the solution $(\mathbf{u}, p)$ of (3.3) can be described as follows.

Algorithm 4.1 (*Uzawa Method (UM)*). Let p_0 be any approximation for p, and for $k = 1, 2, \ldots,$ construct $(\mathbf{u}_k, p_k)$ by

$$\begin{aligned} \mathbf{u}_k &= A^{-1}(\mathbf{f} - B^* p_{k-1}), \\ p_k &= p_{k-1} + \alpha \mathcal{C}^{-1}(B\mathbf{u}_k - g). \end{aligned} \tag{4.1}$$

The convergence of the UM is discussed for particular cases in many publications, see e.g., [5,9,14,15,22]. Included below is a theorem taken from [3] that describes the convergence of the Uzawa algorithm. We will compare the result with our main theorem about the inexact Uzawa algorithm.

Theorem 4.2. *Let $(\mathbf{u}, p)$ be the solution of (3.3) and let $(\mathbf{u}_k, p_k)$ be the sequence of approximations built by the UM (4.1). Then, the following holds.*

(i) *The sequences $\mathbf{u} - \mathbf{u}_k$ and $p - p_k$ satisfy*

$$\begin{aligned} |\mathbf{u} - \mathbf{u}_k|_{\mathbf{V}} &\le M\|p - p_{k-1}\|, \\ \|p - p_k\| &\le \|I - \alpha S_0\|\,\|p - p_{k-1}\|. \end{aligned}$$

(ii) *For $\alpha < \frac{2}{M^2}$, the UM is convergent and*

$$\|I - \alpha S_0\| = \max\{|1 - \alpha m^2|, |1 - \alpha M^2|\} < 1.$$

(iii) *For $\alpha = \frac{1}{M^2}$, the convergence factor is $\|I - \alpha S_0\| = 1 - \frac{m^2}{M^2}$.*

(iv) *The optimal convergence factor is achieved for*

$$\alpha_{opt} := \frac{2}{M^2 + m^2} \quad \textit{and} \quad \|I - \alpha_{opt} S_0\| = \frac{M^2 - m^2}{M^2 + m^2}.$$

Next, following the ideas in [8,13], we will investigate the convergence of an abstract inexact Uzawa algorithm where the exact solution of the elliptic problem (the action of A^{-1}) is replaced by an approximation process.

We describe the approximate process as a map Ψ defined on a subset of $\mathbf{V}^*$, which for $\phi \in \mathbf{V}^*$, returns an approximation of ξ, the solution of $A\xi = \phi$. If $\mathbf{V}$ and Q are finite dimensional spaces, then Ψ can be considered as a linear or nonlinear preconditioner for A (see e.g., [8]). One example of nonlinear process Ψ can be taken as the approximate inverse associated with the preconditioned conjugate gradient algorithm. If $\mathbf{V}$ and Q are not finite dimensional spaces, then $\Psi(\phi)$ can be considered as a discrete Galerkin approximation of the elliptic problem $A\xi = \phi$. The inexact Uzawa algorithm for approximating the solution $(\mathbf{u}, p)$ of (3.3) is as follows.

Algorithm 4.3 (*Inexact Uzawa Method (IUM)*). Let $(\mathbf{u}_0, p_0)$ be any approximation for $(\mathbf{u}, p)$, and for $k = 1, 2, \ldots,$ construct $(\mathbf{u}_k, p_k)$ by

$$\begin{aligned} \mathbf{u}_k &= \mathbf{u}_{k-1} + \Psi(\mathbf{f} - A\mathbf{u}_{k-1} - B^* p_{k-1}), \\ p_k &= p_{k-1} + \alpha \mathcal{C}^{-1}(B\mathbf{u}_k - g). \end{aligned}$$

For $k = 0, 1, \ldots$, let $e_k^{\mathbf{u}} := \mathbf{u} - \mathbf{u}_k$, $e_k^p := p - p_k$, $r_k = \mathbf{f} - A\mathbf{u}_k - B^* p_k$, $e_k^{\mathbf{r}} := e_k^{\mathbf{u}} + A^{-1}B^* e_k^p = A^{-1} r_k$. Next, we present the main result of the paper.

Theorem 4.4. *Let $0 < \alpha < 2/M^2$ and assume that Ψ satisfies*

$$\left|\Psi(r_k) - A^{-1} r_k\right|_{\mathbf{V}} \le \delta \left|A^{-1} r_k\right|_{\mathbf{V}}, \quad k = 0, 1, \ldots, \tag{4.2}$$

with

$$\delta < \frac{2 - \alpha M^2}{2 + \alpha M^2}. \tag{4.3}$$

Then, the **IUM** *converges. There exists $\rho = \rho(\alpha, \delta, m, M) \in (0, 1)$ such that*

$$(\delta |e_k^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_k^p\|_{S_0}^2)^{1/2} \le \rho^k\, (\delta |e_0^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_0^p\|_{S_0}^2)^{1/2} \quad k = 1, 2, \ldots. \tag{4.4}$$

For the particular case when $\mathbf{V}$, Q are finite dimensional spaces, $\alpha = 1$, and $M = 1$, by using singular value decomposition of matrix type operators, a similar result was obtained in [11,16]. The above Algorithm (and Theorem) applies to any symmetric and positive definite saddle point problem no matter the dimension. Theorem 4.4 also improves a similar result presented in [3], where threshold δ_0 depends on the constant m. In our case now, $\delta_0 = \frac{2-\alpha M^2}{2+\alpha M^2}$ is independent of m. If we choose α to be the practical choice $\alpha = \frac{1}{M^2}$, then $\delta_0 = 1/3$ is a universal constant. Thus, if $\alpha = \frac{1}{M^2}$ and the relative error of the approximate process (the amount of inexactness) at each step is smaller than any fixed number smaller than 1/3, then the algorithm converges. The value ρ in (4.4) can be taken exactly the spectral radius of the error operator as can be seen from the proof below. The convergence result for the algorithm for the general infinite dimensional case can be used in building new algorithms for solving saddle point systems with no discrete LBB condition assumption for the discrete spaces (see Section 4.1).

Proof. From the first equation of (3.4) and the first equation of Algorithm 4.3 we have

$$\begin{aligned} e_k^{\mathbf{u}} &= e_{k-1}^{\mathbf{u}} - \Psi(Ae_{k-1}^{\mathbf{u}} + B^*e_{k-1}^p) \\ &= (A^{-1} - \Psi)(Ae_{k-1}^{\mathbf{u}} + B^*e_{k-1}^p) - A^{-1}B^*e_{k-1}^p. \end{aligned} \tag{4.5}$$

From the second equation of (3.4) and the second equation of Algorithm 4.3, we get

$$e_k^p = e_{k-1}^p + \alpha \mathcal{C}^{-1}Be_k^{\mathbf{u}}. \tag{4.6}$$

If we substitute $e_k^{\mathbf{u}}$ from (4.5) in (4.6), then

$$e_k^p = \alpha \mathcal{C}^{-1}B(I - \Psi A)(e_{k-1}^{\mathbf{u}} + A^{-1}B^*e_{k-1}^p) + (I - \alpha\mathcal{C}^{-1}BA^{-1}B^*)e_{k-1}^p, \tag{4.7}$$

and

$$A^{-1}B^*e_k^p = \alpha A^{-1}B^*\mathcal{C}^{-1}B(I - \Psi A)(e_{k-1}^{\mathbf{u}} + A^{-1}B^*e_{k-1}^p) + (I - \alpha A^{-1}B^*\mathcal{C}^{-1}B)A^{-1}B^*e_{k-1}^p. \tag{4.8}$$

Thus,

$$e_k^{\mathbf{u}} + A^{-1}B^*e_k^p = (I + \alpha A^{-1}B^*\mathcal{C}^{-1}B)(I - \Psi A)(e_{k-1}^{\mathbf{u}} + A^{-1}B^*e_{k-1}^p) - \alpha A^{-1}B^*\mathcal{C}^{-1}B(A^{-1}B^*)e_{k-1}^p. \tag{4.9}$$

With the notation of Section 2, since $A = \mathcal{A}$, we have $A^{-1}B^*\mathcal{C}^{-1}B = S$, $\mathbf{V}_1 = A^{-1}B^*(Q)$, and that $S_1 : \mathbf{V}_1 \to \mathbf{V}_1$ is the restriction of S to $\mathbf{V}_1$. Here, we introduce two other closely related operators.

Let $S_{12} : \mathbf{V}_1 \to \mathbf{V}$, $S_{12}\mathbf{v}_1 = S\mathbf{v}_1$ and $S_{21} : \mathbf{V} \to \mathbf{V}_1$, $S_{21}\mathbf{v} = S\mathbf{v}$.

Then, from (4.9) and (4.8), we obtain

$$\begin{pmatrix} e_k^{\mathbf{u}} + A^{-1}B^*e_k^p \\ A^{-1}B^*e_k^p \end{pmatrix} = \begin{pmatrix} I + \alpha S & -\alpha S_{12} \\ \alpha S_{21} & I_1 - \alpha S_1 \end{pmatrix} \begin{pmatrix} (I - \Psi A)(e_{k-1}^{\mathbf{u}} + A^{-1}B^*e_{k-1}^p) \\ A^{-1}B^*e_{k-1}^p \end{pmatrix},$$

where I_1 is the identity on $\mathbf{V}_1$. Using just elementary manipulation, we get

$$\begin{pmatrix} \delta^{1/2}\, e_k^{\mathbf{r}} \\ A^{-1}B^*e_k^p \end{pmatrix} = \begin{pmatrix} \delta(I + \alpha S) & \delta^{1/2}\alpha\, S_{12} \\ \delta^{1/2}\alpha S_{21} & -I_1 + \alpha S_1 \end{pmatrix} \begin{pmatrix} \delta^{-1/2}(I - \Psi A)e_{k-1}^{\mathbf{r}} \\ -A^{-1}B^*e_{k-1}^p \end{pmatrix}.$$

Let $\mathbf{V} \times \mathbf{V}_1$ be the Hilbert space with the standard product inner product with $a_0(\cdot, \cdot)$ as inner product on each component. Then,

$$T := \begin{pmatrix} \delta(I + \alpha S) & \delta^{1/2}\alpha\, S_{12} \\ \delta^{1/2}\alpha\, S_{21} & -I_1 + \alpha S_1 \end{pmatrix}$$

is a symmetric operator on $\mathbf{V} \times \mathbf{V}_1$,

$$\left\| \begin{pmatrix} \delta^{1/2}\, e_k^{\mathbf{r}} \\ A^{-1}B^*e_k^p \end{pmatrix} \right\|_{\mathbf{V}\times\mathbf{V}_1}^2 = \delta|e_k^{\mathbf{r}}|_{\mathbf{V}}^2 + |A^{-1}B^*e_k^p|_{\mathbf{V}}^2 = \delta|e_k^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_k^p\|_{S_0}^2,$$

and, using the assumption (4.2),

$$\begin{aligned} \left\| \begin{pmatrix} \delta^{-1/2}(I - \Psi A)e_{k-1}^{\mathbf{r}} \\ -A^{-1}B^*e_{k-1}^p \end{pmatrix} \right\|_{\mathbf{V}\times\mathbf{V}_1}^2 &= \delta^{-1}|(I - \Psi A)e_{k-1}^{\mathbf{r}}|_{\mathbf{V}}^2 + |A^{-1}B^*e_{k-1}^p|_{\mathbf{V}}^2 \\ &\le \delta^{-1}\delta^2|e_{k-1}^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_{k-1}^p\|_{S_0}^2 \\ &= \delta|e_{k-1}^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_{k-1}^p\|_{S_0}^2. \end{aligned}$$

Thus,

$$(\delta|e_k^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_k^p\|_{S_0}^2)^{1/2} \le \rho(T)\ (\delta|e_{k-1}^{\mathbf{r}}|_{\mathbf{V}}^2 + \|e_{k-1}^p\|_{S_0}^2)^{1/2},$$

where $\rho(T)$ is the spectral radius of T. To complete the proof, we have to show that $\rho(T) < 1$ provided that $0 < \alpha < 2/M^2$ and (4.3) holds. First we will prove that any eigenvalue $\rho \in \sigma(T)$ corresponds to a value $\lambda \in [m^2, M^2]$ and a relation between ρ and λ holds, see (4.11). Then, we will prove that the relation remains valid in the general case when $\rho \in \sigma(T)$ and ρ is not necessarily an eigenvalue.

Let $\rho \in \sigma(T)$ be an eigenvalue and let $\binom{x}{y} \in \mathbf{V} \times \mathbf{V}_1$ be a corresponding eigenvector. Then,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} x \\ y \end{pmatrix},$$

which leads to

$$\begin{aligned} &\delta(I + \alpha S)x + \delta^{1/2}\alpha S y = \rho x, \\ &\delta^{1/2}\alpha S x + (-I + \alpha S)y = \rho y. \end{aligned}$$

Equivalently,

$$\begin{aligned} &S(\delta^{1/2}\alpha x + \alpha y) = \delta^{-1/2}(\rho - \delta)x, \\ &S(\delta^{1/2}\alpha x + \alpha y) = (\rho + 1)y. \end{aligned} \tag{4.10}$$

One can easily see from the above system that, if $x \in \mathbf{V}_0$, $x \neq 0$, then $\binom{x}{0}$ is an eigenvector for T corresponding to $\rho = \delta$. Thus, $\delta \in \sigma(T)$. If $\rho \neq \delta$, then from (4.10), we have that

$$x = \frac{\delta^{1/2}(\rho + 1)}{\rho - \delta} y, \quad \text{with } y \neq 0, \quad \text{and} \quad S_1 y = \frac{(\rho - \delta)(\rho + 1)}{\alpha\rho(\delta + 1)} y.$$

From Lemma 2.3 (v), we deduce that

$$\lambda = \frac{(\rho - \delta)(\rho + 1)}{\alpha\rho(\delta + 1)} \in [m^2, M^2]. \tag{4.11}$$

or,

$$\rho^2 - (\alpha\lambda(\delta + 1) + \delta - 1)\rho - \delta = 0, \quad \text{with } \lambda \in [m^2, M^2]. \tag{4.12}$$

Now let $\rho \in \sigma(T)$, $\rho \neq \delta$ be any spectral value.

According to Proposition 2.2, there exists a sequence $\binom{x_n}{y_n} \in \mathbf{V} \times \mathbf{V}_1$, such that

$$\left\| \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\| = 1 \quad \text{and} \quad \left\| T\begin{pmatrix} x_n \\ y_n \end{pmatrix} - \rho \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\| \to 0, \quad \text{as } n \to \infty.$$

This leads to

$$\begin{aligned} &\delta(I + \alpha S)x_n + \delta^{1/2}\alpha S y_n - \rho\, x_n \to 0, \\ &\delta^{1/2}\alpha S x_n + (-I + \alpha S)y_n - \rho\, y_n \to 0. \end{aligned}$$

Equivalently,

$$\begin{aligned} &S(\delta^{1/2}\alpha x_n + \alpha y_n) - \delta^{-1/2}(\rho - \delta)x_n \to 0, \\ &S(\delta^{1/2}\alpha x_n + \alpha y_n) - (\rho + 1)y_n \to 0, \end{aligned} \tag{4.13}$$

which gives the following convergence on $\mathbf{V}$,

$$\delta^{-1/2}(\rho - \delta)x_n - (\rho + 1)y_n \to 0, \tag{4.14}$$

Since $\|(x_n, y_n)^t\|^2 = |x_n|^2 + |y_n|^2 = 1$, and $\rho - \delta \neq 0$, from (4.14) we can claim that there exists a subsequence of $(x_n, y_n)^t$, for convenience still denoted by $(x_n, y_n)^t$, for which

$$|y_n|^2 \geq \alpha_0 > 0. \tag{4.15}$$

Let $\{z_n\} \subset \mathbf{V}$ be defined by

$$z_n := \alpha\delta^{1/2} x_n - \frac{\alpha\delta(\rho + 1)}{\rho - \delta} y_n. \tag{4.16}$$

From (4.14) we have that $z_n \to 0$. Substituting $\alpha\delta^{1/2}x_n$ from (4.16) in the second part of (4.13), leads to

$$S_1\, y_n - \frac{(\rho-\delta)(\rho+1)}{\alpha\rho(\delta+1)}\, y_n \to 0.$$

Here we used the fact that $z_n \to 0$ and $Sy_n = S_1 y_n$. Due to the estimate (4.15), in light of Proposition 2.2, we have that

$$\lambda := \frac{(\rho-\delta)(\rho+1)}{\alpha\rho(\delta+1)} \in \sigma(S_1).$$

From Lemma 2.3(v), we conclude that (4.11) and consequently (4.12) hold for any $\rho \in \sigma(T)$, $\rho \neq \delta$. Next, based on the spectral informations we have, we will compute $\rho(T)$.

Introducing the real functions $f(x) = \frac{1}{2}(x + \sqrt{x^2+4\delta})$ and $g(x) = \frac{1}{2}(x - \sqrt{x^2+4\delta})$, the roots of (4.12) are

$$\rho_1(\lambda) := f(\alpha\lambda(\delta+1)+\delta-1), \qquad \rho_2(\lambda) := g(\alpha\lambda(\delta+1)+\delta-1),$$

and we have

$$\rho(T) \le \max\left\{\delta, \sup_{\lambda\in[m^2,M^2]} |\rho_1(\lambda)|, \sup_{\lambda\in[m^2,M^2]} |\rho_2(\lambda)|\right\}. \tag{4.17}$$

Since f is an increasing and positive function on $\mathbb{R}$, and g is an increasing and negative function on $\mathbb{R}$, we have that

$$\sup_{\lambda\in[m^2,M^2]} |\rho_1(\lambda)| = f(\alpha M^2(\delta+1)+\delta-1),$$

and

$$\sup_{\lambda\in[m^2,M^2]} |\rho_2(\lambda)| = -g(\alpha m^2(\delta+1)+\delta-1).$$

Elementary calculations show that

$$0 < \delta < f(\alpha M^2(\delta+1)+\delta-1).$$

Moreover, for any $\lambda \in \sigma(S_1)$ we have that $\rho_1(\lambda)$, $\rho_2(\lambda)$, the roots of (4.12), belong to $\sigma(T)$. Indeed, if $\lambda \in \sigma(S_1)$ then, by Proposition 2.2, there exists a sequence $(y_n) \subset \mathbf{V}_1$, such that $\|y_n\| = 1$ for all n, and $S_1\, y_n - \lambda\, y_n \to 0$ as $n \to \infty$. If we define

$$x_n = \frac{\delta^{1/2}(\rho+1)}{\rho-\delta}\, y_n,$$

with $\rho = \rho_1(\lambda)$ or $\rho = \rho_2(\lambda)$, then it is easy to check that

$$\left\|\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\| = \beta_0 > 0 \quad \text{and} \quad \left\| T\begin{pmatrix} x_n \\ y_n \end{pmatrix} - \rho\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\| \to 0, \quad \text{as } n \to \infty.$$

In particular, since m^2 and $M^2 \in \sigma(S_1)$, we obtain that $\rho_1(M^2) = f(\alpha M^2(\delta+1)+\delta-1)$ and $\rho_2(m^2) = g(\alpha m^2(\delta+1)+\delta-1)$ belong to $\sigma(T)$. Therefore,

$$\rho(T) = \max\left\{f(\alpha M^2(\delta+1)+\delta-1), -g(\alpha m^2(\delta+1)+\delta-1)\right\}. \tag{4.18}$$

Using the monotonicity of the two functions f and g, it is easy to verify the following:

$$0 < -g(\alpha m^2(\delta+1)+\delta-1) < 1, \quad \text{for all } \alpha > 0,$$

$$0 < f(\alpha M^2(\delta+1)+\delta-1) < 1, \quad \text{iff } \delta < \frac{2-\alpha M^2}{2+\alpha M^2},$$

$\rho(T)(\alpha)$ is optimal (minimal) for

$$\alpha_{opt} := \frac{1-\delta}{1+\delta}\,\frac{2}{m^2+M^2},$$

$$\rho(T) = -g(\alpha m^2(\delta+1)+\delta-1), \quad \text{for } 0 < \alpha \le \alpha_{opt},$$

and

$$\rho(T) = f(\alpha M^2(\delta+1)+\delta-1), \quad \text{for } \alpha_{opt} \le \alpha < \frac{2}{M^2}.$$

This completes the proof of the theorem. □

Remark 4.5. For $\delta \to 0$, we "recover" the convergence results of UM.

(i) For $\alpha = 1/M^2$ and $\delta \to 0$, we have that $\rho(T)(\delta) \to 1 - \frac{m^2}{M^2}$, (see part (iii) of Theorem 4.2).

(ii) For $\alpha = \alpha_{opt} = \frac{1-\delta}{1+\delta}\,\frac{2}{m^2+M^2}$ and $\delta \to 0$, we have that $\rho(T)(\delta) \to \frac{M^2-m^2}{M^2+m^2}$, (see part (iv) of Theorem 4.2).

The IUM can be applied in particular when $\mathbf{V}$ and Q are finite dimensional spaces. In this case, Ψ can be taken to be a preconditioner for $\mathcal{A}$, and $\mathcal{C}$ can also be replaced with any symmetric and positive definite operator on Q. In particular, $\mathcal{C}$ can be associated with a preconditioner on Q. In this way, by the above theorem we can recover or improve convergence results presented in [8,5,11,16].

4.1. A bridge to adaptive methods

The value of the above theorem resides also in the possibility of solving a saddle point problem by combining the IUM algorithm at the continuous level with standard adaptive methods. The main idea is to build an iterative process of inexact Uzawa type, where the $\mathbf{u}$ variable is updated by solving *adaptively* (on larger and larger subspaces of $\mathbf{V}$) a simple *elliptic, symmetric and positive definite* problem, while the second variable p is updated according to the standard Uzawa algorithm. The main advantage of such a process is that only discrete subspaces of $\mathbf{V}$ play a major role in the algorithm and compatibility conditions for discrete subspaces of $\mathbf{V}$ and Q are not required. To be more precise, we consider that (3.3) is the variational formulation of a boundary value problem on a fixed domain Ω. The Algorithm 4.3 can be used in the following *adaptive way* to approximate the solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$.

First, for a fix $\alpha \in (0, 2/M^2)$ we choose a positive number δ strictly smaller than the computable value $\frac{2-\alpha M^2}{2-\alpha M^2}$ and let $\mathbf{V}_0 \subset \mathbf{V}$ be a finite dimensional space, e.g., the space of continuous piecewise polynomials of certain degree with respect to a given coarse partition $\mathcal{T}_0$ of Ω. Take $\mathbf{u}_0 \in \mathbf{V}_0$ be any approximation of $\mathbf{u}$ and take $p_0 = 0$.

Next, for $k = 1, 2, \ldots$, assuming that $(\mathbf{u}_{k-1}, p_{k-1})$ is determined such that $\mathbf{u}_{k-1} \in \mathbf{V}_{k-1} \subset \mathbf{V}$, to determine $(\mathbf{u}_k, p_k)$ as defined in Algorithm 4.3, we find first $\Psi(\mathbf{f} - A\mathbf{u}_{k-1} - B^* p_{k-1}) = \Psi(r_{k-1}) := \mathbf{w}_k$ as the discrete solution of the following *elliptic, symmetric and positive definite* problem:

$$a(\mathbf{w}_k, \mathbf{v}) = \langle r_{k-1}, v\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}_k. \tag{4.19}$$

The space $\mathbf{V}_k$ will be chosen by using an *adaptive process* such that $\mathbf{V}_{k-1} \subset \mathbf{V}_k \subset \mathbf{V}$, and

$$|\mathbf{w}_k - A^{-1} r_{k-1}|_{\mathbf{V}} \le \delta |r_{k-1}|_{\mathbf{V}^*}. \tag{4.20}$$

More precisely, we let η_k be a computable a posteriori error estimator for the problem (4.19), i.e,

$$|\mathbf{w}_k - A^{-1} r_{k-1}|_{\mathbf{V}} \le \eta_k. \tag{4.21}$$

We assume that η_k is a sum of local error estimators associated with the mesh which defines the space $\mathbf{V}_k$. Since the form $a(\cdot,\cdot)$ in (4.19) gives the inner product on $\mathbf{V}$, and $\mathbf{V}_k$ is a subspace of $\mathbf{V}$, we have

$$|\mathbf{w}_k|_{\mathbf{V}} \le |r_{k-1}|_{\mathbf{V}^*}. \tag{4.22}$$

Thus, to satisfy the sufficient condition (4.20), would be enough to verify that $\eta_k \le \delta |\mathbf{w}_k|_{\mathbf{V}}$, where both η_k and $|\mathbf{w}_k|_{\mathbf{V}}$ are computable quantities. Consequently, at the kth iteration of the algorithm we start by taking $\mathbf{V}_k = \mathbf{V}_{k-1}$ and by solving (4.19) with $\mathbf{V}_k = \mathbf{V}_{k-1}$. If $\eta_k \le \delta|\mathbf{w}_k|_{\mathbf{V}}$, then we compute $\mathbf{u}_k = \mathbf{u}_{k-1} + \mathbf{w}_k$, $p_k = p_{k-1} + \alpha\mathcal{C}^{-1}(B\mathbf{u}_k - g)$, and let $k \to k+1$ (move to the next iteration). If $\eta_k > \delta|\mathbf{w}_k|_{\mathbf{V}}$, then we refine $\mathcal{T}_{k-1}$ according to the information provided by the local error estimators defining η_k and obtain a new space $\mathbf{V}_k$. Then, we solve again (4.19) and verify the validity of $\eta_k \le \delta|\mathbf{w}_k|_{\mathbf{V}}$. The process of refining and solving on a larger space repeats until $\mathbf{V}_k$ is large enough to assure that the sufficient condition $\eta_k \le \delta|\mathbf{w}_k|_{\mathbf{V}}$ is satisfied. Under a minimal regularity assumption for the problem of solving or approximating $A^{-1}r_{k-1}$, we can prove that the process at each step ends, because r_{k-1} in (4.19) is fixed and the spaces $\mathbf{V}_k$'s are allowed to become better and better approximation spaces for $\mathbf{V}$. We can stop the algorithm after a fixed number of iterations given by the rate of convergence of **IUM** or after $|\mathbf{w}_k|_{\mathbf{V}}$ is smaller than a fixed tolerance ϵ.

Let us note that if the update of the p variable ($p_k = p_{k-1} + \alpha\mathcal{C}^{-1}(B\mathbf{u}_k - g)$) can be done at the continuous level, in particular if $\mathcal{C}$ is the identity operator (which is the case for $Q = L^2$), then a sequence of spaces for the p variable is not even needed to be defined in implementing the above algorithm. In any case a discrete LBB condition is not required.

Similar approaches on combining Uzawa algorithms at the continuous level with standard techniques of discretization and a posteriori error estimates can be found in [3,4,12]. Bansch, Morin and Nochetto (see [4]) used a similar adaptive inexact Uzawa algorithm for the Stokes problem and proved a convergence result. Nevertheless, estimates (in terms of the constants α, m, M of the Stokes system) for the amount of inexactness of the approximate inverse or for the convergence factor of the algorithm are not provided in their paper. The precise convergence analysis of the **IUM** algorithm at the continuous level brings more clarity in implementing, and analyzing such combined algorithms.

5. Conclusion

Based on Schur operators on Hilbert spaces, the paper provides new tools in analyzing saddle-point problems. In the author's opinion, the use of the Schur complements in the infinite dimensional case can recover powerful results proved in the finite dimensional setting by means of spectral properties of matrices. As an example, the Inexact Uzawa algorithm at the abstract general level was efficiently analyzed. We proved that for any symmetric saddle point problem, the algorithm converges provided that the inexact process for inverting the residual at each step has the relative error smaller than any fixed number smaller than a computable threshold. The result was known for particular cases and only in the finite dimensional setting. The convergence result for the algorithm at the continuous level, combined with standard techniques of discretization and a posteriori error estimates [4,20,23], could lead to new and efficient algorithms for solving saddle point systems. New applications of the Schur complements, including sharp estimates for Arrow–Hurwicz algorithms for non-symmetric saddle point systems are the focus of the author's work in progress.

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Appendix

In this appendix, we present more of the classical theory for saddle point systems in light of the spectral results of the Schur complements (Lemma 2.3). The first application is the Babušca Lemma (as described in [2]) enriched with Schur stability estimates.

Lemma A.1 (*Babušca*). *Let $b : \mathbf{V} \times Q \to \mathbb{R}$ be a bilinear form satisfying (2.3) and (2.4), and let $F \in \mathbf{V}^*$.*

(i) *The problem: Find $p \in Q$ such that*

$$b(\mathbf{v}, p) = \langle F, \mathbf{v}\rangle, \quad \text{for all } \mathbf{v} \in \mathbf{V} \tag{A.1}$$

has a unique solution if and only if

$$\langle F, \mathbf{v}\rangle = 0, \quad \text{for all } \mathbf{v} \in \mathbf{V}_0. \tag{A.2}$$

If (A.2) holds and p is the solution of (A.1), then

$$m\|p\| \le \|p\|_{S_0} = \|F\|_{\mathbf{V}^*} = \|F\|_{\mathbf{V}_1^*} \le M\|p\|, \quad \text{and} \quad \|p\| = |\mathcal{A}^{-1}F|_{S_1^{-1}}. \tag{A.3}$$

(ii) *Let $F \in \mathbf{V}_1^*$. The problem: Find $p \in Q$ such that*

$$b(\mathbf{v}, p) = \langle F, \mathbf{v}\rangle, \ \mathbf{v} \in \mathbf{V}_1 \tag{A.4}$$

has a unique solution p, and

$$\|p\|_{S_0} = \|F\|_{\mathbf{V}_1^*}. \tag{A.5}$$

(iii) *Assume that the form b, in addition, satisfies the condition*

$$b(\mathbf{v}, p) = 0 \quad \text{for all } p \in Q \text{ implies } \mathbf{v} = 0. \tag{A.6}$$

Then, the problem (A.1) has a unique solution p which satisfies (A.3).

This is a classical result. The improvement brought by the Schur complement approach is the isometric correspondence between the functionals in the subspace $\mathbf{V}_1^*$ of $\mathbf{V}^*$ and the solutions space Q described by (A.3).

Proof. (i) The problem (A.1) reduces to finding $p \in Q$ such that $B^*p = F$ or $\mathcal{A}^{-1}B^*p = \mathcal{A}^{-1}F$. By Lemma 2.3, this is equivalent to $\mathcal{A}^{-1}F \in \mathbf{V}_1 = \mathbf{V}_0^{\perp}$, i.e,

$$a_0(A^{-1}F, \mathbf{v}) = 0, \quad \text{for all } \mathbf{v} \in \mathbf{V}_0,$$

which is exactly the condition (A.2). By part (iii) of Lemma 2.3, we have

$$m\|p\| \le \|p\|_{S_0} = |\mathcal{A}^{-1}B^*p|_{\mathbf{V}} = |\mathcal{A}^{-1}F|_{\mathbf{V}} = \|F\|_{\mathbf{V}^*} = \|F\|_{\mathbf{V}_1^*}.$$

The second part of (A.3) follows from (2.13).

(ii) If F is defined on $\mathbf{V}_1$ only, we can extend F to the entire $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1$ by defining F to be zero on $\mathbf{V}_0$. The extension has the same norm, and we can apply part (i).

(iii) The condition (A.6) implies $V_0 = \{0\}$. Thus, (A.2) holds trivially. The result follows from part (i). □

Remark A.2. Many boundary value problems can be written in the form presented in Lemma A.1 (i). In [6], Bramble and Pasciak introduced a new method for solving the general problem presented in Lemma A.1 (i) based on Riesz representation operators and on reduction to elliptic problems. In light of Lemma 2.3, the Bramble–Pasciak method can be described as follows.

From part (iii) of Lemma 2.3, we have that $\mathbf{V}_1 = \mathcal{A}^{-1}B^*(Q) = \mathbf{V}_0^{\perp}$. Thus, under the assumption (A.2), the problem: Find $p \in Q$ such that (A.1) holds, is equivalent to: Find $p \in Q$ such that

$$b(\mathbf{v}, p) = \langle F, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}_1 = \mathcal{A}^{-1}B^*(Q).$$

or to: Find $p \in Q$ such that

$$(S_0 p, q) = b(\mathcal{A}^{-1}B^*q, p) = \langle F, \mathcal{A}^{-1}B^*q\rangle \quad \text{for all } q \in Q. \tag{A.7}$$

Since S_0 is a symmetric and positive definite operator, once the action of $\mathcal{A}^{-1}$ is available, (A.7) becomes a symmetric and positive definite problem. More details about discretizing (A.7) in general and for solving concrete div–curl systems in particular, can be found in [6].

The proof of Theorem 3.1. First, let us assume the existence of a solution $(\mathbf{u}, p)$ of (3.3), and let us consider the unique decomposition $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$ with $\mathbf{u}_0 \in \mathbf{V}_0$ and $\mathbf{u}_1 \in \mathbf{V}_1$. Then, $\mathbf{u}_1 = \mathcal{A}^{-1}B^*p_1$ with $p_1 \in Q$ and $(\mathbf{u}_1, p_1)$ satisfies

$$\begin{aligned} &\mathcal{A}\mathbf{u}_1 + B^*(-p_1) = 0,\\ &B\mathbf{u}_1 = g, \end{aligned} \tag{A.8}$$

or equivalently,

$$\begin{aligned} &a_0(\mathbf{u}_1, \mathbf{v}) - b(\mathbf{v}, p_1) = 0, \quad \text{for all } \mathbf{v} \in \mathbf{V}, (\text{ or } \mathbf{V}_1),\\ &b(\mathbf{u}_1, q) = \langle g, q\rangle \quad \text{for all } q \in Q. \end{aligned} \tag{A.9}$$

From (3.3), we have that $\mathbf{u}_0$ satisfies

$$a(\mathbf{u}_0, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle - a(\mathbf{u}_1, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{V}_0, \tag{A.10}$$

which is equivalent to

$$\langle \mathbf{f} - A\mathbf{u}, \mathbf{v}\rangle = 0 \quad \text{for all } \mathbf{v} \in \mathbf{V}_0. \tag{A.11}$$

Now, the system (A.8) is equivalent to

$$\begin{aligned} &\mathcal{A}\mathbf{u}_1 = B^*p_1,\\ &\mathcal{C}^{-1}B\mathcal{A}^{-1}B^*p_1 = \mathcal{C}^{-1}g. \end{aligned} \tag{A.12}$$

Since $S_0 : Q \to Q$ and $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}_1$ are isomorphisms (by part (ii) and (iii) of Lemma 2.3), (A.12) has a unique solution depending only on g. Hence, $\mathbf{u}_1$ is unique. Also, by Lax–Milgram lemma, we have that u_0 solving (A.10) is unique, and consequently, $\mathbf{u}$ is unique. From the first equation of (3.4), we have that $B^*p = \mathbf{f} - A\mathbf{u}$. Using that (A.11) holds, from Lemma A.1, we get that p is also unique. To prove the existence, we just notice that if we define $(\mathbf{u}_1, p_1)$ as the unique solution of (A.9), and $\mathbf{u}_0$ as the unique solution of (A.10), then $(\mathbf{u}, p)$ is a solution for (3.3), where $\mathbf{u} := \mathbf{u}_0 + \mathbf{u}_1$ and p is uniquely defined via Lemma A.1, by $B^*p = \mathbf{f} - A\mathbf{u}$. Next, we estimate the norm of the unique solution $(\mathbf{u}, p)$ in terms of $(\mathbf{f}, g)$. From (2.8) and (A.12), we have

$$|\mathbf{u}_1| = |\mathcal{A}^{-1}B^*p_1| = \|p_1\|_{S_0} = \|\mathcal{C}^{-1}g\|_{S_0^{-1}} \le \frac{1}{m}\|g\|_{Q^*}, \tag{A.13}$$

which proves the first part of (3.5). The second part of (3.5) follows immediately from the second equation of (2.11) and (A.8). Using the Lax–Milgram for (A.10), we obtain

$$|\mathbf{u}_0| \le \frac{1}{m_0}\|\mathbf{f} - A\mathbf{u}_1\|_{\mathbf{V}_0^*} \le \frac{1}{m_0}(\|\mathbf{f}\|_{\mathbf{V}_0^*} + M_0|\mathbf{u}_1|) \le \frac{1}{m_0}\left(\|\mathbf{f}\|_{\mathbf{V}_0^*} + \frac{M_0}{m}\|g\|_{Q^*}\right). \tag{A.14}$$

To obtain (3.7), we notice that $B^*p = \mathbf{f} - A\mathbf{u}$ and by Lemma A.1 we have

$$m\|p\| \le \|p\|_{S_0} = \|\mathbf{f} - A\mathbf{u}\|_{\mathbf{V}_1^*} \le \|\mathbf{f}\|_{\mathbf{V}_1^*} + M_0|\mathbf{u}|. \tag{A.15}$$

Combining (A.13)–(A.15), and using $|\mathbf{u}| \le |\mathbf{u}_0| + |\mathbf{u}_1|$, we arrive at (3.8). The estimates can be improved if we use that $|\mathbf{u}|^2 = |\mathbf{u}_0|^2 + |\mathbf{u}_1|^2$. $\square$

Remark A.3. Special case: When the form $a(\cdot,\cdot)$ coincides with the form $a_0(\cdot,\cdot)$ which gives the inner product on $\mathbf{V}$, we have $m_0 = M_0 = 1$, and $A = \mathcal{A}$. Thus, the estimates (3.8) can be made more precise by using Schur norms. Indeed, in this case, (A.10) becomes

$$a_0(\mathbf{u}_0, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}_0. \tag{A.16}$$

Thus,

$$|\mathbf{u}_0| = \|\mathbf{f}\|_{\mathbf{V}_0^*}. \tag{A.17}$$

On the other hand,

$$\|p\|_{S_0} = \|\mathbf{f} - \mathcal{A}\mathbf{u}\|_{\mathbf{V}_1^*} = \|\mathbf{f} - \mathcal{A}\mathbf{u}_1\|_{\mathbf{V}_1^*} = \|\mathbf{f} - B^* p_1\|_{\mathbf{V}_1^*} = \|\mathbf{f} - B^* S_0^{-1}\mathcal{C}^{-1} g\|_{\mathbf{V}_1^*}.$$

Then, the solution $(\mathbf{u}, p)$ satisfies

$$\begin{aligned} &|\mathbf{u}|^2 = \|\mathbf{f}\|^2_{\mathbf{V}_0^*} + \|\mathcal{C}^{-1} g\|^2_{S_0^{-1}}, \qquad |\mathbf{u}_0|^2 + |\mathbf{u}_1|^2_{S_1} = \|\mathbf{f}\|^2_{\mathbf{V}_0^*} + \|g\|^2_{Q^*},\\ &\|p\|_{S_0} = \|\mathbf{f} - \mathcal{A}\mathbf{u}_1\|_{\mathbf{V}_1^*} = \|\mathbf{f} - B^* S_0^{-1}\mathcal{C}^{-1} g\|_{\mathbf{V}_1^*}. \end{aligned} \tag{A.18}$$

By using the spectral properties of S_0 and the estimate

$$\|\mathbf{f} - \mathcal{A}\mathbf{u}_1\|_{\mathbf{V}_1^*} \le \|\mathbf{f}\|_{\mathbf{V}_1^*} + |\mathbf{u}_1|,$$

we get

$$\begin{aligned} &|\mathbf{u}|^2 \le \|\mathbf{f}\|^2_{\mathbf{V}_0^*} + \frac{1}{m^2}\|g\|^2_{Q^*}\\ &\|p\|_{S_0} \le \|\mathbf{f}\|_{\mathbf{V}_1^*} + \frac{1}{m}\|g\|_{Q^*}. \end{aligned} \tag{A.19}$$

Remark A.4. Let $b : \mathbf{V}\times Q \to \mathbb{R}$ be a bilinear form satisfying (2.3) and (2.4). Given $\mathbf{u} \in \mathbf{V}$, according to the abstract Helmholtz decomposition of Lemma 2.4, there exists $\mathbf{u}_0, \mathbf{u}_1 \in \mathbf{V}$ and $p \in Q$ such that $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, $B\mathbf{u}_0 = 0$, and $\mathbf{u}_1 = \mathcal{A}^{-1}B^* p$. We can effectively find $\mathbf{u}_0$, $\mathbf{u}_1$ and p by first solving for $(\mathbf{u}_1, -p)$ the solution of the symmetric saddle point problem

$$\begin{aligned} &a_0(\mathbf{u}_1, \mathbf{v}) + b(\mathbf{v}, -p) = 0, \quad \text{for all } \mathbf{v} \in \mathbf{V},\\ &b(\mathbf{u}_1, q) = b(\mathbf{u}, q), \quad \text{for all } q \in Q, \end{aligned}$$

and then, take $\mathbf{u}_0 = \mathbf{u} - \mathbf{u}_1$.

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