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On the Green's function for the Helmholtz operator in an impedance circular cylindrical waveguide

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ABSTRACT

This paper addresses the problem of finding a series representation for the Green's function of the Helmholtz operator in an infinite circular cylindrical waveguide with impedance boundary condition. Resorting to the Fourier transform, complex analysis techniques and the limiting absorption principle (when the undamped case is analyzed), a detailed deduction of the Green's function is performed, generalizing the results available in the literature for the case of a complex impedance parameter. Procedures to obtain numerical values of the Green's function are also developed in this article.

1. Introduction

The impedance boundary value problem for the Helmholtz operator, sometimes called the Robin boundary value problem, has been used extensively in physics and engineering, due to its wide range of applications. It plays an important role in the fields of acoustics, where it models time-harmonic wave propagation in domains with locally reacting surfaces [1], and electromagnetism, where it is employed to model time-harmonic electromagnetic wave propagation when the impedance boundary condition represents the finite conductivity of the materials, the roughness effect on the surface conductivity and, in general, energy loss through the domain boundaries [2]. In the case of the particular domain considered in this paper – the circular cylindrical waveguide – it is worth mentioning some engineering applications of this problem. For instance, according to Rawlins [3], it arises in electromagnetic communications throughout underground tunnels, in the propagation of waves in fibre-optics waveguides and in the design of mufflers, exhaust and ventilation systems [4].

A powerful mathematical tool to solve impedance boundary value problems for the Helmholtz operator, is the Green's function. It is often employed in wave scattering, resonance and inverse problems defined on bounded and unbounded domains, where it is used as a benchmark solution to test numerical schemes, or it is applied in conjunction with numerical methods such as the boundary element method (BEM) and the mixed boundary element and finite element method (BEM/FEM). For a broader framework about the Green's functions and their use for solving time-harmonic problems, see [5–7].

For the infinite cylindrical waveguide with a Dirichlet or Neumann boundary condition, the Green's function can be obtained by a series provided by the method of separation of variables (see for instance [8] and [9, Section 5.6] for examples of a Neumann and a Dirichlet boundary condition respectively) usually referred to as the eigenfunction expansion. This representation is highly appreciated because it is easy to implement on a computer as it does not require that the Fourier transform be numerically inverted as other Green's functions do in problems arising, for example, on impedance half-planes and half-spaces [10,11]. Moreover, such a representation is suitable to achieve directly the asymptotic behavior at infinity and the radiation condition that the Green's function satisfies, which differs from the classical Sommerfeld radiation condition.

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The separation of variables method works well when the Laplace operator, acting on a subspace of $L^2(\Omega')$, where $\Omega' = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < R\}$ is the waveguide's cross section, is compact and self-adjoint, and consequently its eigenfunctions form a complete orthonormal basis of $L^2(\Omega')$, which is the case when the boundary condition is Dirichlet or Neumann. However, when the impedance boundary condition is imposed, the Laplacian is no longer self-adjoint for all the possible values of the impedance parameter (it is self-adjoint only for negative real values of the impedance parameter). In spite of that, a real impedance parameter leads to eigenfunctions that are still orthogonal and the theory of Dini series [12, chapter 18] may be used in order to apply the separation of variables method to our problem. If instead, the impedance parameter is a proper complex number (i.e. it has non-zero imaginary part), the eigenfunctions are no longer orthogonal and then it is necessary to develop another way to construct the Green's function that does not depend on the orthogonality and completeness of the eigenfunctions.

In this paper we perform a detailed deduction of the eigenfunction expansion of the Green's function for the Helmholtz operator in an infinite impedance circular cylindrical waveguide. This deduction is carried out via a Fourier transform and the theory of Green's functions for non-self-adjoint singular Sturm–Liouville problems. The eigenfunction expansion makes it necessary to study the eigenvalues of the Laplacian generated by the roots of the Dini function $zJ_n'(zR) - \alpha J_n(zR)$, where α is the impedance parameter and $J_n(z)$ is the Bessel function of the first kind with integer order $n \geqslant 0$. Properties of these roots for $\alpha \in \mathbb{R}$, available in [13–15], are used here. Also, new results for the properties of these roots are performed for complex values of α by means of a Mittag–Leffler expansion. Existence of non-simple eigenvalues are reported for particular values of α; some of them are computed using the Delves–Lyness algorithm [16,17]. The limiting absorption principle is employed to analyze the complete undamped case (i.e. real wave number and real impedance parameter), where the existence of propagative modes (which include surface propagative modes) allows one to obtain the far field and the radiation condition that the Green's function satisfies. Finally, basic numerical procedures based on the FEM are stated in order to obtain numerical evaluations of the Green's function.

The structure of this paper is as follows. The next section describes both the problem's set up and the Fourier transform used to compute the Green's function. The spectral Green's function – solution of the problem in the Fourier domain – is also found in this section. In Section 3, the singularities of the spectral Green's function are studied in detail so as to obtain, in Section 4, its series expansion by means of contour integration. Thereafter, Section 5 addresses the inversion of the Fourier transform, using again contour integration and the limiting absorption principle. Finally, Section 6, develops the numerical procedures needed to compute the poles of the spectral Green's function so as to achieve numerical evaluations of the Green's function.

2. The Green's function of the infinite right circular cylinder

The sought Green's function is given by the solution of the following problem in the sense of distributions: Find $G(\boldsymbol{x}, \cdot) \in \mathcal{D}'(\Omega)$ such that:

$$\begin{cases} \Delta_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) + k^2 G(\boldsymbol{x},\boldsymbol{y}) = -\delta_{\boldsymbol{x}}(\boldsymbol{y}), & \boldsymbol{y} \in \Omega, \\ \dfrac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n_{\boldsymbol{y}}} - \alpha G(\boldsymbol{x},\boldsymbol{y}) = 0, & \boldsymbol{y} \in \Gamma, \end{cases} \tag{1}$$

where

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < R^2\right\} \times \mathbb{R}$$

$$\Gamma = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = R^2\right\} \times \mathbb{R}$$

and $\delta_{\boldsymbol{x}} \in \mathcal{D}'(\Omega)$ is the Dirac delta distribution supported at a fixed point $\boldsymbol{x} \in \Omega$ where $\mathcal{D}'(\Omega)$ denotes the dual space of $\mathcal{D}(\Omega)$, i.e., the dual of the space of functions infinitely differentiable and compactly supported in Ω.

Due to physical considerations (cf. [1,2]) the wave number and the impedance parameter are respectively $k \in \mathbb{C}$, $\Im\mathfrak{m}\, k^2 \geqslant 0$ and $\alpha \in \mathbb{C}$, $\Im\mathfrak{m}\, \alpha \geqslant 0$, because the time dependence of the wave is assumed given by $\mathrm{e}^{-\mathrm{i}\omega t}$, where ω is the angular frequency. In absence of dissipation, i.e., when α and k are real numbers, we have to add a radiation condition to (1) in order to obtain a unique outgoing wave solution. Such radiation condition differs greatly from the classical Sommerfeld radiation condition and will be stated later on. Then, for the time being, we assume that k or α are not real numbers.

To achieve a solution of (1), we use the Fourier transform. Hence, first we express (1) in cylindrical coordinates (see Fig. 1) to obtain

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial G}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 G}{\partial \theta^2} + \dfrac{\partial^2 G}{\partial z^2} + k^2 G = -\dfrac{\delta_\rho(r)\,\delta_\zeta(z)\,\delta_\vartheta(\theta)}{r}, & \boldsymbol{y} \in \Omega, \\ \dfrac{\partial G}{\partial r} - \alpha G = 0, & \boldsymbol{y} \in \Gamma, \end{cases} \tag{2}$$

where $\rho \in (0, R)$, $\boldsymbol{y} = (r\cos\theta, r\sin\theta, z)$ is the observation point and $\boldsymbol{x} = (\rho\cos\vartheta, \rho\sin\vartheta, \zeta)$ is the source point.

Then we define a special Fourier transform in the following way: Let $u \in \mathcal{D}((-\pi,\pi) \times \mathbb{R})$, then its Fourier transform $\widehat{u} : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{C}$ is defined by

$$\widehat{u}(n,\xi) = \int_{[-\pi,\pi]\times\mathbb{R}} u(\theta, z)\cos(n\theta)\,\mathrm{e}^{-\mathrm{i}z\xi}\,\mathrm{d}\xi\,\mathrm{d}\theta$$

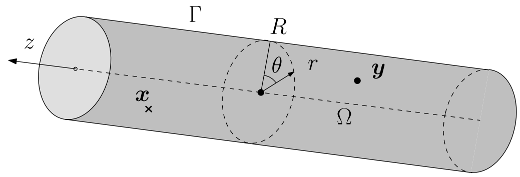

Fig. 1. Domain's geometry.

and its inverse is defined as

$$u(\theta, z) = \frac{1}{4\pi^2}\int_{\mathbb{N}_0\times\mathbb{R}} \widehat{u}(n,\xi)\cos(n\theta)\,\mathrm{e}^{\mathrm{i}z\xi}\epsilon_n\,\mathrm{d}\xi\,\mathrm{d}n$$

where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and

$$\epsilon_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n \geqslant 1. \end{cases}$$

Assuming that for every fixed $\boldsymbol{x} \in \Omega$ and $r \in [0, R]$, $G \in \mathcal{D}'((-\pi,\pi)\times\mathbb{R})$, we can apply the former Fourier transform to (2) to obtain the following integral representation (which, for the time being, is interpreted in the sense of distributions)

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi^2}\int_{\mathbb{N}_0\times\mathbb{R}} g_n(r,\rho;k^2-\xi^2)\cos(n(\theta-\vartheta))\,\mathrm{e}^{\mathrm{i}\xi(z-\zeta)}\epsilon_n\,\mathrm{d}\xi\,\mathrm{d}n \tag{3}$$

where the kernel g_n is referred to as the spectral Green's function and corresponds to the Green's function of the Bessel differential equation with an impedance boundary condition, i.e., the solution of

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\dfrac{\mathrm{d}g_n}{\mathrm{d}r}\right) + \dfrac{n^2}{r}g_n - \lambda r g_n = \delta_\rho, & 0 < r < R \\ \dfrac{\mathrm{d}g_n}{\mathrm{d}r} - \alpha g_n = 0 & r = R \\ \lim\limits_{r\to 0^+} |g_n(r,\rho;\lambda)| < \infty, \end{cases} \tag{4}$$

where $\lambda = k^2 - \xi^2$, $\rho \in (0, R)$ and $n \in \mathbb{N}_0$.

The solution of (4) can be expressed in terms of J_n and Y_n, i.e., the Bessel functions of the first and second kind respectively, with order $n \in \mathbb{N}_0$. Thus, resorting to use standard techniques to find one-dimensional Green's functions (cf. [9]), we obtain that

$$g_n(r,\rho;\lambda) = -\frac{\varphi_1(r_<;\lambda)\varphi_2(r_>;\lambda)}{\rho W[\varphi_1,\varphi_2](\rho)}, \tag{5}$$

where $r_< = \min\{r,\rho\}$, $r_> = \max\{r,\rho\}$,

$$\varphi_1(r;\lambda) = J_n\left(\sqrt{\lambda}r\right), \tag{6a}$$

$$\varphi_2(r;\lambda) = \left[\sqrt{\lambda}J_n'\left(\sqrt{\lambda}R\right) - \alpha J_n\left(\sqrt{\lambda}R\right)\right]Y_n\left(\sqrt{\lambda}r\right) - \left[\sqrt{\lambda}Y_n'\left(\sqrt{\lambda}R\right) - \alpha Y_n\left(\sqrt{\lambda}R\right)\right]J_n\left(\sqrt{\lambda}r\right), \tag{6b}$$

and W is the Wronskian function given by

$$W[\varphi_1,\varphi_2](r) = \frac{2}{\pi r}\left[\sqrt{\lambda}J_n'\left(\sqrt{\lambda}R\right) - \alpha J_n\left(\sqrt{\lambda}R\right)\right]. \tag{7}$$

It is easy to see from (5) that the spectral Green's function can be alternatively written as

$$\begin{aligned} g_n(r,\rho;\lambda) = {} & \frac{\pi}{2}\frac{\sqrt{\lambda}Y_n'\left(\sqrt{\lambda}R\right) - \alpha Y_n\left(\sqrt{\lambda}R\right)}{\sqrt{\lambda}J_n'\left(\sqrt{\lambda}R\right) - \alpha J_n\left(\sqrt{\lambda}R\right)}J_n\left(\sqrt{\lambda}r\right)J_n\left(\sqrt{\lambda}\rho\right) \\ & -\frac{\pi}{2}\begin{cases} Y_n\left(\sqrt{\lambda}\rho\right)J_n\left(\sqrt{\lambda}r\right) & \text{if } 0 < r \leqslant \rho < R, \\ Y_n\left(\sqrt{\lambda}r\right)J_n\left(\sqrt{\lambda}\rho\right) & \text{if } 0 < \rho \leqslant r < R. \end{cases} \end{aligned} \tag{8}$$

Before continuing, there is a need to clarify how the square root in the complex plane will be understood. We define it as the complex map

$$z \mapsto \sqrt{z}, \quad -\pi < \arg z \leqslant \pi \tag{9}$$

and then, its range is $-\pi/2 < \arg\sqrt{z} \leqslant \pi/2$. Let us observe that under our definition of the square root (9), it does not contain the branch cut of Y_n and Y_n', and then $\varphi_2(r;\cdot)$ is analytic in $\{z \in \mathbb{C} : -\pi < \arg z \leqslant \pi\}$.

3. The singularities of the spectral Green's function

Now we shall study the singularities of g_n as a function of λ. There are two sets of candidates to be a singularity of the spectral Green's function. The first one corresponds to the origin in the λ-complex-plane due to the behavior of the Bessel functions of the second kind near the origin; and the second set corresponds to the solutions of $\sqrt{\lambda}J_n'(\lambda R)-\alpha J_n\left(\sqrt{\lambda}R\right)=0$. With respect to the origin we can obtain the following result:

Proposition 3.1. *If $\alpha \neq n/R$ the point $\lambda = 0$ is a removable singularity of $g_n(r,\rho;\cdot)$ while if $\alpha = n/R$ the point $\lambda = 0$ is a simple pole and the residue is given by*

$$\operatorname*{Res}_{\lambda=0} g_n(r,\rho;\lambda) = -\frac{2(n+1)}{R^2}\left(\frac{r}{R}\right)^n\left(\frac{\rho}{R}\right)^n$$

for all $n \in \mathbb{N}_0$, $\rho \in (0,R)$, and $r \in [0,R]$.

Proof. Assume $\alpha \neq n/R$. Hence, by using the recurrence formulas for the Bessel functions and replacing the resultant functions by their asymptotic form for small arguments (cf. [18]), we can compute the limit directly to get

$$\lim_{\lambda\to 0} g_n(r,\rho;\lambda) = \frac{1}{2n}\left(\frac{\rho}{R}\right)^n\left(\frac{r}{R}\right)^n\left(\frac{n+R\alpha}{n-R\alpha}\right)+\frac{1}{2n}\begin{cases}\left(\dfrac{r}{\rho}\right)^n & \text{if } 0<r\leqslant\rho<R,\\ \left(\dfrac{\rho}{r}\right)^n & \text{if } 0<\rho\leqslant r<R,\end{cases} \tag{10}$$

when $n > 0$ and

$$\lim_{\lambda\to 0} g_0(r,\rho;\lambda) = -\frac{1}{R\alpha}-\begin{cases}\ln\left(\dfrac{\rho}{R}\right) & \text{if } 0<r\leqslant\rho<R,\\ \ln\left(\dfrac{r}{R}\right) & \text{if } 0<\rho\leqslant r<R,\end{cases} \tag{11}$$

when $n = 0$. So, as $\rho > 0$, we conclude that the origin $\lambda = 0$ is a removable singularity of g_n for all $n \in \mathbb{N}_0$.

On the other hand, from the previous step we know that

$$\begin{aligned}\sqrt{\lambda}Y_n'\left(\sqrt{\lambda}R\right)-\alpha Y_n\left(\sqrt{\lambda}R\right) &= \left(\frac{n}{R}-\alpha\right)Y_n\left(\sqrt{\lambda}R\right)-\sqrt{\lambda}Y_{n+1}\left(\sqrt{\lambda}R\right),\\ \sqrt{\lambda}J_n'\left(\sqrt{\lambda}R\right)-\alpha J_n\left(\sqrt{\lambda}R\right) &= \left(\frac{n}{R}-\alpha\right)J_n\left(\sqrt{\lambda}R\right)-\sqrt{\lambda}J_{n+1}\left(\sqrt{\lambda}R\right),\end{aligned}$$

and then, if $\alpha = n/R$ it holds

$$g_n(r,\rho;\lambda) = \frac{\pi}{2}\frac{Y_{n+1}\left(\sqrt{\lambda}R\right)}{J_{n+1}\left(\sqrt{\lambda}R\right)}J_n\left(\sqrt{\lambda}r\right)J_n\left(\sqrt{\lambda}\rho\right)-\frac{\pi}{2}\begin{cases}Y_n\left(\sqrt{\lambda}\rho\right)J_n\left(\sqrt{\lambda}r\right) & \text{if } 0<r\leqslant\rho<R,\\ Y_n\left(\sqrt{\lambda}r\right)J_n\left(\sqrt{\lambda}\rho\right) & \text{if } 0<\rho\leqslant r<R.\end{cases} \tag{12}$$

Again substituting the Bessel functions by their asymptotic forms for small arguments in (12) we obtain the residue of g_n at $\lambda = 0$, which is

$$\lim_{\lambda\to 0}\lambda\, g_n(r,\rho;\lambda) = -\frac{2(n+1)}{R^2}\left(\frac{r}{R}\right)^n\left(\frac{\rho}{R}\right)^n$$

for all $n \in \mathbb{N}_0$. □

Continuing with the analysis of the singularities of g_n in the λ-complex-plane, we study the poles of the spectral Green's function given by the non-zero solutions of the equation $\sqrt{\lambda}J_n'\left(\sqrt{\lambda}R\right)-\alpha J_n\left(\sqrt{\lambda}R\right)=0$, which can be expressed equivalently as

$$\begin{aligned}D_n(z,\alpha) &= zJ_n'(zR)-\alpha J_n(zR)\\ &= \left(\frac{n}{R}-\alpha\right)J_n(zR)-zJ_{n+1}(zR)\\ &= zJ_{n-1}(zR)-\left(\frac{n}{R}+\alpha\right)J_n(zR)\\ &= \left(\alpha-\frac{n}{R}\right)J_{n-1}(zR)+\left(\alpha+\frac{n}{R}\right)J_{n+1}(zR)=0,\end{aligned} \tag{13}$$

where $\lambda = z^2$.

Table 1
Summary of results concerning positive and imaginary roots of $zJ_n'(Rz) - \alpha J_n(Rz)$ for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

$\alpha = n/R$	$z_{n,m} = j_{n+1,m}/R$, $m \in \mathbb{N}$
$\alpha = -n/R$	$z_{n,m} = j_{n-1,m}/R$, $m \in \mathbb{N}$
$\alpha > n/R$	$z_{n,1} = \mathrm{i}\, y_n (y_n > 0)$, $0 < z_{n,2} < z_{n,3} < \cdots$
$\alpha < n/R$	$0 < z_{n,1} < z_{n,2} < \cdots$

Let us mention that if z is a solution of (13) for some given $\alpha \in \mathbb{C}$, then $-z$ is a solution too. Therefore, only the non-zero roots located in the complex half-plane $\mathfrak{Re}\, z \geqslant 0$ are analyzed. Also, it is possible to observe that as $D_n(\cdot, \alpha)$ is an analytic function, its zeros are isolated and form a countable set $\{z_{n,m} \in \mathbb{C},\ (n, m) \in \mathbb{N}_0 \times \mathbb{N}\}$. Moreover, for the particular cases $\alpha \neq \pm n/R$, it is possible to achieve explicitly the values of $z_{n,m}$ by replacing α in (13) obtaining that

$$z_{n,m} = j_{n\pm1,m}/R, \quad m \in \mathbb{N},$$

where $j_{n\pm1,m}$ is the m-th positive root of the Bessel function $J_{n\pm1}(z)$. It follows from here that in this particular case, there are infinite simple real roots of (13).

For an arbitrary $\alpha \in \mathbb{R}$, the Eq. (13) is well studied in the literature. The first important result about this equation was made by Dixon [19,12] who proved that (13) has an infinite number of distinct non-zero simple real roots when $n > -1$. Resorting to Dixon's results, Spigler obtained in [13] an asymptotic and series expansion of the roots of (13). Also, he proved the existence of two symmetric purely imaginary roots if and only if $\alpha > n/R > -1/R$. The generalization of [13] for arbitrary cylinder functions instead of J_n, is available in [20]. Properties of convexity and concavity of the zeros depending on $\alpha \in \mathbb{R}$ and $n > 0$ of these general functions are presented in [21]. Bounds for the small positive and imaginary zeros of (13) are developed in [14,22,23] and for the general cylinder functions in [24]. Recent results on the monoticity and multiplicity of the positive roots of (13) for $n, \alpha \in \mathbb{R}$ are presented in [15]. Table 1 outlines the known results for $z_{n,m}$ for $\alpha \in \mathbb{R}$. Also, it establishes the notation assigned to the roots of $D_n(\cdot, \alpha)$ employed throughout the rest of this paper.

The following proposition states some results on the roots of $D_n(\cdot, \alpha)$ for $\alpha \in \mathbb{C}$ that will be used at a later stage in this paper.

Proposition 3.2. (a) *The roots of $D_n(\cdot, \alpha)$ form a countable infinite set and their asymptotic behavior as $m \to \infty$ is given by*

$$z \sim j_{n+1,m} \sim \frac{\pi}{R}\left(\frac{4m + 2n + 1}{4}\right). \tag{14}$$

(b) *Every non-zero root $z_{n,m}$ is simple if either; $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$ with $\alpha \neq \sqrt{n^2/R^2 - z_{n,m}^2}$. Nevertheless, if $z_{n,m}$ is not simple, then it has at most multiplicity two.*
(c) *If $\mathfrak{Im}\, \alpha \neq 0$, then $D_n(\cdot, \alpha)$ has no imaginary and real roots.*
(d) *If $\mathfrak{Im}\, \alpha > 0$, then $\mathfrak{Im}\, z_{n,m}^2 < 0$.*

Proof. (a) From the asymptotic form of the Bessel functions for large arguments (cf. [18]) and (13), we infer straightforwardly that for large $|z|$ it holds

$$D_n(z, \alpha) \sim zJ_{n+1}(zR) \sim \sqrt{\frac{2z}{\pi}} \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad |z| \gg \left|\frac{n}{R} - \alpha\right|.$$

Hence, the zeros of $D_n(\cdot, \alpha)$ have the asymptotic behavior given by (14). From here we can conclude that (13) has infinite number of zeros which have an imaginary part that converges to zero.

(b) In order to prove that the zeros are simple, we shall compute the value of $D_n'(\cdot, \alpha)$ at a non-zero root $z_{n,m}$ and then, the pole is simple if $D_n'(z, \alpha)|_{z=z_{n,m}} \neq 0$ (here D_n' denotes the derivative of D_n with respect to its first variable). Employing the recurrence relations for the Bessel functions (cf. [18]) and the Eq. (13) itself, we find that

$$D_n'(z, \alpha)|_{z=z_{n,m}} = -\frac{J_n(z_{n,m}R)}{z_{n,m}R}\left(R^2 z_{n,m}^2 - n^2 + R^2\alpha^2\right). \tag{15}$$

Therefore, if $\mathfrak{Im}\, \alpha \neq 0$, by the assumption $\alpha \neq \sqrt{n^2/R^2 - z_{n,m}^2}$ we obtain that $D_n'(z, \alpha)|_{z=z_{n,m}} \neq 0$ by virtue of the fact that $z_{n,m}R \neq j_{n,m}$, where $j_{n,m}$ is the j-th positive zero of J_n (it follows directly replacing $z_{n,m}R$ by $j_{n,m}$ in (13) and by taking into account that J_n' and J_n have interlaced roots).

On the other hand, we can note from [19, Eq. (2)] or [13, Eq. II.3] that

$$\int_0^R rJ_n^2(zr)\,\mathrm{d}r = \frac{1}{2}R^2 J_n'^2(zR)\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{J_n(zR)}{zRJ_n'(zR)}\right]. \tag{16}$$

Thus, if $\alpha \in \mathbb{R}$, we employ again the recurrence relations of Bessel functions and (13) in the right side of (16) to obtain that

$$\int_0^R rJ_n^2(z_{n,m}r)\,\mathrm{d}r = \frac{J_n^2(z_{n,m}R)}{2}\left(\frac{R^2 z_{n,m}^2 - n^2 + R^2\alpha^2}{z_{n,m}^2}\right). \tag{17}$$

Then, if $z_{n,m} \in \mathbb{R}$ we have

$$\int_0^R r J_n^2(z_{n,m} r)\,\mathrm{d}r > 0,$$

while if $z_{n,m} = \mathrm{i}\, y_1$ we have $J_n(\mathrm{i}\, y_1 r) = \mathrm{i}^{\,n} I_n(y_1 r)$, where I_n is the modified Bessel function of the first kind. Then

$$(-1)^n \int_0^R r I_n^2(y_1 r)\,\mathrm{d}r = \int_0^R r J_n^2(z_{n,m} r)\,\mathrm{d}r \neq 0$$

and

$$D_n'(z,\alpha)|_{z=z_{n,m}} = -\frac{2z_{n,m}}{R J_n(z_{n,m} R)} \int_0^R r J_n^2(z_{n,m} r)\,\mathrm{d}r \neq 0$$

and consequently, $z_{n,m}$ is simple.

Finally, following [15], we define the function

$$F_n(z) = \frac{z J_n'(zR)}{J_n(zR)},$$

with which we can express (13) as $F_n(z) = \alpha$. Differentiating F_n two times, we get that it satisfies the differential equation (cf. [15, Eq. (34)])

$$F_n''(z) = -2R - \frac{R + 2F_n(z)}{z} F_n'(z).$$

Therefore, if $z_{n,m}$ is a repeated root of $D_n(\cdot,\alpha)$, it satisfies

$$F_n'(z)|_{z=z_{n,m}} = 0,$$

and then $F_n''(z)\big|_{z=z_{n,m}} = -2R \neq 0$. This proves that the multiplicity of any root is less or equal to two.

(c) The Bessel functions can be represented by the following infinite product (cf. [18]):

$$J_n(z) = \frac{(z/2)^n}{n!} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{j_{n,m}^2}\right).$$

It leads to

$$\frac{J_n'(zR)}{J_n(zR)} - \frac{\alpha}{z} = \frac{1}{z}\left(\frac{n}{R} - \alpha\right) + \sum_{m=1}^{\infty} \frac{2zR}{(zR)^2 - j_{n,m}^2} = 0,$$

which is equivalent to the Eq. (13), so that $z_{n,m}$ are the solutions of

$$\sum_{m=1}^{\infty} \frac{2(zR)^2}{(zR)^2 - j_{n,m}^2} = \alpha R - n, \tag{18}$$

for which the left side is the well known Mittag–Leffler expansion of the function $zRJ_n'(zR)/J_n(zR)$, valid for every $z \neq j_{n,m}/R$ (cf. [23–25]). Now, suppose that there is a purely imaginary zero given by $z_{n,m} = \mathrm{i}\, y,\ y \in \mathbb{R}$. Then replacing it in (18) we get

$$\sum_{m=1}^{\infty} \frac{2(yR)^2}{(yR)^2 + j_{n,m}^2} = \alpha R - n. \tag{19}$$

Then, since $\mathfrak{Im}\,\alpha \neq 0$ we have that this equation has no solution $y \in \mathbb{R}$ because the left side of (19) is real while the right side has a non zero imaginary part. It is a contradiction and then $D_n(\cdot,\alpha)$ has no imaginary zeros. The proof that there no exist real roots is completely analogous.

(d) Suppose $\mathfrak{Im}\,\alpha > 0$ and that exists a root z of $D_n(\cdot,\alpha)$ such that $z^2 = (x + \mathrm{i}\, y)/R^2$ with $y \geqslant 0$. Thus, replacing it in (18) we obtain

$$\sum_{m=1}^{\infty} \frac{2(x + \mathrm{i}\, y)(x - j_{n,m}^2 - \mathrm{i}\, y)}{(x - j_{n,m}^2)^2 + y^2} = \alpha R - n.$$

Performing the product in the nominator of each term of the series and taking the imaginary part of the equation, we obtain

$$\sum_{m=1}^{\infty} \frac{-2y j_{n,m}^2}{(x^2 - j_{n,m}^2)^2 + y^2} = R\,\mathfrak{Im}\,\alpha.$$

Therefore, since $j_{n,m}^2 > 0$ and $y \geqslant 0$, we obtain a contradiction and consequently $\mathfrak{Im}\,z^2 < 0$. $\square$

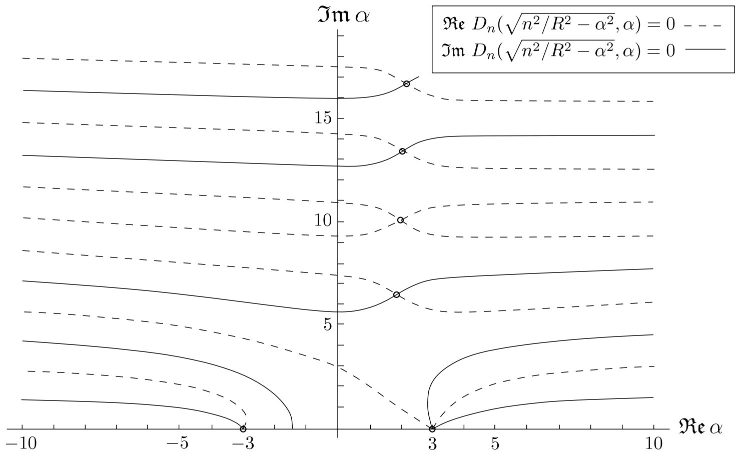

Fig. 2. Location of the values of α leading to a non-simple root of $D_n(\cdot,\alpha)$ for $R=1$ and $n=3$. The non-simple zero due to $\alpha=\pm 3$ corresponds to the root $z=0$, therefore it does not contradict Proposition 3.2(b) in which the case $z\neq 0$ is analyzed.

3.1. The non-simple roots of $zJ_n'(zR)-\alpha J_n(zR)$

The condition $\alpha\neq\sqrt{n^2/R^2-z_{n,m}^2}$ for $\alpha\in\mathbb{C}$, is necessary in order for $z_{n,m}$ to be a simple zero. Moreover, all the values of $\alpha\in\mathbb{C}$ for which it is possible to obtain a non-simple zero, form a countable infinite set. These particular values of α, denoted by $\alpha_{n,l}$, where the subindex l arranges them in ascending order according to their absolute value, can be found searching the roots of the equation

$$D_n\left(\sqrt{\frac{n^2}{R^2}-\alpha^2},\alpha\right)=0 \tag{20}$$

in the α-complex-plane. Fig. 2 shows this situation for a particular case.

Let us add that in order to be in accordance with the physical meaning of the impedance boundary condition, we only have to search the values of α satisfying (20) in the upper complex plane $\mathfrak{Im}\,\alpha>0$. Numerical procedures to obtain the numerical values of $\alpha_{n,l}$ may be performed making use of the so-called *logarithmic residue based quadrature method*. More references about this method can be found in [16], the paper that originated these kinds of methods, and in the book [17]. Table 2 shows the values of $\alpha_{n,l}$ for different values of n computed with the Delves–Lyness algorithm (cf. [16]). From these results we may notice that the non-simple zero arising due to the impedance parameter $\alpha_{n,l}$ is the l-th root of $D_n(\cdot,\alpha_{n,l})$ i.e. $z_{n,m}$ with $m=l$.

4. A series representation of the spectral Green's function

As can be observed in (8), $g_n(r,\rho;\cdot)$ has a branch cut along the negative real line due to the complex square root. However, in the following proposition we show that $g_n(r,\rho;\cdot)$ is a meromorphic function in the whole λ-complex-plane, i.e. it is analytic everywhere except at its poles $\lambda_{n,m}$, $(n,m)\in\mathbb{N}_0\times\mathbb{N}$ given by

$$\lambda_{n,m}=\begin{cases} z_{n,m}^2 & \text{if } n\neq R\alpha,\ m\in\mathbb{N}\\ z_{n,m-1}^2 & \text{if } n=R\alpha,\ m>1\\ 0 & \text{if } n=R\alpha,\ m=1,\end{cases} \tag{21}$$

where $z_{n,m}$, $m\in\mathbb{N}$ are the non-zero roots of $D_n(\cdot,\alpha)$, and are sorted in increasing order according to their absolute value.

Proposition 4.1. *The spectral Green's function (8) is meromorphic in the whole λ-complex-plane for every $\alpha\in\mathbb{C}$, $\rho\in(0,R)$, and $r\in[0,R]$.*

Proof. First, let us observe that $g_n(r,\rho;\cdot)$ is actually meromorphic in the whole complex plane except at the negative real axis. Consequently, to prove this proposition we only have to show that $g_n(r,\rho;\cdot)$ is also meromorphic in a region that contains the negative real axis. Thus, without losing generality, let us assume that $\lambda\neq -y_n^2$ (where $\pm y_n\mathrm{i}$ are the imaginary roots of $D_n(\cdot,\alpha)$ arising when $\alpha>n/R$), and then the limits

$$\lim_{\varepsilon\to 0^+} g_n(r,\rho;|\lambda|\,\mathrm{e}^{\mathrm{i}(\pi-\varepsilon)})\quad\text{and}\quad \lim_{\varepsilon\to 0^+} g_n(r,\rho;|\lambda|\,\mathrm{e}^{-\mathrm{i}(\pi-\varepsilon)}) \tag{22}$$

Table 2
Numerical values of the first (in norm) five impedance parameters $\alpha_{n,l}$ leading to a non-simple root $z_{n,l}$ of the $D_n(\cdot,\alpha)$ computed with the Delves–Lyness algorithm for $R = 1$ and $n = 0, 1, 2, 3, 4$.

n	l	$\alpha_{n,l}$	$z_{n,l}$	$\lvert D_n(z_{n,l},\alpha_{n,l})\rvert$	$\lvert D_n'(z_{n,l},\alpha_{n,l})\rvert$
0	1	$1.2796 + 2.9804i$	$2.9804 - 1.2796i$	$0.0031 \cdot 10^{-6}$	$0.0031\cdot 10^{-6}$
	2	$1.6187 + 6.1752i$	$6.1752 - 1.6187i$	$0.0281 \cdot 10^{-6}$	$0.0281\cdot 10^{-6}$
	3	$1.8189 + 9.3420i$	$9.3420 - 1.8189i$	$0.0788 \cdot 10^{-6}$	$0.0788\cdot 10^{-6}$
	4	$1.9615 + 12.4985i$	$12.4985 - 1.9615i$	$0.1260 \cdot 10^{-6}$	$0.1260\cdot 10^{-6}$
	5	$2.0723 + 15.6501i$	$15.6501 - 2.0723i$	$0.1347 \cdot 10^{-6}$	$0.1347\cdot 10^{-6}$
1	1	$1.5017 + 4.3646i$	$4.4663 - 1.4675i$	$0.0073 \cdot 10^{-5}$	$0.0071\cdot 10^{-5}$
	2	$1.7410 + 7.6320i$	$7.6941 - 1.7270i$	$0.0345 \cdot 10^{-5}$	$0.0342\cdot 10^{-5}$
	3	$1.9028 + 10.8299i$	$10.8746 - 1.8949i$	$0.0862 \cdot 10^{-5}$	$0.0858\cdot 10^{-5}$
	4	$2.0251 + 14.0040i$	$14.0389 - 2.0201i$	$0.1344 \cdot 10^{-5}$	$0.1341\cdot 10^{-5}$
	5	$2.1235 + 17.1669i$	$17.1956 - 2.1199i$	$0.1409 \cdot 10^{-5}$	$0.1407\cdot 10^{-5}$
2	1	$1.6950 + 5.4908i$	$5.8169 - 1.6000i$	$0.0491 \cdot 10^{-6}$	$0.0467\cdot 10^{-6}$
	2	$1.8590 + 8.9054i$	$9.1185 - 1.8155i$	$0.2383 \cdot 10^{-6}$	$0.2331\cdot 10^{-6}$
	3	$1.9870 + 12.1756i$	$12.3346 - 1.9613i$	$0.5460 \cdot 10^{-6}$	$0.5393\cdot 10^{-6}$
	4	$2.0902 + 15.3933i$	$15.5204 - 2.0731i$	$0.7386 \cdot 10^{-6}$	$0.7327\cdot 10^{-6}$
	5	$2.1764 + 18.5855i$	$18.6914 - 2.1640i$	$0.6197 \cdot 10^{-6}$	$0.6162\cdot 10^{-6}$
3	1	$1.8701 + 6.4813i$	$7.1010 - 1.7069i$	$0.0200 \cdot 10^{-5}$	$0.0185\cdot 10^{-5}$
	2	$1.9716 + 10.0606i$	$10.4837 - 1.8921i$	$0.1088 \cdot 10^{-5}$	$0.1047\cdot 10^{-5}$
	3	$2.0698 + 13.4197i$	$13.7437 - 2.0210i$	$0.2691 \cdot 10^{-5}$	$0.2630\cdot 10^{-5}$
	4	$2.1553 + 16.6943i$	$16.9575 - 2.1219i$	$0.3827 \cdot 10^{-5}$	$0.3769\cdot 10^{-5}$
	5	$2.2298 + 19.9263i$	$20.1482 - 2.2053i$	$0.3339 \cdot 10^{-5}$	$0.3303\cdot 10^{-5}$
4	1	$2.0323 + 7.3836i$	$8.3439 - 1.7984i$	$0.0160 \cdot 10^{-4}$	$0.0144\cdot 10^{-4}$
	2	$2.0794 + 11.1310i$	$11.8075 - 1.9603i$	$0.0778 \cdot 10^{-4}$	$0.0736\cdot 10^{-4}$
	3	$2.1506 + 14.5865i$	$15.1145 - 2.0755i$	$0.1657 \cdot 10^{-4}$	$0.1602\cdot 10^{-4}$
	4	$2.2198 + 17.9251i$	$18.3597 - 2.1673i$	$0.1931 \cdot 10^{-4}$	$0.1886\cdot 10^{-4}$
	5	$2.2833 + 21.2031i$	$21.5730 - 2.2441i$	$0.1286 \cdot 10^{-4}$	$0.1265\cdot 10^{-4}$

exist, by virtue of the fact that $g_n(r,\rho;\cdot)$ has no other poles on the negative real line (see Proposition 3.2(c)). Employing (5), we compute the limits in (22), proving that they are equal and then the spectral Green's function is continuous in any region that contains the negative real line (except the poles). Immediately afterwards, a theorem of analytic continuation along curves will ensure that $g_n(r,\rho;\cdot)$ is meromorphic.

First, let us claim that $\varphi_2(r;\cdot)$, defined in (6b), is continuous in the whole complex plane as a function of λ, except at $\lambda = 0$, where it has a singularity due to the Bessel function of the second kind. To show it, we ought to compute

$$\varphi_2(r;\lambda)\big|_{\lambda=|\lambda|\,e^{-i\pi}}^{\lambda=|\lambda|\,e^{i\pi}} = \lim_{\varepsilon\to 0^+}\varphi_2(r;|\lambda|\,e^{i(\pi-\varepsilon)}) - \lim_{\varepsilon\to 0^+}\varphi_2(r;|\lambda|\,e^{-i(\pi-\varepsilon)})$$

where

$$\varphi_2(r;\lambda)\big|_{\lambda=|\lambda|\,e^{-i\pi}}^{\lambda=|\lambda|\,e^{i\pi}} = \left(\beta J_n'(\beta R) - \alpha J_n(\beta R)\right)\,Y_n(\beta r)\big|_{\beta=-i\,|\lambda|^{1/2}}^{\beta=i\,|\lambda|^{1/2}} - \left(\beta Y_n'(\beta R) - \alpha Y_n(\beta R)\right)\,J_n(\beta r)\big|_{\beta=-i\,|\lambda|^{1/2}}^{\beta=i\,|\lambda|^{1/2}}. \tag{23}$$

Replacing Y_n' and J_n' in (23) by their respective recurrence formulas, and observing, with the aid of the analytic continuation formulas (cf. [18]), that

$$\begin{aligned}\beta J_n(\beta r)\,Y_{n+1}(\beta R)\big|_{\beta=-i\,|\alpha|^{1/2}}^{\beta=i\,|\alpha|^{1/2}} &= \beta J_n(\beta R)\,Y_{n+1}(\beta r)\big|_{\beta=-i\,|\alpha|^{1/2}}^{\beta=i\,|\alpha|^{1/2}}\\ &= 2|\lambda|^{1/2}J_n(i\,|\lambda|^{1/2}r)J_{n+1}(i\,|\lambda|^{1/2}R)\end{aligned}$$

and

$$\begin{aligned}\beta J_{n+1}(\beta R)\,Y_n(\beta r)\big|_{\beta=-i\,|\lambda|^{1/2}}^{\beta=i\,|\lambda|^{1/2}} &= \beta J_{n+1}(\beta r)\,Y_n(\beta R)\big|_{\beta=-i\,|\lambda|^{1/2}}^{\beta=i\,|\lambda|^{1/2}}\\ &= 2|\lambda|^{1/2}J_{n+1}(i\,|\lambda|^{1/2}R)J_n(i\,|\lambda|^{1/2}r),\end{aligned}$$

we find

$$\varphi_2(r;\lambda)\big|_{\lambda=|\lambda|\,e^{-i\pi}}^{\lambda=|\lambda|\,e^{i\pi}} = 0.$$

From here we have that $\varphi_2(r;\cdot)$ is continuous in the whole complex plane except at $\lambda = 0$.

On the other hand, evaluating the remaining term of the Green's function in $\lambda = |\lambda|\,e^{i\pi}$ and $\lambda = |\lambda|\,e^{-i\pi}$, we achieve

$$\left.\frac{\varphi_1(r;\lambda)}{\rho W[\varphi_1(r;\lambda),\varphi_2(r;\lambda)](\rho)}\right|_{\lambda=|\lambda|\,e^{i\pi}} = \left.\frac{J_n(\beta r)}{\beta J_n'(\beta R) - \alpha J_n(\beta R)}\right|_{\beta=i\,|\lambda|^{1/2}}$$

$$= \frac{J_n(\mathrm{i}\,|\lambda|^{1/2} r)}{\mathrm{i}\,|\lambda| J_n'\left(\mathrm{i}\,|\lambda|^{1/2} R\right) - \alpha J_n\left(\mathrm{i}\,|\lambda|^{1/2} R\right)} \tag{24}$$

and

$$\left.\frac{\varphi_1(r;\lambda)}{\rho W[\varphi_1(r;\lambda),\varphi_2(r;\lambda)](\rho)}\right|_{\lambda=|\lambda|\,\mathrm{e}^{-\mathrm{i}\pi}} = \left.\frac{J_n(\beta r)}{\beta J_n'(\beta R) - \alpha J_n(\beta R)}\right|_{\beta=-\mathrm{i}\,|\lambda|^{1/2}}$$
$$= \frac{(-1)^n J_n(\mathrm{i}\,|\lambda|^{1/2} r)}{(-1)^n\left[\mathrm{i}\,|\lambda| J_n'\left(\mathrm{i}\,|\lambda|^{1/2} R\right) - \alpha J_n\left(\mathrm{i}\,|\lambda|^{1/2} R\right)\right]}. \tag{25}$$

Therefore, since (24) and (25) are equal and $\varphi_2(r;\cdot)$ is continuous, it follows that $g_n(r,\rho;\cdot)$ is continuous in any region that contains the negative real axis (except the poles).

Now, the fact that $g_n(r,\rho;\cdot)$ is meromorphic can be inferred directly from the following theorem of analytic continuation along curves (cf. [26, p. 206]): If f is a function which is continuous on an open set U and analytic on U except possibly at the points of a simple analytic curve C in U, then f is actually analytic on all of U. □

Having stated the fact that $g_n(r,\rho;\cdot)$ is meromorphic, we proceed to present the main result of this section, which is a series expansion of g_n. To accomplish that, we need first the following lemma.

Lemma 4.2. *There exist positive constants C_1 and C_2 depending on n such that*

$$|g_n(r,\rho;\lambda)| \leqslant \frac{C_1}{|\lambda|^{1/2}\sqrt{r\rho}}$$

provided that: (i) $|\lambda| \geqslant C_2$, *and* (ii) *either* $\Re\mathfrak{e}\sqrt{\lambda} = (4l+2n+3)\pi/4R$ *for some integer l or* $|\Im\mathfrak{m}\sqrt{\lambda}| \geqslant R^{-1}$.

Proof. Let M_1 be a positive constant such that the asymptotic forms of the Bessel functions for large arguments (cf. [18]) are valid when $|\lambda| \geqslant M_1$. Thus, substituting the Bessel functions by their asymptotic forms in (6a) and (6b) we obtain, after some manipulation with the complex trigonometric functions, that the asymptotic form of φ_1 and φ_2 is given by

$$\varphi_1(r;\lambda) \sim \left[\frac{2}{\pi\sqrt{\lambda}r}\right]^{1/2}\cos\left(\sqrt{\lambda}r - \frac{\pi(2n+1)}{4}\right),$$
$$\varphi_2(r;\lambda) \sim \frac{-2}{\pi\sqrt{Rr\lambda}}\left[\sqrt{\lambda}\cos\left(\sqrt{\lambda}(R-r)\right) + \left(\frac{n}{R}-\alpha\right)\sin\left(\sqrt{\lambda}(R-r)\right)\right],$$

for all $|\lambda| > M_1$.

Considering the properties of the complex trigonometric functions, $|\cos(x+\mathrm{i}\,y)| \leqslant 2\,\mathrm{e}^{|y|}$ and $|\sin(x+\mathrm{i}\,y)| \leqslant 2\,\mathrm{e}^{|y|}$ for every $x, y \in \mathbb{R}$, we get that for $|\lambda|$ large enough, the following inequalities hold

$$|\varphi_1(r_<;\lambda)| \leqslant \left[\frac{8}{\pi}\right]^{1/2}\frac{\mathrm{e}^{|\Im\mathfrak{m}\sqrt{\lambda}|r_<}}{|\lambda|^{1/4}\sqrt{r_<}}, \tag{26}$$

$$|\varphi_2(r_>;\lambda)| \leqslant \frac{4}{\pi\sqrt{R}}\left[1+\frac{1}{M_1}\left|\frac{n}{R}-\alpha\right|\right]\frac{\mathrm{e}^{|\Im\mathfrak{m}\sqrt{\lambda}|(R-r_>)}}{\sqrt{r_>}}, \tag{27}$$

for every $|\lambda| > M_1$. Then, multiplying (26) by (27), we obtain

$$|\varphi_1(r_<;\lambda)\varphi_2(r_>;\lambda)| \leqslant \frac{C}{|\lambda|^{1/4}\sqrt{r_<r_>}}\,\mathrm{e}^{\left|\Im\mathfrak{m}\sqrt{\lambda}\right|(R-r_>+r_<)},$$

and since $r_> - r_< = \max(r,\rho) - \min(r,\rho) \geqslant 0$, we have $R - r_> + r_< \leqslant R$ and hence

$$|\varphi_1(r_<;\lambda)\varphi_2(r_>;\lambda)| \leqslant \frac{C}{|\lambda|^{1/4}\sqrt{r_<r_>}}\,\mathrm{e}^{\left|\Im\mathfrak{m}\sqrt{\lambda}\right|R}. \tag{28}$$

On the other hand, the denominator of the spectral Green's function is $D_n(\sqrt{\lambda},\alpha)$ and its asymptotic behavior for large arguments is given by

$$D_n\left(\sqrt{\lambda},\alpha\right) \sim \left[\frac{2}{\pi\sqrt{\lambda}R}\right]^{1/2}\left\{\left(\frac{n}{R}-\alpha\right)\cos\left(\sqrt{\lambda}R - \frac{\pi(2n+1)}{4}\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}R - \frac{\pi(2n+1)}{4}\right)\right\}.$$

So that for all $|\lambda| > M_1$ we have

$$\left|D_n\left(\sqrt{\lambda},\alpha\right)\right| \sim \left|\frac{2\sqrt{\lambda}}{\pi R}\right|^{1/2}\left|\sin\left(\sqrt{\lambda}R - \frac{\pi(2n+1)}{4}\right)\right|\left|\frac{1}{\sqrt{\lambda}}\left(\frac{n}{R}-\alpha\right)\cot\left(\sqrt{\lambda}R - \frac{\pi(2n+1)}{4}\right) - 1\right|. \tag{29}$$

Let us note that for any real x and y, we have

$$|\cos(x+\mathrm{i}\,y)|^2 = \cosh^2(y) - \sin^2(x) \tag{30}$$

$$|\sin(x+\mathrm{i}\,y)|^2 = \cosh^2(y) - \cos^2(x). \tag{31}$$

Thus, since $\cosh y \geqslant \mathrm{e}^{|y|}/2$, we have that placing $\cos x = 0$ in (31) the sine function can be bounded by $|\sin(x+\mathrm{i}\,y)| \geqslant \mathrm{e}^{|y|}/4$. If instead we have $|y| > 1$, the same bound holds due to the inequality

$$|\sin(x+\mathrm{i}\,y)|^2 = \cosh^2(y) - \cos^2(x) \geqslant \cosh^2(y) - 1 \geqslant \frac{\mathrm{e}^{2|y|}}{4} - 1 > \frac{\mathrm{e}^{2|y|}}{16}. \tag{32}$$

Consequently, from (30) and (32), we obtain that

$$|\cot(x+\mathrm{i}\,y)|^2 = \left|\frac{\cos(x+\mathrm{i}\,y)}{\sin(x+\mathrm{i}\,y)}\right|^2 \leqslant \frac{16\cosh^2(y)}{\mathrm{e}^{2|y|}} \leqslant 32\frac{\mathrm{e}^{2|y|}}{\mathrm{e}^{2|y|}}, \tag{33}$$

provided that either $\cos x = 0$ or $|y| > 1$.

Now, setting

$$\sqrt{\lambda}R - \frac{\pi(2n+1)}{4} = x + \mathrm{i}\,y, \tag{34}$$

we can make use of the inequality (33) and then, choosing a constant $C_2 = \max\{M_1, M_2\}$, where

$$M_2 \geqslant 128\left|\frac{n}{R} - \alpha\right|^2,$$

we obtain that for every $|\lambda| \geqslant C_2$ the following inequality holds

$$\left|\frac{1}{\sqrt{\lambda}}\left(\frac{n}{R} - \alpha\right)\cot\left(\sqrt{\lambda}R - \frac{\pi(2n+1)}{4}\right) - 1\right| \geqslant \left|1 - \frac{1}{|\lambda|^{1/2}}\left|\frac{n}{R} - \alpha\right|\left|\cot\left(\sqrt{\lambda}R - \frac{\pi(2n+1)}{4}\right)\right|\right| \geqslant \frac{1}{2}. \tag{35}$$

Then, employing it in (29), we obtain that

$$\left|D_n\left(\sqrt{\lambda}, \alpha\right)\right| \geqslant |\lambda|^{1/4}\frac{\mathrm{e}^{|\Im\mathrm{m}\sqrt{\lambda}|R}}{8}\left|\frac{2}{\pi R}\right|^{1/2}, \tag{36}$$

and thus finally computing the quotient between (28) and (36), we get

$$|g_n(r,\rho;\lambda)| = \left|\frac{\varphi_1(r_<;\lambda)\varphi_2(r_>;\lambda)}{D_n\left(\sqrt{\lambda};\alpha\right)}\right| \leqslant \frac{C_1}{|\lambda|^{1/2}\sqrt{r_<r_>}},$$

where by definition $r_<r_> = \min\{r,\rho\}\max\{r,\rho\} = r\rho$. □

Theorem 4.3. *Let $\lambda \neq \lambda_{n,m}$, then spectral Green's function (8) admits the following representation:*

$$g_n(r,\rho;\lambda) = \sum_{m=1}^{\infty} \operatorname*{Res}_{\nu=\lambda_{n,m}} \frac{g_n(r,\rho;\nu)}{\lambda - \nu},$$

with $n \in \mathbb{N}_0$, $r \in [0,R]$, and $\rho \in (0,R)$.

Proof. Let γ_N be the contour depicted in Fig. 3, where N is a positive integer large enough such that

$$\frac{\pi}{4R}(4N+2n+3) \geqslant C_2$$
$$\frac{\pi}{4R}(4N+2n+3) \geqslant \frac{1}{R}$$
$$N \geqslant 2|\lambda|,$$

where C_2 is the second constant in Lemma 4.2. Note that in the region bounded by γ_N the function g_n is meromorphic as a function of λ, i.e. holomorphic except at the isolated points $\lambda = \lambda_{n,m}$, which are its poles, so that the residue theorem applies. Also, we have that by virtue of Lemma 4.2, on γ_N the function g_n is bounded by $C_1(|\lambda|r\rho)^{-1/2}$ and thus there are no poles on it.

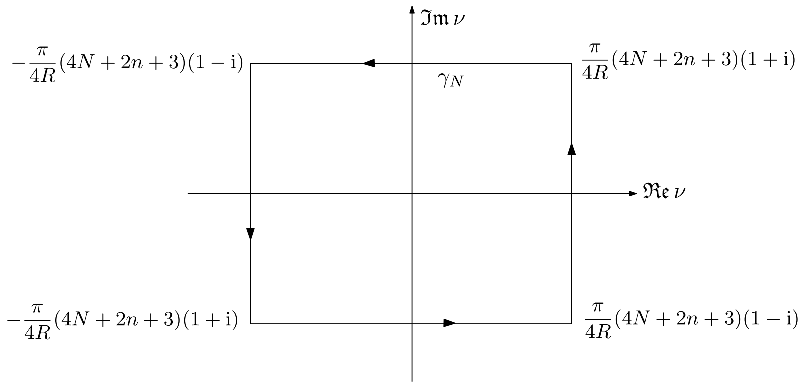

Fig. 3. Integration contour utilized in the proof of Theorem 4.3.

Hence, according to the residue theorem (cf. [26]), the following equality holds for every $\lambda \neq \lambda_{n,m}$:

$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma_N}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}\,\mathrm{d}\nu = -g_n(r,\rho,\lambda) + \sum_{m=1}^{M(N)}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}, \tag{37}$$

where $M(N)$ is the number of poles $\nu = \lambda_{n,m}$ enclosed by γ_N, which is a monotone increasing function of N.

On the other hand, thanks to Lemma 4.2, we have that the integral over γ_N can be bounded in the following way:

$$\left|\int_{\gamma_N}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}\,\mathrm{d}\nu\right| \leqslant \int_{\gamma_N}\left|\frac{g_n(r,\rho;\nu)}{\lambda-\nu}\right|\mathrm{d}\nu \leqslant \frac{C_1}{\sqrt{r\rho}}\int_{\gamma_N}\frac{|\nu|^{-1/2}}{|\nu-\lambda|}\,\mathrm{d}\nu.$$

Then, as $N \geqslant 2|\lambda|$ we have that $|\lambda-\nu| \geqslant N$ and $|\nu| \leqslant \pi\sqrt{2}(4N+2n+3)/4R \leqslant \widetilde{C}N$ for every $\nu \in \gamma_N$. Consequently

$$\frac{C_1|\nu|^{1/2}}{|\lambda-\nu|} \leqslant \frac{\widetilde{C}}{N^{3/2}}, \quad \forall \nu \in \gamma_N$$

and noting that the length of the path γ_N is $l(\gamma_N) \leqslant 2\pi(4N+2n+3)/R$, we get

$$\left|\int_{\gamma_N}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}\,\mathrm{d}\nu\right| \leqslant \frac{1}{\sqrt{r\rho}}\frac{\widehat{C}}{N^{1/2}}. \tag{38}$$

Next, taking the absolute value of (37), we obtain

$$\begin{aligned}\left|g(r,\rho;\lambda) - \sum_{m=1}^{M(N)}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}\right| &\leqslant \frac{1}{2\pi}\left|\int_{\gamma_N}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}\,\mathrm{d}\nu\right| \\ &\leqslant \frac{1}{\sqrt{r\rho}}\frac{\widehat{C}}{N^{1/2}}\end{aligned} \tag{39}$$

and hence

$$\begin{aligned}g_n(r,\rho;\lambda) &= \lim_{N\to\infty}\sum_{m=1}^{M(N)}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\lambda-\nu} \\ &= \sum_{m=1}^{\infty}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}.\end{aligned}$$

Moreover, from (39) it is possible to conclude that

$$\lim_{N\to\infty}\sqrt{r\rho}\left|g_n(r,\rho;\lambda) - \sum_{m=1}^{M(N)}\operatorname*{Res}_{\nu=z_{n,m}^2}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}\right| = 0$$

uniformly for every $r \in [0,R]$ and $\rho \in (0,R)$. $\square$

4.1. Computing the residues of the spectral Green's function

To find the residue of $g_n(r,\rho;\nu)/(\lambda-\nu)$ at $\nu=\lambda_{n,m}$, we will express the spectral Green's function as

$$g_n(r,\rho;\lambda)=\frac{N_n\left(\sqrt{\lambda},\alpha\right)}{D_n\left(\sqrt{\lambda},\alpha\right)}J_n\left(\sqrt{\lambda}r\right)J_n\left(\sqrt{\lambda}\rho\right)-\frac{\pi}{2}J_n\left(\sqrt{\lambda}r_<\right)Y_n\left(\sqrt{\lambda}r_>\right),\tag{40}$$

where D_n is defined in (13) and

$$N_n(z,\alpha)=\frac{\pi}{2}\left[zY_n'(zR)-\alpha Y_n(zR)\right].$$

Let us observe that the last term in (40) contributes only to the residue at the pole $\nu=0$, arising when $\alpha=n/R$, was obtained in Proposition 3.1 and is given by

$$\operatorname*{Res}_{\nu=0}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}=-\frac{2(n+1)}{\lambda R^2}\left(\frac{r}{R}\right)^n\left(\frac{\rho}{R}\right)^n.$$

Therefore, this term in (40) does not exert any influence on the computation of the residues at the non-zero poles. Consequently, all other residues can be obtained with the following formula:

$$\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}=\operatorname*{Res}_{\nu=z_{n,m}^2}\left[\frac{N_n(\sqrt{\nu},\alpha)}{D_n(\sqrt{\nu},\alpha)}\frac{J_n(\sqrt{\nu}r)J_n(\sqrt{\nu}\rho)}{\lambda-\nu}\right].$$

At this stage we ought to distinguish two cases; $\nu=\lambda_{n,m}$ is a simple pole (i.e. $\lambda_{n,m}=z_{n,m}^2$ with $\alpha\in\mathbb{R}$ or $\Im\mathfrak{m}\,\alpha\neq 0$ with $\alpha\neq\sqrt{n^2/R^2-z_{n,m}^2}$); or $\nu=\lambda_{n,m}$ is a double pole (i.e. $\lambda_{n,m}=z_{n,m}^2$ with $\Im\mathfrak{m}\,\alpha\neq 0$ and $\alpha=\sqrt{n^2/R^2-z_{n,m}^2}$).

When $\nu=\lambda_{n,m}$ is a simple pole, the following formula holds

$$\operatorname*{Res}_{\nu=\lambda_{n,m}}\left[\frac{N_n(\sqrt{\nu},\alpha)}{D_n(\sqrt{\nu},\alpha)}\frac{J_n(\sqrt{\nu}r)J_n(\sqrt{\nu}\rho)}{\lambda-\nu}\right]=\lim_{\nu\to z_{n,m}^2}\frac{2\sqrt{\nu}N_n(\sqrt{\nu},\alpha)}{D_n'(\sqrt{\nu},\alpha)}\frac{J_n(\sqrt{\nu}r)J_n(\sqrt{\nu}\rho)}{\lambda-\nu}$$

where D_n' denotes the derivative of D_n with respect to its first variable. Making use of the Wronskian between J_n and Y_n (cf. [18]), we get that

$$\lim_{\nu\to z_{n,m}^2}N_n(\sqrt{\nu},\alpha)=\frac{1}{RJ_n(z_{n,m}R)},$$

and from Proposition 3.2(b) it follows that

$$\lim_{\nu\to z_{n,m}^2}D_n'(\sqrt{\nu},\alpha)=-\frac{J_n(z_{n,m}R)}{z_{n,m}R}\left(R^2z_{n,m}^2-n^2+R^2\alpha^2\right).$$

Therefore

$$\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}=\frac{2z_{n,m}^2J_n(z_{n,m}r)J_n(z_{n,m}\rho)}{J_n^2(z_{n,m}R)(z_{n,m}^2-\lambda)\left(R^2z_{n,m}^2-n^2+R^2\alpha^2\right)}\tag{41}$$

when $\nu=\lambda_{n,m}\neq 0$ is a simple pole.

Now, let us assume that $\nu=\lambda_{n,m}$ is a double pole. Thus, the residue must be computed by the formula

$$\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\nu-\lambda}=\lim_{\nu\to z_{n,m}^2}\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\frac{(\nu-z_{n,m}^2)^2}{(\nu-\lambda)}\frac{N_n(\sqrt{\nu},\alpha)}{D_n(\sqrt{\nu},\alpha)}J_n(\sqrt{\nu}r)J_n(\sqrt{\nu}\rho)\right].$$

After performing algebraic manipulation, employing the properties of the Bessel functions and the equations that define non-simple roots, i.e. $D_n(z_{n,m},\alpha)=0$ and $\alpha^2=n^2/R^2-z_{n,m}^2$, we obtain

$$\begin{aligned}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{\lambda-\nu}=&-\frac{4}{3}\frac{J_n(z_{n,m}r)J_n(z_{n,m}\rho)}{R^2J_n^2(z_{n,m}R)}\left[\frac{3z_{n,m}^2}{(z_{n,m}^2-\lambda)^2}+\frac{(2R\alpha-3n-2)}{z_{n,m}^2-\lambda}\right]\\&-\frac{J_{n+1}(z_{n,m}r)J_n(z_{n,m}\rho)}{R^2J_n^2(z_{n,m}R)}\frac{2z_{n,m}r}{z_{n,m}^2-\lambda}-\frac{J_n(z_{n,m}r)J_{n+1}(z_{n,m}\rho)}{R^2J_n^2(z_{n,m}R)}\frac{2z_{n,m}\rho}{z_{n,m}^2-\lambda}.\end{aligned}\tag{42}$$

5. The spatial Green's function

Herein we recover the spatial Green's function from the integral representation and the spectral Green's function introduced in (3). Two different expressions of the spatial Green's function are obtained. The first one corresponds to a series expansion deduced as a result of Theorem 4.3, and the second one shows explicitly the behavior of the Green's function at the source point.

5.1. Series representation

Thanks to Theorem 4.3, we have that if $k^2-\xi^2 \neq \lambda_{n,m}$, then the spectral Green's function can be expressed as

$$g_n(r,\rho;k^2-\xi^2)=\sum_{m=1}^{\infty}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{k^2-\xi^2-\nu}. \tag{43}$$

Paying attention to the condition $k^2-\xi^2\neq\lambda_{n,m}$ and to Proposition 3.2(d), it may be noted that such condition holds if $\mathfrak{Im}\,k^2>0$ or $\mathfrak{Im}\,\alpha>0$, by virtue of the fact that under either of those conditions, $\mathfrak{Im}(k^2-\xi^2)$ and $\mathfrak{Im}\,\lambda_{n,m}$ have opposite signs. This guarantees that every term in the series (43) belongs to $L^1(\mathbb{R})$ when it is viewed as a function of ξ, due to the fact that the functions $(k^2-\xi^2-z_{n,m}^2)^{-1}$ and $(k^2-\xi^2-z_{n,m}^2)^{-2}$, that appear in (41) and (42), have no singularities on the real line.

Therefore, assuming that $\mathfrak{Im}\,k^2>0$ or $\mathfrak{Im}\,\alpha>0$, we utilize (3) and Theorem 4.3 to obtain the following integral representation of the Green's function with the aid of the Lebesgue convergence theorem

$$G(\boldsymbol{x},\boldsymbol{y})=\frac{1}{4\pi^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\epsilon_n\cos(n(\theta-\vartheta))\int_{\mathbb{R}}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{k^2-\xi^2-\nu}\,e^{i\xi(z-\zeta)}\,d\xi. \tag{44}$$

To compute the integral in (44), we need the following inverse Fourier transforms (cf. [27, p. 116])

$$\int_{\mathbb{R}}\frac{e^{iz\xi}}{\xi^2+z_{n,m}^2-k^2}\,d\xi=i\pi\frac{e^{i\sqrt{k^2-z_{n,m}^2}|z|}}{\sqrt{k^2-z_{n,m}^2}} \tag{45}$$

$$\int_{\mathbb{R}}\frac{e^{iz\xi}}{(\xi^2+z_{n,m}^2-k^2)^2}\,d\xi=i\pi\frac{e^{i\sqrt{k^2-z_{n,m}^2}|z|}}{2(k^2-z_{n,m}^2)^{3/2}}\left[i\sqrt{k^2-z_{n,m}^2}|z|-1\right]. \tag{46}$$

The formulas (45) and (46) together with (41) and (42) lead to:

- If the pole of the spectral Green's function is $\lambda_{n,1}=0$, which occurs when $\alpha=n/R$, then we have that

$$\int_{\mathbb{R}}\operatorname*{Res}_{\nu=\lambda_{n,1}}\frac{g_n(r,\rho;\nu)}{k^2-\xi^2-\nu}\,e^{i\xi(z-\zeta)}\,d\xi=\frac{2\pi i\,(n+1)}{R^2}\left(\frac{r}{R}\right)^n\left(\frac{\rho}{R}\right)^n\frac{e^{ik|z-\zeta|}}{k}. \tag{47}$$

- If $\lambda_{n,m}=z_{n,m}^2$ is a simple pole of the spectral Green's function, i.e. $D_n(z_{n,m},\alpha)=0$ and $D_n'(z_{n,m},\alpha)\neq0$, then we obtain that

$$\int_{\mathbb{R}}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{k^2-\xi^2-\nu}\,e^{i\xi(z-\zeta)}\,d\xi=\frac{2\pi i\,z_{n,m}^2}{J_n^2(z_{n,m}R)\left(R^2z_{n,m}^2-n^2+R^2\alpha^2\right)}J_n(z_{n,m}r)J_n(z_{n,m}\rho)\frac{e^{i\sqrt{k^2-z_{n,m}^2}|z-\zeta|}}{\sqrt{k^2-z_{n,m}^2}}. \tag{48}$$

- When $\lambda_{n,m}=z_{n,m}^2$ is a double pole of the spectral Green's function, i.e. $D_n(z_{n,m},\alpha)=0$ and $D_n'(z_{n,m},\alpha)=0$, we find that

$$\begin{aligned}\int_{\mathbb{R}}\operatorname*{Res}_{\nu=\lambda_{n,m}}\frac{g_n(r,\rho;\nu)}{k^2-\xi^2-\nu}\,e^{i\xi(z-\zeta)}\,d\xi=\frac{1}{2R^2J_n^2(z_{n,m}R)}\Bigg[&z_{n,m}^2J_n(z_{n,m}r)J_n(z_{n,m}\rho)\frac{\left[i\sqrt{k^2-z_{n,m}^2}|z-\zeta|-1\right]}{(k^2-z_{n,m}^2)}\\&+z_{n,m}J_{n+1}(z_{n,m}r)J_n(z_{n,m}\rho)r+z_{n,m}J_n(z_{n,m}r)J_{n+1}(z_{n,m}\rho)\rho\\&+\frac{2}{3}(2R\alpha-3n-2)J_n(z_{n,m}r)J_n(z_{n,m}\rho)\Bigg]\frac{e^{i\sqrt{k^2-z_{n,m}^2}|z-\zeta|}}{i\sqrt{k^2-z_{n,m}^2}}.\end{aligned} \tag{49}$$

Replacing (47)–(49) in (44) when it corresponds, we get the series expansion of the Green's function. It is important to note that under the assumption $\mathfrak{Im}\,k^2>0$ or $\mathfrak{Im}\,\alpha>0$, every term in the series representation of Green's function decays exponentially in the direction of the axis of the waveguide (see Proposition 3.2(c)).

Results on the convergence of the series (44) for the particular case $\alpha = 0$ can be found in [8]. More specifically, in that work it is proved that for $(\boldsymbol{x}, \boldsymbol{y}) \in \overline{\Omega} \times \overline{\Omega} \setminus \{z = \zeta\}$ and $\Im\mathrm{m}\, k^2 > 0$, the series converges uniformly. A generalization of this result for $\alpha \in \mathbb{C}$, $\Im\mathrm{m}\,\alpha \geqslant 0$ seems to be possible to obtain from this work, but it is out of the scope of this paper. Anyway, we will assume that the uniform convergence of the series (44) for the domain stated above holds for an arbitrary impedance parameter α with $\Im\mathrm{m}\,\alpha \geqslant 0$.

5.2. Local behavior at the source point

Let us replace the Bessel function of second kind, Y_n, by

$$Y_n(z) = \mathrm{i}\left[J_n(z) - H_n^{(1)}(z)\right]$$

in (8), where $H_n^{(1)}(z)$ is the Hankel function of the first kind (cf. [18]). It leads to

$$\begin{aligned} g_n(r,\rho;\lambda) &= -\frac{\mathrm{i}\,\pi}{2}\frac{\sqrt{\lambda}H_n^{(1)'}\left(\sqrt{\lambda}R\right) - \alpha H_n^{(1)}\left(\sqrt{\lambda}R\right)}{\sqrt{\lambda}J_n'\left(\sqrt{\lambda}R\right) - \alpha J_n\left(\sqrt{\lambda}R\right)} J_n\left(\sqrt{\lambda}r\right) J_n\left(\sqrt{\lambda}\rho\right) \\ &\quad + \frac{\mathrm{i}\,\pi}{2}\begin{cases} J_n\left(\sqrt{\lambda}r\right) H_n^{(1)}\left(\sqrt{\lambda}\rho\right) & \text{if } 0 < r \leqslant \rho < R, \\ J_n\left(\sqrt{\lambda}\rho\right) H_n^{(1)}\left(\sqrt{\lambda}r\right) & \text{if } 0 < \rho \leqslant r < R, \end{cases} \end{aligned}$$

which may be written equivalently as

$$g_n(r,\rho;\lambda) = g_n^{\infty}(r,\rho;\lambda) + g_n^{\mathrm{c}}(r,\rho;\lambda),$$

where

$$g_n^{\infty}(r,\rho;\lambda) = \frac{\mathrm{i}\,\pi}{2}\begin{cases} J_n\left(\sqrt{\lambda}r\right) H_n^{(1)}\left(\sqrt{\lambda}\rho\right) & \text{if } 0 < r \leqslant \rho < R, \\ J_n\left(\sqrt{\lambda}\rho\right) H_n^{(1)}\left(\sqrt{\lambda}r\right) & \text{if } 0 < \rho \leqslant r < R, \end{cases} \tag{50}$$

and

$$g_n^{\mathrm{c}}(r,\rho;\lambda) = -\frac{\mathrm{i}\,\pi}{2}\frac{\sqrt{\lambda}H_n^{(1)'}(\lambda R) - \alpha H_n^{(1)}(\lambda R)}{\sqrt{\lambda}J_n'(\lambda R) - \alpha J_n(\lambda R)} J_n(\lambda r) J_n(\lambda\rho). \tag{51}$$

Now, setting (50) and (51) in the integral representation of the Green's function (3), we define

$$G^{\infty}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi^2}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\,\xi(z-\zeta)} \sum_{n\in\mathbb{N}_0} g_n^{\infty}(r,\rho;k^2-\xi^2)\cos(n(\theta-\vartheta))\epsilon_n\,\mathrm{d}\xi \tag{52}$$

and

$$G^{\mathrm{c}}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi^2}\int_{\mathbb{N}_0\times\mathbb{R}} g_n^{\mathrm{c}}(r,\rho;k^2-\xi^2)\cos(n(\theta-\vartheta))\epsilon_n\,\mathrm{d}\xi\,\mathrm{d}n, \tag{53}$$

from which the Green's function may be expressed as $G = G^{\infty} + G^{\mathrm{c}}$. Hence, as a result of Graf's addition theorem (cf. [27, p. 21, Eq. (3b)]) we find that

$$H_0^{(1)}\left(\sqrt{k^2-\xi^2}\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\vartheta)}\right) = \sum_{n\in\mathbb{N}_0}\epsilon_n H_n^{(1)}\left(\sqrt{k^2-\xi^2}\rho\right) J_n\left(\sqrt{k^2-\xi^2}r\right)\cos(n(\theta-\vartheta)). \tag{54}$$

Therefore, replacing (54) in (52), we obtain

$$G^{\infty}(\boldsymbol{x},\boldsymbol{y}) = \frac{\mathrm{i}}{8\pi}\int_{\mathbb{R}} H_0^{(1)}\left(\sqrt{k^2-\xi^2}\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\vartheta)}\right) \mathrm{e}^{\mathrm{i}\,\xi(z-\zeta)}\,\mathrm{d}\xi, \tag{55}$$

and then, thanks to Weyrich's formula [27, p. 34],

$$\frac{\mathrm{i}}{2}\int_{\mathbb{R}} H_0^{(1)}\left(\sqrt{k^2-\xi^2}\varrho\right) \mathrm{e}^{\mathrm{i}\,\xi z}\,\mathrm{d}\xi = \frac{\mathrm{e}^{\mathrm{i}\,k\sqrt{\varrho^2+z^2}}}{\sqrt{\varrho^2+z^2}},$$

which is valid for every $k \in \mathbb{C}$ such that $0 \leqslant \arg\sqrt{k^2-\xi^2} < \pi$, $0 \leqslant \arg(k) < \pi$, which is our case. We get that the integral in (55) can be solved exactly to obtain

$$G^{\infty}(\boldsymbol{x},\boldsymbol{y}) = \frac{\mathrm{e}^{\mathrm{i}\,k\sqrt{(z-\zeta)^2+r^2+\rho^2-2r\rho\cos(\theta-\vartheta)}}}{4\pi\sqrt{(z-\zeta)^2+r^2+\rho^2-2r\rho\cos(\theta-\vartheta)}},$$

where $G^{\infty}(\boldsymbol{x},\boldsymbol{y}) = \Phi(\boldsymbol{x}-\boldsymbol{y})$, with

$$\Phi(\boldsymbol{x}) = \frac{\mathrm{e}^{\mathrm{i}\,k|\boldsymbol{x}|}}{4\pi\,|\boldsymbol{x}|},$$

the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$. It can be noted that G^{∞} contains the unique singularity of the Green's function located at $\boldsymbol{y} = \boldsymbol{x}$. Therefore, the remaining term G^{c} is "regular", in the sense that numerical techniques such as the Inverse Fast Fourier Transform may be applied to approximate it (cf. [9–11]).

5.3. Undamped wave propagation and the radiation condition

Now we turn our attention to the complete undamped case, i.e. when the wave number and the impedance parameter are such that $\mathfrak{Im}\,k^2 = \mathfrak{Im}\,\alpha = 0$ and, consequently, all the poles of the spectral Green's function $\lambda_{n,m}$ are simple and lie on the real line.

When these conditions hold, the functions in the inverse Fourier transforms (45) and (46) no longer belong to $L^1(\mathbb{R})$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, because, when $k^2 > z_{n,m}^2$ for some m and n, they have poles on the real line at $\sqrt{k^2 - z_{n,m}^2}$ and $-\sqrt{k^2 - z_{n,m}^2}$. Consequently, the integrals (45) and (46) lack meaning within the frame of the Lebesgue integral and must be understood in a broader sense. To face that problem we appeal to the limiting absorption principle (cf. [28,29]), a mathematical procedure that allows us to obtain a suitable meaning in physical terms for (45) and (46) in a way that leads to an unique outgoing wave solution of (1).

To apply the limiting absorption principle we ought to add dissipation to the system, perturbing either the wave number designated as $k_\varepsilon = k + \mathrm{i}\,\varepsilon,\ \varepsilon > 0$ or the impedance parameter as $\alpha_\varepsilon = \alpha + \mathrm{i}\,\varepsilon,\ \varepsilon > 0$. Subsequently, (1) can be solved resorting to the procedure described above to obtain an ε-dependent Green's function G_ε. Finally, taking the limit assuming the uniform convergence of the series

$$G = \lim_{\varepsilon\to 0} G_\varepsilon \tag{56}$$

we get the unique outgoing wave solution of (1) that can be represented by the following series

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{\mathrm{i}}{4\pi}\sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}} \epsilon_n \cos(n(\theta-\vartheta))\varphi_{n,m}(r)\varphi_{n,m}(\rho)\frac{\mathrm{e}^{\mathrm{i}\sqrt{k^2-\lambda_{n,m}}|z-\zeta|}}{\sqrt{k^2-\lambda_{n,m}}}, \tag{57}$$

where

$$\varphi_{n,m}(r) = \frac{\sqrt{2}z_{n,m}J_n\left(z_{n,m}r\right)}{J_n\left(z_{n,m}R\right)\sqrt{R^2z_{n,m}^2 - n^2 + R^2\alpha^2}}, \quad m \in \mathbb{N}, \tag{58}$$

if $\alpha \neq n/R,\ \lambda_{n,m} = z_{n,m}^2$ and

$$\varphi_{n,m}(r) = \begin{cases} \dfrac{\sqrt{2}J_n(j_{n+1,m-1}r/R)}{J_n\left(j_{n+1,m-1}\right)R} & \text{if } m > 1,\\[2ex] \dfrac{\sqrt{2n+2}}{R}\left(\dfrac{r}{R}\right)^n & \text{if } m = 1, \end{cases} \tag{59}$$

if $\alpha = n/R$ for some $n \in \mathbb{N}_0$. Let us observe that the set of functions $\{\sqrt{r}\varphi_{n,m}(r)\}$ forms a complete orthonormal basis of $L^2((0,R))$. It can be deduced from the theory of Dini series [12] or, equivalently, from the theory of singular self-adjoint Sturm–Liouville problems [30].

Unlike the damped case, in the undamped case the Green's function does not decay exponentially in the direction of the waveguide's axis. In fact, we have that the term $\mathrm{i}\sqrt{k^2-\lambda_{n,m}}$ in (57) can be either purely real or purely imaginary. Thus, when $k^2 > \lambda_{n,m}$, it holds that $\sqrt{k^2-\lambda_{n,m}} > 0$, and then the mode in the series representation of the time-harmonic Green's function $\mathfrak{Re}(G(\boldsymbol{x},\boldsymbol{y})\,\mathrm{e}^{-\mathrm{i}\,\omega t})$, that is

$$\mathfrak{Re}\left[\cos(n(\theta-\vartheta))\varphi_{n,m}(r)\varphi_{n,m}(\rho)\,\mathrm{e}^{\mathrm{i}\left[\sqrt{k^2-\lambda_{n,m}}|z-\zeta|-\omega t\right]}\right], \tag{60}$$

represents an unattenuated propagative mode that travels out from the point source located at $\boldsymbol{x} = (\rho\cos\vartheta, \rho\sin\vartheta, \zeta)$ in the direction of the waveguide's axis, with velocity $c_{n,m} = \omega/\sqrt{k^2-\lambda_{n,m}}$. On the other hand, when $k^2 < \lambda_{n,m}$ we have that $\mathrm{i}\sqrt{k^2-\lambda_{n,m}} < 0$, thus the mode (60) decays exponentially while it travels out from the source. The latter modes are the so-called evanescent modes and the former correspond to the propagative modes.

The following result states that in absence of dissipation the number of propagative modes is finite while the number of evanescent modes is infinite.

Proposition 5.1. *The cardinality of the set*

$$\Lambda = \left\{(n, m) \in \mathbb{N}_0 \times \mathbb{N} : D\left(\sqrt{\lambda_{n,m}}, \alpha\right) = 0 \text{ and } \lambda_{n,m} < k^2\right\} \tag{61}$$

is finite.

Proof. First, let us observe that when $\alpha \in \mathbb{R}$ we have that $\lambda_{n,m} \in \mathbb{R}$ and $\lambda_{n,m} \to \infty$ as $m \to \infty$ for every $n \in \mathbb{N}_0$. Then, there exists $M \in \mathbb{N}$ such that $\lambda_{n,m} > k^2$ for every $m > M$ and $n \in \mathbb{N}_0$. On the other hand, according to [24] the following bound holds:

$$x_1^2 > \frac{(Rn-\alpha)(n+1)}{R(2R+nR-\alpha)}, \quad \forall n > \alpha/R,$$

where x_1 is the first positive (real) root of $D_n(\cdot, \alpha)$. According to that, we can choose a $N > \max\{\alpha/R, 0\}$ such that

$$\lambda_{n,1} > \frac{(Rn-\alpha)(n+1)}{R(2R+nR-\alpha)} > k^2, \quad \forall n > N,$$

and hence we obtain that $|\Lambda| < MN < \infty$. $\square$

Now, since the evanescent modes decay exponentially, the far-field form of the Green's function is only composed by the propagative modes. Hence it is given by

$$G^{ff}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\mathrm{i}}{4\pi} \sum_{(n,m)\in\Lambda} \epsilon_n \cos(n(\theta - \vartheta))\varphi_{n,m}(r)\varphi_{n,m}(\rho) \frac{\mathrm{e}^{\mathrm{i}\sqrt{k^2-\lambda_{n,m}}|z-\zeta|}}{\sqrt{k^2 - z_{n,m}^2}}.$$

This far-field form of the Green's function and the orthogonality of the functions

$$\left\{\sqrt{r}\varphi_{n,m}(r)\,\mathrm{e}^{\mathrm{i}n\theta}, (n, m) \in \mathbb{N}_0 \times \mathbb{N}\right\}$$

in the $L^2(\Omega')$ inner product (where Ω' is the waveguide's cross section), allow us to obtain directly the radiation condition, which can be written as

$$\lim_{|L|\to\infty} \int_{\Gamma_L} \left\{\frac{\partial G}{\partial z} - \mathrm{i}\sqrt{k^2 - \lambda_{n,m}}\operatorname{sign}(z-\zeta)G\right\} \mathrm{e}^{\mathrm{i}n\theta}\varphi_{n,m}(r)\,\mathrm{d}\sigma_{\boldsymbol{y}} = 0, \quad \forall (n, m) \in \Lambda$$

where the convergence of the limit is uniformly for every $\boldsymbol{x} \in \Omega$, the surface Γ_L is defined by

$$\Gamma_L = \left\{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2, x_3 = L\right\} = \Omega' \times \{L\}$$

and the set of indices Λ is defined in (61).

An interesting phenomenon arising only when a non-dissipative impedance boundary condition is imposed on the waveguide's walls, is the propagation of a surface wave. This kind of wave decays exponentially to the interior of the waveguide, transporting energy only near the waveguide's surface. This wave is the result of the superposition of the surface wave modes that comprise the series expansion of the Green's function. These modes appear for any real impedance parameter $\alpha > 0$ due the negative poles of the spectral Green's function that arise when a purely imaginary root of $D_n(\cdot, \alpha)$ exists. It leads to the fact that the radial component of these modes (58) is

$$\varphi_{n,1}(r) = \frac{\sqrt{2}y_n I_n(y_n r)}{I_n(y_n R)\sqrt{R^2 y_n^2 + n^2 - R^2\alpha^2}},$$

where I_n are the modified Bessel functions of the first kind (cf. [18]), which are monotonically increasing functions.

6. Numerical procedures

In this section we state some basic procedures to obtain numerical evaluations of the Green's function (44). To do that, we present two different methods to compute the zeros of $D_n(\cdot, \alpha)$, based on standard algorithms for finding the roots of a real valued function when the impedance parameter is real, and based on the finite element method when the impedance parameter is a proper complex number. We are particularly interested in obtaining accurate approximations for the smallest $z_{n,m}$, since the larger ones can be approximated by the asymptotic formula obtained in Proposition 3.2.

6.1. Real impedance case

As was discussed above, in this case there exist an infinite number of real positive roots of $D_n(\cdot, \alpha)$ and only one purely imaginary root in the positive imaginary axis. Therefore, the search for the imaginary root does not present difficulties

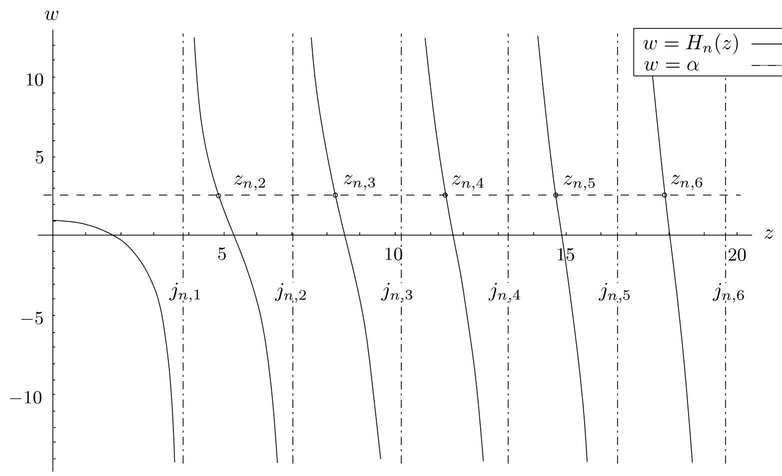

Fig. 4. Location of $z_{n,m}$ for $\alpha = 2.5$, $R = 1$ and $n = 1$.

because there are standard methods to solve that kind of problem. However, the search for the positive real roots produce more complications. To face them, let us note that the equation $D_n(z, \alpha) = 0$ can be expressed as

$$F_n(z) = \alpha,$$

so that to achieve a good approximation of $z_{n,m}$ we have to compute the intersection points of the horizontal line $w = \alpha$ and the real valued function $w = F_n(z)$. Thus, since $F_n'(z) < 0$ for all $0 < z \neq j_{n,m}/R$ and $n > -1$ (cf. [15]) and F_n has poles in $j_{n,m}$, we have that there is at most one intersection point in every interval $[j_{n,m}/R, j_{m+1,n}/R]$. So that, to obtain the positive real roots of $D_n(\cdot, \alpha)$ we only have to search for them in every interval $[j_{n,m}/R, j_{m+1,n}/R]$. Fig. 4 shows the situation.

From a numerical point of view, to obtain the value of $z_{n,m}$ we need first to have $j_{n,m}$, the positive zeros of the Bessel function $J_n(x)$. To compute them there exist several efficient numerical methods. Some of them can be found in [31,32]. Once the numbers $j_{n,m}$ are found, we may obtain $z_{n,m}$ making use of standard algorithms to find zeros of functions that change their sign once in a given interval, such as can be found in [33] (which is implemented in Matlab).

6.2. Complex impedance case

When the impedance parameter is a proper complex number we cannot use the method described above because nothing guarantees that the zeros are real. In fact, they have a non-zero imaginary part, as can be seen from Fig. 5. Consequently, we have to face the problem of finding $z_{n,m}$ on the whole complex plane.

To find these values we appeal to the Bessel differential equation. Thus, let us note that the zeros of $D_n(\cdot, \alpha)$ can be characterized as the eigenvalues of the following differential problem for $\varphi \in C^1[0, R] \cap C^2(0, R)$

$$\begin{cases} -\left(r\varphi'\right)' + \dfrac{n^2}{r}\varphi = r\lambda\varphi, & 0 < r < R, \\ \varphi' - \alpha\varphi = 0, & r = R, \\ \lim\limits_{r\to 0^+} \varphi(r) < \infty, & \end{cases} \tag{62}$$

where the eigenvalues and the eigenfunctions are $\lambda = z_{n,m}^2$ and $\varphi_{n,m}$ respectively and where the functions $\varphi_{n,m}$ are defined in (58) and (59).

To achieve the numerical values of $z_{n,m}$, we transform the differential eigenvalue problem (62) into a generalized matrix eigenvalue problem by means of the finite element method or the finite difference method. To employ the finite element method, we discretize the interval $(0, R)$ at N regular subintervals $[r_n, r_{n+1}]$ of length $h = R/N$, where $r_n = nR/N,\ 0 \leqslant n \leqslant N$, and we construct basis functions that are linear at every subinterval and satisfy $\psi_n(r_m) = \delta_{n,m}$. Thus, multiplying (62) by $r\psi \in H^1(0, R)$, and using integration by parts, the variational formulation is obtained. Next, making use of the Galerkin method with basis functions ψ_n, the discretized version of (62) is achieved and reads as follows:

$$\begin{cases} \text{Find } 0 \neq \lambda^{(h)} \in \mathbb{C} \quad \text{and} \quad \mathbf{0} \neq \mathbf{x} \in \mathbb{C}^N \text{ such that:} \\ \mathbf{A}_n\mathbf{x} = \lambda^{(h)}\mathbf{B}\mathbf{x}, \end{cases} \tag{63}$$

where the matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{N\times N}$ are sparse and are defined by:

$$[\mathbf{A}_n]_{i,j} = -\alpha R^2\psi_i(R)\psi_j(R) + \int_0^R r^2\psi_i(r)\psi_j'(r)\,\mathrm{d}r + n^2\int_0^R \psi_i(r)\psi_j(r)\,\mathrm{d}r$$

$$[\mathbf{B}]_{i,j} = \int_0^R r^2\psi_i(r)\psi_j(r)\,\mathrm{d}r.$$

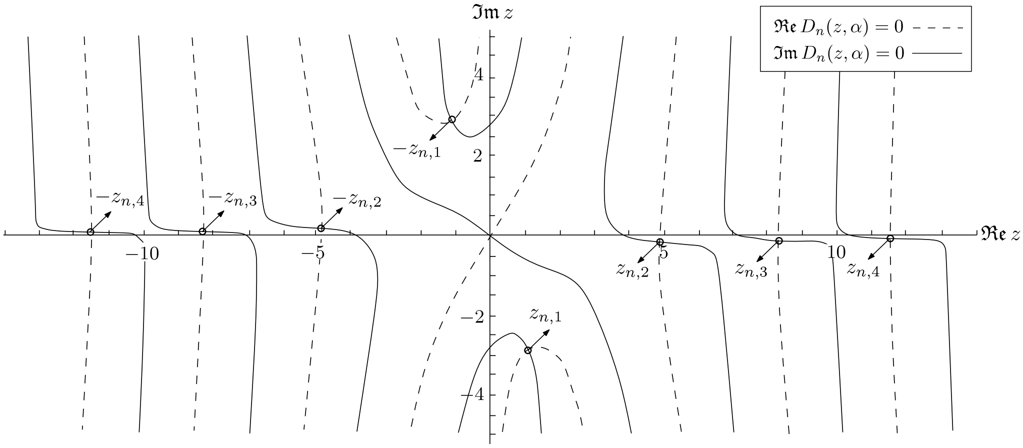

Fig. 5. Location of $z_{n,m}$ for $\alpha = 2.5 + \mathrm{i}$, $R = 1$ and $n = 1$.

Table 3
Numerical results of the finite element method applied to find the first five zeros of $D_n(\cdot, \alpha)$ for $R = 1$, $n = 0, 1, 2, 3, 4$ and $\alpha = 2.5 + \mathrm{i}$. All the values were computed with $h = 10^{-5}$.

n	m	$z_{n,m}^{(h)}$	$j_{n+1,m}/R$	$\lvert D_n(z_{n,m}^{(h)}, \alpha)\rvert$
0	1	0.9701 − 3.0594i	3.8317	$0.0834 \cdot 10^{-5}$
	2	3.1926 − 0.1935i	7.0156	$0.0055 \cdot 10^{-5}$
	3	6.6600 − 0.1322i	10.1735	$0.0025 \cdot 10^{-5}$
	4	9.9279 − 0.0950i	13.3237	$0.0057 \cdot 10^{-5}$
	5	13.1361 − 0.0736i	16.4706	$0.0042 \cdot 10^{-5}$
1	1	1.1116 − 2.8342i	5.1356	$0.5515 \cdot 10^{-5}$
	2	4.8468 − 0.1694i	8.4172	$0.0464 \cdot 10^{-5}$
	3	8.2396 − 0.1131i	11.6198	$0.0394 \cdot 10^{-5}$
	4	11.4909 − 0.0839i	14.7960	$0.0295 \cdot 10^{-5}$
	5	14.6947 − 0.0665i	17.9598	$0.0281 \cdot 10^{-5}$
2	1	1.5439 − 2.1842i	6.3802	$0.7108 \cdot 10^{-5}$
	2	6.2959 − 0.1511i	9.7610	$0.1933 \cdot 10^{-5}$
	3	9.7081 − 0.1009i	13.0152	$0.1558 \cdot 10^{-5}$
	4	12.9761 − 0.0762i	16.2235	$0.1356 \cdot 10^{-5}$
	5	16.1923 − 0.0613i	19.4094	$0.1206 \cdot 10^{-5}$
3	1	2.5572 − 1.4060i	7.5883	$0.5930 \cdot 10^{-5}$
	2	7.6507 − 0.1378i	11.0647	$0.4173 \cdot 10^{-5}$
	3	11.1088 − 0.0923i	14.3725	$0.3600 \cdot 10^{-5}$
	4	14.4068 − 0.0704i	17.6160	$0.3167 \cdot 10^{-5}$
	5	17.6441 − 0.0572i	20.8269	$0.2794 \cdot 10^{-5}$
4	1	3.8601 − 0.9877i	8.7715	$0.8437 \cdot 10^{-5}$
	2	8.9496 − 0.1277i	12.3386	$0.6867 \cdot 10^{-5}$
	3	12.4627 − 0.0858i	15.7002	$0.5919 \cdot 10^{-5}$
	4	15.7969 − 0.0660i	18.9801	$0.5265 \cdot 10^{-5}$
	5	19.0599 − 0.0540i	22.2178	$0.4756 \cdot 10^{-5}$

The generalized matrix eigenvalue problem (63) can be solved with ARPACK (or the MATLAB version of it, implemented in the function `eigs.m`) to obtain the eigenvalues $\lambda^{(h)}$ with smallest absolute value. Thus, with this method we can achieve the first N smallest approximated zeros of $D_n(\cdot, \alpha)$, which are $z_{n,m}^{(h)} = \sqrt{\lambda^{(h)}}$.

Table 3 shows the results for the particular case $n = 1$, where the error is measured by $\lvert D_n(z_{n,m}^{(h)}, \alpha)\rvert$.

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