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IDENTIFICATION OF TIME-DEPENDENT CONVECTION COEFFICIENT IN A TIME-FRACTIONAL DIFFUSION EQUATION

LIANGLIANG SUN, XIONGBIN YAN, TING WEI*

ABSTRACT. In the present paper, we devote our effort to solve a nonlinear inverse problem for identifying a time-dependent convection coefficient in a time-fractional diffusion equation from the measured data at an interior point for one-dimensional case. We prove the existence, uniqueness and regularity of solution for the direct problem by using the fixed point theorem. The stability of inverse convection coefficient problem is obtained based on the regularity of solution for the direct problem and some generalized Gronwall's inequalities. We use a modified optimal perturbation regularization algorithm to solve the inverse convection coefficient problem. Two numerical examples are provided to show the effectiveness of the proposed method.

1. INTRODUCTION

Let $0 < \alpha < 1$, $\Omega = (0, 1)$, $Q_T = \Omega \times (0, T]$. Consider the following initial boundary value problem (IBVP) for a time-fractional diffusion equation with a convection term

$$(1.1) \quad \begin{cases} \partial_t^\alpha u(x, t) + Au(x, t) - p(t)q(x, t)u_x(x, t) - c(x, t)u(x, t) = 0, & (x, t) \in Q_T, \\ u(0, t) = u(1, t) = 0, & t \in (0, T], \\ u(x, 0) = \varphi(x), & x \in \Omega, \end{cases}$$

where ∂_t^α denotes the Caputo fractional left-sided derivative defined by

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t - s)^\alpha}, \quad t > 0,$$

in which $\Gamma(\cdot)$ is the Gamma function (see Kilbas et al. [16] and Podlubny [25]) and the differential operator A is defined by

$$Au(x, t) = -\frac{\partial}{\partial x} \left(a(x) \frac{\partial u}{\partial x}(x, t) \right),$$

where the coefficients satisfy

$$a(x) \geq \mu, \quad x \in \bar{\Omega}, \quad a \in C^1(\bar{\Omega})$$

Key words and phrases. Fractional diffusion equation; Inverse problem; Convection coefficient; Modified optimal perturbation algorithm; Stability.

for some $\mu > 0$. If all functions $a(x)$, $p(t)$, $q(x, t)$, $c(x, t)$ and $\varphi(x)$ are given appropriately, the problem (1.1) is a direct problem. Here the time-dependent convection coefficient $p(t)$ is unknown, and we need to identify it based on an additional data. The inverse problem here is to determine the convection coefficient $p(t)$ in problem (1.1) by additional data

$$(1.2) \quad u(x_0, t) = h(t), \quad x_0 \in \Omega, \quad 0 < t < T.$$

Partial differential equations with time fractional derivatives such as (1.1) are proposed as new models for describing anomalous diffusion and/ or convection phenomena. For example, Adams and Gelhar [1] points out that field data in a saturated zone of a highly heterogeneous aquifer indicate a long-tailed profile in the spatial distribution of densities as the time passes, which is difficult to be interpreted by the classical advection diffusion equation, and the data indicate ‘slower’ diffusion than the classical one. Since then the partial differential equations with fractional orders begin to play an important role in modeling of the so called anomalous phenomena and in the theory of the complex systems (see e.g. [1, 12, 8, 21] and references therein) during the last few decades.

Here we refer to several works on the mathematical treatments for fractional diffusion equations. In the cases of the convection term vanished ($p(t)q(x, t) = 0$) and/ or all coefficients only depending on spatial variable x in (1.1), there are a lot of study results from different aspects. For example, for the direct problems, Sakamoto and Yamamoto [26] gave a comprehensive analysis including the well-posedness for fractional diffusion-wave equations as well as the long-time asymptotic behavior of the solution. Gorenflo et al. [11] extended the definition of the Caputo derivative in the fractional Sobolev spaces, and provided the maximal regularity of the solutions for some initial-boundary value problems. On the other hand, for the inverse problem, we here only mention some nonlinear inverse coefficient problems. Cheng et al. [4] and Li et al. [20] established the uniqueness in determining fractional order α and space-dependent diffusion coefficient with the Dirac delta function and a smooth function as initial conditions based on the Gel’fand-Levitan theory, respectively. For the inverse zeroth-order coefficient problem, Jin et al. [15, 14] obtained a uniqueness result in determining the zeroth-order term coefficient from the flux measurements in one dimension with the Dirichlet boundary conditions. Yamamoto and Zhang [31] gave a conditional stability estimate in determining the zeroth-order coefficient in a half-order fractional diffusion equation by a Carleman estimate. Miller et al. [22] discussed an inverse problem of determining the zeroth-order term coefficient $p(x)$ and fractional order

α from the internal data. Tuan [28] proved a uniqueness of the zeroth-order term coefficient by taking suitable initial distributions only finitely many measurements on the boundary. Sun et al. [27] investigated the uniqueness in determining the fractional order α and the zeroth-order coefficient simultaneously in a time-fractional diffusion equation, and gave a valid numerical method.

Nevertheless, to the authors' best knowledge, the published works on time-fractional diffusion equations with a convection term or time-dependent coefficients are quite few. Gorenflo et al. [11] considered a time-fractional diffusion equation with a convection term but all the coefficients are spatially dependent. It considered only the direct problem without inverse problems. On the other hand, for nonlinear inverse time-dependent coefficients problems, Zhang [32] considered an inverse time-dependent diffusion coefficient problem without a convection term. Wei et al. [30] studied a nonlinear inverse Robin coefficient problem in a fractional diffusion equation from a boundary measurement. Very recently, Fujishiro et al. [6] considered two kinds of inverse time-dependent parameter problems with the parameter in a source term or a zeroth-order coefficient term from observations of the solution at an interior or a boundary point, and obtained the stability of inverse problems.

In this paper, we focus on the inverse time-dependent convection coefficient $p(t)$ by measurements $u(x_0, t)$, $x_0 \in \Omega$, $0 < t < T$. To the authors' best knowledge, there are no works on the inverse convection coefficient problem at present for a time-fractional diffusion equation. In this study, we obtain the existence, uniqueness and some regularities of the solution for the direct problem and also obtain the stability of the inverse problem. Moreover, we propose a numerical method for solving the inverse coefficient problem which is a modification of the conventional optimal perturbation algorithm. The numerical results for two examples are provided to show the effectiveness and robustness of the proposed methods. The main contributions are to obtain the uniqueness result of the inverse problem deduced directly by the stability of the inverse convection coefficient and to give firstly an effect algorithm to obtain the good numerical approximations.

The main result in this paper is the following stability result for the inverse convection coefficient problem.

Theorem 1.1. Assume $q(x, t) \in L^\infty(0, T; H_0^1(\Omega))$, $c(x, t) \in L^\infty(0, T; H^1(\Omega))$, $\varphi(x) \in H^3(\Omega) \cap H_0^1(\Omega)$, $A\varphi(0) = A\varphi(1) = 0$. Let u_i be the solution of (1.1) for $p = p_i \in L^\infty(0, T)$ with

$\|p_i\|_\infty \leq M$ ($i = 1, 2$). Assume that there exist $x_0 \in \Omega$ and $v > 0$ such that

$$(1.3) \quad \left| q(x_0, t) \frac{\partial}{\partial x} u_2(x_0, t) \right| \geq v, \quad a.e. \quad t \in (0, T).$$

Then there exists a constant $C > 0$ depending on $M, T, \alpha, \Omega, v, \|q\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|c\|_{L^\infty(0,T;H^1(\Omega))}$ such that

$$(1.4) \quad C^{-1} \|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_\infty \leq \|p_1 - p_2\|_\infty \leq C \|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_\infty,$$

and

$$(1.5) \quad \|u_1(x, t) - u_2(x, t)\|_{L^2(0,T;D(A))} \leq C \|p_1 - p_2\|_{L^2(0,T)}.$$

The remainder of this paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we give the existence, uniqueness and regularity of solution for the direct problem. In Section 4, we give the proof of the main result, i.e., the stability of the inverse convection coefficient problem. In Section 5, we use the modified optimal perturbation algorithm to solve the inverse coefficient problem. Numerical results for two examples are investigated in Section 6. Finally, we give a brief conclusion in Section 7.

2. PRELIMINARIES

Denote $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T)}$, (\cdot, \cdot) as the inner product of $L^2(\Omega)$ and $H^s(\Omega)$, $s \in \mathbb{R}$ is the Sobolev spaces (see Adams [2]). Throughout this paper, we always assume the following conditions hold

$$(2.1) \quad p(t) \in L^\infty(0, T),$$

$$(2.2) \quad q(x, t) \in L^\infty(0, T; H_0^1(\Omega)),$$

$$(2.3) \quad c(x, t) \in L^\infty(0, T; H^1(\Omega)),$$

$$(2.4) \quad \varphi(x) \in H^3(\Omega) \cap H_0^1(\Omega), \quad A\varphi(0) = A\varphi(1) = 0.$$

Define $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Noting that A is a self adjoint and positive operator. Let $\{\lambda_k, \phi_k\}_{k=1}^\infty$ be an eigensystem of A in $D(A)$. Then we know $0 < \lambda_1 < \lambda_2 < \dots$, $\lim_{k \rightarrow \infty} \lambda_k = \infty$, $A\phi_k = \lambda_k \phi_k$, and $\{\phi_k\}_{k=1}^\infty \subset H^2(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$. We can define

the Hilbert scale space $D(A^\gamma)$ for $\gamma \geq 0$ (see, e.g., [24]) by

$$D(A^\gamma) = \left\{ \psi \in L^2(\Omega); \sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(\psi, \phi_k)|^2 < \infty \right\},$$

$$A^\gamma \psi = \sum_{k=1}^{\infty} \lambda_k^\gamma (\psi, \phi_k) \phi_k, \quad \psi \in D(A^\gamma),$$

equipped with the norm $\|\psi\|_{D(A^\gamma)} = \|A^\gamma \psi\|$. It is easy to obtain $A^{\alpha+\beta} \psi = A^\alpha (A^\beta \psi) = A^\beta (A^\alpha \psi)$ for $\alpha, \beta \geq 0$ and $\psi \in D(A^{\alpha+\beta})$. According to [7, 10], we have

$$(2.5) \quad D(A^\gamma) \subset H^{2\gamma}(\Omega), \quad 0 \leq \gamma \leq 1,$$

$$(2.6) \quad C_1 \|\psi\|_{H^{2\gamma}(\Omega)} \leq \|\psi\|_{D(A^\gamma)} \leq C_2 \|\psi\|_{H^{2\gamma}(\Omega)}, \quad \psi \in D(A^\gamma), 0 \leq \gamma \leq 1, \gamma \neq \frac{1}{4}$$

$$(2.7) \quad D(A^{\frac{1}{2}}) = H_0^1(\Omega).$$

Proposition 2.1. [16] *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a constant $c = c(\alpha, \beta, \mu) > 0$ such that*

$$(2.8) \quad |E_{\alpha, \beta}(z)| \leq \frac{c}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi.$$

Lemma 2.2. [3] *Let $f \in L^p(0, T)$ and $g \in L^q(0, T)$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Then the function $f * g$ defined by $f * g(t) = \int_0^t f(t-s)g(s)ds$ belongs to $C[0, T]$ and satisfies*

$$|f * g(t)| \leq \|f\|_{L^p(0, t)} \|g\|_{L^q(0, t)}, \quad t \in [0, T].$$

Lemma 2.3. [3] *Let $u, v \in H^1(\Omega)$. Then $uv \in H^1(\Omega)$ with the estimate*

$$(2.9) \quad \|uv\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

with $C > 0$ depending on $\|u\|_{H^1(\Omega)}$.

3. EXISTENCE, UNIQUENESS AND REGULARITY OF SOLUTION FOR THE DIRECT PROBLEM

In this section we will obtain the existence, uniqueness and regularity of solution for the direct problem (1.1).

Theorem 3.1. *Let conditions (2.1)-(2.4) hold. Then the IBVP (1.1) has a unique solution $u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfying*

$$Au \in C([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha u \in L^\infty(0, T; H^{2\gamma}(\Omega))$$

for $0 \leq \gamma < 1/2$, $\gamma \neq 1/4$. Moreover, we have the following estimate

$$(3.1) \quad \|Au\|_{C([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha u\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} \leq C \|\varphi\|_{H^3(\Omega)}$$

with $C > 0$ depending on Ω , T , α , γ , $\|p\|_\infty$, $\|q\|_{L^\infty(0,T;H^1(\Omega))}$, $\|c\|_{L^\infty(0,T;H^1(\Omega))}$.

In order to obtain the above existence, uniqueness and regularity results, we firstly study the following problems and give their existence, uniqueness and regularity of solutions

$$(3.2) \quad \begin{cases} \partial_t^\alpha v(x, t) + Av(x, t) - b(x, t)v_x(x, t) - c(x, t)v(x, t) = F(x, t), & (x, t) \in Q_T, \\ v(0, t) = v(1, t) = 0, & t \in (0, T], \\ v(x, 0) = 0, & x \in \Omega, \end{cases}$$

and

$$(3.3) \quad \begin{cases} \partial_t^\alpha w(x, t) + Aw(x, t) - b(x, t)w_x(x, t) - c(x, t)w(x, t) = 0, & (x, t) \in Q_T, \\ w(0, t) = w(1, t) = 0, & t \in (0, T], \\ w(x, 0) = \varphi(x), & x \in \Omega, \end{cases}$$

where we assume

$$(3.4) \quad b(x, t) \in L^\infty(0, T; H_0^1(\Omega)),$$

$$(3.5) \quad F(x, t) \in L^\infty(0, T; H_0^1(\Omega)).$$

By the fixed point theorem, we can obtain the following existence, uniqueness and regularity results for problem (3.2) and (3.3), respectively.

Lemma 3.2. *Let (2.3), (3.4) and (3.5) hold. Then the IBVP (3.2) exists a unique solution $v \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfying*

$$Av \in C([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0, T; H^{2\gamma}(\Omega))$$

for $0 \leq \gamma < 1/2$, $\gamma \neq 1/4$. Moreover, we have

$$(3.6) \quad \|Av\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \leq C\|F\|_{L^\infty(0,T;H^1(\Omega))}$$

with $C > 0$ depending on Ω , T , α , γ , $\|b\|_{L^\infty(0,T;H^1(\Omega))}$, $\|c\|_{L^\infty(0,T;H^1(\Omega))}$.

Lemma 3.3. *Let (2.3), (2.4) and (3.4) hold. Then the IBVP (3.3) exists a unique solution $w \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfying*

$$Aw \in C([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha w \in L^\infty(0, T; H^{2\gamma}(\Omega))$$

for $0 \leq \gamma < 1/2$, $\gamma \neq 1/4$. Moreover, we have

$$(3.7) \quad \|Aw\|_{C([0,T];H^{2\gamma}(\Omega))} + \|\partial_t^\alpha w\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \leq C\|\varphi\|_{H^3(\Omega)}$$

with $C > 0$ depending on Ω , T , α , γ , $\|b\|_{L^\infty(0,T;H^1(\Omega))}$, $\|c\|_{L^\infty(0,T;H^1(\Omega))}$.

Note that if we set $b(x, t) = p(t)q(x, t)$, then conditions (2.1) and (2.2) are equivalent to (3.4). Therefore, it is sufficient to prove Lemma 3.3 in order to obtain Theorem 3.1.

In order to prove Lemmas 3.2-3.3, we define the operator valued function $K(t)$ by

$$K(t)\psi = \sum_{k=1}^{\infty} (\psi, \phi_k) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k t^{\alpha}) \phi_k, \quad \psi \in L^2(\Omega), \quad t > 0,$$

with the Mittag-Leffler function given by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \quad \alpha > 0, \quad \beta \in \mathbb{R}.$$

It is easy to obtain that $K(t) \in L^1(0, T; B(L^2(\Omega)))$, where $B(L^2(\Omega))$ denote the bounded linear operator in $L^2(\Omega)$.

From $A^{\gamma} K(t)\psi = \sum_{k=1}^{\infty} \lambda_k^{\gamma} (\psi, \phi_k) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k t^{\alpha}) \phi_k$ and Proposition 2.1, we have

$$\begin{aligned} \|A^{\gamma} K(t)\psi\| &= \left(\sum_{k=1}^{\infty} \left[\lambda_k^{\gamma} (\psi, \phi_k) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k t^{\alpha}) \right]^2 \right)^{1/2} \\ &\leq C \left(\sum_{k=1}^{\infty} \left(\frac{\lambda_k^{\gamma} t^{\alpha-1}}{1 + \lambda_k t^{\alpha}} (\psi, \phi_k) \right)^2 \right)^{1/2} \\ (3.8) \quad &\leq C t^{\alpha(1-\gamma)-1} \|\psi\|, \quad \psi \in L^2(\Omega), \quad t > 0. \end{aligned}$$

In particular, if $\gamma < 1$, then the mapping $t \mapsto A^{\gamma} K(t)$ belongs to $L^1(0, T; B(L^2(\Omega)))$.

We consider the following Cauchy problem in $L^2(\Omega)$

$$(3.9) \quad \begin{cases} \partial_t^{\alpha} \omega(t) + A\omega(t) = F(t), & t \in (0, T], \\ \omega(0) = 0. \end{cases}$$

By Theorem 2.2 in [26], for $F \in L^{\infty}(0, T; L^2(\Omega))$, (3.9) admits a unique solution given by

$$(3.10) \quad \omega(t) = \int_0^t K(t-s) F(s) ds.$$

Noting that $A^{\frac{1}{2}}$ and $K(t)$ can commute, we obtain that for $F \in L^{\infty}(0, T; D(A^{\frac{1}{2}}))$,

$$A\omega(t) = \int_0^t A^{\frac{1}{2}} K(t-s) A^{\frac{1}{2}} F(s) ds.$$

By (3.8), the map $t \mapsto A^{\frac{1}{2}}K(t)$ belongs to $L^1(0, T; B(L^2(\Omega)))$. Thus by Lemma 2.2, we have $\omega \in C([0, T]; D(A))$ and

$$\begin{aligned} \|\omega(t)\|_{D(A)} = \|A\omega(t)\| &\leq \int_0^t \|A^{\frac{1}{2}}K(t-s)\| \|A^{\frac{1}{2}}F(s)\| ds \\ &\leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|F(s)\|_{D(A^{\frac{1}{2}})} ds \\ (3.11) \qquad \qquad \qquad &\leq CT^{\frac{\alpha}{2}} \|F\|_{L^\infty(0, T; D(A^{\frac{1}{2}}))}. \end{aligned}$$

Here we define the map $H : L^\infty(0, T; D(A^{\frac{1}{2}})) \rightarrow C([0, T]; D(A))$ by

$$(3.12) \qquad Hf(t) = \int_0^t K(t-s)f(s)ds, \quad f \in L^\infty(0, T; D(A^{\frac{1}{2}})).$$

Therefore, we have

$$(3.13) \qquad \|HF\|_{C([0, T]; D(A))} \leq CT^{\frac{\alpha}{2}} \|F\|_{L^\infty(0, T; D(A^{\frac{1}{2}}))}.$$

Next we give the proofs of Lemmas 3.2-3.3.

Proof of Lemma 3.2. The IBVP (3.2) could be written as

$$(3.14) \qquad \begin{cases} \partial_t^\alpha v(t) + Av(t) = b(t)\frac{\partial}{\partial x}v(t) + c(t)v(t) + F(t), & t \in (0, T], \\ v(0) = 0, \end{cases}$$

where $v(t) = v(\cdot, t)$, $b(t) = b(\cdot, t)$, $c(t) = c(\cdot, t)$ and $F(t) = F(\cdot, t)$. We see from (3.10) that the solution v of (3.14) can be written as

$$(3.15) \qquad v(t) = \int_0^t K(t-s) \left(b(s)\frac{\partial}{\partial x}v(s) + c(s)v(s) \right) ds + \int_0^t K(t-s)F(s)ds.$$

Thus we will look for a fixed point of the operator $G : C([0, T]; D(A)) \rightarrow C([0, T]; D(A))$ defined by

$$G(v)(t) = Qv(t) + HF(t), \quad v \in C([0, T]; D(A)), \quad t \in (0, T],$$

where

$$Qv(t) = \int_0^t K(t-s) \left(b(s)\frac{\partial}{\partial x}v(s) + c(s)v(s) \right) ds, \quad t \in (0, T].$$

By induction, we have

$$G^m(v) = Q^m v + \sum_{k=0}^{m-1} Q^k HF.$$

Here we denote $Q^0 = I$.

By Lemma 2.3, we know that $b\frac{\partial}{\partial x}v$, $cv \in L^\infty(0, T; D(A^{\frac{1}{2}}))$ and

$$(3.16) \qquad \left\| b(s)\frac{\partial}{\partial x}v(s) \right\|_{D(A^{\frac{1}{2}})} \leq C\|v(s)\|_{D(A)},$$

$$(3.17) \qquad \|c(s)v(s)\|_{D(A^{\frac{1}{2}})} \leq C\|v(s)\|_{D(A^{\frac{1}{2}})},$$

where $C > 0$ is depending on $\|b\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|c\|_{L^\infty(0,T;H^1(\Omega))}$. By (3.8), (3.16) and (3.17), we obtain

$$\begin{aligned}
 \|Qv(t)\|_{D(A)} &= \left\| \int_0^t A^{\frac{1}{2}} K(t-s) A^{\frac{1}{2}} \left(b(s) \frac{\partial}{\partial x} v(s) + c(s) v(s) \right) ds \right\| \\
 &\leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} \left\| b(s) \frac{\partial}{\partial x} v(s) + c(s) v(s) \right\|_{D(A^{\frac{1}{2}})} ds \\
 (3.18) \quad &\leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|v(s)\|_{D(A)} ds.
 \end{aligned}$$

By Lemma 2.2, for $v \in C([0, T]; D(A))$, we have $Qv \in C([0, T]; D(A))$ and the estimate

$$\|Qv\|_{C([0,T];D(A))} \leq CT^{\alpha/2} \|v\|_{C([0,T];D(A))}.$$

Therefore we can see that Q maps $C([0, T]; D(A))$ into itself. Combing $HF \in C([0, T]; D(A))$, we obtain that the operator G also maps $C([0, T]; D(A))$ into itself. Repeating the similar calculation, we have

$$\begin{aligned}
 \|Q^2 v(t)\|_{D(A)} &= \|Q(Qv(t))\|_{D(A)} \leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|Qv(s)\|_{D(A)} ds \\
 &\leq C^2 \int_0^t (t-s)^{\frac{\alpha}{2}-1} \left(\int_0^s (s-\tau)^{\frac{\alpha}{2}-1} \|v(\tau)\|_{D(A)} d\tau \right) ds \\
 &= C^2 \int_0^t \left(\int_\tau^t (t-s)^{\frac{\alpha}{2}-1} (s-\tau)^{\frac{\alpha}{2}-1} ds \right) \|v(\tau)\|_{D(A)} d\tau \\
 &= \frac{(C\Gamma(\alpha/2))^2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|v(\tau)\|_{D(A)} d\tau.
 \end{aligned}$$

By induction, we have

$$\|Q^m v(t)\|_{D(A)} \leq \frac{(C\Gamma(\alpha/2))^m}{\Gamma(m\alpha/2)} \int_0^t (t-\tau)^{m\alpha/2-1} \|v(\tau)\|_{D(A)} d\tau, \quad v \in C([0, T]; D(A)).$$

By Lemma 2.2, we have $Q^m v \in C([0, T]; D(A))$ and the estimate

$$(3.19) \quad \|Q^m v\|_{C([0,T];D(A))} \leq \rho_m \|v\|_{C([0,T];D(A))}, \quad v \in C([0, T]; D(A)).$$

where $\rho_m = \frac{(C\Gamma(\alpha/2)T^{\alpha/2})^m}{\Gamma(m\alpha/2+1)}$. Therefore, for $v_1, v_2 \in C([0, T]; D(A))$, we obtain

$$\begin{aligned}
 \|G^m(v_1) - G^m(v_2)\|_{C([0,T];D(A))} &= \|Q^m(v_1 - v_2)\|_{C([0,T];D(A))} \\
 &\leq \rho_m \|v_1 - v_2\|_{C([0,T];D(A))}.
 \end{aligned}$$

It is easy to verify $\rho_m \rightarrow 0$ as $m \rightarrow \infty$. Therefore, we have $|\rho_m| < 1$ for large $m \in \mathbb{N}$. Therefore, the operator G^m is a contraction mapping from $C([0, T]; D(A))$ into itself. Hence the mapping G^m has a unique fixed point still denoted by $v \in C([0, T]; D(A))$, that is, $G^m(v) = v$. Since $G^m(G(v)) = G^{m+1}(v) = G(G^m(v)) = G(v)$, the point $G(v)$ is also a

fixed point of the mapping G^m . By the uniqueness of the fixed point of G^m , we have $Qv + HF = G(v) = v$, that is, the equation $v = Qv + HF$ has a unique solution v in $C([0, T]; D(A))$. Moreover, we have

$$v = G(v) = G^m(v) = Q^m v + \sum_{k=0}^{m-1} Q^k HF.$$

As $HF \in C([0, T]; D(A))$, by (3.19) and (3.13), we have

$$\begin{aligned} \|v\|_{C([0, T]; D(A))} &\leq \|Q^m v\|_{C([0, T]; D(A))} + \sum_{k=0}^{m-1} \|Q^k HF\|_{C([0, T]; D(A))} \\ &\leq \rho_m \|v\|_{C([0, T]; D(A))} + \sum_{k=0}^{m-1} \rho_k \|HF\|_{C([0, T]; D(A))} \\ &\leq \rho_m \|v\|_{C([0, T]; D(A))} + \sum_{k=0}^{m-1} \rho_k CT^{\alpha/2} \|F\|_{L^\infty(0, T; D(A^{\frac{1}{2}}))}. \end{aligned}$$

By take sufficiently large $m \in \mathbb{N}$ such that $\rho_m < 1$, we have

$$(3.20) \quad \|v\|_{C([0, T]; D(A))} \leq C \|F\|_{L^\infty(0, T; D(A^{\frac{1}{2}}))}$$

with $C > 0$ depending on T , Ω , α , $\|b\|_{L^\infty(0, T; H^1(\Omega))}$ and $\|c\|_{L^\infty(0, T; H^1(\Omega))}$.

Now we fix $0 \leq \gamma < 1/2$. Similar to the treatment of (3.18), we obtain

$$\begin{aligned} \|Av(t)\|_{D(A^\gamma)} &= \left\| \int_0^t A^{\gamma+\frac{1}{2}} K(t-s) A^{\frac{1}{2}} \left(b(s) \frac{\partial}{\partial x} v(s) + c(s) v(s) \right) ds \right\| \\ &\quad + \left\| \int_0^t A^{\gamma+\frac{1}{2}} K(t-s) A^{\frac{1}{2}} F(s) ds \right\| \\ (3.21) \quad &\leq C \int_0^t (t-s)^{\alpha(\frac{1}{2}-\gamma)-1} \left(\|v(s)\|_{D(A)} + \|F(s)\|_{D(A^{\frac{1}{2}})} \right) ds. \end{aligned}$$

By Lemma 2.2, we have $Av \in C([0, T]; D(A^\gamma))$ and the following estimate from (3.20) and (3.21)

$$\begin{aligned} \|Av\|_{C([0, T]; D(A^\gamma))} &\leq CT^{\alpha(\frac{1}{2}-\gamma)} \left(\|v\|_{C([0, T]; D(A))} + \|F\|_{L^\infty(0, T; D(A^{\frac{1}{2}}))} \right) \\ (3.22) \quad &\leq C \|F\|_{L^\infty(0, T; D(A^{\frac{1}{2}}))}. \end{aligned}$$

Therefore, we have $Av \in C([0, T]; H^{2\gamma}(\Omega))$ with $0 \leq \gamma < 1/2$ from (2.5), and

$$(3.23) \quad \|Av\|_{C([0, T]; H^{2\gamma}(\Omega))} \leq C \|F\|_{L^\infty(0, T; H^1(\Omega))}$$

from (2.6) and (2.7). By the original equation $\partial_t^\alpha v = -Av + bv_x + cv + F$, combining (3.16), (3.17), (3.20) and (3.23), we see that $\partial_t^\alpha v \in L^\infty(0, T; H^{2\gamma}(\Omega))$ with the estimate

$$\begin{aligned} \|\partial_t^\alpha v\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} &\leq C\|F\|_{L^\infty(0, T; H^1(\Omega))} + \|bv_x + cv\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} + \|F\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} \\ &\leq C\|F\|_{L^\infty(0, T; H^1(\Omega))}. \end{aligned}$$

Thus we complete the proof. \square

Proof of Lemma 3.3. We split the solution w of (3.3) into $w = w_0 + \varphi$, where w_0 solves (3.24)

$$\begin{cases} \partial_t^\alpha w_0(x, t) + Aw_0(x, t) - b(x, t)(w_0)_x(x, t) - c(x, t)w_0(x, t) = F_0(x, t), & (x, t) \in Q_T, \\ w_0(0, t) = w_0(1, t) = 0, & t \in (0, T], \\ w_0(x, 0) = 0, & x \in \Omega, \end{cases}$$

with $F_0(x, t) = -A\varphi(x) + b(x, t)\varphi'(x) + c(x, t)\varphi(x)$. By (2.3), (2.4) and (3.4), we have $F_0 \in L^\infty(0, T; H_0^1(\Omega))$, and the estimate

$$\|F_0\|_{L^\infty(0, T; H^1(\Omega))} \leq C\|\varphi\|_{H^3(\Omega)}.$$

By Lemma 3.2, the IBVP (3.24) exists a unique solution $w_0 \in C([0, T]; D(A))$ satisfying

$$Aw_0 \in C([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha w_0 \in L^\infty(0, T; H^{2\gamma}(\Omega))$$

for $0 \leq \gamma < 1/2$. Moreover,

$$\|Aw_0\|_{C([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha w_0\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} \leq \|F_0\|_{L^\infty(0, T; H^1(\Omega))} \leq C\|\varphi\|_{H^3(\Omega)}.$$

Therefore, the IBVP (3.3) admits a unique solution $w = w_0 + \varphi \in C([0, T]; D(A))$ satisfying

$$Aw \in C([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha w \in L^\infty(0, T; H^{2\gamma}(\Omega)),$$

and the estimate (3.7) holds. \square

4. STABILITY OF THE SOLUTION FOR THE INVERSE CONVECTION COEFFICIENT PROBLEM

In this section, we give the proof of our main Theorem 1.1. To this end, we prepare the following Lemmas with the Gronwall type inequalities.

Lemma 4.1. Let $C, \alpha > 0$ and $u, d \in L^1(0, T)$ be nonnegative functions satisfying

$$u(t) \leq Cd(t) + C \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in (0, T).$$

Then we have

$$u(t) \leq Cd(t) + C \int_0^t (t-s)^{\alpha-1} d(s) ds, \quad t \in (0, T).$$

Lemma 4.2. Let $a, b, \alpha > 0$ and $u \in L^1(0, T)$ be nonnegative functions satisfying

$$u(t) \leq a + b \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad a.e. t \in (0, T).$$

Then we have

$$u(t) \leq aE_{\alpha,1}((b\Gamma(\alpha))^{1/\alpha} t^\alpha), \quad a.e. t \in (0, T).$$

For the proofs of above two Lemmas, please refer to Lemma 7.1.1 and Lemma 7.1.2 on p.188-189 of [13].

Proof of Theorem 1.1. Let u_i be the solutions to (1.1) corresponding to $p = p_i$ ($i = 1, 2$).

We set $u = u_1 - u_2$ and $p = p_1 - p_2$. Then u solves

$$(4.1) \quad \begin{cases} \partial_t^\alpha u(x, t) + Au(x, t) - p_1(t)q(x, t)\frac{\partial}{\partial x}u(x, t) - c(x, t)u(x, t) = p(t)q(x, t)\frac{\partial}{\partial x}u_2(x, t), & (x, t) \in Q_T, \\ u(0, t) = u(1, t) = 0, & t \in (0, T], \\ u(x, 0) = 0, & x \in \Omega, \end{cases}$$

denote $b(x, t) = p_1(t)q(x, t)$ and $F(x, t) = p(t)q(x, t)\frac{\partial}{\partial x}u_2(x, t)$, and $u(x, t)$ is given by

$$u(t) = \int_0^t K(t-s)(b(s)u_x(s) + c(s)u(s))ds + \int_0^t p(s)K(t-s)R(s)ds,$$

where $R(s) = q(\cdot, s)\frac{\partial}{\partial x}u_2(\cdot, s)$.

First we estimate $\|u(t)\|_{D(A)}$. By Lemma 2.3, we see that $R = q\frac{\partial}{\partial x}u_2 \in L^\infty(0, T; H_0^1(\Omega))$, and the estimate from (3.16) and (3.20)

$$(4.2) \quad \|R\|_{L^\infty(0, T; H_0^1(\Omega))} \leq C\|u_2\|_{C([0, T]; D(A))} \leq C\|\varphi\|_{H^3(\Omega)},$$

with $C > 0$ depending on $\Omega, T, \|p_2\|_\infty, \|q\|_{L^\infty(0, T; H^1(\Omega))}$ and $\|c\|_{L^\infty(0, T; H^1(\Omega))}$. Similar to the argument of (3.18), we have from (4.2) that

$$(4.3) \quad \begin{aligned} \|u(t)\|_{D(A)} &= \left\| \int_0^t A^{\frac{1}{2}} K(t-s) A^{\frac{1}{2}} (b(s)u_x(s) + c(s)u(s)) ds \right\| \\ &\quad + \left\| \int_0^t p(s) A^{\frac{1}{2}} K(t-s) A^{\frac{1}{2}} R(s) ds \right\| \\ &\leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|u(s)\|_{D(A)} ds + C \int_0^t (t-s)^{\frac{\alpha}{2}-1} |p(s)| ds, \end{aligned}$$

with $C > 0$ depending on $\Omega, T, M, \|c\|_{L^\infty(0, T; H^1(\Omega))}, \|q\|_{L^\infty(0, T; H^1(\Omega))}$ and $\|\varphi\|_{H^3(\Omega)}$.

Denote $d(t) = \int_0^t (t-s)^{\frac{\alpha}{2}-1} |p(s)| ds$, then by Lemma 4.1, we obtain

$$\|u(t)\|_{D(A)} \leq Cd(t) + \int_0^t (t-s)^{\frac{\alpha}{2}-1} d(s) ds, \quad t \in (0, T).$$

Since

$$\begin{aligned}
 \int_0^t (t-s)^{\frac{\alpha}{2}-1} d(s) ds &= \int_0^t (t-s)^{\frac{\alpha}{2}-1} \left(\int_0^s (s-\tau)^{\frac{\alpha}{2}-1} |p(\tau)| d\tau \right) ds \\
 &= \int_0^t \left(\int_\tau^t (t-s)^{\frac{\alpha}{2}-1} (s-\tau)^{\frac{\alpha}{2}-1} ds \right) |p(\tau)| d\tau \\
 &= B(\alpha/2, \alpha/2) \int_0^t (t-\tau)^{\alpha-1} |p(\tau)| d\tau \\
 &\leq T^{\alpha/2} B(\alpha/2, \alpha/2) \int_0^t (t-\tau)^{\alpha/2-1} |p(\tau)| d\tau \\
 &\leq Cd(t),
 \end{aligned}$$

thus we obtain $\|u(t)\|_{D(A)} \leq Cd(t)$, $t \in (0, T)$, that is,

$$\|u(t)\|_{D(A)} \leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} |p(s)| ds, \quad t \in (0, T).$$

By the Young inequality for the convolution, we have

$$(4.4) \quad \int_0^T \|u(t)\|_{D(A)}^2 dt \leq C \int_0^T \left(\int_0^t (t-s)^{\frac{\alpha}{2}-1} |p(s)| ds \right)^2 dt \leq CT^\alpha \|p\|_{L^2(0,T)}^2.$$

That means (1.5) is true.

In the same way, we have that for $0 \leq \gamma < 1/2$

$$\|Au(t)\|_{D(A^\gamma)} \leq C \int_0^t (t-s)^{\alpha(\frac{1}{2}-\gamma)-1} |p(s)| ds, \quad t \in (0, T).$$

Let $\frac{1}{4} < \gamma < \frac{1}{2}$. By the Sobolev embedding, we have

$$\begin{aligned}
 &|Au(x_0, t) - b(x_0, t)u_x(x_0, t) - c(x_0, t)u(x_0, t)| \\
 &\leq C \|Au(\cdot, t) - b(\cdot, t)u_x(\cdot, t) - c(\cdot, t)u(\cdot, t)\|_{H^{2\gamma}(\Omega)} \\
 &\leq C \|Au(\cdot, t)\|_{H^{2\gamma}(\Omega)} + C \|u_x(\cdot, t)\|_{H^{2\gamma}(\Omega)} + C \|u(\cdot, t)\|_{H^{2\gamma}(\Omega)} \\
 (4.5) \quad &\leq C \|Au(\cdot, t)\|_{D(A^\gamma)} \leq C \int_0^t (t-s)^{\alpha(\frac{1}{2}-\gamma)-1} |p(s)| ds, \quad t \in (0, T).
 \end{aligned}$$

In the first equation of (4.1), let $x = x_0$, we have

$$(4.6) \quad p(t)R(x_0, t) = \partial_t^2 u(x_0, t) + Au(x_0, t) - b(x_0, t)u_x(x_0, t) - c(x_0, t)u(x_0, t), \quad a.e. t \in (0, T).$$

From condition (1.3), we obtain

$$|R(x_0, t)| = |q(x_0, t) \frac{\partial}{\partial x} u_2(x_0, t)| > \nu > 0.$$

Therefore, combining (4.5) and (4.6), we arrive at

$$\begin{aligned} |p(t)| &\leq C |\partial_t^\alpha u(x_0, t)| + C |Au(x_0, t) - b(x_0, t)u_x(x_0, t) - c(x_0, t)u(x_0, t)| \\ &\leq C \|\partial_t^\alpha u(x_0, t)\|_{L^\infty(0, T)} + C \int_0^t (t-s)^{\alpha(\frac{1}{2}-\gamma)-1} |p(s)| ds, \quad a.e. t \in (0, T). \end{aligned}$$

Applying Lemma 4.2, we have

$$|p(t)| \leq C \|\partial_t^\alpha u(x_0, t)\|_{L^\infty(0, T)}.$$

Thus we have proved the right hand side of (1.4). On the other hand, from (4.5), (4.6), combining the Sobolev embedding, we arrive at

$$\begin{aligned} |\partial_t^\alpha u(x_0, t)| &\leq |p(t)R(x_0, t)| + |Au(x_0, t) - b(x_0, t)u_x(x_0, t) - c(x_0, t)u(x_0, t)| \\ &\leq C |p(t)| \|R(\cdot, t)\|_{D(A^{\frac{1}{2}})} + C \int_0^t (t-s)^{\alpha(\frac{1}{2}-\gamma)-1} |p(s)| ds \\ &\leq C \left(\|R\|_{L^\infty(0, T; D(A^{\frac{1}{2}}))} + T^{\alpha(\frac{1}{2}-\gamma)} \right) \|p\|_{L^\infty(0, T)}. \end{aligned}$$

Thus we complete the proof. \square

Remark 1. Under the assumptions of Theorem 1.1, from Theorem 3.1 we know that the IBVP (1.1) admits a unique solution $u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ with $\partial_t^\alpha u \in L^\infty(0, T; H^{2\gamma}(\Omega))$ for some $0 \leq \gamma < \frac{1}{2}$ and $\gamma \neq \frac{1}{4}$. By the Sobolev embedding theorem, for $\frac{d}{4} < \gamma < \frac{1}{2}$ and $x_0 \in \Omega$, we have $\partial_t^\alpha u(x_0, \cdot) \in L^\infty(0, T)$ and $\frac{\partial}{\partial x_0} u(x_0, t) \in C[0, T]$. Therefore, our result only holds in one-dimensional case.

Remark 2. Compared our present results with the one in paper [6], there are two main differences. First, the regularity of the solution for the direct problem is weak (see Theorem 3.1). Second, the stability for the inverse problem is only valid in one-dimensional case (see Theorem 1.1). The main reasons for these discrepancy are that the convection term has a higher order derivative than the potential term considered in [6] and the regularity on x of $b(s)\frac{\partial}{\partial x}v(s)$ in (3.15) is reduced by one order. We find a feasible way by dividing the operator $A\psi = A^{1/2}(A^{1/2}\psi)$ (e.g. see (3.18)) for dealing with the regularity of $b(s)\frac{\partial}{\partial x}v(s)$. Finally, we provide an efficient inversion algorithm to recover the convection coefficient $p(t)$, whereas the paper [6] is only concerned with the theoretical results.

5. NUMERICAL ALGORITHM

In the following, we solve numerically the inverse problem of identifying the time-dependent convection coefficient $p(t)$. As we know, most of inversion algorithms are based

on regularization strategies so as to overcome ill-posedness of inverse problems, and different kinds of inverse problems may need different approximate methods on the basis of conditional well-posedness analysis. It is notable that the optimal perturbation algorithm has been testified to be effective for solving inverse problems for the integer-order as well as fractional diffusion equations (see, e.g., [19, 9, 20, 18, 5, 29]). In this paper, we modify it to obtain a good approximation to the convection coefficient $p(t)$.

Based on Theorem 3.1, we can define a forward operator

$$(5.1) \quad \mathcal{F} : p(t) \in H^1(0, T) \rightarrow u(x_0, t; p) \in L^2(0, T).$$

Thus the inverse problem is translated into solving the following abstract operator equation

$$(5.2) \quad \mathcal{F}(p) = h(t) \triangleq u(x_0, t).$$

Let $u = v + \varphi$ in (1.1) where v solves (3.2). By Lemma 3.2, we have $\partial_t^\alpha v(x_0, t; p) \in L^\infty(0, T)$, note that $v(x_0, 0; p) = 0$, by the book [16], we know $v(x_0, t; p) = I_{0+}^\alpha \psi(t)$ where $\psi(t) \in L^\infty(0, T)$. By the paper [11], we have $v \in H^\alpha(0, T)$ and further $u(x_0, t; p) = v(x_0, t; p) + \varphi(x_0) \in H^\alpha(0, T)$. As $H^\alpha(0, T) \hookrightarrow L^2(0, T)$ compactly, thus the operator $\mathcal{F} : H^1(0, T) \rightarrow L^2(0, T)$ is compact. The inverse convection coefficient problem is ill-posed. Let $p^* \in H^1(0, T)$ be a suitable guess of p , in order to ensure a stable numerical reconstruction of $p(t)$, we give the following minimization problem with a high order Tikhonov regularization term

$$(5.3) \quad \min_{p \in H^1(0, T)} J(p) = \|u(x_0, t; p) - h^\delta(t)\|_{L^2(0, T)}^2 + \mu \|p - p^*\|_{H^1(0, T)}^2,$$

where $\mu > 0$ is a regularization parameter, and h^δ is the noisy function of h .

Proposition 5.1. *Under the conditions of Theorem 1.1, there exists at least one minimizer $p_\mu^\delta \in H^1(0, T)$ for the variational problem (5.3).*

Proof. Since the functional J is nonnegative, there exists $d = \inf_{p \in H^1(0, T)} J(p)$. Thus, there exist a sequence $p_n \in H^1(0, T)$ such that $J(p_n) \rightarrow d$. Therefore, we obtain $\mu \|p_n - p^*\|_{H^1(0, T)}^2$ is bounded. That deduce $\{p_n\}$ is bounded in $H^1(0, T)$, then there exists a subsequence, still denoted by p_n , such that $p_n \rightharpoonup p_\mu^\delta$ in $H^1(0, T)$ and $p_n \rightarrow p_\mu^\delta$ in $L^2(0, T)$. Based on (1.5) of Theorem 1.1 and the Sobolev embedding, we have $u(x_0, t; p_n) \rightarrow u(x_0, t; p_\mu^\delta)$ in $L^2(0, T)$. That is

$$\|u(x_0, t; p_n) - h^\delta(t)\|_{L^2(0, T)}^2 \rightarrow \|u(x_0, t; p_\mu^\delta) - h^\delta(t)\|_{L^2(0, T)}^2, \quad n \rightarrow \infty.$$

Based on the weak lower semicontinuity of H^1 -norm, we have

$$(5.4) \quad \mu \|p_\mu^\delta - p^*\|_{H^1(0,T)}^2 \leq \liminf_{n \rightarrow \infty} \mu \|p_n - p^*\|_{H^1(0,1)}^2.$$

Therefore, we have

$$d \leq J(p_\mu^\delta) \leq \liminf_{n \rightarrow \infty} J(p_n) = d.$$

Then p_μ^δ is a minimizer. \square

Now we consider the discretization of the minimization problem. We define an admissible set of unknown coefficients $p(t)$. Suppose that $\{\varphi_s(t), s = 1, 2, \dots, \infty\}$ is a basis in $H^1(0, T)$, let

$$(5.5) \quad p(t) \approx p^S(t) = \sum_{s=1}^S a_s \varphi_s(t) \text{ and } p^*(t) \approx p^{*S}(t) = \sum_{s=1}^S a_s^* \varphi_s(t),$$

where p^S is the S -dimensional approximation solution to $p(t)$ and $S \in \mathbb{N}$ is a truncated level of $p(t)$, and $a_s, s = 1, 2, \dots, S$ are the expansion coefficients. It is convenient to set a finite dimensional space as

$$(5.6) \quad \Phi^S = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_S\},$$

and S -dimensional vector $a = (a_1, a_2, \dots, a_S) \in \mathbb{R}^S$. We identify an approximation $p^S(t) \in \Phi^S$ with a vector $a \in \mathbb{R}^S$.

Based on the above discussions, by setting $u(x, t; a) = u(x, t; p^S)$ as a unique solution of the forward problem, a feasible way for numerical solution to solve the following minimization problem

$$(5.7) \quad \min_{a \in \mathbb{R}^S} \left\{ \|u(x_0, t; a) - h(t)\|_{L^2(0,T)}^2 + \mu(a - a^*)A(a - a^*)^T \right\},$$

where $A = ((\varphi_i, \varphi_j)_{H^1})_{S \times S}$, $a^* = (a_1^*, a_2^*, \dots, a_S^*)$ and a^T denotes the transpose of a .

In the following, we give an inversion algorithm for determining the coefficient. For any known $a_k \in \mathbb{R}^S$, set

$$(5.8) \quad a_{k+1} = a_k + \delta a_k, \quad k = 0, 1, \dots,$$

where δa_k is called a perturbation for given a_k . Thus, in order to get a_{k+1} from the given a_k , we only need to get an optimal perturbation δa_k . For convenience of writing, a_k and δa_k are abbreviated as a and δa , respectively.

By the Taylor expansion for $u(x_0, t; a + \delta a)$ at a and ignoring the higher orders terms, we can obtain

$$(5.9) \quad u(x_0, t; a + \delta a) \approx u(x_0, t; a) + \nabla_a^T u(x_0, t; a) \cdot \delta a.$$

Take $a^* = a_k$ at each step, the functional (5.7) becomes

$$(5.10) \quad F(\delta a) \simeq \|\nabla_a^T u(x_0, t; a) \cdot \delta a - [h(t) - u(x_0, t; a)]\|_{L^2(0, T)}^2 + \mu(\delta a)A(\delta a)^T.$$

Now, discretizing the time domain $[0, T]$ with $0 = t_1 < t_2 < \dots < t_K = T$, where K denotes the number of grid points, the above L^2 norm can be reduced to the discrete Euclidean norm given as

$$(5.11) \quad F(\delta a) \simeq \frac{T}{K} \|\delta a B^T - (\eta - \xi)\|_2^2 + \mu(\delta a)A(\delta a)^T,$$

where

$$(5.12) \quad B = (b_{ks})_{K \times S}, \quad b_{ks} = \frac{u(x_0, t_k; a_1, \dots, a_s + \tau, \dots, a_S) - u(x_0, t_k; a)}{\tau}, \quad k = 1, 2, \dots, K,$$

and τ is the numerical differentiation step, where

$$(5.13) \quad \xi = (u(x_0, t_1; a), u(x_0, t_2; a), \dots, u(x_0, t_K; a)), \quad \eta = (h(t_1), h(t_2), \dots, h(t_K)).$$

We readily verify that the minimization of (5.10) is reduced to the following normal equation(see[17])

$$(5.14) \quad \left(\frac{\mu K}{T} A + B^T B\right) \delta a = B^T (\eta - \xi).$$

Hence, an optimal perturbation can be solved via the formula

$$(5.15) \quad \delta a = \left(\frac{\mu K}{T} A + B^T B\right)^{-1} B^T (\eta - \xi),$$

with which an optimal inversion solution can be approximated by the iteration procedure as long as arriving at the given number of iterations, or the perturbation satisfying a prescribed convergent precision given by

$$\|\delta a\| \leq eps,$$

where eps is a given convergent precision.

6. NUMERICAL EXPERIMENTS

In this section, we present two examples to verify the effectiveness of the algorithm. In numerical computations, we always set $T = 1$. The grid points on $[0, 1]$ and $[0, T]$ are both 21 when solving the direct problem by finite difference method in [23]. Here we point out that the elliptic operator A is the Laplace operator and the measured point $x_0 = 0.7$ in the following two examples. The noisy is generated by adding a random perturbation, i.e

$$(6.1) \quad h^\delta = h + \epsilon h \cdot (2 \cdot \text{rand}(\text{size}(h)) - 1).$$

The corresponding noise level is calculated by $\delta = \|h^\delta - h\|_{L^2(0,T)}$.

To show the accuracy of the numerical solution we compute the approximate error by

$$(6.2) \quad e_k = \|p_k(t) - p(t)\|_{L^2(0,T)},$$

where $p_k(t)$ is the reconstructed coefficients at the k th iteration, and $p(t)$ is the exact solution.

Example 1. Suppose $p(t) = e^{-t^2}$ and the additional data $u(x_0, t)$ are obtained by solving the direct problem (1.1) with $q(x, t) = \exp(-t) \sin(2\pi x)$, $c(x, t) = \exp(t) \cos(\pi x)$ and initial value $\varphi(x) = \sin(\pi x)$. The numerical results for Example 1 for various noise levels in the cases of $\alpha = 0.3, \alpha = 0.8$ are shown in Figure 1(a) and 1(b) respectively by taking a precision $\epsilon ps = 2 \times 10^{-6}$ and a numerical differentiation step size $\tau = 0.2$. Here we choose the initial guess as $a_0 = 0$ and $\Phi^4 = \text{span}\{1, \cos(\pi t), \cos(2\pi t), \cos(3\pi t)\}$. We can see that the numerical results for Example 1 match the exact ones quite well even up to 1% noise added in the “exact” data $u(x_0, t)$.

Example 2. Suppose $p(t) = t \exp(-t^2) \sin(2\pi t)$ and the additional data $u(x_0, t)$ are obtained by solving the direct problem (1.1) with $q(x, t) = \exp(-t) \sin(2\pi x)$, $c(x, t) = \exp(t) \cos(\pi x)$ and initial value $\varphi(x) = \sin(\pi x)$. The numerical results for Example 2 for various noise levels in the cases of $\alpha = 0.1, \alpha = 0.9$ are shown in Figure 2(a) and 2(b) respectively by taking a precision $\epsilon ps = 2 \times 10^{-4}$ and a numerical differentiation step size $\tau = 0.1$. Here we choose the initial guess as $a_0 = 0$, and $\Phi^4 = \text{span}\{\sin \pi t, \sin(2\pi t), \sin(3\pi t), \sin(4\pi t)\}$. We can see that the numerical results for Example 2 match the exact ones quite well even up to 1% noise added in the “exact” data $u(x_0, t)$. In Table 1, we show the numerical errors e_k and the stop steps in parentheses for Example 2 with different α and ϵ . It can be seen that the numerical results become a little worse when the relative noise levels increase and are not sensitive to the fractional order α .

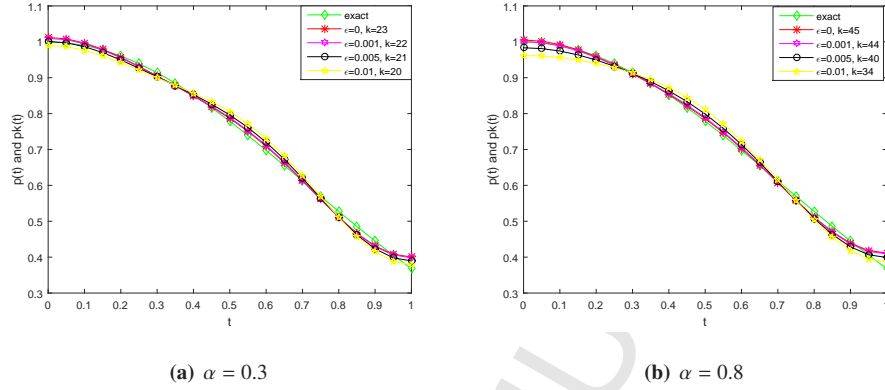


FIGURE 1. The numerical results for Example 1 for various noise levels with $\mu = \delta^{\frac{14}{3}}$.

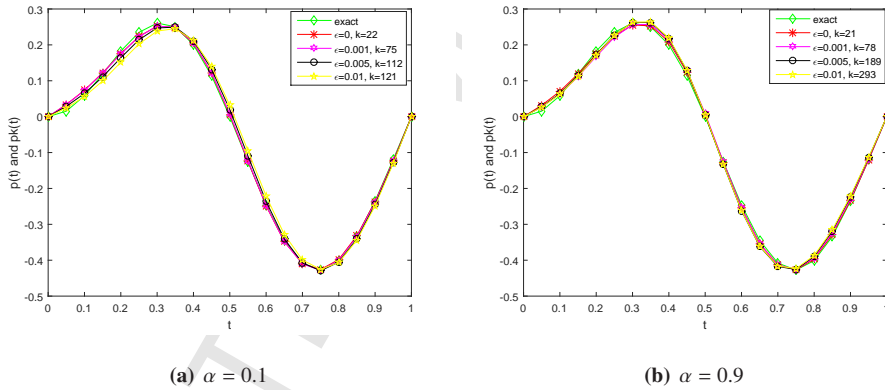


FIGURE 2. The numerical results for Example 2 for various noise levels with $\mu = \delta$.

TABLE 1. The error e_k and stop steps with different α and ϵ in Example 2.

$\alpha \backslash \epsilon$	0	0.001	0.003	0.005	0.01
$\alpha = 0.1$	0.0071(22)	0.0073(75)	0.0096(98)	0.0115(112)	0.0189(121)
$\alpha = 0.3$	0.0070(22)	0.0073(69)	0.0098(90)	0.0122(103)	0.0197(111)
$\alpha = 0.5$	0.0070(23)	0.0073(69)	0.0099(89)	0.0106(89)	0.0218(107)
$\alpha = 0.7$	0.0070(20)	0.0075(68)	0.0089(84)	0.0130(94)	0.0226(100)
$\alpha = 0.9$	0.0072(21)	0.0082(78)	0.0102(132)	0.0106(189)	0.0115(293)

7. CONCLUSIONS

In this paper, we devote to identify the time-dependent convection coefficient $p(t)$ in a time-fractional diffusion equation for one dimensional case. The existence, uniqueness and regularity of the solution for the direct problem is obtained (see e.g. Theorem 3.1). Then the stability of the solution for the inverse problem is provided by using the regularity of the corresponding direct problem and some generalized Gronwall's inequalities (see e.g. Theorem 1.1). Finally, we transform the inverse convection coefficient problem into a variational problem. Employing the modified optimal perturbation algorithm, the variational problem is solved. Compared the present paper with the published papers, there are two factors that bring new difficulties. One is the convective term, which makes the original elliptic operator non self-adjoint. Thus the method of eigenfunction expansion is failure. The other is that all coefficients are not just spatially dependent. Therefore, we can not use the conventional Fourier method that transform the problem to ordinary equations of fractional order to solve the problem directly (see e.g. [26]). The previous results obtained by Fourier series, such as the existence and regularity of the solution for the direct problem are no longer true. Since we consider the determination of the time-dependent convection coefficient, in order to overcome the difficulty, we apply the fixed point theorem to show the existence of the solution.

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