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Asymptotic analysis of perturbed Takens–Bogdanov points

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Abstract

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Takens–Bogdanov points are limit points with a spectral degeneracy. Subjected to a perturbation, they may give birth to Hopf bifurcation points. The aim of this paper is to contribute to the constructive analysis of this phenomenon. Numerical applications are hinted.

Keywords: Nonlinear equations, bifurcation point, Takens–Bogdanov point, imperfect bifurcation diagrams, qualitative analysis, approximation.

1. Introduction

We consider a smooth, parameter-dependent mapping $F: \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^N$, $F = F(u, \beta)$, $\beta = (\lambda, \alpha) \in \mathbb{R}^1 \times \mathbb{R}^k = \mathbb{R}^n$. The equation

$$F(u, \lambda, \alpha) = 0 \tag{1.1}$$

is understood as an implicit definition of the dependence of $u \in \mathbb{R}^N$ (the state variable) on $\lambda \in \mathbb{R}^1$ (the control parameter) with $\alpha \in \mathbb{R}^k$ being fixed (the imperfection).

We discuss a particular singular solution to (1.1) called Takens–Bogdanov point, see, e.g., [3].

Definition 1.1. We say that $(u^*, \beta^*) \in \mathbb{R}^N \times \mathbb{R}^n$ is a *TB-point* (i.e., *Takens–Bogdanov point*) provided that

$$F(u^*, \beta^*) = 0 \tag{1.2}$$

and there exist a $\xi^* \in \mathbb{R}^N$ and a linear functional $L \in \mathscr{L}(\mathbb{R}^N, \mathbb{R}^1)$ such that

$$F_u^2(u^*, \beta^*)\xi^* = 0, \tag{1.3}$$

$$L\xi^* = 0, \qquad LF_u(u^*, \beta^*)\xi^* = 1, \tag{1.4}$$

$$\operatorname{Ker} L \cap \operatorname{Ker} F_u(u^*, \beta^*) = \{0\}, \tag{1.5}$$

$$\xi^* \notin \operatorname{Im} F_u(u^*, \beta^*). \tag{A.0}$$

(F_u denotes partial differential of F with respect to $u \in \mathbb{R}^N$.)

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The condition (1.5) implies dim Ker $F_u(u^*, \beta^*) \leqslant 1$. Thus, due to (1.3), (1.4), Ker $F_u =$ span$[\eta^*]$, $L\eta^* = 1$, where

$$\eta^* = F_u(u^*, \beta^*)\xi^*. \tag{1.6}$$

Consequently, there exists a Jordan cell $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in the canonical form of the matrix $F_u(u^*, \beta^*)$. The size 2×2 of this cell is guaranteed by the *nondegeneracy condition* (A.0).

It is clear that L plays the role of a normalising vector and the above definition can be made independent of the choice of L. Nevertheless, we consider L being fixed and think that (u^*, β^*) is chosen a priori from the (open) set

$$\mathcal{M} = \{(u, \beta) \in \mathbb{R}^N \times \mathbb{R}^n\colon \text{Ker } L \cap \text{Ker } F_u(u, \beta) = \{0\}\}.$$

We also *assume throughout this paper* that any point $(u, \beta) \in \mathbb{R}^N \times \mathbb{R}^n$ we shall deal with belongs to $\mathcal{M}$.

The significance of TB-points consists in the fact that, after a perturbation, Hopf bifurcation points may appear in a neighbourhood of (u^*, β^*). In order to analyse this phenomenon, one has to specify (u^*, β^*) as a "bifurcation singularity" according to the standard classification, see [2]. Essentially, such an assumption defines a qualitative behaviour of the solution set to the equation $F(u, \lambda, \alpha^*) = 0$ in a neighbourhood of (u^*, λ^*). Here, we used the natural notation $(\lambda^*, \alpha^*) \in \mathbb{R}^{1+k}$ for the splitting of β^*.

In particular, we assume (u^*, β^*) to be *limit point*, i.e., the singularity of the smallest possible (bifurcation) codimension. The formulation of this assumption is postponed till Section 3.

Our aim is to give a "first-order" analysis of singular points in a neighbourhood of $(u^*, \lambda^*, \alpha^*)$ provided that α^* is subjected to a small perturbation. In principle, we shall reproduce (and slightly enhance) the results presented in [5]. We use a different technical approach which, as we believe, might be of some interest.

We tried to use the concept of Liapunov–Schmidt reduction (see [4] for its "numerical" version). We generalized the procedure slightly and constructed a "first-order approximation" to the Jordan–Arnold canonical form (see [1]) of all imperfections to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Even though this point will not be proved explicitly, our results in Section 3 have such an interpretation.

Practical applications of the asymptotic analysis were already suggested in [5]. Given a TB-point $(u^*, \lambda^*, \alpha^*)$ and a perturbation $\delta\alpha$ of α^*, one can compute a good initial guess for a numerical approximation of "lower" singularities (namely, limit and Hopf points) in a neighbourhood of (u^*, λ^*). We give an example in Section 5.

For a complete list of references concerning numerical treatment of TB-points, we refer to [5].

2. On a dimensional reduction

Given $(u, \beta) \in \mathbb{R}^N \times \mathbb{R}^n$, we define $M \in \mathbb{R}^N$, $w \in \mathbb{R}^N$, $\sigma \in \mathbb{R}^1$ and $s \in \mathbb{R}^1$ as the solution to

$$\begin{pmatrix} F_u(u, \beta) & M \\ L & 0 \end{pmatrix}\begin{pmatrix} w \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.1}$$

$$\begin{pmatrix} F_u(u, \beta) & M \\ L & 0 \end{pmatrix}\begin{pmatrix} M \\ \sigma \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix}. \tag{2.2}$$

We note that at (u^*, β^*), the conditions (2.1), (2.2) can be satisfied taking $s = \sigma = 0$, $M = \xi^*$ and $w = \eta^*$. Moreover, the matrix

$$\begin{pmatrix} F_u(u^*, \beta) & \xi^* \\ L & 0 \end{pmatrix} \in \mathscr{L}(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$$

is regular. Then, by virtue of the Implicit Function Theorem, the solution M, w, σ, s to (2.1), (2.2) can be locally parametrised by (u, β). Thus,

$$\begin{aligned} &M: \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^N, \qquad && w: \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^N, \\ &s: \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^1, \qquad && \sigma: \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^1 \end{aligned} \tag{2.3}$$

are germs of smooth mappings.

We define germs of smooth mappings $Q: \mathbb{R}^N \times \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^N, \mathbb{R}^N)$ and $Q^c: \mathbb{R}^N \times \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^N, \mathbb{R}^1)$, setting

$$Q = I - cMM^{\mathrm{T}}, \qquad Q^c = cM^{\mathrm{T}}, \qquad c = (M^{\mathrm{T}}M)^{-1}, \tag{2.4}$$

at each (u, β) from a sufficiently small neighbourhood of (u^*, β^*). By means of Q and Q^c we introduce a kind of Liapunov–Schmidt reduction.

Given $(x, y) \in \mathbb{R}^1 \times \mathbb{R}^n$, we define $v \in \mathbb{R}^N$ by the condition

$$\begin{aligned} &Q(u^* + v, \beta^* + y) F(u^* + v, \beta^* + y) = 0, \\ &Lv = x. \end{aligned} \tag{2.5}$$

Due to the Implicit Function Theorem, $v: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^N$ is a germ of smooth mapping, $v(0) = 0$. Let $g: \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$, $\phi: \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$ be the germs of smooth mappings which are defined as follows:

$$g(x, y) = Q^c(u^* + v(x, y), \beta^* + y) F(u^* + v(x, y), \beta^* + y), \tag{2.6}$$

$$\phi(x, y) = -\sigma(u^* + v(x, y), \beta^* + y), \tag{2.7}$$

for $\sigma = \sigma(u, \beta)$, see (2.1), (2.2).

It can easily be verified that

$$g = g_x = \phi = 0 \quad \text{at the origin } 0 \in \mathbb{R}^1 \times \mathbb{R}^n.$$

Theorem 2.1. *The roots of F and g are locally isomorphic, namely, $F(u, \beta) = 0$ iff*

$$u = u^* + v(x, y), \qquad \beta = \beta^* + y, \tag{2.8}$$

$$g(x, y) = 0, \tag{2.9}$$

in the obvious local sense (i.e., for (u, β) and (x, y) from sufficiently small neighbourhoods of (u^*, β^*) and $(0, 0)$, respectively).

The manifold of u's and β's parametrised by (2.8) is called *critical manifold* of F. The operator g is (a kind of) *Liapunov–Schmidt reduction* of F at (u^*, β^*).

We try to track all "singularities" of F in a small neighbourhood of the "organising centre" (u^*, β^*). We shall have in mind such roots of F where dynamical stability is likely to be changed due to the fact that the relevant differential F_u has eigenvalues on imaginary axes. In principle, we distinguish two cases having different dynamical consequences.

Definition 2.2. A point $(u, \beta) \in \mathbb{R}^N \times \mathbb{R}^n$ is *singular point* of F provided that $F(u, \beta) = 0$, and

$$0 \in \text{spectrum of } F_u(u, \beta).$$

Definition 2.3. A point $(u, \beta) \in \mathbb{R}^N \times \mathbb{R}^n$ is *Hopf point* (with frequency $\nu > 0$) provided that $F(u, \beta) = 0$, and

$$i\nu \in \text{spectrum of } F_u(u, \beta), \quad \nu > 0.$$

We note that *Hopf bifurcation points*, see, e.g., [3] (and *limit points*, see, e.g., [2], respectively) are Hopf points (and singular points) satisfying certain nondegeneracy conditions.

Theorem 2.4. *When restricted to a sufficiently small neighbourhood of* (u^*, β^*), *a point* $(u, \beta) \in \mathbb{R}^N \times \mathbb{R}^n$ *is a Hopf point with a sufficiently small frequency* $\nu > 0$ *iff* (2.8) *holds and*

$$\begin{aligned} &g(x, y) = 0, \qquad \phi(x, y) = 0, \\ &g_x(x, y) + \mu = 0, \quad \mu = \nu^2 > 0. \end{aligned} \tag{2.10}$$

Theorem 2.5. *In the above local sense,* $(u, \beta) \in \mathbb{R}^N \times \mathbb{R}^n$ *is singular point of F iff* (2.8) *holds and*

$$g(x, y) = g_x(x, y) = 0. \tag{2.11}$$

Moreover, $\phi(x, y)$ *is (a real) eigenvalue of* $F_u(u, \beta)$.

Remark 2.6. Since $\phi(0) = 0$, the eigenvalue $\phi = \phi(x, y)$ mentioned in Theorem 2.5 is the second smallest (in modulus) eigenvalue of F_u, next to the zero eigenvalue.

3. Asymptotic analysis of singularities

We recall the "bifurcation context" in which we wanted to discuss (1.1). Theorem 2.1 makes the link between (1.1) and the bifurcation equation (2.9). The splitting $\beta = (\lambda, \alpha) \in \mathbb{R}^{1+k}$ of the parameter $\beta \in \mathbb{R}^n$ induces the splitting $y = (t, z) \in \mathbb{R}^{1+k}$ of the increment $y = \beta - \beta^*$ in (2.5), (2.7), namely,

$$t = \lambda - \lambda^*, \qquad z = \alpha - \alpha^*.$$

Our aim is to describe positions of both kinds of singularities (i.e., singular and Hopf points) as α (i.e., z) is perturbed. To this end, we try to apply the Implicit Function Theorem to (2.10) and (2.11).

For example, let us consider the solution set $\{x, t, z, \mu\}$ to (2.10), omitting the requirement $\mu > 0$. Due to the Implicit Function Theorem, the solution set can be locally (i.e., in a neighbourhood of the origin $0 \in \mathbb{R}^{k+2}$) parametrised by $z \in \mathbb{R}^k$ provided that

$$\det \begin{pmatrix} g_x & g_t & 0 \\ g_{xx} & g_{xt} & 1 \\ \phi_x & \phi_t & 0 \end{pmatrix} \neq 0, \quad \text{at the origin.}$$

Since $g_x = 0$, the above assumption reads as $g_t \phi_x \neq 0$ at the origin. Thus, $x = x(z)$, $t = t(z)$, $\mu = \mu(z)$ is the parametrisation of the above considered solution set. Taylor expansion of $x(z)$, $t(z)$, $\mu(z)$ can be found by formal differentiation of (2.10) at the origin. The (local) validity of

the restriction $\mu > 0$ is equivalent to the requirement $\mu_z(0)z > 0$, which imposes restrictions on the imperfection z.

The practical limit on such an asymptotic analysis is the available information concerning differentials of g and ϕ at the origin. We assume that the following $3 \times (k+2)$ matrix is known:

$$B = \begin{pmatrix} g_x & g_t & g_z \\ g_{xx} & g_{xt} & g_{xz} \\ \phi_x & \phi_t & \phi_z \end{pmatrix}, \quad \text{at } (x, t, z) = 0. \tag{3.1}$$

It represents, in a sense, minimal data needed for our asymptotic analysis. In Section 4 we show how it is linked with differentials of the equations (1.2)–(1.4).

We review, without further comments, asymptotic formulae exploiting data (3.1).

Let the following assumptions be satisfied:

$$p = \operatorname{sgn} g_{xx}(0) \neq 0, \qquad q = \operatorname{sgn} g_t(0) \neq 0, \tag{A.1}$$

$$\phi_x(0) \neq 0. \tag{A.2}$$

Remark 3.1. The assumption (A.1) represents the nondegeneracy conditions for (u^*, β^*) to be *limit point*, see [2]. Obviously, each singular point (see Definition 2.2) which is close enough to (u^*, β^*) is a limit point.

In order to simplify formulations, we assume $k = 1$ without loss of generality. All the partials of g, ϕ and v which participate in the coming formulae are evaluated at the origin $(x, t, z) = 0$.

Theorem 3.2. *Hopf points* $(u_H, \lambda_H, \alpha_H)$ *in a neighbourhood of the organising centre* $(u^*, \lambda^*, \alpha^*)$, *with sufficiently small frequencies* $\nu > 0$, *create a smooth manifold which can be parametrised by* $z \in \mathbb{R}^1$ *satisfying*

$$-\frac{\det B}{g_t \phi_x} z > 0. \tag{3.2}$$

Then,

$$\begin{aligned} u_H &= u^* + x v_x + t v_t + z v_z + O(z^2), \\ \lambda_H &= \lambda^* + t, \qquad \alpha_H = \alpha^* + z, \end{aligned} \tag{3.3}$$

where

$$x = -\frac{z}{g_t \phi_x} \det \begin{pmatrix} g_t & g_z \\ \phi_t & \phi_z \end{pmatrix} + O(z^2), \tag{3.4}$$

$$t = -\frac{g_z}{g_t} z + O(z^2), \tag{3.5}$$

as $z \to 0$ *and satisfies* (3.2). *Moreover*,

$$\nu^2 = -\frac{\det B}{g_t \phi_x} z + O(z^2). \tag{3.6}$$

Theorem 3.3. *Limit points* $(u_L, \lambda_H, \alpha_H)$ *in a neighbourhood of* $(u^*, \lambda^*, \alpha^*)$ *create a smooth manifold which can be parametrised by* z. *Namely,*

$$\begin{aligned} u_L &= u^* + xv_x + tv_t + zv_z + O(z^2), \\ \lambda_L &= \lambda^* + t, \qquad \alpha_L = \alpha^* + z, \end{aligned} \tag{3.7}$$

where

$$x = -\frac{z}{g_{xx}g_t}\det\begin{pmatrix} g_t & g_z \\ gxt & g_{xx} \end{pmatrix} + O(z^2), \tag{3.8}$$

and t *satisfies* (3.5). *Moreover, the relevant eigenvalue* ϕ *next to the zero eigenvalue, can be estimated as*

$$\phi = -\frac{\det B}{g_{xx}g_t}z + O(z^2), \tag{3.9}$$

for $z \to 0$.

The first-order analysis does not distinguish λ-coordinates of Hopf and limit points which are due to the same imperfection z.

Theorem 3.4. *Let* λ_L *and* λ_H *be the* λ*-coordinate of limit and Hopf point, respectively, which are due to an imperfection* z *satisfying* (3.2). *Then the distance* $\mathrm{dist} = \lambda_H - \lambda_L$ *can be estimated as follows*:

$$\mathrm{dist} = -\frac{1}{2g_{xx}g_t}\left(\frac{\det B}{\phi_x\phi_t}\right)^2 z^2 + O(z^3), \tag{3.10}$$

as $z \to 0$.

Higher-order terms in all the above estimates require higher derivatives of g and ϕ than those listed in (3.1).

4. Data for the qualitative analysis

Any differential of g, ϕ or v at the origin $0 \in \mathbb{R}^{2+k}$ can be calculated by means of the chain rule straight from the relevant definition (2.5)–(2.7). In order to present the resulting formulae, we introduce two operators, $F_u^+ \in \mathscr{L}(\mathbb{R}^N, \mathbb{R}^N)$ and $Z \in \mathscr{L}(\mathbb{R}^N, \mathbb{R}^1)$. Given any $r \in \mathbb{R}^N$, we define $F_u^+ r \in \mathbb{R}^N$ and $Zr \in \mathbb{R}^1$ to be the solution to

$$\begin{pmatrix} F_u & \xi^* \\ L & 0 \end{pmatrix}\begin{pmatrix} F_u^+ v \\ Zr \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}. \tag{4.1}$$

Here and in the coming formulae, F and its partial derivatives are evaluated at $(u^*, \lambda^*, \alpha^*)$. Direct calculation yields

$$v_x = \eta^*, \qquad v_t = -F_u^+ F_\lambda, \qquad v_z = -F_u^+ F_\alpha, \tag{4.2}$$

and

$$g_x = 0, \qquad g_t = ZF_\lambda, \qquad g_z = ZF_\alpha, \tag{4.3}$$

The row vector Z can be defined explicitly, i.e.,

$$(Z, -g_x)\begin{pmatrix} F_u & \xi^* \\ L & 0 \end{pmatrix} = (0, 1). \tag{4.4}$$

In fact, due to (4.3), Z is the left eigenvector of F_u. We also need a generalised left eigenvector $R \in \mathscr{L}(\mathbb{R}^N, \mathbb{R}^1)$,

$$(R, \phi)\begin{pmatrix} F_u & \xi^* \\ L & 0 \end{pmatrix} = (Z, 0). \tag{4.5}$$

Let us note that $\phi = 0$ at the origin.

Now we can complete the list of data (3.1):

$$\begin{aligned} g_{xx} &= ZF_{uu}v_x v_x, \\ g_{xt} &= Z(F_{uu}v_x v_t + F_{u\lambda}v_x), \\ g_{xz} &= Z(F_{uu}v_x v_\alpha + F_{u\alpha}v_x), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \phi_x &= ZF_{uu}v_x\xi^* + RF_{uu}v_x v_x, \\ \phi_t &= Z(F_{uu}v_t\xi^* + F_{u\lambda}\xi^*) + R(F_{uu}v_t v_x + F_{u\lambda}v_x), \\ \phi_z &= Z(F_{uu}v_z\xi^* + F_{u\alpha}\xi^*) + R(F_{uu}v_z v_x + F_{u\alpha}v_x). \end{aligned} \tag{4.7}$$

At the first glimpse, the formulae look nasty and the cost of B seems to be enormous. Nevertheless, we want to comment on the fact that B is closely related to an evaluation of one Newton step for a solution to the defining equations (1.2)–(1.4) of our organising centre (u^*, β^*). In other words, once we find (u^*, β^*) via Newton iterations, the data B can be obtained at the cost of one additional Newton step at (u^*, β^*).

Our statement can be proved by direct computation. Since we are limited by the size of our communication, we explain just the idea behind. At any rate, the suggested computation of B is incorporated into the code we are demonstrating in Section 5.

Takens–Bogdanov points can be characterised as Hopf points with "zero frequency" (plus nondegeneracy conditions). In other words, TB-points appear generically at the intersection of both limit and Hopf point manifolds. Due to Theorem 2.4, (u, β) is a TB-point in a neighbourhood of (u^*, β^*) iff (2.8) holds and

$$g(x, y) = g_x(x, y) = \phi(x, y) = 0. \tag{4.8}$$

Thus, the matrix B is differential of the "reduced defining equation" (4.8) for TB-points. It can be proved that (u^*, β^*) is a regular root of (1.2)–(1.4), i.e., the differential of the "extended system" (1.2)–(1.5) at (u^*, β^*) is surjective, iff B has full rank.

Let

$$\text{rank } B = 3, \tag{A.3}$$

where B is defined at (3.1). Consequently, all TB-points in a neighbourhood of (u^*, β^*) are a $(k-1)$-dimensional manifold.

Remark 4.1. If $k = 1$, then the assumption (A.3) is equivalent to the key assumption [5, (2.11)]. It can be checked that $\det B$ is proportional to the quantity $\mathrm{d}/\mathrm{d}\alpha\ (\psi_1^{\mathrm{T}}\phi_1)\,|_{\alpha=\alpha_1}$ in the quoted assumption. It is clear from our analysis in Section 3 that (A.3) makes it possible to parametrise the path of Hopf points by the square of frequency $\nu^2 \geqslant 0$. In this sense, (A.3) may replace (A.2).

5. Example

We consider $f: \mathbb{R}^4 \times \mathbb{R}^{1+3} \to \mathbb{R}^4$,

$$F(u, \lambda, \alpha) = \begin{cases} D(w^{(2)} - w^{(1)}) + \lambda f(w^{(1)}), \\ D(w^{(1)} - w^{(2)}) + \lambda f(w^{(2)}), \end{cases} \tag{5.1}$$

where

$$u = \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix}, \qquad \lambda \in \mathbb{R}^1, \qquad \alpha = (B, A) \in \mathbb{R}^2,$$

and

$$f(w) = \begin{pmatrix} A - (B+1)x + x^2 y \\ Bx - x^2 y \end{pmatrix}, \quad \text{at } w = (x, y) \in \mathbb{R}^2,$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}.$$

The equation $F(u, \lambda, \alpha) = 0$ defines steady-state concentrations $w^{(1)}$, $w^{(2)}$ in the 2-box Brusselator reaction model for given parameter λ (= control) and α (= the imperfection).

Fixing $\alpha_2^* = A^2 = 2$, the point

$$u^* = (3.14186494, 1.83942439, 0.858135058, 4.58112097),$$
$$\lambda^* = 22.010690558532,$$
$$\alpha_1^* = B^* = 5.3827640836414,$$
$$\xi^* = (-0.00949103, 0.00692515, -0.01644783, 0.00692515)$$

is a regular solution to (1.2), (1.3) with

$$L = (0, -1, 0, 1),$$

which satisfies (1.5). The nondegeneracy condition (A.0) can be verified by means of the Fredholm alternative. The data for our qualitative analysis are as follows:

$$B = \begin{pmatrix} 0.0 & -62.14540927 & 490.54603830 & -1587.80653465 \\ -395.2040935 & 40.65439874 & 236.18195510 & 337.42499965 \\ -36.74669778 & 5.85357041 & 20.84313230 & 42.45923784 \end{pmatrix},$$

$$v_x = \begin{pmatrix} 0.41648114 \\ -0.21453376 \\ -0.41648114 \\ 0.78546623 \end{pmatrix}, \qquad v_t = \begin{pmatrix} -0.05561912 \\ 0.03104419 \\ 0.02148804 \\ 0.03104419 \end{pmatrix},$$

$$v_{z_1} = \begin{pmatrix} 0.06364227 \\ 0.26283138 \\ 0.20577202 \\ 0.26283138 \end{pmatrix}, \qquad v_{z_2} = \begin{pmatrix} 0.7940017 \\ -0.4461346 \\ 0.3339541 \\ -0.4461346 \end{pmatrix}.$$

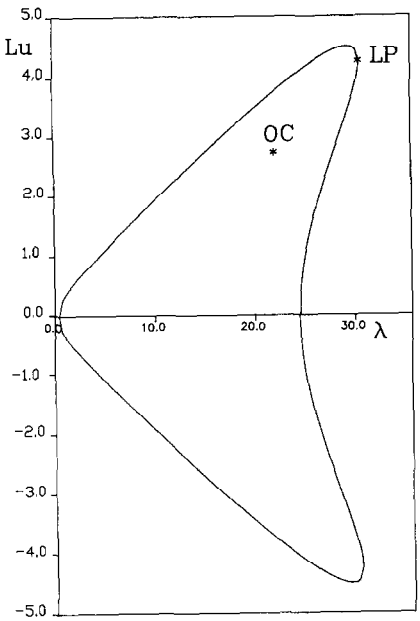


Fig. 1.

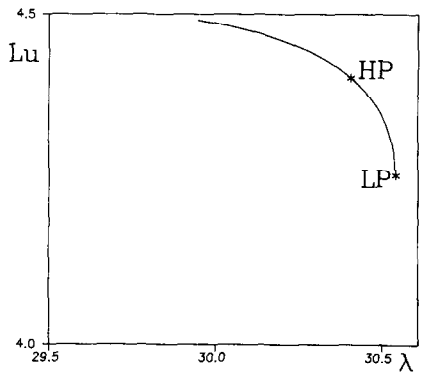


Fig. 2.

At this moment, neither the last column of B nor v_{z_2} are relevant. It can be readily verified that assumptions (A.1)–(A.3) are satisfied. Checking the condition (3.2), it appears that *positive perturbations* z of B^* imply the occurrence of a Hopf point in a neighbourhood of our organising centre.

Let us, for example, consider the perturbation $z = 1$. The corresponding bifurcation diagram (i.e., Lu versus λ as a projection of the solution set $F(u, \lambda, \alpha) = 0$, where $\alpha = B^* + 1$ is fixed) is depicted in Fig. 1. One limit point $\mathrm{LP} \equiv (\lambda_L = 30.54178216,\ x_L = 4.25685055)$ was detected and one Hopf point $\mathrm{HP} \equiv (\lambda_H = 30.4063006,\ x_H = 4.403338966)$ in a neighbourhood of our organising centre OC. The relevant detail of Fig. 1 is plotted in Fig. 2.

In Fig. 3, there is an "eigenvalue-movie" which might illustrate the birth of the above singularities by means of a spectral degeneracy. Namely, two eigenvalues of $F_u(u, \beta)$ which are

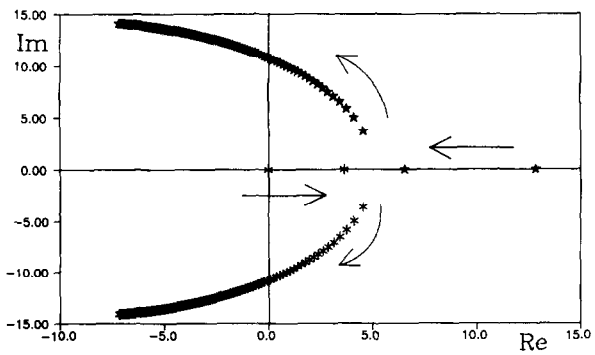


Fig. 3.

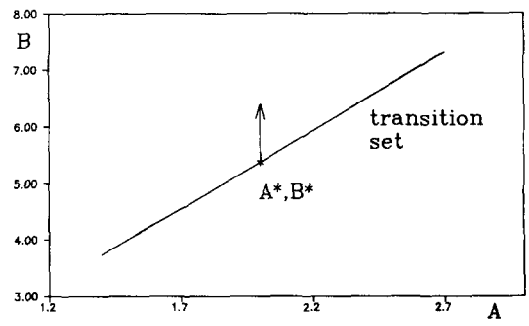


Fig. 4.

Table 1
Numerically detected Hopf points (x_H, λ_H) with frequency ν, which are due to the imperfection z of B^*

z	λ_H	x_H	ν
1.0	30.40630055	4.40333896	10.80628127
0.1	22.80207932	2.91876827	3.90221652
0.01	22.08963634	2.75987700	1.27526208
0.001	22.01858417	2.74352052	0.40480267
0.0001	22.01147991	2.74187903	0.12805905
0.00001	22.01076949	2.74171482	0.04049673

Table 2
Numerically detected limit point (x_L, λ_L), which are due to the imperfection z of B^*

z	λ_L	x_L
1.0	30.54178216	4.25685054
0.1	22.80643992	2.88364522
0.01	22.08968975	2.75580256
0.001	22.01858471	2.74310629
0.0001	22.01147991	2.74183754
0.00001	22.01076949	2.74171067

Table 3
Asymptotic analysis of Hopf points

z	$(\lambda_H - \lambda^*)/z$	$(\lambda_H - \lambda_L)/z^2$	$(x_H - x^*)/z$	ν^2/z
1.0	8.39560999	−0.13548160	1.66164238	116.77571493
0.1	7.91388767	−0.43606017	1.77071697	152.27293783
0.01	7.89457904	−0.53410416	1.81811225	162.62933772
0.001	7.89361393	−0.54621177	1.82394713	163.86520328
0.0001	7.89352937	−0.54745115	1.82454395	163.99120785
0.00001	7.89352104	−0.54750870	1.82459045	163.99857206
Predicted limit as $z \to 0$	7.89352011	−0.547589329	1.82461043	164.00524205

Table 4
Asymptotic analysis of limit points

z	$(\lambda_L - \lambda^*)/z$	$(x_L - x^*)/z$	ϕ/z
1.0	8.53109160	1.51515396	12.86316748
0.1	7.95749369	1.41948648	14.76370490
0.01	7.89992001	1.41059849	15.19561924
0.001	7.89416014	1.40971928	15.24402157
0.0001	7.89358412	1.40963146	15.2489232
0.00001	7.89352651	1.40962268	15.2496988
Predicted limit as $z \to 0$	7.89352011	1.40962170	15.24946518

"responsible" for both LP and HP are traced in complex plane as (u, λ) varies along Fig. 2, starting from LP. The λ-coordinate is decreasing with a constant speed while one hundred points along this trajectory are taken and processed. The corresponding motion of two selected eigenvalues ($*$ and $\star$ in Fig. 3) is marked by arrows. Obviously, both eigenvalues "crash" and change qualitatively. This change is sudden.

In order to test our asymptotic formulae we consider a sequence $\{10^{-i}\}_{i=0}^{5}$ of perturbations z. For each perturbation, Hopf point and limit point are detected by a Newton procedure, see Tables 1 and 2.

In Tables 3 and 4 the asymptotic formulae are tested. The predicted limit values were obtained by means of (3.5), (3.10), (3.4), (3.6) and (3.5), (3.8), (3.9) as $z \to 0_+$.

At the end, we relax the parameter A ($= \alpha_2$). Thus $k = 2$. Due to (A.3), there exists a curve of TB-points passing through our organising centre. The curve can be parametrised by A. Applying the Implicit Function Theorem, and using data B (in this case, the whole matrix 3×4 is considered), we can easily derive the first-order approximation of the mentioned curve of TB-points in a neighbourhood of our organising centre. The projection of this curve onto the (A,B)-parameter plane is called *transition set*. The first-order analysis predicts also the tangent to the transition set at (A^*, B^*). We refer to Fig. 4, where this tangent is depicted. The actual transition set and its approximation cannot graphically be distinguished in the scale of Fig. 4. The arrow shows the direction of all imperfections considered in Tables 1–4. In the obvious local meaning, all the imperfections from the half plane containing the arrow lead to the "birth" of a Hopf point with positive frequency in a neighbourhood of our organising centre.

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