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PII: S0377-0427(16)30141-8
DOI: http://dx.doi.org/10.1016/j.cam.2016.03.021
Reference: CAM 10575

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 14 October 2015
Revised date: 20 March 2016

Please cite this article as: L. Liu, Q. Zhu, Mean square stability of two classes of theta method for neutral stochastic differential delay equations, *Journal of Computational and Applied Mathematics* (2016), http://dx.doi.org/10.1016/j.cam.2016.03.021

Mean square stability of two classes of theta method for neutral stochastic differential delay equations *

Linna Liu[1] and Quanxin Zhu[1,2†]
[1] School of Mathematical Sciences and Institute of Finance and Statistics, Nanjing Normal University, Nanjing, 210023, Jiangsu, China
[2]Department of Mathematics, University of Bielefeld, Bielefeld D-33615, Germany

Abstract

In this paper, a stochastic linear theta (SLT) method is introduced and analyzed for neutral stochastic differential delay equations (NSDDEs). We give some conditions on neutral item, drift and diffusion coefficients, which admit that the diffusion coefficient can be highly nonlinear and does not necessarily satisfy a linear growth or global Lipschitz condition. It is proved that, for all positive stepsizes, the SLT method with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean stable and so is $\theta \in [0, \frac{1}{2})$ under a stronger assumption. Furthermore, we consider the split-step theta (SST) method and obtain a similar but better result. That is, the SST method with $\theta \in [\frac{1}{2}, 1]$ is exponentially mean stable and so is $\theta \in [0, \frac{1}{2})$. Finally, two numerical examples are given to show the efficiency of the obtained results.

Key Words: Neutral stochastic differential delay equation; mean square stability; exponential stability; stochastic linear theta method; split-step theta method.

1 Introduction

As is well known, there has appeared a large number of works on neutral stochastic differential delay equations (NSDDEs) (see [1–5]) since they have been widely applied to many fields such as economics, finance, physics, biology, medicine, and other science. The

*This work was jointly supported by the Alexander von Humboldt Foundation of Germany (Fellowship CHN/1163390), the National Natural Science Foundation of China (61374080), the Priority Academic Program Development of Jiangsu Higher Education Institutions.

†The correspondent author's e-mail: zqx22@126.com.

stability issue of NSDDEs is one of the most important problems in their research field. Recently, various stability theorems of stochastic differential systems, for example, moment stability (M-stability, see [6, 7]) and almost sure stability (or the trajectory stability (T-stability), see [8]), have been reputed in the literature. Some of the stability criteria related neutral stochastic functional differential equations (NSFDEs) were considered in [2, 4, 5, 9–12] and the references therein. On the other hand, many NSDDEs may not have explicit solutions. Therefore, it seems to be interesting and necessary to study the numerical solutions of NSDDEs (see [13–18]). However, there has been very few works to consider the theta methods on NSDDEs, despite its practical importance and more extensive.

Luckily, there have appeared some results on the numerical solutions about theta methods of stochastic ordinary differential equations (SODEs). Stochastic linear theta (SLT) method is the simplest method, and it has been widely used in the literature. For example, the mean square stability of the SLT method was investigated in [8, 19–22] for linear SODEs and in [23] for nonlinear SODEs. For stochastic differential delay equations (SDDEs), Huang [24] investigated the exponential mean square stability of SLT method and so was Mao in [31]. Zong et al. in [32] proved that the SLT method can inherit the exponential mean square stability of the exact solution for SODEs and SDDEs. Besides, Huang also introduced another theta method called the split-step theta (SST) method in [33]. For the special case of $\theta = 0$, this approximation is EM approximation, and for the case of $\theta = 1$, this approximation is equivalent to the split-step backward Euler (SSBE) method. Both for SLT and SST methods, Huang in [32] revealed that the linear growth condition on the drift coefficient is necessary with $\theta \in [0, \frac{1}{2})$ to be mean square stable, but for $\theta \in [\frac{1}{2}, 1]$, two methods can reproduce the exponential mean square stability without the linear growth condition. Also, Liu et al. [34] studied the mean-square stability of the stochastic theta method for linear scalar model equations. Baker and Buckwar [35] analyzed the exponential stability in p-th moment of the stochastic theta method by using the Halanay inequality. Wang and Gan [36] investigated the mean-square exponential stability of a split-step Euler method. However, all of the above results are derived from SDDEs in which the diffusion coefficient need to satisfy a linear growth or global Lipschitz condition. Moreover, these results ignored the effect of the neutral term, which often yields much difficulty.

Motivated by the above discussion, in this paper, we study the stability of numerical methods for NSDDEs under some conditions on the drift coefficient, diffusion coefficient and neutral term. These conditions admit that the diffusion coefficient is highly nonlinear, and it does not necessarily satisfy the linear growth or global Lipschitz condition. To the best of our knowledge, there is only one paper [37] studying the stability of SST and SLT methods for NSDDES. However, in this paper we propose some weaker assumptions on the drift and diffusion coefficients than those in [37]. Indeed, we do not require the condition that f and g need to satisfy the global Lipschitz condition. Moreover, the SLT and SST methods presented in this paper generalize and improve those given in [37]. In the paper, we prove that, for all positive stepsizes, the SLT method with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean square stable and so is $\theta \in [0, \frac{1}{2})$ under a stronger assumption. Furthermore, we also establish the SST method method with $\theta \in [\frac{1}{2}, 1]$ is exponentially mean stable and so is $\theta \in [0, \frac{1}{2})$. Hence, we can see that the SST method has a better exponential stability property than the SLT method.

The rest of the paper is arranged as follows. In Section 2, we introduce some notations, assumptions and preliminary lemmas. In Section 3, we use the SLT method to discuss the mean square stability of numerical solutions to NSDDEs. In Section 4, we use the SST method to investigate the mean square stability of numerical solutions to NSDDEs. After some numerical examples are provided to illustrate the obtained results in Section 5, we conclude the paper with some general remarks in Section 6.

2 Notations, assumptions and lemmas

Throughout this paper, we use the following notations. If A is a vector or matrix, its transpose is denoted by A^T. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace norm in $\mathbb{R}^{n\times d}$(denoted by $|A| = \sqrt{trace(A^TA)}$). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\{w(t), t \geq 0\}$ be a d-dimensional Brownian motion defined on the probability space.

Let $D : \mathbb{R}^n \longmapsto \mathbb{R}^n, f : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{R}^n, g : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{R}^{n\times d}$ be Borel measurable functions. Let us consider the following neutral stochastic differential delay equation

$$d[y(t) - D(y(t-\tau))] = f(y(t), y(t-\tau))dt + g(y(t), y(t-\tau))dw(t), t > 0, \tag{2.1}$$

with initial data $y(t) = \Phi(t) \in C([-\tau, 0]; \mathbb{R}^n)$ satisfying

$$\sup_{-\tau\leq t\leq 0} \mathbb{E}[\Phi^T(t)\Phi(t)] < +\infty. \tag{2.2}$$

For the purpose of stability, assume that $D(0) = f(0, 0) = 0$, $g(0, 0) = 0$. This implies that system (2.1) admits a trivial solution.

There exist many numerical schemes for stochastic differential equations in the literature. If an appropriate interpolation procedure for the delay argument is employed, these schemes can be adapted to solve NSDDEs. An adaptation of the classic stochastic theta method to system (2.1) leads to

$$\begin{aligned} y_{n+1} = y_n &+ D(\bar{y}_{n+1}) - D(\bar{y}_n) \\ &+ \theta\Delta f(y_{n+1}, \bar{y}_{n+1}) + (1-\theta)\Delta f(y_n, \bar{y}_n) + g(y_n, \bar{y}_n)\Delta w_n, \end{aligned} \tag{2.3}$$

where $\Delta > 0$ is the time stepsize, y_n is an approximation to $y(t_n), \theta \in [0, 1]$ is a fixed parameter, $\Delta w_n = w(t_{n+1}) - w(t_n)$, and $\bar{y}_n$ denotes an approximation to the delay argument $y(t_n - \tau)$.

For an arbitrarily fixed time stepsize Δ, there exist a unique positive integer m and a real number $\delta \in [0, 1)$ such that $\tau = (m - \delta)\Delta$. This implies that $y(t_n - \tau) = y(t_{n-m} + \delta\Delta)$.

Therefore, it is natural to define y_n by the linear interpolation

$$\bar{y}_n = \delta y_{n-m+1} + (1-\delta) y_{n-m}, \tag{2.4}$$

where $\bar{y}_n = \Phi(t_n)$ for $n \leq 0$.

In order to distinguish this method and another method with parameter θ below, we will refer to (2.3) as the stochastic linear theta (SLT) method following the notation in [33]. An adaptation of the split-step theta (SST) method in [33] to system (2.1) leads to

$$Y_n = y_n - D(\bar{y}_n) + D(\bar{Y}_n) + \theta \Delta f(Y_n, \bar{Y}_n), \tag{2.5}$$

$$\bar{y}_n = y_{n-m}, \bar{Y}_n = \delta Y_{n-m+1} + (1-\delta) Y_{n-m}, \tag{2.6}$$

$$y_{n+1} = y_n - D(\bar{y}_n) + D(\bar{y}_{n+1}) + \Delta f(Y_n, \bar{Y}_n) + g(Y_n, \bar{Y}_n)\Delta w_n. \tag{2.7}$$

Here we use the equi-stage linear interpolation technique [25] to approximate the delay argument. In the case of deterministic delay equations (i.e., $g = 0$) , it is known that this interpolation can lead to some desirable linear and nonlinear stability properties (see [25] and [26]). We naturally hope that it will have a good performance for stochastic equations.

In the special case of $\theta = 1$, this method is equivalent to the split-step backward Euler method, which was firstly proposed for stochastic ordinary differential equations in [27]. We also mention that there exist some other types of split methods with the parameter θ in the literature ([28], [29], [30]). The reason why we consider scheme (2.5)-(2.7) is that we can establish some useful stability results for it. In particular, this scheme possesses a better exponential mean square stability property than the classic SLT method.

Then, let us give some stability concepts for numerical methods.

Definition 2.1. *For a given stepsize Δ, a numerical method is said to be exponentially stable in mean square if there is a pair of positive constants γ and C such that for any initial data $\Phi(t)$, the numerical solution y_n produced by the method satisfies*

$$\mathbb{E}[y_n^T y_n] \leq C e^{-\gamma t_n} \cdot \sup_{-\tau \leq t \leq 0} [\Phi^T(t)\Phi(t)], \forall n \geq 0.$$

Definition 2.2. *For a given stepsize Δ, a numerical method is said to be asymptotically stable in mean square if for any initial data $\Phi(t)$, the numerical solution y_n produced by the method satisfies*

$$\lim_{n\to\infty} \mathbb{E}[y_n^T y_n] = 0.$$

To discuss the stability of numerical methods, we need to impose the following necessary assumptions on system (2.1).

Assumption 1. There exists a symmetric and positive definite $n \times n$ matrix Q such that

$$D(v)^T Q D(v) \leq \tilde{\gamma} v^T Q v, \tag{2.8}$$

where $\tilde{\gamma} \in (0, 1)$.

Assumption 2. There exist a symmetric, positive definite $n \times n$ matrix Q and two positive constants K_1 and K_2 such that

$$f^T(u, v) Q f(u, v) \leq K_1 u^T Q u + K_2 v^T Q v, (u, v) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.9}$$

The following two lemmas will play an important role in this paper.

Lemma 2.3. *([3]) Assume that there exist a symmetric, positive definite $n \times n$ matrix Q and two constants $\tilde{\alpha}$ and two $\tilde{\beta}$ such that for all $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$,*

$$(u - D(v))^T Q f(u, v) + \frac{1}{2} trace[g^T(u, v) Q g(u, v)] \leq \tilde{\alpha} u^T Q u + \tilde{\beta} v^T Q v, \tag{2.10}$$

with $\tilde{\alpha} + \tilde{\beta} < 0$, then the trivial solution of system (2.1) is exponentially mean square stable.

Lemma 2.4. *([3]) Condition (2.10) implies $\tilde{\beta} \geq 0$.*

In the following sections, we will employ these lemmas and assumptions to establish the mean square stability theorem for SLT approximation and SST approximation, respectively.

3 Stability analysis of the SLT approximation

Let us firstly investigate the stability of SLT approximation $\{y_n\}_{n \geq 0}$.

Theorem 3.1. *Assume that system (2.1) satisfies (2.10) with $\tilde{\alpha} + \tilde{\beta} < 0$.*

(1) Then the SLT method (2.3)-(2.4) with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean square stable for all $\Delta > 0$.

(2) If system (2.1) satisfies (2.9), then for any $\theta \in [0, \frac{1}{2})$, there exists a constant Δt_0 depending on θ such that the SLT method is asymptotically mean square stable for $\Delta \in (0, \Delta t_0)$.

Proof. It follows from (2.3) and a direct computation that

$$\begin{aligned}
&(z_{n+1} - \theta \Delta f(y_{n+1}, \bar{y}_{n+1}))^T Q (z_{n+1} - \theta \Delta f(y_{n+1}, \bar{y}_{n+1})) \\
&= (z_n + (1-\theta)\Delta f(y_n, \bar{y}_n) + g(y_n, \bar{y}_n)\Delta w_n)^T Q (z_n + (1-\theta)\Delta f(y_n, \bar{y}_n) + g(y_n, \bar{y}_n)\Delta w_n) \\
&= (z_n - \theta \Delta f(y_n, \bar{y}_n))^T Q (z_n - \theta \Delta f(y_n, \bar{y}_n)) + (1 - 2\theta)\Delta^2 f^T(y_n, \bar{y}_n) Q f(y_n, \bar{y}_n) \\
&\quad + 2\Delta z_n^T Q f(y_n, \bar{y}_n) + \Delta w_n^T g^T(y_n, \bar{y}_n) Q g(y_n, \bar{y}_n) \Delta w_n \\
&\quad + 2\Delta w_n^T g^T(y_n, \bar{y}_n) Q (z_n + (1-\theta)\Delta f(y_n, \bar{y}_n)),
\end{aligned} \tag{3.1}$$

where $z_n = y_n - D(\bar{y}_n)$.

Since $w(t)$ is a standard d-dimensional Brownian motion, we have

$$\Delta w_n = w(n+1) - w(n) \sim N(0, \Delta I_d),$$

where I_d is the identity matrix. Noting that $g^T(y_n, \bar{y}_n)Qg(y_n, \bar{y}_n)$ is independent of Δw_n, we obtain

$$\mathbb{E}[\Delta w_n^T g^T(y_n, \bar{y}_n)Qg(y_n, \bar{y}_n)\Delta w_n] = \Delta\mathbb{E}[trace(g^T(y_n, \bar{y}_n)Qg(y_n, \bar{y}_n))].$$

Taking expectation on both sides of (3.1), we get

$$\begin{aligned}\tilde{u}_{n+1} :&= \mathbb{E}[(z_{n+1} - \theta\Delta f(y_{n+1}, \bar{y}_{n+1}))^T Q(z_{n+1} - \theta\Delta f(y_{n+1}, \bar{y}_{n+1}))] \\ &= \tilde{u}_n + (1-2\theta)\Delta^2\mathbb{E}f^T(y_n, \bar{y}_n)Qf(y_n, \bar{y}_n) \\ &\quad + 2\Delta\mathbb{E}[z_n^T Qf(y_n, \bar{y}_n) + \frac{1}{2}trace(g^T(y_n, \bar{y}_n)Qg(y_n, \bar{y}_n))].\end{aligned}$$

Using (2.10), we have

$$\begin{aligned}\tilde{u}_{n+1} &\le \tilde{u}_n + (1-2\theta)\Delta^2\mathbb{E}[f^T(y_n, \bar{y}_n)Qf(y_n, \bar{y}_n)] \\ &\quad + 2\Delta\mathbb{E}[\tilde{\alpha}y_n^T Qy_n + \tilde{\beta}\bar{y}_n^T Q\bar{y}_n] \\ &\le \tilde{u}_0 + (1-2\theta)\Delta^2\sum_{j=0}^{n}\mathbb{E}[f^T(y_j, \bar{y}_j)Qf(y_j, \bar{y}_j)] \\ &\quad + 2\Delta\sum_{j=0}^{n}\mathbb{E}[\tilde{\alpha}y_j^T Qy_j + \tilde{\beta}\bar{y}_j^T Q\bar{y}_j].\end{aligned} \tag{3.2}$$

Then, it follows from (2.4) that

$$\bar{y}_j{}^T Q\bar{y}_j \le \delta y_{j-m+1}^T Qy_{j-m+1} + (1-\delta)y_{j-m}^T Qy_{j-m},$$

which gives

$$\sum_{j=0}^{n}\bar{y}_j{}^T Q\bar{y}_j \le \sum_{j=-m+1}^{n-m+1} y_j^T Qy_j + (1-\delta)y_{-m}^T Qy_{-m}. \tag{3.3}$$

Besides, due to $n-m+1 \leq n$ and $\delta \in [0,1)$, we have

$$
\begin{aligned}
&\sum_{j=-m+1}^{n-m+1} y_j^T Q y_j + (1-\delta) y_{-m}^T Q y_{-m} \\
&= \sum_{j=-m+1}^{-1} y_j^T Q y_j + \sum_{j=0}^{n-m+1} y_j^T Q y_j + (1-\delta) y_{-m}^T Q y_{-m} \\
&\leq \sum_{j=-m+1}^{-1} y_j^T Q y_j + \sum_{j=0}^{n} y_j^T Q y_j + (1-\delta) \max_{-m \leq j \leq -1} y_j^T Q y_j \\
&\leq \sum_{j=0}^{n} y_j^T Q y_j + (m-\delta) \max_{-m \leq j \leq -1} y_j^T Q y_j.
\end{aligned}
$$

Since (2.10) implies $\tilde{\beta} \geq 0$, substituting (3.3) into (3.2) yields

$$
\begin{aligned}
&2\Delta \sum_{j=0}^{n} \mathbb{E}[\tilde{\alpha} y_j^T Q y_j + \tilde{\beta} \bar{y}_j^T Q \bar{y}_j] \\
&= 2\Delta\tilde{\alpha} \sum_{j=0}^{n} \mathbb{E}[y_j^T Q y_j] + 2\Delta\tilde{\beta} \sum_{j=0}^{n} \mathbb{E}[\bar{y}_j^T Q \bar{y}_j] \\
&\leq 2\Delta\tilde{\alpha} \sum_{j=0}^{n} \mathbb{E}[y_j^T Q y_j] + 2\Delta\tilde{\beta} \mathbb{E}[\sum_{j=-m+1}^{n-m+1} y_j^T Q y_j + (1-\delta) y_{-m}^T Q y_{-m}] \\
&\leq 2\Delta(\tilde{\alpha}+\tilde{\beta}) \sum_{j=0}^{n} \mathbb{E}(y_j^T Q y_j) + 2\Delta\tilde{\beta}(m-\delta) \max_{-m \leq j \leq -1} \mathbb{E}(y_j^T Q y_j) \\
&= 2\Delta(\tilde{\alpha}+\tilde{\beta}) \sum_{j=0}^{n} \mathbb{E}(y_j^T Q y_j) + 2\tilde{\beta}\tau \max_{-m \leq j \leq -1} \mathbb{E}(y_j^T Q y_j),
\end{aligned}
$$

where the last equality follows from the fact $(m-\delta)\Delta = \tau$. Therefore,

$$
\begin{aligned}
\tilde{u}_{n+1} \leq \tilde{u}_0 &+ (1-2\theta)\Delta^2 \sum_{j=0}^{n} \mathbb{E}[f^T(y_j,\bar{y}_j) Q f(y_j,\bar{y}_j)] + 2\Delta(\tilde{\alpha}+\tilde{\beta}) \sum_{j=0}^{n} \mathbb{E}(y_j^T Q y_j) \\
&+ 2\tilde{\beta}\tau \max_{-m \leq j \leq -1} \mathbb{E}(y_j^T Q y_j).
\end{aligned} \tag{3.4}
$$

We now prove part (1). Noting that $\theta \in [\frac{1}{2}, 1]$ and $\tilde{\alpha}+\tilde{\beta} < 0$, it follows from (3.4) that

$$
\sum_{j=0}^{n} \mathbb{E}(y_j^T Q y_j) \leq \frac{\tilde{u}_0}{-2\Delta(\tilde{\alpha}+\tilde{\beta})} + \frac{\tilde{\beta}\tau \max_{-m \leq j \leq -1} \mathbb{E}(y_j^T Q y_j)}{-\Delta(\tilde{\alpha}+\tilde{\beta})}. \tag{3.5}
$$

By (3.5), we see that $\sum_{j=0}^{\infty} \mathbb{E}(y_j^T Q y_j) < \infty$. So we have $\lim_{n\to\infty} \mathbb{E}[y_n^T Q y_n] = 0$, which implies that the scheme is asymptotically mean square stable for all $\Delta > 0$.

Next, we will prove part (2). In fact, it follows condition (2.9) and $\theta \in [0, \frac{1}{2})$ that

$$\begin{aligned}
&\sum_{j=0}^{n} \mathbb{E}[f^T(y_j, \bar{y}_j) Q f(y_j, \bar{y}_j)] \\
&\le K_1 \sum_{j=0}^{n} \mathbb{E}[y_j^T Q y_j] + K_2 \sum_{j=0}^{n} \mathbb{E}[\bar{y}_j^T Q \bar{y}_j] \\
&\le K_1 \sum_{j=0}^{n} \mathbb{E}[y_j^T Q y_j] + K_2 (\sum_{j=-m+1}^{n-m+1} y_j^T Q y_j + (1-\delta) y_{-m}^T Q y_{-m}) \\
&\le (K_1 + K_2) \sum_{j=0}^{n} \mathbb{E}[y_j^T Q y_j] + K_2 (m-\delta) \max_{-m \le j \le -1} \mathbb{E}(y_j^T Q y_j),
\end{aligned}$$

which together with (3.4) gives

$$\begin{aligned}
\tilde{u}_{n+1} \le\ & \tilde{u}_0 + (1-2\theta)\Delta^2 \Sigma_{j=0}^{n} \mathbb{E}[(K_1+K_2) \sum_{j=0}^{n} \mathbb{E}[y_j^T Q y_j] + K_2(m-\delta) \max_{-m\le j\le -1} \mathbb{E}(y_j^T Q y_j)] \\
& + 2\Delta(\tilde{\alpha} + \tilde{\beta}) \sum_{j=0}^{n} \mathbb{E}(y_j^T Q y_j) + 2\tilde{\beta}\tau \max_{-m\le j\le -1} \mathbb{E}(y_j^T Q y_j) \\
\le\ & \tilde{u}_0 + \Delta[(1-2\theta)(K_1+K_2)\Delta + 2(\tilde{\alpha}+\tilde{\beta})] \sum_{j=0}^{n} \mathbb{E}(y_j^T Q y_j) \\
& + [2\tilde{\beta}\tau + (1-2\theta)\Delta^2 K_2 (m-\delta) \max_{-m\le j\le -1} \mathbb{E}(y_j^T Q y_j)].
\end{aligned}$$

Setting $\Delta t_0 = \frac{-2(\tilde{\alpha}+\tilde{\beta})}{(1-2\theta)(K_1+K_2)}$, we have that $\Delta[(1-2\theta)(K_1+K_2)\Delta + 2(\tilde{\alpha}+\tilde{\beta})] < 0$ for any $\Delta \in (0, \Delta t_0)$. Similar to the proof of (3.5) in part (1), we obtain

$$\begin{aligned}
\sum_{j=0}^{n} \mathbb{E}(y_j^T Q y_j) \le\ & \frac{\tilde{u}_0}{-\Delta[(1-2\theta)(K_1+K_2)\Delta + 2(\tilde{\alpha}+\tilde{\beta})]} \\
& + \frac{2\tilde{\beta}\tau + (1-2\theta)\Delta^2 K_2(m-\delta) \max_{-m\le j\le -1} \mathbb{E}(y_j^T Q y_j)}{-\Delta[(1-2\theta)(K_1+K_2)\Delta + 2(\tilde{\alpha}+\tilde{\beta})]}.
\end{aligned} \tag{3.6}$$

By (3.6), we see that $\sum_{j=0}^{\infty} \mathbb{E}(y_j^T Q y_j) < \infty$. So we have $\lim_{n\to\infty} \mathbb{E}[y_n^T Q y_n] = 0$, which implies that the scheme is asymptotically mean square stable for any $\Delta \in (0, \Delta t_0)$. This completes the proof of Theorem 3.1.

Remark 3.2. *In Theorem 3.1, we introduce a symmetric and positive definite $n \times n$ matrix Q to discuss the SLT method of n-dimensional NSDDEs. Letting $Q = I$, then our result can be reduced to the case of one-dimensional NSDDEs, which was studied in [37]. Therefore, Theorem 3.1 generalizes and improves that given in [37].*

Remark 3.3. *It is clear that (2.3) and (2.4) can be reduced to (2.9) in [37] when $\delta = 0$. Hence, our SLT method is more general than that given in [37].*

Remark 3.4. *Indeed, the SLT method includes the EM method ($\theta = 0$), the trapezoidal method ($\theta = \frac{1}{2}$) and the BEM method ($\theta = 1$).*

4 Stability analysis of the SST approximation

In this section, we will study the stability of SST approximation $\{y_n\}_{n\geq 0}$.

Theorem 4.1. *Assume that system (2.1) satisfies (2.10) with $\tilde{\alpha} + \tilde{\beta} < 0$.*

(1) Then the SST method (2.5)-(2.7) with $\theta \in [\frac{1}{2}, 1]$ is exponentially mean square stable for all $\Delta > 0$.

(2) If system (2.1) satisfies (2.9), then for any $\theta \in (0, \frac{1}{2}]$, there exists a constant Δt_0 depending on θ such that the SST method is exponentially mean square stable for $\Delta \in (0, \Delta t_0)$.

Proof. By (2.7), we have

$$\begin{aligned} z_{n+1}^T Q z_{n+1} &= (z_n + \Delta f(Y_n, \bar{Y}_n) + g(Y_n, \bar{Y}_n)\Delta w_n)^T Q (z_n + \Delta f(Y_n, \bar{Y}_n) + g(Y_n, \bar{Y}_n)\Delta w_n) \\ &= z_n^T Q z_n + \Delta^2 f^T(Y_n, \bar{Y}_n) Q f(Y_n, \bar{Y}_n) + \Delta w_n^T g^T(Y_n, \bar{Y}_n) Q g(Y_n, \bar{Y}_n)\Delta w_n \\ &\quad + 2\Delta z_n^T Q f(Y_n, \bar{Y}_n) + 2 z_n^T Q g(Y_n, \bar{Y}_n)\Delta w_n + 2\Delta f^T(Y_n, \bar{Y}_n) g(Y_n, \bar{Y}_n)\Delta w_n, \end{aligned}$$

where $z_n = y_n - D(\bar{y}_n)$. Taking expectation on both sides of the above equality, it follows from (2.5) that

$$\begin{aligned} \mathbb{E}(z_{n+1}^T Q z_{n+1}) = \mathbb{E}(z_n^T Q z_n) &+ (1-2\theta)\Delta^2 \mathbb{E}(f^T(Y_n, \bar{Y}_n) Q f(Y_n, \bar{Y}_n)) + \\ &+ 2\Delta \mathbb{E}(c_n^T Q f(Y_n, \bar{Y}_n)) + \Delta \mathbb{E}(trace(g^T(Y_n, \bar{Y}_n) Q g(Y_n, \bar{Y}_n)), \end{aligned}$$

where $c_n = z_n + \theta f(Y_n, \bar{Y}_n)\Delta = Y_n - D(\bar{Y}_n)$. Using (2.10), we obtain

$$\begin{aligned} \mathbb{E}(z_{n+1}^T Q z_{n+1}) \leq \mathbb{E}(z_n^T Q z_n) &+ 2\Delta \mathbb{E}(\tilde{\alpha} Y_n^T Q Y_n + \tilde{\beta} \bar{Y}_n^T Q \tilde{\beta} \bar{Y}_n) \\ &+ (1-2\theta)\Delta^2 \mathbb{E}(f^T(Y_n, \bar{Y}_n) Q f(Y_n, \bar{Y}_n)). \end{aligned} \tag{4.1}$$

We first prove part (1). Noting that $\theta \in [\frac{1}{2}, 1]$,

$$\mathbb{E}(z_{n+1}^T Q z_{n+1}) \leq \mathbb{E}(z_n^T Q z_n) + 2\Delta \mathbb{E}(\tilde{\alpha} Y_n^T Q Y_n + \tilde{\beta} \bar{Y}_n^T Q \tilde{\beta} \bar{Y}_n).$$

By (2.6), we get

$$\bar{Y}_j^T Q \bar{Y}_j \leq \delta Y_{j-m+1}^T Q Y_{j-m+1} + (1-\delta) Y_{j-m}^T Q Y_{j-m},$$

which gives

$$\sum_{j=0}^{n} \bar{Y}_j^T Q \bar{Y}_j \leq \sum_{j=-m+1}^{n-m+1} Y_j^T Q Y_j + (1-\delta) Y_{-m}^T Q Y_{-m}.$$

Besides, due to $n-m+1 \leq n$ and $\delta \in [0,1)$, we have

$$\begin{aligned}
&\sum_{j=-m+1}^{n-m+1} Y_j^T Q Y_j + (1-\delta) Y_{-m}^T Q Y_{-m} \\
&= \sum_{j=-m+1}^{-1} Y_j^T Q Y_j + \sum_{j=0}^{n-m+1} Y_j^T Q Y_j + (1-\delta) Y_{-m}^T Q Y_{-m} \\
&\leq \sum_{j=-m+1}^{-1} Y_j^T Q Y_j + \sum_{j=0}^{n} Y_j^T Q Y_j + (1-\delta) \max_{-m \leq j \leq -1} Y_j^T Q Y_j \\
&\leq \sum_{j=0}^{n} Y_j^T Q Y_j + (m-\delta) \max_{-m \leq j \leq -1} Y_j^T Q Y_j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(z_{n+1}^T Q z_{n+1}) &\leq \mathbb{E}(z_n^T Q z_n) + 2\Delta\tilde{\alpha}\mathbb{E}(Y_n^T Q Y_n) + 2\Delta\tilde{\beta}\mathbb{E}(\bar{Y}_n^T Q \bar{Y}_n) \\
&\leq \mathbb{E}(z_0^T Q z_0) + 2\Delta\tilde{\alpha} \sum_{j=0}^{n} \mathbb{E}[Y_j^T Q Y_j] + 2\Delta\tilde{\beta}\mathbb{E}[\sum_{j=-m+1}^{n-m+1} Y_j^T Q Y_j + (1-\delta) Y_{-m}^T Q Y_{-m}] \\
&\leq \mathbb{E}(z_0^T Q z_0) + 2\Delta(\tilde{\alpha}+\tilde{\beta}) \sum_{j=0}^{n} \mathbb{E}(Y_j^T Q Y_j) + 2\Delta\tilde{\beta}(m-\delta) \max_{-m \leq j \leq -1} \mathbb{E}(Y_j^T Q Y_j) \\
&= \mathbb{E}(z_0^T Q z_0) + 2\Delta(\tilde{\alpha}+\tilde{\beta}) \sum_{j=0}^{n} \mathbb{E}(Y_j^T Q Y_j) + 2\tilde{\beta}\tau \cdot \max_{-m \leq j \leq -1} \mathbb{E}(Y_j^T Q Y_j).
\end{aligned}$$

Noting that $\tilde{\alpha} + \tilde{\beta} < 0$, we have $\mathbb{E}(z_{n+1}^T Q z_{n+1}) \leq C(\Phi)$, where $C(\Phi)$ is a function of initial value. Since $z_i = y_i - D(\bar{y}_i)$, then it follows from the condition (2.8) and the definition of $\bar{y}_i$ that for any $\epsilon > 0$ and $0 \leq i \leq k$,

$$\begin{aligned}
\mathbb{E}(y_i^T Q y_i) &= \mathbb{E}(z_i + D(\bar{y}_i))^T Q (z_i + D(\bar{y}_i)) \\
&\leq (1+\epsilon)\mathbb{E}(z_i^T Q z_i) + (1+\frac{1}{\epsilon})\mathbb{E}[D(\bar{y}_i)^T Q D(\bar{y}_i)] \\
&\leq (1+\epsilon) C(\Phi) + (1+\frac{1}{\epsilon})\tilde{\gamma}\mathbb{E}(\bar{y}_{i-m}^T Q \bar{y}_{i-m}) \\
&\leq (1+\epsilon) C(\Phi) + (1+\frac{1}{\epsilon})\tilde{\gamma} \sup_{-m \leq j \leq k} \mathbb{E}(y_j^T Q y_j).
\end{aligned}$$

Obviously, this inequality also holds for all $-m \leq i \leq 0$. Thus, we have,

$$\sup_{-m\leq j\leq k} \mathbb{E}(y_j^T Q y_j) \leq (1+\epsilon)C(\Phi) + (1+\frac{1}{\epsilon})\tilde{\gamma} \sup_{-m\leq j\leq k} \mathbb{E}(y_j^T Q y_j),$$

and so

$$\sup_{-m\leq j\leq k} \mathbb{E}(y_j^T Q y_j) \leq \frac{1+\epsilon}{1-(1+\frac{1}{\epsilon})\bar{\gamma}} C(\Phi),$$

when we choose $\epsilon > \frac{\bar{\gamma}}{1-\bar{\gamma}}$, which means $\mathbb{E}(y_k^T Q y_k) \leq \frac{1+\epsilon}{1-(1+\frac{1}{\epsilon})\bar{\gamma}} C(\Phi)$. We immediately know that the scheme is exponentially mean square stable for all $\Delta > 0$.

Next, we will prove part (2). Noting that $\theta \in [0, \frac{1}{2})$, it follows from (2.9) and (4.1) that

$$\begin{aligned}
\mathbb{E}(z_{n+1}^T Q z_{n+1}) &\leq \mathbb{E}(z_n^T Q z_n) + 2\Delta \mathbb{E}(\tilde{\alpha} Y_n^T Q Y_n + \tilde{\beta} \bar{Y}_n^T Q \tilde{\beta} \bar{Y}_n) \\
&\quad + (1-2\theta)\Delta^2 \mathbb{E}(K_1 Y_n^T Q Y_n + K_2 \bar{Y}_n^T Q \bar{Y}_n) \\
&= \mathbb{E}(z_n^T Q z_n) + \Delta[(1-2\theta)\Delta K_1 + 2\tilde{\alpha}]\mathbb{E}(Y_n^T Q Y_n) \\
&\quad + \Delta[(1-2\theta)\Delta K_2 + 2\tilde{\beta}]\mathbb{E}(\bar{Y}_n^T Q \bar{Y}_n).
\end{aligned}$$

By (2.5), (2.8) and (2.9), we get

$$\begin{aligned}
z_n^T Q z_n &= (c_n - \theta\Delta f(Y_n, \bar{Y}_n))^T Q (c_n - \theta\Delta f(Y_n, \bar{Y}_n)) \\
&= c_n^T Q c_n - 2c_n^T Q \Delta f(Y_n, \bar{Y}_n) + (\theta\Delta)^2 f^T(Y_n, \bar{Y}_n) Q f(Y_n, \bar{Y}_n) \\
&\leq (1+\theta\Delta) c_n^T Q c_n + \theta\Delta(1+\theta\Delta) f^T(Y_n, \bar{Y}_n) Q f(Y_n, \bar{Y}_n) \\
&\leq (1+\theta\Delta) c_n^T Q c_n + \theta\Delta(1+\theta\Delta)(K_1 Y_n^T Q Y_n + K_2 \bar{Y}_n Q \bar{Y}_n) \\
&\leq (1+\theta\Delta)(2+K_1\theta\Delta) Y_n^T Q Y_n + (1+\theta\Delta)(2\tilde{\gamma} + \theta\Delta K_2)\bar{Y}_n Q \bar{Y}_n.
\end{aligned}$$

Thus, we have

$$z_n^T Q z_n \leq L_1 Y_n^T Q Y_n + L_2 \bar{Y}_n Q \bar{Y}_n,$$

where $L_1 = (1+\theta\Delta)(2+K_1\theta\Delta), L_2 = (1+\theta\Delta)(2\tilde{\gamma}+\theta\Delta K_2)$. Let

$$\Delta t_0 = \begin{cases} +\infty, & \theta = \frac{1}{2}, \\ \frac{-2(\tilde{\alpha}+\tilde{\beta})}{(1-2\theta)(K_1+K_2)}, & \theta \in [0, \frac{1}{2}). \end{cases}$$

Then for any fixed $\Delta \in (0, \Delta t_0)$, we get

$$2(\tilde{\alpha}+\tilde{\beta}) + (1-2\theta)(K_1+K_2) < 0,$$

and there exists a small positive number ε such that

$$2(\tilde{\alpha}+\tilde{\beta}) + (1-2\theta)(K_1+K_2) + \frac{L_1+L_2}{\Delta}\varepsilon < 0.$$

Hence, we obtain

$$
\begin{aligned}
\mathbb{E}(z_{n+1}^T Q z_{n+1}) &\le (1-\varepsilon)\mathbb{E}(z_n^T Q z_n) + \Delta[(1-2\theta)\Delta K_1 + 2\tilde{\alpha} + \frac{L_1}{\Delta}\varepsilon]\mathbb{E}(Y_n^T Q Y_n) \\
&\quad + \Delta[(1-2\theta)\Delta K_2 + 2\tilde{\beta} + \frac{L_2}{\Delta}\varepsilon]\mathbb{E}(\bar{Y}_n^T Q \tilde{\beta}\bar{Y}_n) \\
&\le \tilde{k}^{n+1}\mathbb{E}(z_0^T Q z_0) + \Delta[(1-2\theta)\Delta K_1 + 2\tilde{\alpha} + \frac{L_1}{\Delta}\varepsilon]\sum_{j=0}^{n}\tilde{k}^{n-j}\mathbb{E}(Y_j^T Q Y_j) \\
&\quad + \Delta[(1-2\theta)\Delta K_2 + 2\tilde{\beta} + \frac{L_2}{\Delta}\varepsilon]\sum_{j=0}^{n}\tilde{k}^{n-j}\mathbb{E}(\bar{Y}_j^T Q \tilde{\beta}\bar{Y}_j) \\
&\le \tilde{k}^{n+1}\mathbb{E}(z_0^T Q z_0) + \Delta[(1-2\theta)\Delta K_1 + 2\tilde{\alpha} + \frac{L_1}{\Delta}\varepsilon]\sum_{j=0}^{n}\tilde{k}^{n-j}\mathbb{E}(Y_j^T Q Y_j) \\
&\quad + \Delta[(1-2\theta)\Delta K_2 + 2\tilde{\beta} + \frac{L_2}{\Delta}\varepsilon][\tilde{k}^{-m}\sum_{j=0}^{n-m+1}\tilde{k}^{n-j}\mathbb{E}(\bar{Y}_j^T Q \bar{Y}_j) \\
&\quad + (m-\delta)\tilde{k}^{n-m+1}\max_{-m\le j\le -1}\mathbb{E}(Y_j^T Q Y_j)] \\
&\le \tilde{k}^{n+1}[\mathbb{E}(z_0^T Q z_0) + \tau((1-2\theta)\Delta K_2 + 2\tilde{\beta} + \frac{L_2}{\Delta}\varepsilon)\tilde{k}^{-m}\max_{-m\le j\le -1}\mathbb{E}(Y_j^T Q Y_j)] \\
&\quad + \Delta[(1-2\theta)\Delta K_1 + 2\tilde{\alpha} + \frac{L_1}{\Delta}\varepsilon + ((1-2\theta)\Delta K_2 \\
&\quad + 2\tilde{\beta} + \frac{L_2}{\Delta}\varepsilon)\tilde{k}^{-m}]\sum_{j=0}^{n-m+1}\tilde{k}^{n-j}\mathbb{E}(\bar{Y}_j^T Q \bar{Y}_j),
\end{aligned}
$$

where $\tilde{k} = \max\left\{1-\varepsilon, (\frac{(1-2\theta)\Delta K_2 + 2\tilde{\beta} + \frac{L_2}{\Delta}\varepsilon}{-((1-2\theta)\Delta K_1 + 2\tilde{\alpha} + \frac{L_1}{\Delta}\varepsilon)})^{\frac{1}{m}}\right\}$. It is easy to prove $0 < \tilde{k} < 1$. Therefore, we have

$$
\mathbb{E}(z_{n+1}^T Q z_{n+1}) \le \tilde{k}^{n+1}[\mathbb{E}(z_0^T Q z_0) + \tilde{L}\tilde{k}^{-m}\max_{-m\le j\le -1}\mathbb{E}(Y_j^T Q Y_j)].
$$

where $\tilde{L} = \tau((1-2\theta)\Delta K_2 + 2\tilde{\beta} + \frac{L_2}{\Delta}\varepsilon)$. Then, we have that $\mathbb{E}(z_{n+1}^T Q z_{n+1}) \le C'(\Phi)$, where $C'(\Phi)$ is a function of initial value. Hence, similar to the proof of part (1), we see that the SST method is exponentially mean square for $\Delta \in (0, \Delta t_0)$.

Remark 4.2. *For the special case $\theta = 0$, this approximation is actually the EM approximation, and for the case $\theta = 1$, this approximation is the split-step backward Euler method, which can be treated as an extension of the split-step backward Euler method for SODEs or SDDEs in [23, 36].*

Remark 4.3. *It is known that the exponential mean square stability implies the asymptotic mean square stability and almost sure stability. Hence, under the condition of Theorem 4.1, the SST method is also asymptotically mean square stable and almost sure stable. However,* *we can not prove that the SLT method is also exponentially mean square stable. Hence, the SST method is better than the SLT method.*

Remark 4.4. *For Theorems 3.1 and 4.1, when $D(y(t-\tau)) = 0$, NSDDEs can be reduced to SDDEs. Besides, the mean square stability of theta methods was studied in [20, 21, 23] without delays. Hence, Theorems 3.1 and 4.1 can be regarded as an extension of those results in [20, 21, 23].*

Remark 4.5. *Let us consider the following general form of linear scalar equation*

$$d[y(t) - a_0 y(t-\tau)] = [a_1 y(t) + a_2 y(t-\tau)]dt + [a_3 y(t) + a_4 y(t-\tau)]dw(t), \tag{4.2}$$

where $a_i \in \mathbb{R}$. Obviously, this equation satisfies condition (2.10) with

$$\tilde{\alpha} = a_1 - \frac{1}{2}a_0 a_1 + \frac{1}{2}a_3^2 + \frac{1}{2}|a_2 + a_3 a_4|, \ \tilde{\beta} = \frac{1}{2}a_4^2 - \frac{1}{2}a_0 a_1 - a_0 a_2 + \frac{1}{2}|a_2 + a_3 a_4|. \tag{4.3}$$

Hence, the equation is exponentially mean-square stable if

$$\tilde{\alpha} + \tilde{\beta} = a_1 - a_0 a_1 - a_0 a_2 + \frac{1}{2}(a_3^2 + a_4^2) + |a_2 + a_3 a_4| < 0. \tag{4.4}$$

Also, the neutral term and drift coefficient satisfy condition (2.8) and (2.9) with $\tilde{\gamma} = a_0^2$, $K_1 = 2a_1^2$ and $K_2 = 2a_2^2$. Set

$$\Delta t_0 = \begin{cases} +\infty, & \theta \in [\frac{1}{2}, 1], \\ \frac{-(a_1 - a_0 a_1 - a_0 a_2 + \frac{1}{2}(a_3^2 + a_4^2) + |a_2 + a_3 a_4|)}{(1-2\theta)(a_1^2 + a_2^2)}, & \theta \in [0, \frac{1}{2}). \end{cases} \tag{4.5}$$

Then for any $\Delta \in (0, \Delta t_0)$, the SLT method (2.3)-(2.4) applied to (4.2) is asymptotically mean square stable and the SST method (2.5)-(2.7) applied to (4.2) is exponentially mean square stable. In the case of linear equations, our above results are new. In fact, the traditional numerical stability analysis based on the model (4.2) usually needs the following condition:

$$a_1 - a_0 a_1 - a_0 a_2 + |a_2| + \frac{1}{2}(|a_3| + |a_4|)^2 < 0. \tag{4.6}$$

Obviously, this condition is stronger than the condition (4.4).

Remark 4.6. *Another obvious difference between our result and the existing ones on the stability of numerical methods is that our result can be applied to some equations in which the diffusion coefficient is highly nonlinear.*

5 Numerical examples

In this section, we present some numerical examples to illustrate our theoretical results. First, we study the linear scalar equation in Example 5.1.

Example 5.1. *Consider the following linear NSDDE:*

$$d[y(t) + \frac{1}{8}y(t-1)] = [-16y(t) - 8y(t-1)]dt + [2y(t) + 3y(t-1)]dw(t), t > 0 \tag{5.1}$$

with $y(t) = 1, t \in [-1, 0]$.

By Remark 4.5, we can obtain $a_0 = -\frac{1}{8}, a_1 = -16, a_2 = -8, a_3 = 2, a_4 = 3$. It is easy to compute that $\tilde{\alpha} = -14$, $\tilde{\beta} = 3.5$ satisfying $\tilde{\alpha} + \tilde{\beta} = -10.5 < 0$ but they do not satisfy inequality (4.6) with $a_1 - a_0 a_1 - a_0 a_2 + |a_2| + \frac{1}{2}(|a_3| + |a_4|)^2 = 1.5 > 0$. Hence, we see that the SLT method with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean stable for all Δ and so is $\theta \in [0, \frac{1}{2})$ for $\Delta \in (0, \Delta t_0)$. To show it more clearly, we give some remarks below.

Remark 5.2. *Figures 1-5 are all for Example 5.1. Figure 1 reveals that when $\theta = 0.1$, the SLT method will tend to zero on $\Delta = 0.1$, but will not tend to zero on $\Delta = 1$. While $\theta = 0.6$, the SLT method will tend to zero not only on $\theta = 0.1$, bue also tend to zero on $\Delta = 1$ in Figure 2. The SST method have the same results in Figures 3 and 4.*

Remark 5.3. *More clearly, even when both of the two methods converge to zero, the SST method converges to zero more quickly than the SLT method under the same θ and Δ, such as $\theta = 0.6$ and $\Delta = 0.1$. We show this fact in Figure 5.*

Next, let us discuss an example of nonlinear equations.

Example 5.4. *Consider the following nonlinear NSDDE:*

$$\begin{aligned} d[y(t) - \frac{1}{6}\sin y(t-1)] = [-2y(t) - 2y^5(t) - \frac{1}{3}\sin y(t-1)]dt \\ + \frac{y^2(t)}{1 + y^2(t-1)}dw(t), t > 0. \end{aligned} \tag{5.2}$$

$y(t) = 1$ *when* $t \in [-1, 0]$.

It is easy to check that for any $u, v \in \mathbb{R}$,

$$\begin{aligned} &(u - \frac{1}{6}\sin v)^T Q(-2u - 2u^5 - \frac{1}{3}\sin v) + \frac{1}{2}trace[(\frac{u^2}{1+v^2})^T Q \frac{u^2}{1+v^2}] \\ &\leq -2u^T Qu - 2u^T Qu^5 + \frac{1}{3}\sin^T vQu^5 + \frac{1}{18}\sin^T vQ\sin v + \frac{1}{2}u^T Qu \\ &\leq -2u^T Qu + \frac{1}{9}v^T Qv + \frac{1}{2}u^T Qu \\ &= -\frac{3}{2}u^T Qu + \frac{1}{9}v^T Qv. \end{aligned}$$

Hence, we get $\tilde{\alpha} = -\frac{3}{2}$ and $\tilde{\beta} = \frac{1}{9}$ satisfying $\tilde{\alpha} + \tilde{\beta} = -\frac{25}{18} < 0$. Thus, we see that the SLT method with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean stable for all Δ and so is $\theta \in [0, \frac{1}{2})$ for $\Delta \in (0, \Delta t_0)$.

Remark 5.5. *In Example 5.4, the figures are similar to those in Example 5.1, and we show these figures in Figure 6-Figure 9.*

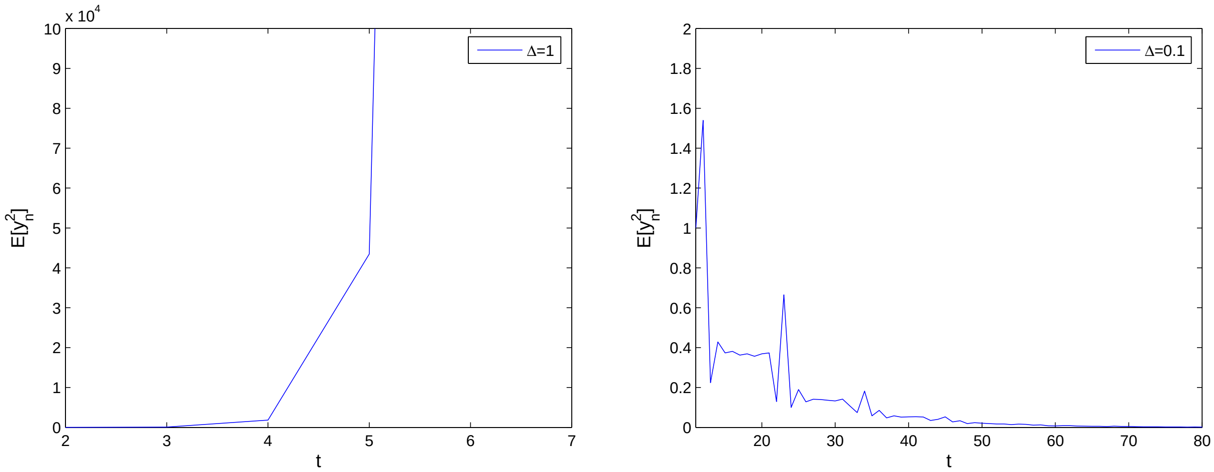

Figure 1: Numerical simulation of SLT method for different stepsizes with $\theta = 0.1$.

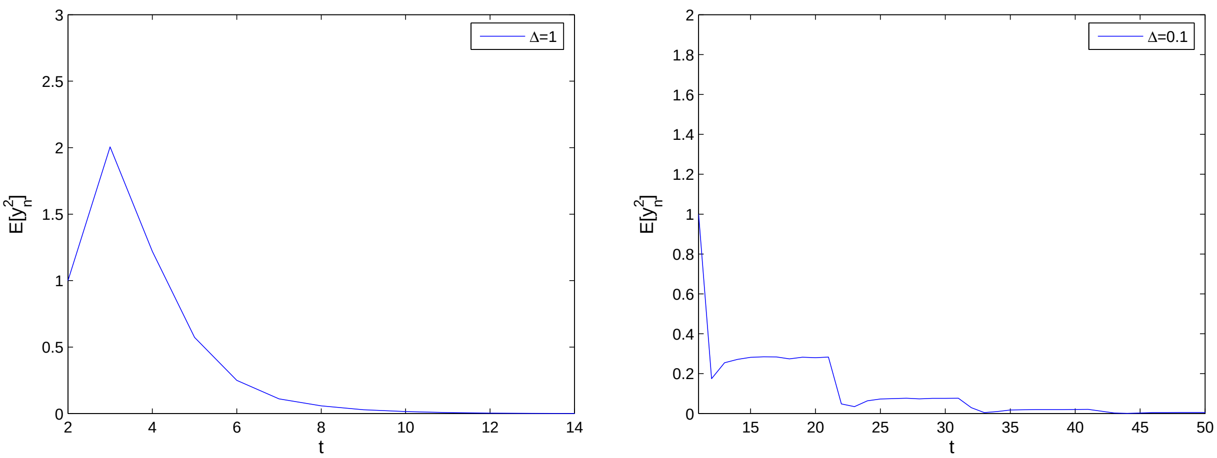

Figure 2: Numerical simulation of SLT method for different stepsizes with $\theta = 0.6$.

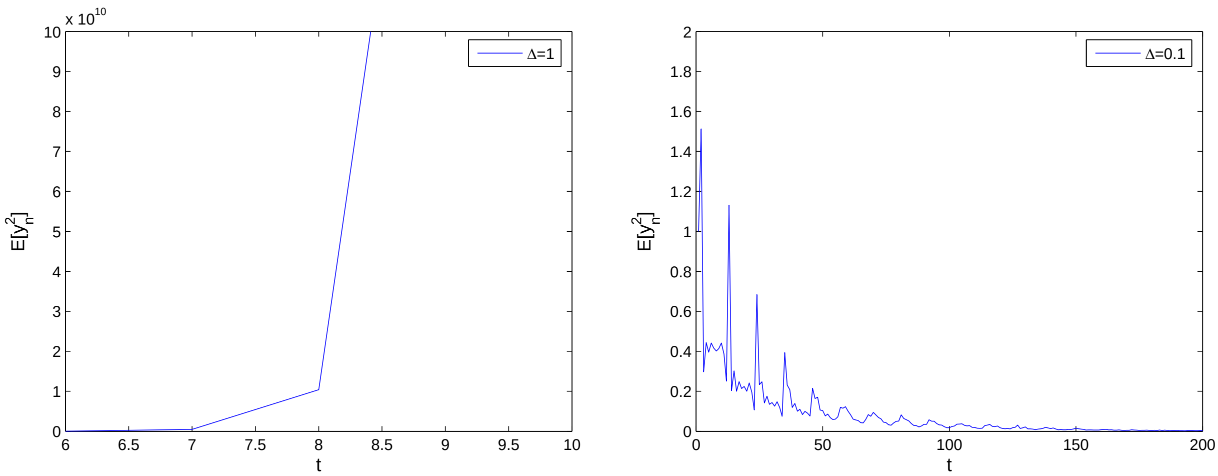

Figure 3: Numerical simulation of SST method for different stepsizes with $\theta = 0.1$.

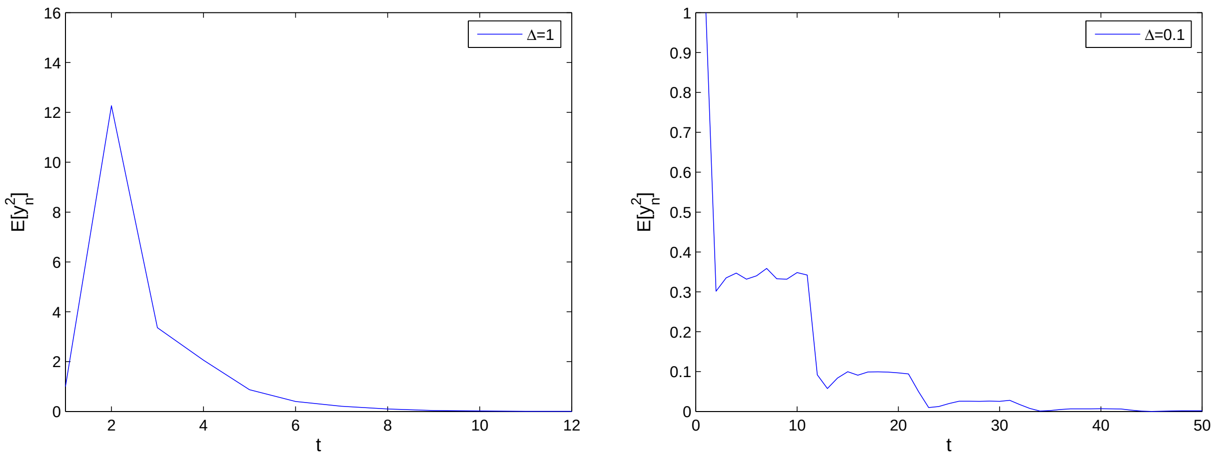

Figure 4: Numerical simulation of SST method for different stepsizes with $\theta = 0.6$.

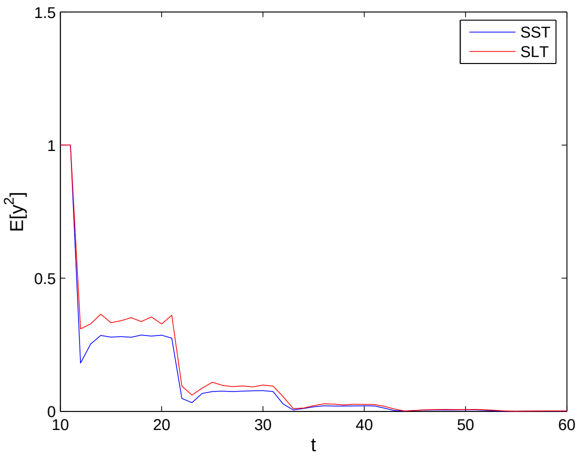

Figure 5: Compare SLT method with SST method with $\theta = 0.6$ and $\Delta = 0.1$.

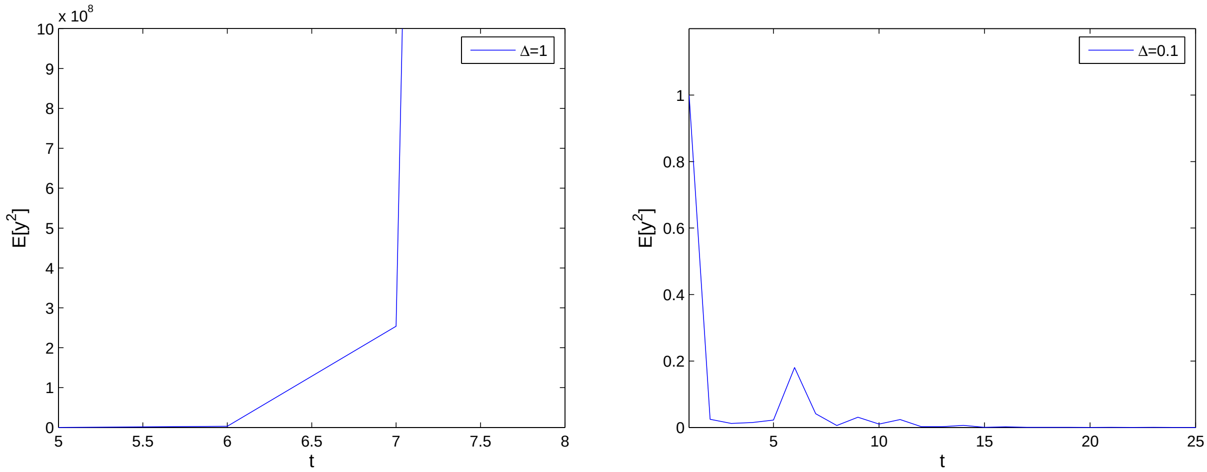

Figure 6: Numerical simulation of SLT method for different stepsizes with $\theta = 0.1$.

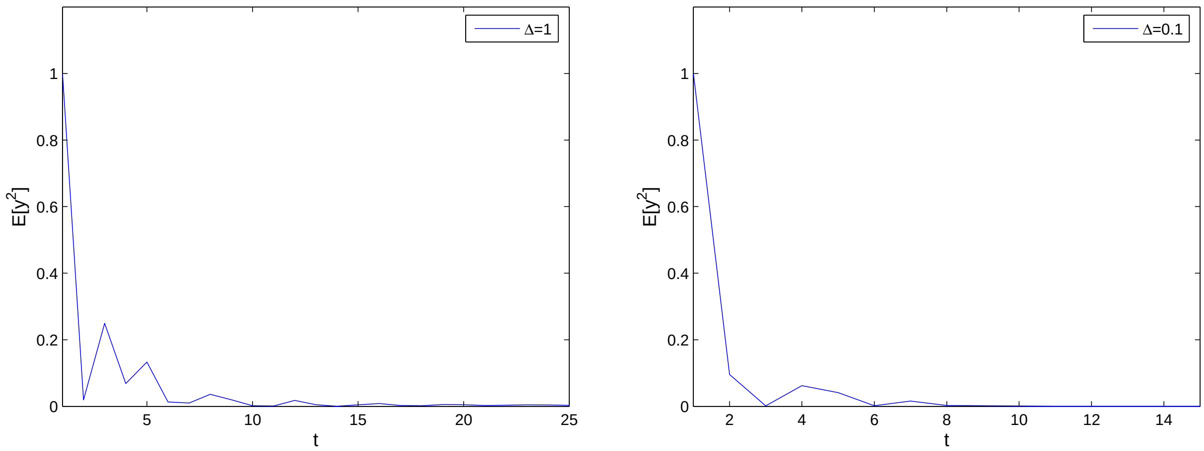

Figure 7: Numerical simulation of SLT method for different stepsizes with $\theta = 0.6$.

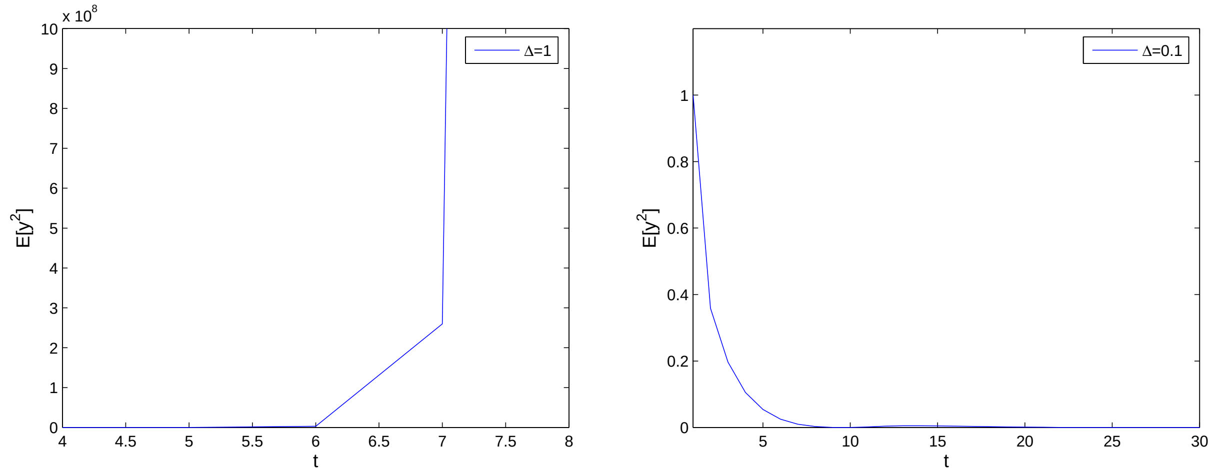

Figure 8: Numerical simulation of SST method for different stepsizes with $\theta = 0.1$.

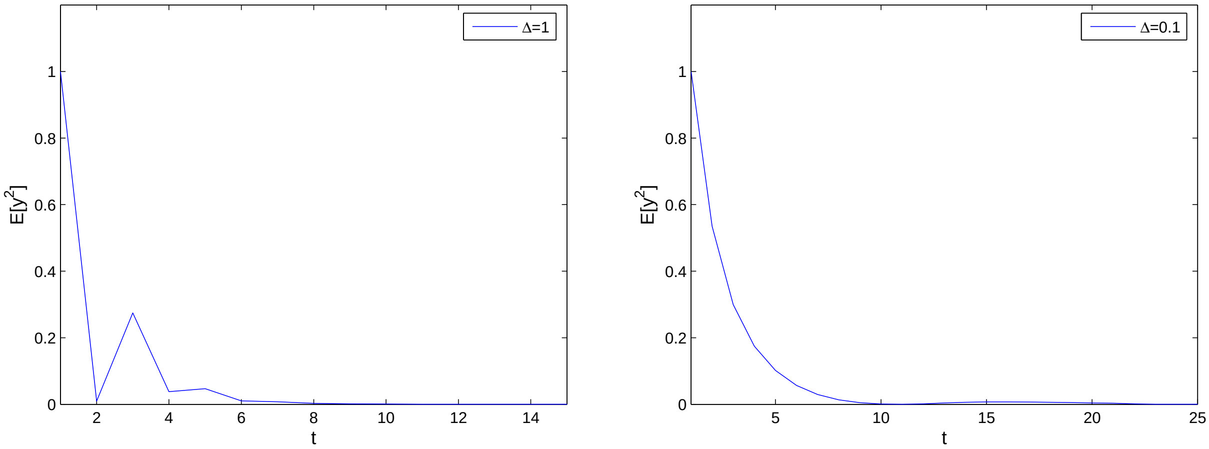

Figure 9: Numerical simulation of SST method for different stepsizes with $\theta = 0.6$.

6 Conclusion

For most NSDDEs, we can not get their explicit solutions. Therefore, it is important to develop some appropriate numerical schemes such as the Euler scheme even more general method like theta method to study the properties of NSDDEs. In this paper, both the SLT and SST methods are discussed. The main aim of this paper is to show that both the SLT and SST methods are mean square stable under some reasonable conditions. Moreover, our research reveals that the SST method has a stronger property than the SLT method.

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