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Superconvergence of Discontinuous Galerkin Methods for Nonlinear Delay Differential Equations with Vanishing Delay

Xiuxiu Xu, Qiumei Huang*
Beijing Institute for Scientific and Engineering Computing, Beijing University of Technology, Beijing 100124, China
Email: xuxiuxiu@emails.bjut.edu.cn, qmhuang@bjut.edu.cn

Abstract. In this paper, we investigate the local superconvergence of the discontinuous Galerkin (DG) solutions on quasi-graded meshes for nonlinear delay differential equations with vanishing delay. It is shown that the optimal order of the DG solution at the mesh points is $O(h^{2m+1})$. By analyzing the supercloseness between the DG solution and the interpolation $\Pi_h u$ of the exact solution, we get the optimal order $O(h^{m+2})$ of the DG solution at characteristic points. We then extend the convergence results of DG solutions to state dependent delay differential equations. Numerical examples are provided to illustrate the theoretical results.

Keywords. *superconvergence, nonlinear delay differential equations, discontinuous Galerkin methods, vanishing delay, state dependent delay .*

1 Introduction

We consider the following nonlinear vanishing delay differential equation (DDE)

$$\begin{aligned} &u'(t) = f(t, u(t), u(\theta(t))), \qquad t \in J = [0, T], \\ &u(0) = u_0, \end{aligned} \tag{1.1}$$

where the delay function θ is subject to the following conditions:

1. $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$,
2. $\min_{t \in J} \theta'(t) =: q_0 > 0$,

and the function $f \in C(J)$.

Delay differential equations are important mathematical models that describe various real-life phenomena such as biological, physical and chemical systems. The exact analytical solutions of the DDEs are not available in general, one has to rely on numerical methods to find approximate solutions. There are many numerical methods for DDEs including Runge-Kutta methods ([2, 19, 30]), linear multi-step methods [25], θ-methods [20],

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collocation methods [6, 29], discontinuous (continuous) Galerkin methods [11, 17, 32, 33], and spectral methods [1, 24].

Discontinuous Galerkin (DG) methods are efficient numerical methods in solving various partial differential equations. There are extensive research works on DG methods. See, for example, the papers [9, 28] and the monograph [10]. DG methods are also applied successfully to ordinary differential equations (ODEs) [13, 27], DDEs with constant delay [21, 22], and DDEs with (non-) linear vanishing delay [7, 16].

For DDEs of pantograph type it was shown in [7] that on uniform meshes J_h the optimal order of the DG solution U at the mesh points in the space $S_m^{-1}(J_h)$ of (discontinuous) piecewise polynomial of degree $m \geq 1$ cannot exceed $O(h^{m+2})$. This is in sharp contrast to DG solutions for ODEs [13] and DDEs of constant delay [21] where DG solution at the grid points leads to $O(h^{2m+1})$ superconvergence. A plausible explanation of the convergence order reduction is that the delay term qt maps current mesh into two former adjacent subintervals which leads to the decrease of the regularity of the auxiliary problem. This also increased the computational complexity and difficulties of theoretical analysis. To avoid this phenomenon and reduce the computational complexity, different meshes relating to the delay coefficient were proposed [3, 4, 8].

Bellen et al. [3] showed that, for functional differential equations with vanishing delay, the optimal superconvergence of collocation solutions are valid for suitably chosen quasi-graded meshes. Recently, Huang et al. [17] studied the optimal order of global convergence and local superconvergence of CG methods on quasi-geometric meshes for linear pantograph-type DDEs, and showed that the optimal order at the mesh points is $O(h^{2m})$ in the space $S_m^0(J_h)$ of piecewise polynomial of degree $m \geq 2$. Xu and Huang [31] applied DG methods to solve linear DDEs with vanishing delay, they showed that the optimal order at the mesh points in the space $S_m^{-1}(J_h)$ of piecewise polynomial of degree $m \geq 1$ is $O(h^{2m+1})$ on quasi-graded meshes. However, the superconvergence analysis of these methods remain largely open in nonlinear DDEs. For DG methods of nonlinear DDEs of constant delay, Li and Zhang [22] showed that the global convergence order is $O(h^{m+1})$, but the superconvergence results were not mentioned. We therefore want to obtain the superconvergecne of DG solutions for nonlinear DDEs of vanishing delay.

It is the aim of this paper to present global convergence and superconvergence analysis of DG solution for nonlinear DDEs of time dependent vanishing delay (1.1). The superconvergence results are also extended to DDEs with state dependent vanishing delay. For investigation to the numerical literature on this problem we refer the reader to papers [5, 12, 14, 15, 18].

The outline of the paper is as follows: In section 2, we introduce the DG method for (1.1) under quasi-graded meshes. The main results on the optimal order of global convergence and local superconvergence of the DG solution are stated in section 3. Section 4 presents the DG scheme of state dependent DDEs and describes how to get the suitable partition that results in the optimal nodal superconvergence. In section 5, we provide results of numerical experiments to illustrate our theory. Conclusions and plans for future research work are summarized in the final section 6.

2 DG Method for Nonlinear DDEs

Suppose that on a given (small) initial subinterval $J_0 = [0, t_0]$ of $[0, T]$, $t_0 = \theta^k(T)$ for a suitable value of k, we have already computed the approximation $\phi(t)$ by the DG method or by the truncation of the Taylor expansion of the exact solution $u(t)$, such that

$$\|u(t) - \phi(t)\|_\infty = \max_{t\in[0,t_0]} \mid u(t) - \phi(t) \mid \leq C_0 t_0^p, \tag{2.2}$$

here

$$\theta^k(T) := (\underbrace{\theta \circ \theta \circ \ldots \circ \theta}_{k})(T).$$

Subsequently, we solve the following equation:

$$\begin{aligned} &u'(t) = f(t, u(t), u(\theta(t))), \qquad t \in J = [t_0, T], \\ &u(t) = \phi(t), \qquad \theta(t_0) \leq t \leq t_0. \end{aligned} \tag{2.3}$$

We assume that the nonlinear term f of (2.3) satisfies

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|), \quad L > 0, \tag{2.4}$$

then we get the existence and uniqueness for the solution of (2.3).

Remark 2.1. *Suppose the above condition 2.4 holds, and assume that the initial function $\phi(t)$ is Lipschitz continuous for t. Then the problem (2.3) possesses a unique solution which depends continuously on the initial data (see chapter 2 of [2] for details).*

On the interval $[t_0, T]$, we introduce the macro-mesh $\{\xi_\mu\}$ by setting

$$t_0 = \xi_0 < \xi_1 < \cdots < \xi_k = T, \quad \xi_\mu := \theta^{k-\mu}(T)\ (0 \leq \mu \leq k),$$

with increasing size $H_\mu := \xi_\mu - \xi_{\mu-1}$ $(\mu = 1, \cdots, k)$ denoting the macro-steps. In the subinterval $I_\mu := [\xi_{\mu-1}, \xi_\mu]$ $(\mu = 1, \cdots, k)$, we insert $l - 1$ nodes

$$t_0 = \xi_0 < t_1 < \cdots < t_l = \xi_1 < \cdots < t_{2l} = \xi_2 < \cdots < t_{kl} = \xi_k = T.$$

defined recursively by

$$t_{n-l} = \theta(t_n), \quad n = l, \cdots, kl.$$

In the last macro subinterval I_k, $t_{(k-1)l+1}, \cdots, t_{kl}$ can be chosen arbitrarily (see [3] for details). $J_h : t_0 = \xi_0 < t_1 < \cdots < t_l = \xi_1 < \cdots < t_{2l} = \xi_2 < \cdots < t_{kl} = \xi_k = T$ is called quasi-graded mesh. We will use the notation

$$N = kl, \quad I_n := [t_{n-1}, t_n], \quad h_n := t_n - t_{n-1}, \quad h := \max_{1\leq n\leq N} h_n \quad (1 \leq n \leq N).$$

Define the corresponding DG finite element space

$$S_m^{(-1)}(J_h) = \{v \in L^2(J) : v|_{I_n} \in P_m, 1 \leq n \leq N\},$$

where P_m denotes the space of (real) polynomials of degree up to m, with $m \geq 0$.

At the nodes $\{t_n\}_{n=0}^N$, the left-hand and right-hand limits (i.e., the jump of discontinuities) of $v \in S_m^{(-1)}(J_h)$ are defined by

$$v_n^+ := \lim_{s\to 0, s>0} v(t_n + s), \quad 0 \leq n \leq N-1,$$
$$v_n^- := \lim_{s\to 0, s>0} v(t_n - s), \quad 1 \leq n \leq N,$$

and set $[v]_n := v_n^+ - v_n^-$.

The DG method for (2.3) read as: find $U \in S_m^{(-1)}(J_n)$ such that

$$\sum_{n=1}^{N} \int_{I_n} [U'(t) - f(t, U(t), U(\theta(t)))]v(t)dt + \sum_{n=1}^{N-1} [U]_n v_n^+ + U_0^+ v_0^+ = u(t_0)v_0^+, \ \forall v \in S_m^{(-1)}(J_h). \tag{2.5}$$

Here we set $U(t) = \phi(t)$ $(\theta(t_0) \leq t \leq t_0)$. Note that the exact solution u of (2.3) also satisfies (2.5),

$$\sum_{n=1}^{N} \int_{I_n} [u'(t) - f(t, u(t), u(\theta(t)))]v(t)dt + \sum_{n=1}^{N-1} [u]_n v_n^+ + u_0^+ v_0^+ = u(t_0)v_0^+, \ \forall v \in S_m^{(-1)}(J_h). \tag{2.6}$$

Setting $e(t) = u(t) - U(t)$ and $U_0^- := u(t_0)$, then subtracting (2.6) from (2.5) gives

$$\begin{aligned} B_{DG}(e, v) \ := \ & \sum_{n=1}^{N} \int_{I_n} \big(e'(t) - [f(t, u(t), u(\theta(t))) - f(t, U(t), U(\theta(t)))]\big)v(t)dt \\ & + \sum_{n=1}^{N} [e]_{n-1} v_{n-1}^+ = 0, \quad \forall v \in S_m^{(-1)}(J_h). \end{aligned} \tag{2.7}$$

It is obvious that the error satisfy

$$\begin{aligned} & \int_{I_n} \big(e'(t) - [f(t, u(t), u(\theta(t))) - f(t, U(t), U(\theta(t)))]\big)v(t)dt \\ & + [e]_{n-1} v_{n-1}^+ = 0. \quad \forall v \in S_m^{(-1)}(J_h). \end{aligned} \tag{2.8}$$

The DG method (2.5) can be interpreted as a time-stepping scheme. If U is known on the time intervals I_k, $1 \leq k \leq n-1$, we find $U|_{I_n} \in P_m(I_n)$ by solving

$$\int_{I_n} [U'(t) - f(t, U(t), U(\theta(t)))]v(t)dt + [U_{n-1}]v_{n-1}^+ = 0, \quad \forall v \in P_m(I_n). \tag{2.9}$$

Here, we use again $U_0^- := u(t_0)$. For a continuous function u, we define the interpolation operator $\Pi_h : C[0, 1] \mapsto S_m^{(-1)}(J_h)$ by

$$\Pi_h u(t_n^-) \ = \ u(t_n^-); \tag{2.10}$$
$$\int_{I_n} \Pi_h uv dt \ = \ \int_{I_n} uv dt, \quad \forall v \in \mathcal{P}_{m-1}(I_n), \quad m \geq 1. \tag{2.11}$$

This interpolation operator was used to get error estimates of the DG solution (see for example [27, 28]). It is well known that this interpolation operator admits the error estimate

$$\|u-\Pi_h u\|_{I_n,\infty} \quad \leq \quad Ch^{m+1}\|u\|_{I_n,m+1,\infty}. \tag{2.12}$$

3 Error analysis

In this section, we present main results of the paper. We first describe optimal global order of the DG solution and then briefly discuss the existence and uniqueness of this solution in Remark. The superconvergence at quasi-graded mesh points is given by subsection 3.2. We then prove the supercloseness between the DG solution and the interpolant of the exact solution to get all the superconvergence points.

3.1 Global convergence of the DG solution

This subsection gives the error bound for the DG solution.

Theorem 3.1. *Assume the following.*

1. *the functions $f, \phi(t)$ are in $C^m(J)$ in DDE (2.3).*
2. *$u \in W^{m+1,\infty}(J)$ is the exact solution of the initial value problem for the DDE (2.3).*
3. *$U \in S_m^{(-1)}(J_h)$ defined in (2.5) is the DG solution of u.*
4. *J_h is a quasi-graded partition for $[t_0, T]$ and h is sufficiently small.*
5. *the nonlinear term f of (2.3) satisfies the condition (2.4).*

Then the m-degree DG solution U satisfies

$$\|u-U\|_\infty \leq Ch^{m+1}, \tag{3.1}$$

where the constant C depends on $t, \theta(t), L, u$ and its derivatives but is independent of h.

The proof of this theorem is similar to the proof of Theorem 3.1 in [22], we leave it the reader.

3.2 Nodal Superconvergence of the DG solution

The following theorem gives the superconvergence result of the DG solution at mesh points.

Theorem 3.2. *Assume that the conditions of Theorem 3.1 hold, then the attainable superconvergence order of DG solution at the mesh point t_n is*

$$|(u-U)(t_n^-)| = O(h^{2m+1}), \quad n = 1, \cdots, N, \quad (m \geq 1). \tag{3.2}$$

Proof. Expanding the nonlinear term, we obtain

$$\begin{aligned} f(t,u(t),u(\theta(t))) - f(t,U(t),U(\theta(t))) &= \frac{\partial}{\partial u(t)} f(t,U(t),U(\theta(t)))e(t) \\ &+ \frac{\partial}{\partial u(\theta(t))} f(t,U(t),U(\theta(t)))e(\theta(t)) \\ &+ O(e^2(t)+e^2(\theta(t))). \end{aligned}$$

Setting $a(t) = \frac{\partial}{\partial u(t)} f(t,U(t),U(\theta(t)))$, $b(t) = \frac{\partial}{\partial u(\theta(t))} f(t,U(t),U(\theta(t)))$ and $R(t) = O(e^2(t)+e^2(\theta(t)))$, (2.7) becomes

$$\begin{aligned} B_{DG}(e,v) &:= \sum_{n=1}^{N} \int_{I_n} \big(e'(t) - a(t)e(t) - b(t)e(\theta(t))\big)v(t)dt + \sum_{n=1}^{N} [e]_{n-1} v_{n-1}^{+} \\ &\quad - \sum_{n=1}^{N} \int_{I_n} R(t)v(t)dt = 0. \quad \forall v \in P_m(I_n). \end{aligned} \tag{3.3}$$

In each subinterval, (3.3) can be written as the following form:

$$\begin{aligned} &\int_{I_n} \big(e'(t) - a(t)e(t) - b(t)e(\theta(t))\big)v(t)dt + [e]_{n-1} v_{n-1}^{+} \\ &- \int_{I_n} R(t)v(t)dt = 0. \quad \forall v \in P_m(I_n). \end{aligned} \tag{3.4}$$

When $n \le l$, equation (2.3) can be seen as the following ordinary differential equation:

$$u'(t) = f(t,u(t),\phi(t)), \quad t \in [t_0, t_{n-l}].$$

Then $|(u-U)(t_n^-)| = O(h^{2m+1}), \;\; n = 1, \cdots, l$. This estimate has been shown in [13].

When $n > l$, we construct the auxiliary problem :

$$\begin{aligned} \varphi'(t) + a(t)\varphi(t) + \tilde{b}(t)\varphi(\theta^{-1}(t)) &= 0, \quad t \in [t_0, t_n), \\ \varphi(t_n) &= \alpha := e_n^-, \end{aligned}$$

where $\tilde{b}(t)$ is defined by

$$\tilde{b}(t) := \begin{cases} \theta'(\theta^{-1}(t)) b(\theta^{-1}(t)), & t_0 \le t \le \theta(t_n), \\ 0, & \theta(t_n) < t \le t_n. \end{cases}$$

Then one has error estimate $\|\varphi\|_q \le C|\alpha|, \; 0 \le q \le m+1$. Therefore, from the initial condition $e(t) = u(t) - U(t) = \phi(t) - \phi(t) = 0$, $\theta(t_0) \le t \le t_0$, we have

$$\begin{aligned} B(e,\varphi) &:= \sum_{j=1}^{n} \Big\{ \int_{I_j} (e'(s) - a(s)e(s) - b(s)e(\theta(s)))\varphi(s)ds + [e]_{j-1}\varphi_{j-1}^{+} \Big\} \\ &= \sum_{j=1}^{n} \Big\{ (e\varphi)_j^- - (e\varphi)_{j-1}^- - \int_{I_j} e(s)(\varphi'(s) + a(s)\varphi(s) + \tilde{b}(s)\varphi(\theta^{-1}(s)))ds \Big\} \\ &= (e\varphi)_n^- = |e_n^-|^2. \end{aligned}$$

We assume $\varphi_h \in S_m^{(-1)}(J_h)$ $(m \geq 1)$ is the mth degree piecewise polynomial interpolation of φ (with $(\varphi - \varphi_h)_j^+ = 0$). Combining with (3.4), we obtain

$$\begin{aligned}
|e_n^-|^2 &= B(e, \varphi - \varphi_h) + \sum_{j=1}^{n} \int_{I_j} R(t)\varphi(t)dt \\
&= \sum_{j=1}^{n} \int_{I_j} (e'(s) - a(s)e(s) - b(s)e(\theta(s)))(\varphi(s) - \varphi_h(s))ds \\
&\quad + \sum_{j=1}^{n} \int_{I_n} R(t)\varphi(t)dt \\
&= \sum_{j=1}^{n-l} \int_{I_j} (e'(s) - a(s)e(s) - b(s)e(\theta(s)))(\varphi(s) - \varphi_h(s))ds \\
&\quad + \sum_{j=n-l+1}^{n} \int_{I_j} (e'(s) - a(s)e(s) - b(s)e(\theta(s)))(\varphi(s) - \varphi_h(s))ds \\
&\quad + \sum_{j=1}^{n} \int_{I_n} R(t)\varphi(t)dt \\
&\leq C\|e\|_{1,\infty,[t_0,t_{n-l})}\|\varphi - \varphi_h\|_{0,1,[t_0,t_{n-l})} \\
&\quad + C\|e\|_{1,\infty,[t_{n-l+1},t_n]}\|\varphi - \varphi_h\|_{0,1,[t_{n-l+1},t_n]} + C||e||^2_{0,\infty,[t_0,t_n]} \max_{t\in[t_0,t_n]} ||\varphi|| \\
&\leq Ch^m h^{m+1}\|u\|_{m+1,\infty,[t_0,t_{n-l})}|e_n^-| + Ch^m h^{m+1}\|u\|_{m+1,\infty,[t_{n-l+1},t_n]}|e_n^-| \\
&\quad + Ch^{2m+2}\|u\|_{m+1,\infty,[t_0,t_n]}|e_n^-| \\
&\leq Ch^{2m+1}\|u\|_{m+1,\infty,[t_0,t_n]}|e_n^-|. \qquad (3.5)
\end{aligned}$$

The estimate (3.5) means that

$$|e_n^-| \leq Ch^{2m+1}\|u\|_{m+1,\infty}, n = l + 1, \ldots, N. \qquad (3.6)$$

The proof is completed. $\square$

Remark 3.1. *From the auxiliary problem, we observe that $\tilde{b}(t)$ is discontinuous at $\theta(t_n)$, and so $\varphi'(t)$ is discontinuous at $t = \theta(t_n)$. Therefore, the convergence $\|\varphi-\varphi_h\|_{0,1,[t_{n^*},t_{n^*+1}]} = Ch^{m+1}\|\varphi\|_{m+1,\infty,[t_{n^*},t_{n^*+1}]}$ $(n^* = l, \ldots, n-1)$ is valid if and only if $\theta(t_n) \notin (t_{n^*}, t_{n^*+1})$. That is, $\theta(t)$ maps the current mesh points onto some previous ones. This is the purpose of choosing the quasi-graded mesh.*

To illustrate the effectiveness of DG method, we make a comparison between DG method and collocation method. For the m-point (arbitrary set) collocation method for the equation (2.3) on quasi-graded meshes, the order of the collocation solution $V \in S_m^0(J_h)$ is given by (see [3] for details)

$$\|u - V\|_\infty \leq Ch^m\|u\|_{m+1,\infty}. \qquad (3.7)$$

If the m collocation parameters c_i are subject to the orthogonality condition

$$\int_0^1 \prod_{j=1}^{m}(s-c_j)\,ds = 0 \tag{3.8}$$

Then there follows the global convergence estimation

$$\|u-V\|_\infty \leq Ch^{m+1}\|u\|_{m+2,\infty}. \tag{3.9}$$

and the optimal nodal superconvergence

$$|(u-U)(t_n)| \leq Ch^{2m}\|u\|_{m+2,\infty}. \tag{3.10}$$

To obtain the same global convergence order, the regularity requirements for the collocation method are more than those for DG method. Moreover, the optimal nodal superconvergence order of DG solution is higher than that of collocation solution on quasi-graded meshes.

3.3 Supercloseness analysis for U and $\Pi_h u$

In order to derive all the superconvergent points, we first analyze the supercloseness between the DG solution U and the interpolation $\Pi_h u$ of the exact solution u.

In general, there is a crucial phenomenon in finite element error estimation: the L_∞ norm of error for DG solution U and the interpolation $\Pi_h u$ of the exact solution u is much less than that of U and the exact solution u, that is,

$$\|\Pi_h u - U\|_\infty \leq Ch^\alpha \|u-U\|_\infty, \qquad \alpha > 0.$$

This phenomenon is called the "supercloseness".

Theorem 3.3. *Suppose the conditions of Theorem 3.1 hold. Then there follows the supercloseness result:*

$$\|\Pi_h u - U\|_\infty \leq Ch^{m+2}\|u\|_{m+1,\infty}, \qquad (m \geq 2). \tag{3.11}$$

Proof. Let $\zeta = \Pi_h u - U$, $\eta = u - \Pi_h u$, then $e = \eta + \zeta$. By (3.4), we obtain

$$\begin{aligned}&\int_{I_n}(\zeta'(t) - a(t)\zeta(t) - b(t)\zeta(\theta(t)))v(t)dt + [\zeta]_{n-1}v_{n-1}^+ \\ &= -\int_{I_n}(\eta'(t) - a(t)\eta(t) - b(t)\eta(\theta(t)))v(t)dt - [\eta]_{n-1}v_{n-1}^+ + \int_{I_n} R(t)v(t)dt.\end{aligned} \tag{3.12}$$

Integrating by parts and combining with (2.11), we have

$$\begin{aligned}-\int_{I_n}\eta'(t)v(t)dt - [\eta]_{n-1}v_{n-1}^+ &= -[\eta v]|_{t_{n-1}^+}^{t_n^-} + \int_{I_n}\eta(t)v'(t)dt - [\eta]_{n-1}v_{n-1}^+ \\ &= -\eta_n^- v_n^- + \eta_{n-1}^- v_{n-1}^+.\end{aligned} \tag{3.13}$$

From the definition of $\Pi_h u$, we have $\eta_n^- = \eta_{n-1}^- = 0$. Hence, (3.12) becomes

$$\begin{aligned}&\int_{I_n} (\zeta'(t) - a(t)\zeta(t) - b(t)\zeta(\theta(t)))v(t)dt + [\zeta]_{n-1}v_{n-1}^+ \\ &= \int_{I_n} (a(t)\eta(t) + b(t)\eta(\theta(t)))v(t)dt + \int_{I_n} R(t)v(t)dt,\end{aligned}$$

and there follows

$$\int_{I_n} \zeta'(t)v(t)dt + [\zeta]_{n-1}v_{n-1}^+ = \int_{I_n} (a(t)e(t) + b(t)e(\theta(t)))v(t)dt + \int_{I_n} R(t)v(t)dt.$$

Choose $v(t) = (t - t_{n-1})\zeta'(t)$ and we get $v_{n-1}^+ = 0$, then we have

$$\begin{aligned}\int_{I_n} (t - t_{n-1})|\zeta'(t)|^2 dt \quad &= \quad \int_{I_n} (a(t)e(t) + b(t)e(\theta(t)))(t - t_{n-1})\zeta'(t)dt \\ &\quad + \int_{I_n} R(t)(t - t_{n-1})\zeta'(t)dt, \qquad (3.14)\end{aligned}$$

and by the mean value inequality $\alpha\beta \leq \varepsilon\alpha^2 + \frac{\beta^2}{4\varepsilon}$ (with $\varepsilon > 0$ being an arbitrary constant), we obtain

$$\begin{aligned}&\int_{I_n} (a(t)e(t) + b(t)e(\theta(t)))(t - t_{n-1})\zeta'(t)dt \\ \leq\quad &\varepsilon \int_{I_n} (t - t_{n-1})|\zeta'(t)|^2 dt + \frac{1}{4\varepsilon}\int_{I_n} \big(a(t)e(t) + b(t)e(\theta(t))\big)^2 (t - t_{n-1})dt. \qquad (3.15)\end{aligned}$$

$$\int_{I_n} R(t)(t - t_{n-1})\zeta'(t)dt \leq \varepsilon \int_{I_n} (t - t_{n-1})|\zeta'(t)|^2 dt + \frac{1}{4\varepsilon}\int_{I_n} R^2(t)(t - t_{n-1})dt. \qquad (3.16)$$

We now combine (3.14), (3.15), (3.16) and Theorem 3.1 with $\varepsilon \neq \frac{1}{2}$, and obtain

$$\begin{aligned}\int_{I_n} (t - t_{n-1})|\zeta'(t)|^2 dt \quad &\leq \quad C\int_{I_n} \big(a(t)e(t) + b(t)e(\theta(t))\big)^2 (t - t_{n-1})dt + C\int_{I_n} R^2(t)(t - t_{n-1})dt \\ &\leq \quad Ch^{2m+4}\|u\|_{m+1,\infty}. \qquad (3.17)\end{aligned}$$

It can be proved that the two norms $h\int_{I_n} |\zeta'(s)|^2 ds$ and $\int_{I_n} (t - t_{n-1})|\zeta'(t)|^2 dt$ are equivalent (see [16] for details). Hence,

$$\Big|\int_{t_n^-}^{t} \zeta'(s)ds\Big|^2 \leq C\int_{I_n} (t - t_{n-1})|\zeta'(t)|^2 dt. \qquad (3.18)$$

Moreover, by the definition of $\Pi_h u$ and Theorem 3.2, we have

$$|\zeta(t_n^-)| = |\Pi_h u(t_n^-) - U(t_n^-)| = |u(t_n^-) - U(t_n^-)| \leq Ch^{2m+1}\|u\|_{m+1,\infty}. \qquad (3.19)$$

Writing

$$\zeta(t) = \zeta(t_n^-) + \int_{t_n^-}^{t} \zeta'(s)ds.$$

and combining (3.17), (3.18) and (3.19), we obtain that

$$\|\zeta\|_{I_n,\infty} \leq |\zeta(t_n^-)| + \left\| \int_{t_n^-}^{t} \zeta'(s)ds \right\|_{I_n,\infty} \leq Ch^{m+2}\|u\|_{m+1,\infty}.$$

This completes the proof of Theorem 3.3.

□

3.4 Superconvergent points of the DG approximation

On the basis of the supercloseness between U and $\Pi_h u$, we now discuss the superconvergent points of the DG approximation. To determine all the superconvergence points, we introduce Legendre's polynomials in interval $[-1, 1]$,

$$l_n = \frac{1}{2^n n!}\frac{d^n}{ds^n}(s^2-1)^n, \quad n = 0, 1, 2, \cdots.$$

We recall the definition of the Radau II polynomials:

$$\varphi_0(s) = 1, \quad \varphi_i(s) = l_i(s) - l_{i-1}(s), \quad i \geq 1.$$

Then the zeros of the polynomials $\varphi_i(s)$ define the i Radau II points s_l $(l = 1, \cdots, i)$ in $[-1, 1]$.

Theorem 3.4. *Assume the conditions of Theorem 3.1 hold, and let $u \in W^{m+2,\infty}(J)$. Then the $m+1$ Radau II points in each interval I_n are the superconvergence points of the DG solution and*

$$|u(t_{nr}) - U(t_{nr})| \leq \quad Ch^{m+2}\|u\|_{m+2,\infty},$$

where t_{nr} stands for any of the Radau II points in I_n $(1 \leq n \leq N, 1 \leq r \leq m+1)$.

The proof of this theorem is similar with the proof of Theorem (3.3) in [7].

Remark 3.2. *Observe that the maximum stepsize of the partition J_h for $(t_0, T]$ is attained in the last interval I_k, if we want DG solutions described above to be global convergent and local superconvergent on the original interval $[0, T]$, we must consider how to choose t_0. It is suggested that t_0 is chosen, such that*

$$t_0 = \theta^\kappa(T) \leq h, \tag{3.20}$$

Here, κ is the minimum integer for which (3.20) holds.

When $p = m+1, 2m+1, m+2, m+2$, Theorems 3.1, 3.2, 3.3, 3.4 hold in $[0, T]$, respectively.

4 DG for State dependent vanishing DDEs

We can extend the time dependent delay term $\theta(t)$ to more complicated state dependent delay $\alpha(t, u(t))$. That is, state dependent DDEs

$$\begin{aligned} &u'(t) = f(t, u(t), u(\alpha(t, u(t)))), \qquad t \in J = [0, T], \\ &u(0) = u_0, \end{aligned} \tag{4.1}$$

and the delay function $\alpha(t, u(t))$ is subject to the conditions 1 and 2. We require the approximation $\phi(t)$ of the solution $u(t)$ in some initial interval $[0, t_0]$ furnished by any approximation to order p, and we solve the following problem

$$\begin{aligned} &u'(t) = f(t, u(t), u(\alpha(t, u(t)))), \qquad t \in J = (t_0, T], \\ &u(t) = \phi(t), \qquad\qquad\qquad\qquad 0 \le t \le t_0 \end{aligned} \tag{4.2}$$

When we choose suitable partition that the delay term $\alpha(t, u(t))$ maps the current mesh points onto some previous ones, the optimal global convergence and local superconvergence results of the DG solutions remain valid. Because the delay is state dependent, the mesh points are only known after solving (4.2) numerically (using DG method) and cannot be included directly in the partition. We first propose the DG scheme of state dependent DDEs and then describe how to get the suitable partition that $\alpha(t, u(t))$ maps the current mesh points onto some previous ones.

4.1 DG scheme for state dependent DDEs

For a given step size h_n, On $I_n = (t_{n-1}, t_{n-1} + h_n]$, the DG solution $U_n(t)$ can be written as

$$U_n(t) = \sum_{i=1}^{m+1} u_{n,i} l_{n,i}(t) = \sum_{i=1}^{m+1} u_{n,i} L_i(\frac{t - t_{n-1}}{h_n}).$$

where $L_i(s)$ are given Lagrange basis functions in $[0, 1]$. We assume that there are two integers $\alpha_{n,0}$ and $\alpha_{n,1}$ such that $\alpha(t_{n-1}, U_n(t_{n-1})) \in I_{\alpha_{n,0}}$ and $\alpha(t_{n-1}+h_n, U_n(t_{n-1}+h_n)) \in I_{\alpha_{n,1}}$. By (2.9) and replaced $\theta(t)$ by $\alpha(t, u(t))$, we solve $u_{n,i}$ by

$$\begin{aligned} &\sum_{i=1}^{m+1} u_{n,i} L_i(0) L_j(0) + \sum_{i=1}^{m+1} \int_0^1 u_{n,i} L_i'(s) L_j(s) ds \\ &= h_n \sum_{\lambda=1}^{\alpha_{n,1}-\alpha_{n,0}+1} \int_{s^*_{\lambda-1}}^{s^*_\lambda} f(t_{n-1} + sh_n, \sum_{i=1}^{m+1} u_{n,i} L_i(s), U_{\alpha_{n,0}+\lambda-1}(s)) L_j(s) ds \\ &+ \sum_{i=1}^{m+1} u_{n-1,i} L_i(1) L_j(0) \qquad j = 1, \ldots, m. \end{aligned} \tag{4.3}$$

Here, let $s^*_0 = 0, s^*_{\alpha_{n,1}-\alpha_{n,0}+1} = 1$, and $0 < s^*_1 < \ldots < s^*_{\alpha_{n,1}-\alpha_{n,0}} < 1$ satisfying

$$\alpha(t_{n-1} + s^*_\lambda h_n, U_n(t_{n-1} + s^*_\lambda h_n)) = t_{\alpha_{n,0}+\lambda-1}, \quad \lambda = 1, \ldots, \alpha_{n,1} - \alpha_{n,0}.$$

The computational scheme of delay terms $U_{\alpha_{n,0}+\lambda-1}(s)$ passes through the two distinct phases described below.

- *Phase 1:* If $\alpha(t_{n-1}, U_n(t_{n-1})) \leq t_0$, let $I_0 = [0, t_0]$

$$U_{\alpha_{n,0}}(s) = \phi(\alpha(t_{n-1} + sh_n, \sum_{i=1}^{m+1} u_{n,i} L_i(s))).$$

$$U_{\alpha_{n,0}+\lambda-1}(s) = \sum_{i=1}^{m+1} u_{\alpha_{n,0}+\lambda-1,i} L_i(\frac{\alpha(t_{n-1} + sh_n, \sum_{i=1}^{m+1} u_{n,i} L_i(s)) - t_{\alpha_{n,0}+\lambda-2}}{h_{\alpha_{n,0}+\lambda-1}}),$$

$$\lambda = 2, \ldots, \alpha_{n,1} - \alpha_{n,0} + 1.$$

Here, $U_{\alpha_{n,0}+\lambda-1}(s) = 0$ if $\alpha(t_n, U_n(t_n)) \leq t_0$.

- *Phase 2:* If $\alpha(t_{n-1}, U_n(t_{n-1})) \geq t_0$,

$$U_{\alpha_{n,0}+\lambda-1}(s) = \sum_{i=1}^{m+1} u_{\alpha_{n,0}+\lambda-1,i} L_i(\frac{\alpha(t_{n-1} + sh_n, \sum_{i=1}^{m+1} u_{n,i} L_i(s)) - t_{\alpha_{n,0}+\lambda-2}}{h_{\alpha_{n,0}+\lambda-1}}),$$

$$\lambda = 1, \ldots, \alpha_{n,1} - \alpha_{n,0} + 1.$$

4.2 Methods of generating the partition

Since superconvergence occurred in mesh points which $\alpha(t, u(t))$ maps the current mesh points onto some previous ones, we expect to choose the mesh points $t_n < T$ satisfying

$$\alpha(t_n, U_n(t_n)) = t_q, \quad q < n.$$

We start with $\xi_0 = t_0$. Assume ξ_1 to be found and satisfy $\alpha(\xi_1, U_1(\xi_1)) = \xi_0$, we insert arbitrarily $l - 1$ nodes

$$\xi_0 = t_0 < t_1 < \ldots < t_{l-1} < t_l = \xi_1,$$

in the interval $(\xi_0, \xi_1]$. Once t_q $(0 < q \leq l)$ are given in the interval $(\xi_0, \xi_1]$, we then take $t_n < T$ that $\alpha(t_n, U_n(t_n)) = t_{n-l}$ $(n > l)$ to be mesh points. In order to compute the mesh points, algorithm is described as following.
Assume that the mesh points are found successfully until t_{n-1} and a step size h_n is proposed for the next step.

1. The Newton iteration is employed to solve the system (4.3) to derive approximation solution $U_n(t)$. We look for the zeros of the function

$$d_\xi(t) = \alpha(t, U_n(t)) - t_{n-l}, \quad n \geq l. \tag{4.4}$$

2. If $d_\xi(t_{n-1}) \cdot d_\xi(t_{n-1}+h_n) > 0$, we let $h_n = 2h_n$, and then solve the system (4.3) to derive approximation solution $U_n(t)$. Repeat this process until solution $U_n(t)$ satisfying $d_\xi(t_{n-1}) \cdot d_\xi(t_{n-1}+h_n) \leq 0$ be calculated, then there exists a zero point ζ_n of the function $d_\xi(t)$ in the interval $(t_{n-1}, t_{n-1}+h_n]$.

Once this point is detected inside the interval $(t_{n-1}, t_{n-1}+h_n]$, we expect to compute it more accurately. The main idea comes from recent work [5, 14]. Starting with $h_n^{[0]} = \zeta_n^{[0]} - t_{n-1}$, where $\zeta_n^{[0]}$ is obtained by solving (4.3), we consider the following algorithm.

Algorithm 4.1. *1. Newton iteration is applied to solve the system (4.3) with respect to the variables $u_{n,i}^{[k]}$ with fixed $h_n = h_n^{[k]}$.*

2. By extrapolation from the previous step, we provide an approximation function

$$\tilde{U}_n^{[k]}(t_{n-1}+sh_n^{[k]}) = \sum_{i=1}^{m+1} u_{n,i}^{[k]} L_i(s) \quad s > 0.$$

3. Compute $h_n^{[k+1]}$ by solving equation $\alpha(t_{n-1}+h_n^{[k+1]}, \tilde{U}_n^{[k]}(t_{n-1}+h_n^{[k+1]})) - t_{n-l} = 0$.

If iterative scheme is converges (its efficiency depends on the speed of convergence), we then let point $t_n = t_{n-1} + h_n^{[k]}$ be a mesh point. In fact, this iterative method converges (see [14] for Lemma 3.4). We also make a flowchart of the partition generating method in Figure 1.

Remark 4.1. *In fact, strategies of generating the mesh points are similar to the methods of tracking the jump discontinuities of state DDEs [5, 14]. The difference is that we add some inner nodes satisfying (4.4) between every two discontinuous points to form the new partition.*

5 Numerical Experiments

In this section, two examples are given to illustrate the theory established in the previous section. In the following tables we use the notations

$$e = ||u - U||_\infty, \qquad R = \frac{log(e_{N1}/e_{N2})}{log(h_{N1}/h_{N2})},$$

$$e_\pi = ||\Pi_h u - U||_\infty, \qquad R_\pi = \frac{log((e_\pi)_{N1}/(e_\pi)_{N2})}{log(h_{N1}/h_{N2})},$$

$$e_r = \max_{\substack{1\leq n\leq N \\ 1\leq r\leq m+1}} |u(t_{nr}) - U(t_{nr})|, \qquad R_r = \frac{log((e_r)_{N1}/(e_r)_{N2})}{log(h_{N1}/h_{N2})},$$

$$e_n = \max_{1\leq n\leq N} |u(t_n) - U(t_n)|, \qquad R_n = \frac{log((e_n)_{N1}/(e_n)_{N2})}{log(h_{N1}/h_{N2})},$$

where t_{nr} denote Radau II points, t_n denote nodal points.

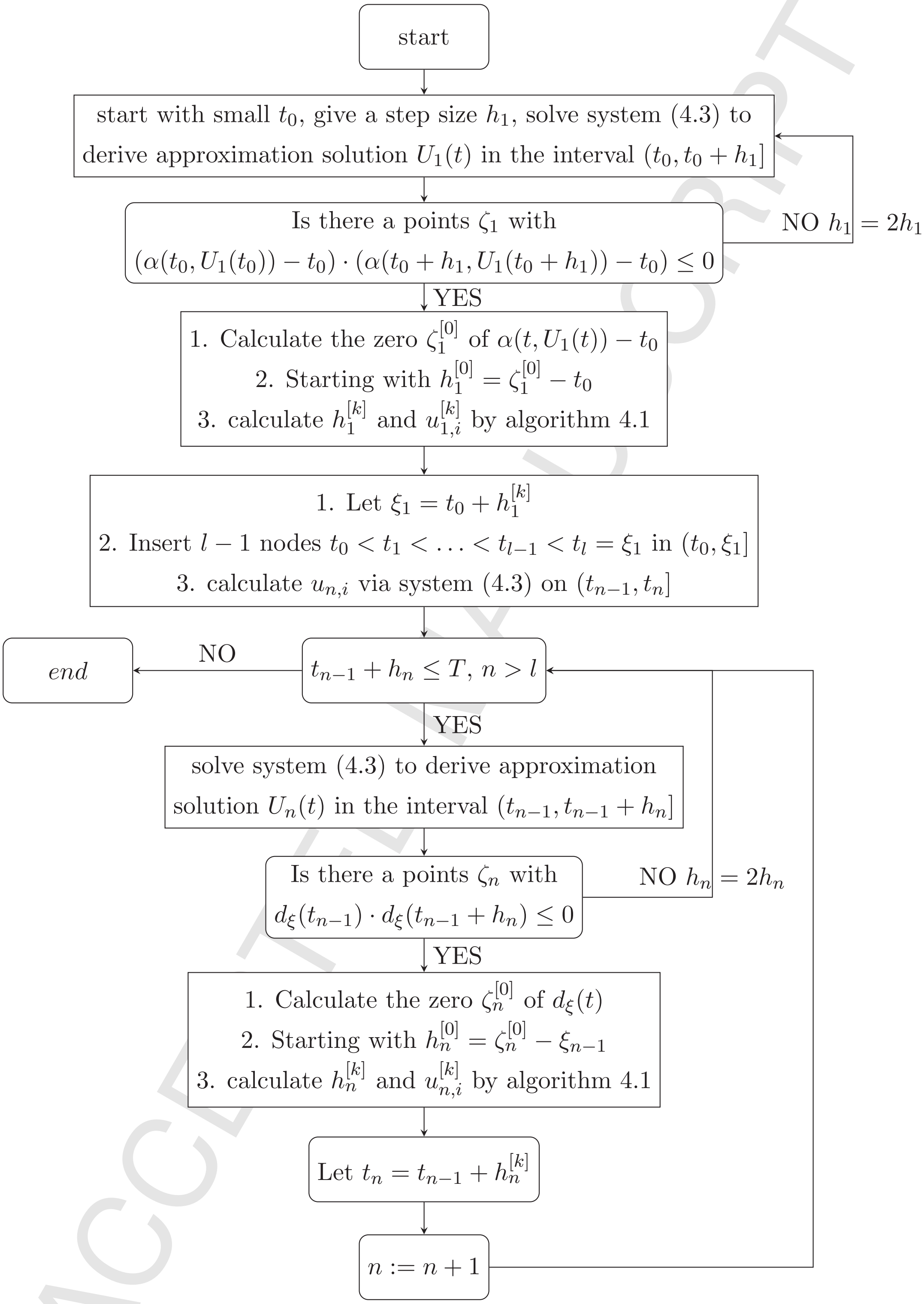

Figure 1: Flowchart of strategy

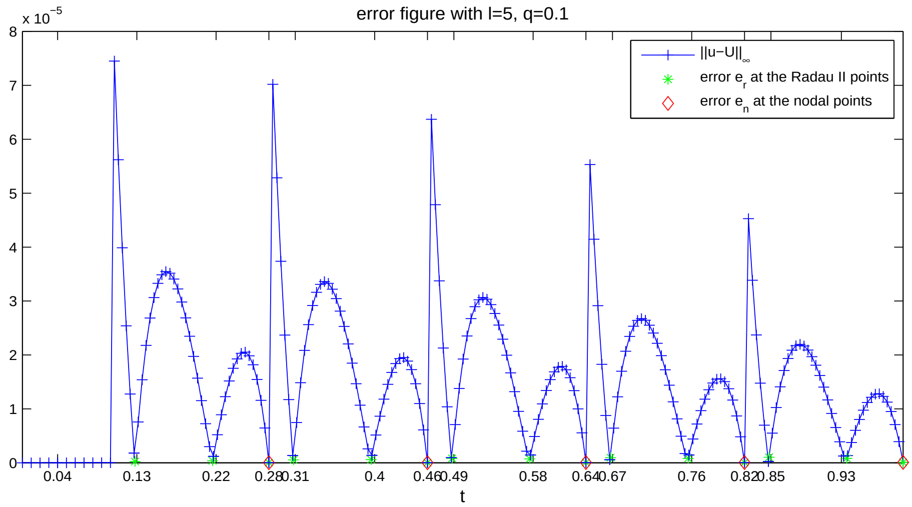

Figure 2: Errors for the DDE of Example 5.1

5.1 Nonlinear vanishing delay case

Example 5.1. (nonlinear proportional delay) We first use the DG method to solve the following nonlinear proportional DDE:

$$\begin{aligned} &u'(t) = au^2(t) + bu(qt) + cos(t) - asin^2(t) - bsin(qt), \quad 0 \le t \le 1, \\ &u(0) = 0. \end{aligned} \tag{5.1}$$

Its exact solution is $u(t) = \sin(t)$ for any $0 < q < 1$.

In the initial subinterval $J_0 = [0, t_0]$, we select $t_0 = q^k$ with $k = \kappa + 1$, and the approximation $\phi(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}$ to the exact solution $u(t)$ by Taylor expansion.

We choose a quasi-graded mesh J_h with last $l + 1$ nodes being assigned equidistant, $\tilde{N} = N + 1 = kl + 1$. In this example we use the Newton-iterative method to solve the nonlinear term.

The numerical results are obtained by piecewise quadratic DG solution ($m = 2$). In the first interval of quasi-graded mesh J_h the initial values are $[0.1, 0.2, 0.4]$, and for the following intervals, their initial values are acquired from their previous intervals values $U_{n-1} := (u_{n-1,1}, \ldots, u_{n-1,m+1})$ which have been calculated.

The following Figure 2 exhibits the behavior of the error of the piecewise quadratic DG solution for Example 5.1. We find that superconvergence occurs at the Radau II points and the error of DG solution at the nodal points is smaller than that at the other Radau II points.

In Tables 1, 2, 3, we present the error and convergence order of the piecewise quadratic DG solution for (5.1), with $a = -1, b = 0.5, q = 0.1,\ 0.5,\ 0.9$.

Table 1. Errors of the piecewise quadratic DG solution $q = 0.1$

l	$\tilde{N}$	h	e	R	e_r	R_r	e_n	R_n	e_π	R_π
8	9	0.11	1.68e-05		1.57e-07		5.49e-09		2.23e-07	
16	33	0.05	2.11e-06	2.99	1.01e-08	3.97	1.74e-10	4.98	1.42e-08	3.97
32	65	0.02	2.64e-07	2.99	6.38e-10	3.98	5.51e-12	4.99	8.99e-10	3.99
64	129	0.01	3.31e-08	2.99	4.01e-11	3.99	1.89e-13	4.87	5.64e-11	3.99

Table 2. Errors of the piecewise quadratic DG solution $q = 0.5$

l	$\tilde{N}$	h	e	R	e_r	R_r	e_n	R_n	e_π	R_π
4	17	0.13	1.95e-05		2.27e-07		6.86e-09		3.16e-07	
8	41	0.06	2.51e-06	2.96	1.46e-08	3.96	2.18e-10	4.98	2.03e-08	3.96
16	97	0.03	3.17e-07	2.98	9.28e-10	3.98	6.88e-12	4.99	1.29e-09	3.98
32	225	0.01	3.99e-08	2.99	5.29e-11	3.99	2.35e-13	4.87	8.10e-11	3.99

Table 3. Errors of the piecewise quadratic DG solution $q = 0.9$

l	$\tilde{N}$	h	e	R	e_r	R_r	e_n	R_n	e_π	R_π
2	59	0.05	8.95e-07		5.11e-09		2.21e-11		6.34e-09	
3	100	0.03	2.69e-07	2.97	1.02e-09	3.98	2.93e-12	4.98	1.27e-09	3.97
4	145	0.025	1.14e-07	2.98	3.24e-10	3.98	7.09e-13	4.93	4.03e-10	3.98
5	191	0.02	5.87e-09	2.98	1.33e-10	3.99	2.47e-13	4.73	1.66e-10	3.99

From numerical results of the example above, we see that for $m = 2$:

$$\begin{aligned} \|u - U\|_\infty &= O(h^{m+1}), & \|\Pi_h u - U\|_\infty &= O(h^{m+2}), \\ \max_{\substack{1\le n\le N \\ 1\le r\le m+1}} |u(t_{nr}) - U(t_{nr})| &= O(h^{m+2}), & \max_{1\le n\le N} |u(t_n) - U(t_n)| &= O(h^{2m+1}). \end{aligned}$$

These illustrate the correctness of the main theoretical results.

Example 5.2. (vanishing delay system) Here we solve a nonlinear delay differential system with nonlinear vanishing delay,

$$\begin{aligned} &u_1'(t) = au_1(t) + bu_2(t) + cu_1^2(\theta(t)) + f_1(t),\ 0 < t \le 2, \\ &u_2'(t) = bu_1(t) + au_2(t) + cu_2^2(\theta(t)) + f_2(t),\ 0 < t \le 2, \\ &u_1(0) = 1, \quad u_2(0) = 1, \end{aligned} \tag{5.2}$$

with $\theta(t) = \arctan(t)$. We set $f_1(t), f_2(t)$ to make the exact solution $u_1(t) = e^{-t}, u_2(t) = e^{-2t}$. In initial subinterval $J_0 = [0, t_0]$, the approximations $\phi_1(t), \phi_2(t)$ of the exact solutions $u_1(t), u_2(t)$ are provided by Taylor expansion. We choose a quasi-graded mesh J_h with $l - 1$ nodes in the last subinterval being assigned equidistant, $\tilde{N} = N + 1 = \kappa l + 1$.

In Tables 4, 5 we show the error behavior of the piecewise quadratic DG solutions ($m = 2$) for (5.2), with $a = -6,\ b = 1,\ c = 10$.

Table 4. Errors of the piecewise quadratic CG solution for $u_1(t)$

l	$\tilde{N}$	h	e	R	e_r	R_r	e_n	R_n
5	236	0.1792	1.9432e-05		6.6405e-07		7.3167e-08	
8	961	0.1120	5.0216e-06	2.8785	1.0873e-07	3.8492	5.5412e-09	5.4894
10	1881	0.0894	2.6203e-06	2.8913	4.5258e-08	3.8960	1.6892e-09	5.2802
13	4135	0.0687	1.2137e-06	2.9235	1.6049e-08	3.9383	4.6467e-10	4.9031

Table 5. Errors of the piecewise quadratic CG solution for $u_2(t)$

l	$\tilde{N}$	h	e	R	e_r	R_r	e_n	R_n
5	236	0.1792	4.4948e-05		2.7600e-06		2.7626e-07	
8	961	0.1120	1.2170e-05	2.7792	4.8642e-07	3.6927	2.9157e-08	4.7834
10	1881	0.0894	6.4552e-06	2.8186	2.0895e-07	3.7560	9.9850e-09	4.7632
13	4135	0.0687	3.0366e-06	2.8649	7.6393e-08	3.8222	2.7238e-09	4.9348

Numerical data are demonstrated in Tables 4, 5 and convergence rates are $m+1$ for e, $m+2$ for e_r and $2m+1$ for e_n, respectively. These results verify our theoretical findings.

5.2 DG vs collocation

To illustrate the effectiveness of DG method for nonlinear DDEs with nonlinear vanishing delay, we compare the global convergence and local superconvergence at mesh points of our method with those of the collocation method in [3] under quasi-graded mesh. We take the above Example 5.2 to illustrate these differences. The collocation points employed to compute $V_h \in S_m^{(0)}(J_h)$ $(m = 2)$ are the arbitrary points $(c_1 = \frac{1}{4}, c_2 = \frac{3}{4})$ and Gaussian points $(c_1 = \frac{3-\sqrt{3}}{6}, c_2 = \frac{3+\sqrt{3}}{6})$.

Table 6. Errors of the collocation solution for $u_1(t)$

	$c_1 = \frac{1}{4}, c_2 = \frac{3}{4}$				$c_1 = \frac{3-\sqrt{3}}{6}, c_2 = \frac{3+\sqrt{3}}{6}$			
l	e	R	e_n	R_n	e	R	e_n	R_n
5	1.7202e-04		2.3896e-05		1.5024e-05		1.6816e-06	
8	6.8677e-05	1.9532	8.9714e-06	2.0840	3.6782e-06	2.9935	2.6453e-07	3.9344
10	4.4218e-05	1.9570	5.6744e-06	2.0361	1.8824e-06	2.9776	1.0889e-07	3.9456
13	2.6295e-05	1.9744	3.3208e-06	2.0351	8.5624e-07	2.9925	3.8323e-08	3.9668

Table 7. Errors of the collocation solution for $u_2(t)$

	$c_1 = \frac{1}{4}, c_2 = \frac{3}{4}$				$c_1 = \frac{3-\sqrt{3}}{6}, c_2 = \frac{3+\sqrt{3}}{6}$			
l	e	R	e_n	R_n	e	R	e_n	R_n
5	2.1639e-04		4.0669e-05		3.6985e-05		4.8935e-06	
8	8.8298e-05	1.9068	1.4420e-05	2.2057	9.4436e-06	2.9041	7.4599e-07	4.0012
10	5.7301e-05	1.9220	9.0370e-06	2.0769	4.8886e-06	2.9267	3.0778e-07	3.9353
13	3.4341e-05	1.9448	5.2532e-06	2.0608	2.2426e-06	2.9602	1.0763e-08	3.9912

We observe Tables 6, 7 and find the global convergence rate of order $m+1$, and local superconvergence rate $2m$ at nodal points only if collocation parameters satisfy the orthogonality condition (3.8). Furthermore, note that the superconvergence rate $2m+1$ of DG solution at nodal points is 1 order higher than those of the collocation solution.

5.3 State dependent delay case

Example 5.3. We present numerical results of the DG solutions for the state dependent delay DDE with vanishing delay. See the following example.

$$
\begin{aligned}
&u'(t) = u(u(t)) + \tfrac{\pi}{4}cos(\tfrac{\pi}{4}t) - sin(\tfrac{\pi}{4}sin(\tfrac{\pi}{4}t)), \qquad t \in J = [0,1],\\
&u(0) = 0,
\end{aligned}
$$

where $u(t) = sin(\frac{\pi}{4}t)$ is the exact solution. We use the piecewise linear DG method to approximate the exact solution. The mesh points are derived by §4.2 with $t_0 = 0.01$. The global convergence of order $m+1$ and local superconvergence of order $2m+1$ at the nodal points are shown in Table 8.

Table 8. Errors of piecewise linear DG solution

l	N	e	R	e_n	R_n
3	54	5.6257e-04		1.1280e-06	
5	90	2.1453e-04	1.9819	2.4027e-07	3.1792
7	126	1.1229e-04	1.9885	8.3923e-08	3.2312
9	162	6.8923e-05	1.9916	3.6808e-08	3.3626

6 Concluding remarks

The following three problems remain to be addressed in future research work:

- CG/DG solutions for state dependent systems.
- Postprocessing of the CG/DG solutions for pantograph-type DDEs with multiple delays.
- Analysis of the attainable order of global convergence and local superconvergence of the DG method for delay reaction-diffusion equations.

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