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Journal of Computational and Applied Mathematics 178 (2005) 437–452

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Two linear transformations each tridiagonal with respect to an eigenbasis of the other: comments on the split decomposition

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Received 22 September 2003; received in revised form 23 April 2004

Abstract

Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both conditions below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call such a pair a *Leonard pair* on V. Referring to the above Leonard pair, it is known there exists a decomposition of V into a direct sum of one-dimensional subspaces, on which A acts in a lower bidiagonal fashion and A^* acts in an upper bidiagonal fashion. This is called the *split decomposition*. In this paper, we give two characterizations of a Leonard pair that involve the split decomposition.

MSC: 05E30; 05E35; 33C45; 33D45

Keywords: Leonard pair; Tridiagonal pair; q-Racah polynomial

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doi:10.1016/j.cam.2004.04.017

1. Leonard pairs and Leonard systems

We begin by recalling the notion of a *Leonard pair* [6,12–18]. We will use the following terms. Let X denote a square matrix. Then X is called *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is called *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 (*Terwilliger [13, Definition 1.1]*). Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfies both (i) and (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2. According to a common notational convention A^* denotes the conjugate transpose of A. We are not using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

Our use of the name “Leonard pair” is motivated by a connection to a theorem of Leonard [2, p. 260]; [9], which involves the q-Racah polynomials [1]; [3, p. 162] and some related polynomials of the Askey scheme [7]. This connection is discussed in [13, Appendix A] and [15, Section 16]. See [4,5,8,10,19] for related topics.

In this paper, we obtain two characterizations of a Leonard pair. These characterizations are based on a concept which we call the *split decomposition*. We will formally define the split decomposition in Section 2, but roughly speaking, this is a decomposition of the underlying vector space into a direct sum of one-dimensional subspaces, with respect to which one element of the pair acts in a lower bidiagonal fashion and the other element of the pair acts in an upper bidiagonal fashion. In [13] we showed that every Leonard pair has a split decomposition. In the present paper, we consider a pair of linear transformations that is not necessarily a Leonard pair. We find a necessary and sufficient condition for this pair to have a split decomposition. Our main result along this line is Theorem 4.1. Now assuming the pair has a split decomposition, we give two necessary and sufficient conditions for this pair to be a Leonard pair. These conditions are stated in Theorems 5.1 and 5.2. These conditions are restated for a more concrete setting in Theorems 6.3 and 6.4.

When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract concept called a *Leonard system*. In order to define this we recall a few more terms. Let d denote a nonnegative integer. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices which have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. Let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of all $d+1$ by 1 matrices which have entries in $\mathbb{K}$. We index the rows by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$ under matrix multiplication. We observe this module is irreducible. For the rest of this paper we let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. When we refer to an $\mathcal{A}$-module we mean a left $\mathcal{A}$-module. Let V denote an irreducible $\mathcal{A}$-module. We remark that

V is unique up to isomorphism of $\mathscr{A}$-modules and that V has dimension $d+1$. Let $v_0, v_1, \ldots, v_d$ denote a basis for V. For $X \in \mathscr{A}$ and for $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say Y *represents* X *with respect to* $v_0, v_1, \ldots, v_d$ whenever $Xv_j = \sum_{i=0}^{d} Y_{ij} v_i$ for $0 \leqslant j \leqslant d$. Let A denote an element of $\mathscr{A}$. We say A is *multiplicity-free* whenever it has $d+1$ distinct eigenvalues in $\mathbb{K}$. Assume A is multiplicity-free. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \leqslant i \leqslant d$ put

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j},$$

where I denotes the identity of $\mathscr{A}$. We observe (i) $AE_i = \theta_i E_i$ $(0 \leqslant i \leqslant d)$, (ii) $E_i E_j = \delta_{ij} E_i$ $(0 \leqslant i, j \leqslant d)$, (iii) $\sum_{i=0}^{d} E_i = I$, (iv) $A = \sum_{i=0}^{d} \theta_i E_i$. Let $\mathscr{D}$ denote the subalgebra of $\mathscr{A}$ generated by A. Using (i)–(iv) we find $E_0, E_1, \ldots, E_d$ is a basis for the $\mathbb{K}$-vector space $\mathscr{D}$. We call E_i the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0 V + E_1 V + \cdots + E_d V \quad \text{(direct sum)}.$$

For $0 \leqslant i \leqslant d$, $E_i V$ is the (one dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace. We remark that the sequence $\{A^i \mid 0 \leqslant i \leqslant d\}$ is a basis for the $\mathbb{K}$-vector space $\mathscr{D}$ and that $\prod_{i=0}^{d}(A - \theta_i I) = 0$. By a *Leonard pair in* $\mathscr{A}$ we mean an ordered pair of elements taken from $\mathscr{A}$ which act on V as a Leonard pair in the sense of Definition 1.1. We now define a Leonard system.

Definition 1.3 (*Terwilliger [13, Definition 1.4]*). By a *Leonard system* in $\mathscr{A}$, we mean a sequence $(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ which satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element of $\mathscr{A}$.
(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of A.
(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of A^*.
(iv)

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$$

(v)

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$$

We comment on how Leonard pairs and Leonard systems are related. In the following discussion V denotes an irreducible $\mathscr{A}$-module. Let $(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ denote a Leonard system in $\mathscr{A}$. For $0 \leqslant i \leqslant d$ let v_i denote a nonzero vector in $E_i V$. Then the sequence $v_0, v_1, \ldots, v_d$ is a basis for V which satisfies Definition 1.1(ii). For $0 \leqslant i \leqslant d$ let v_i^* denote a nonzero vector in $E_i^* V$. Then the sequence $v_0^*, v_1^*, \ldots, v_d^*$ is a basis for V which satisfies Definition 1.1(i). By these comments the pair A, A^* is a Leonard pair in $\mathscr{A}$. Conversely let A, A^* denote a Leonard pair in $\mathscr{A}$. By [13, Lemma 1.3] each of A, A^* is multiplicity-free. Let $v_0, v_1, \ldots, v_d$ denote a basis for V which satisfies Definition 1.1(ii). For $0 \leqslant i \leqslant d$ the vector v_i is an eigenvector for A; let E_i denote the corresponding primitive idempotent. Let $v_0^*, v_1^*, \ldots, v_d^*$

denote a basis for V which satisfies Definition 1.1(i). For $0 \leqslant i \leqslant d$ the vector v_i^* is an eigenvector for A^*; let E_i^* denote the corresponding primitive idempotent. Then $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system in $\mathscr{A}$. In summary we have the following.

Lemma 1.4. *Let A and A^* denote elements in $\mathscr{A}$. Then the pair A, A^* is a Leonard pair in $\mathscr{A}$ if and only if the following* (i) *and* (ii) *hold.*

(i) *Each of A, A^* is multiplicity-free.*
(ii) *There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of A and there exists an ordering $E_0^*, E_1^*, \ldots, E_d^*$ of the primitive idempotents of A^* such that $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system in $\mathscr{A}$.*

2. The split decomposition

In [13] we introduced the split decomposition for Leonard systems and in [15] we discussed this decomposition in detail. For our present purposes it is useful to define the split decomposition in a more general context. We will refer to the following set-up.

Definition 2.1. Let A and A^* denote multiplicity-free elements in $\mathscr{A}$. Let $E_0, E_1, \ldots, E_d$ denote an ordering of the primitive idempotents of A and for $0 \leqslant i \leqslant d$ let θ_i denote the eigenvalue of A for E_i. Let $E_0^*, E_1^*, \ldots, E_d^*$ denote an ordering of the primitive idempotents of A^* and for $0 \leqslant i \leqslant d$ let θ_i^* denote the eigenvalue of A^* for E_i^*. We let $\mathscr{D}$ (respectively $\mathscr{D}^*$) denote the subalgebra of $\mathscr{A}$ generated by A (respectively A^*). We let V denote an irreducible $\mathscr{A}$-module.

With reference to Definition 2.1, by a *decomposition* of V we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of one-dimensional subspaces of V such that

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)}.$$

Definition 2.2. With reference to Definition 2.1, let $U_0, U_1, \ldots, U_d$ denote a decomposition of V. We say this decomposition is *split* (with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$) whenever both

$$(A - \theta_i I)U_i = U_{i+1} \quad (0 \leqslant i \leqslant d-1), \quad (A - \theta_d I)U_d = 0, \tag{1}$$

$$(A^* - \theta_i^* I)U_i = U_{i-1} \quad (1 \leqslant i \leqslant d), \quad (A^* - \theta_0^* I)U_0 = 0. \tag{2}$$

Later in this paper we will obtain two characterizations of a Leonard system which involve the split decomposition. For the time being we consider the existence and uniqueness of the split decomposition. We start with uniqueness.

Lemma 2.3. *With reference to Definition* 2.1, *the following* (i), (ii) *hold.*

(i) *Assume there exists a decomposition $U_0, U_1, \ldots, U_d$ of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Then $U_i = \prod_{h=0}^{i-1}(A - \theta_h I)E_0^* V$ and $U_i = \prod_{h=i+1}^{d}(A^* - \theta_h^* I)E_d V$ for $0 \leqslant i \leqslant d$.*

(ii) *There exists at most one decomposition of V which is split with respect to the orderings* $E_0, E_1, \ldots, E_d$ *and* $E^*_0, E^*_1, \ldots, E^*_d$.

Proof. (i) From the equation on the right in (2) we find $U_0 = E^*_0 V$. Using this and (1) we obtain $U_i = \prod_{h=0}^{i-1}(A - \theta_h I)E^*_0 V$ for $0 \leqslant i \leqslant d$. From the equation on the right in (1) we find $U_d = E_d V$. Using this and (2) we obtain $U_i = \prod_{h=i+1}^{d}(A^* - \theta^*_h I)E_d V$ for $0 \leqslant i \leqslant d$.

(ii) Immediate from (i) above. □

We turn our attention to the existence of the split decomposition. In Section 4, we will give a necessary and sufficient condition for this existence. We will use the following result.

Lemma 2.4. *With reference to Definition* 2.1, *assume there exists a decomposition* $U_0, U_1, \ldots, U_d$ *of V which is split with respect to the orderings* $E_0, E_1, \ldots, E_d$ *and* $E^*_0, E^*_1, \ldots, E^*_d$. *Then the following* (i)–(v) *hold for* $0 \leqslant i \leqslant d$.

(i) $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} A^h E^*_0 V$.
(ii) $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} E^*_h V$.
(iii) $\sum_{h=i}^{d} U_h = \sum_{h=0}^{d-i} A^{*h} E_d V$.
(iv) $\sum_{h=i}^{d} U_h = \sum_{h=i}^{d} E_h V$.
(v) $U_i = (E^*_0 V + E^*_1 V + \cdots + E^*_i V) \cap (E_i V + E_{i+1} V + \cdots + E_d V)$.

Proof. (i) For $0 \leqslant j \leqslant d$ we have $U_j = \prod_{h=0}^{j-1}(A - \theta_h I)E^*_0 V$ by Lemma 2.3(i) so $U_j \subseteq \sum_{h=0}^{j} A^h E^*_0 V$. Apparently $\sum_{h=0}^{i} U_h \subseteq \sum_{h=0}^{i} A^h E^*_0 V$. In this inclusion the sum on the left has dimension $i+1$ since $U_0, U_1, \ldots, U_d$ is a decomposition. The sum on the right has dimension at most $i+1$. Therefore $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} A^h E^*_0 V$.

(ii) For $0 \leqslant j \leqslant d$ we have $\prod_{h=0}^{j}(A^* - \theta^*_h I)U_j = 0$ by (2) so $U_j \subseteq \sum_{h=0}^{j} E^*_h V$. Apparently $\sum_{h=0}^{i} U_h \subseteq \sum_{h=0}^{i} E^*_h V$. In this inclusion each side has dimension $i+1$ so equality holds.

(iii) Similar to the proof of (i) above.

(iv) Similar to the proof of (ii) above.

(v) Combine (ii) and (iv) above. □

3. Some products

Our next goal is to display a necessary and sufficient condition for the existence of the split decomposition. With reference to Definition 2.1, consider the products

$$E^*_i A E^*_j, \quad E_i A^* E_j \quad (0 \leqslant i, j \leqslant d).$$

Our condition has to do with which of these products is 0. In order to motivate our result we initially consider just one of these products.

Lemma 3.1. *With reference to Definition* 2.1, *for* $0 \leqslant i \leqslant d$ *let* v^*_i *denote a nonzero vector in* $E^*_i V$ *and observe* $v^*_0, v^*_1, \ldots, v^*_d$ *is a basis for V. Let B denote the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *which represents A with* *respect to this basis, so that*

$$Av_j^* = \sum_{i=0}^{d} B_{ij} v_i^* \quad (0 \leqslant j \leqslant d). \tag{3}$$

Then for $0 \leqslant i, j \leqslant d$ *the following are equivalent*: (i) $E_i^* A E_j^* = 0$, (ii) $B_{ij} = 0$.

Proof. Let the integers i, j be given. Observe $E_r^* v_s^* = \delta_{rs} v_s^*$ for $0 \leqslant r, s \leqslant d$. By this and (3) we find $E_i^* A E_j^* V$ is spanned by $B_{ij} v_i^*$. The result follows. □

In the next lemma we consider a certain pattern of vanishing products among the $E_i^* A E_j^*$. We will use the following notation. Let λ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in λ which have coefficients in $\mathbb{K}$. Let $f_0, f_1, \ldots, f_d$ denote a sequence of polynomials taken from $\mathbb{K}[\lambda]$. We say this sequence is *graded* whenever f_i has degree exactly i for $0 \leqslant i \leqslant d$.

Lemma 3.2. *With reference to Definition* 2.1, *the following* (i)–(iii) *are equivalent.*

(i) $$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i - j > 1 \\ \neq 0 & \text{if } i - j = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$$

(ii) *There exists a graded sequence of polynomials* $f_0, f_1, \ldots, f_d$ *taken from* $\mathbb{K}[\lambda]$ *such that* $E_i^* V = f_i(A) E_0^* V$ *for* $0 \leqslant i \leqslant d$.

(iii) *For* $0 \leqslant i \leqslant d$,

$$\sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i} A^h E_0^* V. \tag{4}$$

Proof. (i) ⇒ (ii) For $0 \leqslant i \leqslant d$ let v_i^* denote a nonzero vector in $E_i^* V$ and observe $v_0^*, v_1^*, \ldots, v_d^*$ is a basis for V. Let B denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which represents A with respect to this basis. By Lemma 3.1,

$$B_{ij} = \begin{cases} 0 & \text{if } i - j > 1 \\ \neq 0 & \text{if } i - j = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d). \tag{5}$$

Let $f_0, f_1, \ldots, f_d$ denote the polynomials in $\mathbb{K}[\lambda]$ which satisfy $f_0 = 1$ and

$$\lambda f_j = \sum_{i=0}^{j+1} B_{ij} f_i \quad (0 \leqslant j \leqslant d - 1). \tag{6}$$

We observe f_i has degree exactly i for $0 \leqslant i \leqslant d$ so the sequence $f_0, f_1, \ldots, f_d$ is graded. Comparing (3) and (6) in light of (5) we find $v_i^* = f_i(A) v_0^*$ for $0 \leqslant i \leqslant d$. It follows $E_i^* V = f_i(A) E_0^* V$ for $0 \leqslant i \leqslant d$.

(ii) ⇒ (iii) For $0 \leqslant j \leqslant d$ we have $E_j^* V = f_j(A) E_0^* V$. The degree of f_j is j so $E_j^* V \subseteq \sum_{h=0}^{j} A^h E_0^* V$. Apparently $\sum_{h=0}^{i} E_h^* V \subseteq \sum_{h=0}^{i} A^h E_0^* V$. In this inclusion the sum on the left has dimension $i + 1$ and the sum on the right has dimension at most $i + 1$. Therefore $\sum_{h=0}^{i} E_h^* V = \sum_{h=0}^{i} A^h E_0^* V$.

(iii) $\Rightarrow$ (i) For $0 \leqslant i \leqslant d$ let V_i denote the subspace on the left or right in (4). From the right-hand side of (4) we find $V_i + AV_i = V_{i+1}$ for $0 \leqslant i \leqslant d-1$. From the left-hand side of (4) we find $E^*_r V_s = 0$ for $0 \leqslant s < r \leqslant d$. Let i, j denote integers $(0 \leqslant i, j \leqslant d)$ and first assume $i - j > 1$. We show $E^*_i A E^*_j = 0$. Observe $E^*_j V \subseteq V_j$ and $AV_j \subseteq V_{j+1}$ so $AE^*_j V \subseteq V_{j+1}$. However, $E^*_i V_{j+1} = 0$ since $i - j > 1$ so $E^*_i A E^*_j V = 0$. It follows $E^*_i A E^*_j = 0$. Next we assume $i - j = 1$ and show $E^*_i A E^*_j \neq 0$. Suppose $E^*_i A E^*_j = 0$. Then by our previous remarks $E^*_i A E^*_h = 0$ for $0 \leqslant h \leqslant j$. By this and since $V_j = \sum_{h=0}^{j} E^*_h V$ we find $E^*_i AV_j = 0$. However, $V_i = V_j + AV_j$ and $E^*_i V_j = 0$ so $E^*_i V_i = 0$. This contradicts the construction so $E^*_i A E^*_j \neq 0$. □

Corollary 3.3. *With reference to Definition* 2.1, *let* v^*_0 *denote a nonzero vector in* $E^*_0 V$ *and consider the* $\mathbb{K}$*-linear transformation from* $\mathscr{D}$ *to* V *which sends* X *to* Xv^*_0 *for all* $X \in \mathscr{D}$. *Assume the equivalent conditions* (i)–(iii) *hold in Lemma* 3.2. *Then this linear transformation is an isomorphism.*

Proof. Since the $\mathbb{K}$-vector spaces $\mathscr{D}$ and V have the same dimension it suffices to show the linear transformation is surjective. Setting $i = d$ in (4) we find $V = \mathscr{D} v^*_0$. Therefore, the linear transformation is surjective. □

Replacing $(A; A^*; \{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ by $(A^*; A; \{E^*_{d-i}\}_{i=0}^d; \{E_{d-i}\}_{i=0}^d)$ in Lemma 3.2 and Corollary 3.3 we routinely obtain the following results.

Lemma 3.4. *With reference to Definition* 2.1, *the following* (i)–(iii) *are equivalent.*

(i) $E_i A^* E_j = \begin{cases} 0 & \text{if } j - i > 1 \\ \neq 0 & \text{if } j - i = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$

(ii) *There exists a graded sequence of polynomials* $f^*_0, f^*_1, \ldots, f^*_d$ *taken from* $\mathbb{K}[\lambda]$ *such that* $E_i V = f^*_{d-i}(A^*) E_d V$ *for* $0 \leqslant i \leqslant d$.

(iii) *For* $0 \leqslant i \leqslant d$,

$$\sum_{h=i}^{d} E_h V = \sum_{h=0}^{d-i} A^{*h} E_d V.$$

Corollary 3.5. *With reference to Definition* 2.1, *let* v_d *denote a nonzero vector in* $E_d V$ *and consider the* $\mathbb{K}$*-linear transformation from* $\mathscr{D}^*$ *to* V *which sends* X *to* Xv_d *for all* $X \in \mathscr{D}^*$. *Assume the equivalent conditions* (i)–(iii) *hold in Lemma* 3.4. *Then this linear transformation is an isomorphism.*

4. The existence of the split decomposition

We now display a necessary and sufficient condition for the existence of the split decomposition.

Theorem 4.1. *With reference to Definition* 2.1, *the following* (i), (ii) *are equivalent.*

(i) *There exists a decomposition of V which is split with respect to the orderings* $E_0, E_1, \ldots, E_d$ *and* $E^*_0, E^*_1, \ldots, E^*_d$.
(ii) *Both*

$$E^*_i A E^*_j = \begin{cases} 0 & \text{if } i-j>1 \\ \neq 0 & \text{if } i-j=1 \end{cases} \quad (0 \leqslant i, j \leqslant d), \tag{7}$$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } j-i>1 \\ \neq 0 & \text{if } j-i=1 \end{cases} \quad (0 \leqslant i, j \leqslant d). \tag{8}$$

Proof. (i) $\Rightarrow$ (ii) By assumption there exists a decomposition $U_0, U_1, \ldots, U_d$ of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E^*_0, E^*_1, \ldots, E^*_d$. For $0 \leqslant i \leqslant d$ we have $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} A^h E^*_0 V$ by Lemma 2.4(i) and $\sum_{h=0}^{i} U_h = \sum_{h=0}^{i} E^*_h V$ by Lemma 2.4(ii) so $\sum_{h=0}^{i} E^*_h V = \sum_{h=0}^{i} A^h E^*_0 V$. This gives Lemma 3.2(iii). Applying that lemma we obtain (7). For $0 \leqslant i \leqslant d$ we have $\sum_{h=i}^{d} U_h = \sum_{h=0}^{d-i} A^{*h} E_d V$ by Lemma 2.4(iii) and $\sum_{h=i}^{d} U_h = \sum_{h=i}^{d} E_h V$ by Lemma 2.4(iv) so $\sum_{h=i}^{d} E_h V = \sum_{h=0}^{d-i} A^{*h} E_d V$. This gives Lemma 3.4(iii). Applying that lemma we obtain (8).

(ii) $\Rightarrow$ (i) For $0 \leqslant i \leqslant d$ we define $\tau_i = \prod_{h=0}^{i-1} (A - \theta_h I)$. We observe $\tau_0, \tau_1, \ldots, \tau_d$ is a basis for the $\mathbb{K}$-vector space $\mathscr{D}$. Let v^*_0 denote a nonzero vector in $E^*_0 V$. Observe Lemma 3.2(i) holds by (7) so Corollary 3.3 applies; by that corollary $\tau_i v^*_0$ $(0 \leqslant i \leqslant d)$ is a basis for V. We define $U_i = \text{Span}(\tau_i v^*_0)$ for $0 \leqslant i \leqslant d$ and observe $U_0, U_1, \ldots, U_d$ is a decomposition of V. We show this decomposition is split with respect to $E_0, E_1, \ldots, E_d$ and $E^*_0, E^*_1, \ldots, E^*_d$. To do this we show the sequence $U_0, U_1, \ldots, U_d$ satisfies (1) and (2). Concerning (1), from the construction $(A - \theta_i I)\tau_i = \tau_{i+1}$ for $0 \leqslant i \leqslant d-1$ and $(A - \theta_d I)\tau_d = 0$. Applying both sides of theseequations to v^*_0 we find $(A - \theta_i I)U_i = U_{i+1}$ for $0 \leqslant i \leqslant d-1$ and $(A - \theta_d I)U_d = 0$. We have now shown (1). Concerning (2), this will follow if we can show

(a) $(A^* - \theta^*_i I)U_i \subseteq \sum_{h=0}^{i-1} U_h$ for $0 \leqslant i \leqslant d$,
(b) $(A^* - \theta^*_i I)U_i \subseteq \sum_{h=i-1}^{d} U_h$ for $1 \leqslant i \leqslant d$,
(c) $(A^* - \theta^*_i I)U_i \neq 0$ for $1 \leqslant i \leqslant d$.

We begin with (a). For $0 \leqslant j \leqslant d$ the elements $\{\tau_h | 0 \leqslant h \leqslant j\}$ and the elements $\{A^h | 0 \leqslant h \leqslant j\}$ span the same subspace of $\mathscr{D}$. Therefore $\sum_{h=0}^{j} U_h = \sum_{h=0}^{j} A^h E^*_0 V$. We mentioned Lemma 3.2(i) holds so Lemma 3.2(iii) holds; therefore $\sum_{h=0}^{j} E^*_h V = \sum_{h=0}^{j} A^h E^*_0 V$ so $\sum_{h=0}^{j} U_h = \sum_{h=0}^{j} E^*_h V$. Observe $(A^* - \theta^*_i I) \sum_{h=0}^{i} E^*_h V = \sum_{h=0}^{i-1} E^*_h V$ for $0 \leqslant i \leqslant d$. Combining these comments we find $(A^* - \theta^*_i I)U_i \subseteq \sum_{h=0}^{i-1} U_h$ for $0 \leqslant i \leqslant d$.We now have (a). Next we prove (b). From the construction, for $0 \leqslant j \leqslant d$ we have $\prod_{h=j}^{d} (A - \theta_h I)\tau_j = 0$ so $\prod_{h=j}^{d} (A - \theta_h I)U_j = 0$. From this we find $U_j \subseteq \sum_{h=j}^{d} E_h V$. Apparently $\sum_{h=i}^{d} U_h \subseteq \sum_{h=i}^{d} E_h V$ for $0 \leqslant i \leqslant d$. By this and since $U_0, U_1, \ldots, U_d$ is a decomposition we find $\sum_{h=i}^{d} U_h = \sum_{h=i}^{d} E_h V$ for $0 \leqslant i \leqslant d$. From (8) we find $A^* E_j V \subseteq \sum_{h=j-1}^{d} E_h V$ for $1 \leqslant j \leqslant d$. Therefore $(A^* - \theta^*_j I) \sum_{h=j}^{d} E_h V \subseteq \sum_{h=j-1}^{d} E_h V$ for $1 \leqslant j \leqslant d$. From these comments we find $(A^* - \theta^*_j I)U_j \subseteq \sum_{h=j-1}^{d} U_h$for $1 \leqslant j \leqslant d$. We now have (b). Next we show (c). Suppose there exists an integer i $(1 \leqslant i \leqslant d)$ such that $(A^* - \theta^*_i I)U_i = 0$. We assume i is maximal subject to this. We obtain a contradiction as follows. For $i < j \leqslant d$ we find $(A^* - \theta^*_j I)U_j \subseteq U_{j-1}$ by (a), (b). In this inclusion the left-hand side is nonzero and the right-hand side has dimension 1 so we have equality. We mentioned earlier $(A - \theta_d I)U_d = 0$ so $U_d = E_d V$. Apparently $U_j = \prod_{h=j+1}^{d} (A^* - \theta^*_h I) E_d V$ for $i \leqslant j \leqslant d$. In particular $U_i = \prod_{h=i+1}^{d} (A^* - \theta^*_h I) E_d V$.

Combining this with $(A^* - \theta_i^* I)U_i = 0$ we obtain $0 = \prod_{h=i}^{d} (A^* - \theta_h^* I)E_d V$. Let v_d denote a nonzero vector in $E_d V$ and observe $0 = \prod_{h=i}^{d} (A^* - \theta_h^* I)v_d$. This is inconsistent with Corollary 3.5 and the fact that $0 \neq \prod_{h=i}^{d} (A^* - \theta_h^* I)$.We now have a contradiction and (c) is proved. Combining (a)–(c) we obtain (2). We have shown the decomposition $U_0, U_1, \ldots, U_d$ satisfies (1), (2). Applying Definition 2.2 we find $U_0, U_1, \ldots, U_d$ is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. □

5. Two characterizations of a Leonard system

In this section, we obtain two characterizations of a Leonard system, both of which involve the split decomposition. We will first state the characterizations, then prove a few lemmas, and then prove the characterizations. Our first characterization is stated as follows.

Theorem 5.1. *With reference to Definition* 2.1, *the sequence* $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ *is a Leonard system if and only if both* (i), (ii) *hold below.*

(i) *There exists a decomposition of V which is split with respect to the orderings* $E_0, E_1, \ldots, E_d$ *and* $E_0^*, E_1^*, \ldots, E_d^*$.
(ii) *There exists a decomposition of V which is split with respect to the orderings* $E_d, E_{d-1}, \ldots, E_0$ *and* $E_0^*, E_1^*, \ldots, E_d^*$.

In order to state our second characterization we recall a definition. Let $\sigma : \mathcal{A} \to \mathcal{A}$ denote any map. We call σ an *antiautomorphism* of $\mathcal{A}$ whenever σ is an isomorphism of $\mathbb{K}$-vector spaces and $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \mathcal{A}$. For example assume $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathbb{K})$. Then σ is an antiautomorphism of $\mathcal{A}$ if and only if there exists an invertible $R \in \mathcal{A}$ such that $X^\sigma = R^{-1}X^t R$ for all $X \in \mathcal{A}$, where t denotes transpose. This follows from the Skolem–Noether Theorem [11, Corollary 9.122].

We now state our second characterization of a Leonard system.

Theorem 5.2. *With reference to Definition* 2.1, *the sequence* $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ *is a Leonard system if and only if both* (i), (ii) *hold below.*

(i) *There exists a decomposition of V which is split with respect to the orderings* $E_0, E_1, \ldots, E_d$ *and* $E_0^*, E_1^*, \ldots, E_d^*$.
(ii) *There exists an antiautomorphism* † *of* $\mathcal{A}$ *such that* $A^\dagger = A$ *and* $A^{*\dagger} = A^*$.

We now prove some lemmas which we will use to obtain Theorems 5.1 and 5.2. We have a preliminary remark. With reference to Definition 2.1, we consider the following four conditions:

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i - j > 1 \\ \neq 0 & \text{if } i - j = 1 \end{cases} \qquad (0 \leqslant i, j \leqslant d), \tag{9}$$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } j - i > 1 \\ \neq 0 & \text{if } j - i = 1 \end{cases} \qquad (0 \leqslant i, j \leqslant d), \tag{10}$$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } i - j > 1 \\ \neq 0 & \text{if } i - j = 1 \end{cases} \qquad (0 \leqslant i, j \leqslant d), \tag{11}$$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } j-i>1 \\ \neq 0 & \text{if } j-i=1 \end{cases} \quad (0 \leqslant i, j \leqslant d). \tag{12}$$

We observe $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if each of (9)–(12) holds.

Lemma 5.3. *With reference to Definition* 2.1, *assume conditions* (9) *and* (10) *hold. Then* A, E_0^* *together generate* $\mathscr{A}$. *Moreover* A, A^* *together generate* $\mathscr{A}$.

Proof. Examining the proof of [15, Lemma 3.1] we find that the elements $A^r E_0^* A^s$ $(0 \leqslant r, s \leqslant d)$ form a basis for the $\mathbb{K}$-vector space $\mathscr{A}$. It follows that A, E_0^* together generate $\mathscr{A}$. The elements A, A^* together generate $\mathscr{A}$ since E_0^* is a polynomial in A^*. □

Lemma 5.4. *With reference to Definition* 2.1, *assume conditions* (9) *and* (10) *hold. Then there exists a unique antiautomorphism* † *of* $\mathscr{A}$ *such that* $A^\dagger = A$ *and* $A^{*\dagger} = A^*$. *Moreover* $X^{\dagger\dagger} = X$ *for all* $X \in \mathscr{A}$.

Proof. Concerning the existence of †, for $0 \leqslant i \leqslant d$ let v_i^* denote a nonzero element of $E_i^* V$ and recall $v_0^*, v_1^*, \ldots, v_d^*$ is a basis for V. For $X \in \mathscr{A}$ let $X^\flat$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which represents X with respect to the basis $v_0^*, v_1^*, \ldots, v_d^*$. We observe $\flat : \mathscr{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of $\mathbb{K}$-algebras. We abbreviate $B = A^\flat$ and $B^* = A^{*\flat}$. We observe B is irreducible tridiagonal and $B^* = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$. Let D denote the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has *ii* entry

$$D_{ii} = \frac{B_{01} B_{12} \cdots B_{i-1,i}}{B_{10} B_{21} \cdots B_{i,i-1}} \quad (0 \leqslant i \leqslant d).$$

It is routine to verify $D^{-1} B^t D = B$. Each of D, B^* is diagonal so $DB^* = B^* D$; also $B^{*t} = B^*$ so $D^{-1} B^{*t} D = B^*$. Let $\sigma : \mathrm{Mat}_{d+1}(\mathbb{K}) \to \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the map which satisfies $X^\sigma = D^{-1} X^t D$ for all $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. We observe σ is an antiautomorphism of $\mathrm{Mat}_{d+1}(\mathbb{K})$ such that $B^\sigma = B$ and $B^{*\sigma} = B^*$. We define the map $\dagger : \mathscr{A} \to \mathscr{A}$ to be the composition $\dagger := \flat \sigma \flat^{-1}$. We observe † is an antiautomorphism of $\mathscr{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. We have now shown there exists an antiautomorphism † of $\mathscr{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. This antiautomorphism is unique since A, A^* together generate $\mathscr{A}$. The map $X \to X^{\dagger\dagger}$ is an isomorphism of $\mathbb{K}$-algebras from $\mathscr{A}$ to itself. This map is the identity since $A^{\dagger\dagger} = A$, $A^{*\dagger\dagger} = A^*$, and since A, A^* together generate $\mathscr{A}$. □

Lemma 5.5. *With reference to Definition* 2.1, *assume there exists an antiautomorphism* † *of* $\mathscr{A}$ *such that* $A^\dagger = A$ *and* $A^{*\dagger} = A^*$. *Then* $E_i^\dagger = E_i$ *and* $E_i^{*\dagger} = E_i^*$ *for* $0 \leqslant i \leqslant d$.

Proof. Recall E_i (respectively E_i^*) is a polynomial in A (respectively A^*) for $0 \leqslant i \leqslant d$. □

Lemma 5.6. *With reference to Definition* 2.1, *assume there exists an antiautomorphism* † *of* $\mathscr{A}$ *such that* $A^\dagger = A$ *and* $A^{*\dagger} = A^*$. *Then for* $0 \leqslant i, j \leqslant d$, (i) $E_i^* A E_j^* = 0$ *if and only if* $E_j^* A E_i^* = 0$; *and* (ii) $E_i A^* E_j = 0$ *if and only if* $E_j A^* E_i = 0$.

Proof. By Lemma 5.5 and since † is an antiautomorphism,

$$(E_i^* A E_j^*)^\dagger = E_j^* A E_i^* \quad (0 \leqslant i, j \leqslant d).$$

Assertion (i) follows since $\dagger : \mathscr{A} \to \mathscr{A}$ is a bijection. To obtain (ii) interchange the roles of A and A^* in the proof of (i). □

Lemma 5.7. *With reference to Definition* 2.1, *assume at least three of* (9)–(12) *hold*. *Then each of* (9)–(12) *hold*; *in other words* $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ *is a Leonard system*.

Proof. Interchanging A and A^* if necessary, we may assume without loss of generality that (9) and (10) hold. By Lemma 5.4 there exists an antiautomorphism † of $\mathscr{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. By assumption at least one of (11), (12) holds. Combining this with Lemma 5.6 we find (11), (12) both hold. The result follows. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. By Theorem 4.1 we find (i) holds if and only if each of (9), (12) holds. Applying Theorem 4.1 again, this time with $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ replaced by $(A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$, we find (ii) holds if and only if each of (9), (11) holds. Suppose $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. Then each of (9)–(12) holds. In particular each of (9), (11), (12) holds so (i), (ii) hold by our above remarks. Conversely suppose (i), (ii) hold. Then each of (9), (11), (12) holds. At least three of (9)–(12)hold so $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system by Lemma 5.7. □

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. First assume $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. Then (i) holds by Theorem 5.1 and (ii) holds by Lemma 5.4. Conversely assume (i), (ii) hold. Combining (i) and Theorem 4.1 we obtain (9), (12). Combining this with (ii) and using Lemma 5.6 we obtain (10), (11). Now each of (9)–(12) holds so $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. □

We would like to emphasize the following fact.

Theorem 5.8. *Let* A, A^* *denote a Leonard pair in* $\mathscr{A}$. *Then there exists a unique antiautomorphism* † *of* $\mathscr{A}$ *such that* $A^\dagger = A$ *and* $A^{*\dagger} = A^*$. *Moreover* $X^{\dagger\dagger} = X$ *for all* $X \in \mathscr{A}$.

Proof. Since A, A^* is a Leonard pair there exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of A and an ordering $E_0^*, E_1^*, \ldots, E_d^*$ of the primitive idempotents of A^* such that $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. These orderings satisfy (9)–(12). In particular (9), (10) are satisfied so the result follows by Lemma 5.4. □

We finish this section with a comment.

Lemma 5.9. *With reference to Definition* 2.1, *assume there exists a decomposition of V which is split with respect to the orderings* $E_0, E_1, \ldots, E_d$ *and* $E_0^*, E_1^*, \ldots, E_d^*$. *Then the following* (i), (ii) *are equivalent*.

(i) *The pair* A, A^* *is a Leonard pair*.
(ii) *The sequence* $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ *is a Leonard system*.

Proof. (i) $\Rightarrow$ (ii) We assume there exists a decomposition of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Therefore, each of (9), (12) holds by Theorem 4.1. Since A, A^* is a Leonard pair there exists an antiautomorphism $\dagger$ of $\mathscr{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Applying Lemma 5.6 we find each of (10), (11) holds. Now each of (9)–(12) holds so $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.

(ii) $\Rightarrow$ (i) Clear. $\square$

6. The two characterizations in terms of matrices

In this section, we restate Theorems 5.1 and 5.2 in terms of matrices. We first set some notation. With reference to Definition 2.1, suppose there exists a decomposition $U_0, U_1, \ldots, U_d$ of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Pick an integer i $(1 \leqslant i \leqslant d)$. By (2) we find $(A^* - \theta_i^* I)U_i = U_{i-1}$ and by (1) we find $(A - \theta_{i-1} I)U_{i-1} = U_i$. Apparently U_i is an eigenspace for $(A - \theta_{i-1} I)(A^* - \theta_i^* I)$ and the corresponding eigenvalue is a nonzero element of $\mathbb{K}$. Let us denote this eigenvalue by φ_i. We call $\varphi_1, \varphi_2, \ldots, \varphi_d$ the *split sequence* for A, A^* with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. The split sequence has the following interpretation. For $0 \leqslant i \leqslant d$ let u_i denote a nonzero vector in U_i and recall $u_0, u_1, \ldots, u_d$ is a basis for V. We normalize the u_i so that $(A - \theta_i I)u_i = u_{i+1}$ for $0 \leqslant i \leqslant d-1$. With respect to the basis $u_0, u_1, \ldots, u_d$ the matrices which represent A and A^* are as follows.

$$A: \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \quad A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot \;\; \varphi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}.$$

Motivated by this we consider the following set-up.

Definition 6.1. Let d denote a nonnegative integer. Let A and A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$ of the form

$$A = \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot \;\; \varphi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix},$$

where

$$\theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if } i \neq j, \quad (0 \leqslant i, j \leqslant d),$$
$$\varphi_i \neq 0, \quad (1 \leqslant i \leqslant d).$$

We observe A (respectively A^*) is multiplicity-free, with eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$). For $0 \leqslant i \leqslant d$ we let E_i (respectively E_i^*) denote the primitive idempotent for A (respectively A^*) associated with θ_i (respectively θ_i^*).

We have some comments. With reference to Definition 6.1, for $0 \leqslant i \leqslant d$ let u_i denote the vector in $\mathbb{K}^{d+1}$ which has ith entry 1 and all other entries 0. We observe $u_0, u_1, \ldots, u_d$ is a basis for $\mathbb{K}^{d+1}$. From the form of A we have $(A - \theta_i I)u_i = u_{i+1}$ for $0 \leqslant i \leqslant d-1$ and $(A - \theta_d I)u_d = 0$. From the form of A^* we have $(A^* - \theta^*_i I)u_i = \varphi_i u_{i-1}$ for $1 \leqslant i \leqslant d$ and $(A^* - \theta^*_0 I)u_0 = 0$. For $0 \leqslant i \leqslant d$ let U_i denote the subspace of $\mathbb{K}^{d+1}$ spanned by u_i. Then $U_0, U_1, \ldots, U_d$ is a decomposition of $\mathbb{K}^{d+1}$. This decomposition satisfies $(A - \theta_i I)U_i = U_{i+1}$ for $0 \leqslant i \leqslant d-1$ and $(A - \theta_d I)U_d = 0$. Similarly $(A^* - \theta^*_i I)U_i = U_{i-1}$ for $1 \leqslant i \leqslant d$ and $(A^* - \theta^*_0 I)U_0 = 0$. In other words the decomposition $U_0, U_1, \ldots, U_d$ is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E^*_0, E^*_1, \ldots, E^*_d$. We observe $\varphi_1, \varphi_2, \ldots, \varphi_d$ is the corresponding split sequence for A, A^*. We now consider when is the pair A, A^* a Leonard pair. We begin with a remark.

Lemma 6.2. *With reference to Definition* 6.1, *the following* (i), (ii) *are equivalent.*

(i) *The pair A, A^* is a Leonard pair.*
(ii) *The sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ is a Leonard system.*

Proof. We mentioned there exists a decomposition of $\mathbb{K}^{d+1}$ which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E^*_0, E^*_1, \ldots, E^*_d$. Therefore, Lemma 5.9 applies and the result follows. □

We now give a matrix version of Theorem 5.1.

Theorem 6.3. *Referring to Definition* 6.1, *the following* (i), (ii) *are equivalent.*

(i) *The pair A, A^* is a Leonard pair.*
(ii) *There exists an invertible $G \in \mathrm{Mat}_{d+1}(\mathbb{K})$ and there exists nonzero $\phi_i \in \mathbb{K}$ $(1 \leqslant i \leqslant d)$ such that*

$$G^{-1}AG = \begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & & \\ & 1 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix}, \quad G^{-1}A^*G = \begin{pmatrix} \theta^*_0 & \phi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \phi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \phi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}.$$

Suppose (i), (ii) *hold. Then the sequence $\phi_1, \phi_2, \ldots, \phi_d$ is the split sequence for A, A^* associated with the orderings $E_d, E_{d-1}, \ldots, E_0$ and $E^*_0, E^*_1, \ldots, E^*_d$.*

Proof. (i) ⇒ (ii) The sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ is a Leonard system by Lemma 6.2. By Theorem 5.1 there exists a decomposition of $\mathbb{K}^{d+1}$ which is split with respect to the orderings $E_d, E_{d-1}, \ldots, E_0$ and $E^*_0, E^*_1, \ldots, E^*_d$. Let $V_0, V_1, \ldots, V_d$ denote this decomposition. By the definition of a split decomposition we have $(A - \theta_{d-i} I)V_i = V_{i+1}$ for $0 \leqslant i \leqslant d-1$ and $(A - \theta_0 I)V_d = 0$. Moreover $(A^* - \theta^*_i I)V_i = V_{i-1}$ for $1 \leqslant i \leqslant d$ and $(A^* - \theta^*_0 I)V_0 = 0$. For $0 \leqslant i \leqslant d$ let v_i denote a nonzero vector in V_i and observe $v_0, v_1, \ldots, v_d$ is a basis for $\mathbb{K}^{d+1}$. We normalize the v_i so that $(A - \theta_{d-i} I)v_i = v_{i+1}$ for $0 \leqslant i \leqslant d-1$. Let $\phi_1, \phi_2, \ldots, \phi_d$ denote the split sequence for A, A^* with respect to the orderings $E_d, E_{d-1}, \ldots, E_0$ and $E^*_0, E^*_1, \ldots, E^*_d$. Then $\phi_i \neq 0$ $(1 \leqslant i \leqslant d)$ and moreover $(A^* - \theta^*_i I)v_i = \phi_i v_{i-1}$ $(1 \leqslant i \leqslant d)$, $(A^* - \theta^*_0 I)v_0 = 0$. Let G denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has column i equal to v_i for $0 \leqslant i \leqslant d$. We observe G is invertible. Moreover, the matrices $G^{-1}AG$ and $G^{-1}A^*G$ have the form shown above.

(ii) $\Rightarrow$ (i) We show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In order to do this we apply Theorem 5.1. In the paragraph after Definition 6.1 we mentioned there exists a decomposition of $\mathbb{K}^{d+1}$ which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Therefore, Theorem 5.1(i) holds. We show Theorem 5.1(ii) holds. For $0 \leqslant i \leqslant d$ let v_i denote column i of G and observe $v_0, v_1, \ldots, v_d$ is a basis for $\mathbb{K}^{d+1}$. From the form of $G^{-1}AG$ we find $(A - \theta_{d-i}I)v_i = v_{i+1}$ for $0 \leqslant i \leqslant d-1$ and $(A - \theta_0 I)v_d = 0$. From the form of $G^{-1}A^*G$ we find $(A^* - \theta_i^* I)v_i = \phi_i v_{i-1}$ for $1 \leqslant i \leqslant d$ and $(A^* - \theta_0^* I)v_0 = 0$. For $0 \leqslant i \leqslant d$ let V_i denote the subspace of $\mathbb{K}^{d+1}$ spanned by v_i. Then $V_0, V_1, \ldots, V_d$ is a decomposition of $\mathbb{K}^{d+1}$. Also $(A - \theta_{d-i}I)V_i = V_{i+1}$ for $0 \leqslant i \leqslant d-1$ and $(A - \theta_0 I)V_d = 0$. Moreover $(A^* - \theta_i^* I)V_i = V_{i-1}$ for $1 \leqslant i \leqslant d$ and $(A^* - \theta_0^* I)V_0 = 0$. Apparently $V_0, V_1, \ldots, V_d$ is split with respect to the orderings $E_d, E_{d-1}, \ldots, E_0$ and $E_0^*, E_1^*, \ldots, E_d^*$. Now Theorem 5.1(ii) holds; applying that theorem we find $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In particular A, A^* is a Leonard pair.

Assume (i), (ii) both hold. From the proof of (ii) $\Rightarrow$ (i) we find that for $1 \leqslant i \leqslant d$, ϕ_i is the eigenvalue of $(A - \theta_{d-i+1}I)(A^* - \theta_i^* I)$ associated with V_i. Therefore $\phi_1, \phi_2, \ldots, \phi_d$ is the split sequence for A, A^* associated with the orderings $E_d, E_{d-1}, \ldots, E_0$ and $E_0^*, E_1^*, \ldots, E_d^*$. □

We now give a matrix version of Theorem 5.2.

Theorem 6.4. *Referring to Definition* 6.1, *the following* (i), (ii) *are equivalent*.

(i) *The pair* A, A^* *is a Leonard pair*.
(ii) *There exists an invertible* $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *such that*

$$H^{-1}A^t H = A, \quad H^{-1}A^{*t}H = A^*.$$

Proof. (i) $\Rightarrow$ (ii) By Theorem 5.8 there exists an antiautomorphism † of $\mathrm{Mat}_{d+1}(\mathbb{K})$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Since † is an antiautomorphism there exists an invertible $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $X^\dagger = H^{-1}X^t H$ for all $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Setting $X = A$ we have $H^{-1}A^t H = A$. Setting $X = A^*$ we have $H^{-1}A^{*t}H = A^*$.

(ii) $\Rightarrow$ (i) We show $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In order to do this we apply Theorem 5.2. In the paragraph after Definition 6.1 we mentioned there exists a decomposition of $\mathbb{K}^{d+1}$ which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Therefore, Theorem 5.2(i) holds. Let $\dagger : \mathrm{Mat}_{d+1}(\mathbb{K}) \to \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the map which satisfies $X^\dagger = H^{-1}X^t H$ for all $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Then † is an antiautomorphism of $\mathrm{Mat}_{d+1}(\mathbb{K})$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Now Theorem 5.2(ii) holds; applying that theorem we find $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. In particular A, A^* is a Leonard pair. □

7. Remarks

Referring to Definition 6.1, presumably condition (ii) of Theorems 6.3 or 6.4 can be translated into a condition on the entries of A and A^*. We obtained such a condition in [13]; we cite it here for the sake of completeness.

Theorem 7.1 (*Terwilliger [13, Corollary 14.2]*). *With reference to Definition* 6.1, *the pair* A, A^* *is a Leonard pair if and only if there exists nonzero* $\phi_i \in \mathbb{K}$ $(1 \leqslant i \leqslant d)$ *such that* (i)–(iii) *hold below*.

(i) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d), \quad (1 \leqslant i \leqslant d).$

(ii) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0), \quad (1 \leqslant i \leqslant d).$

(iii) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leqslant i \leqslant d - 1$.

Suppose (i)–(iii) *hold. Then* $\phi_1, \phi_2, \ldots, \phi_d$ *is the split sequence for* A, A^* *with respect to the orderings* $E_d, E_{d-1}, \ldots, E_0$ *and* $E^*_0, E^*_1, \ldots, E^*_d$.

Acknowledgements

The author would like to thank Brian Curtin, Eric Egge, Mark MacLean, Arlene Pascasio, and Chih-wen Weng for giving the manuscript a close reading and offering many valuable suggestions.

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