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Approximation of discontinuous curves and surfaces with tangent conditions

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Abstract

We deal with a smoothing method of constructing some discontinuous curve or surface from a Lagrangean data and tangent conditions. Such method is based on the theory of smoothing variational splines conveniently adapted to introduce the tangent condition and the discontinuity set. To show the efficiency of our method we finish this work by a convergence result and some numerical and graphical examples.

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1. Introduction

The problem of construction of curves and surfaces which present some discontinuity from a set of points (Lagrange data) and other of tangent spaces—known as tangent conditions—is frequently

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encountered in CAGD, Geology and other Earth Sciences. The work [9] can be considered as a prior one on variational surface smoothing involving discontinuities. It proposes a general class of controlled-continuity stabilizers that provides the necessary control over smoothness. In the context of computational vision, these nonquadratic stabilizing functionals may be thought of as controlled-continuity constraints.

The smoothing variational spline is introduced in [3] by minimizing a quadratic functional in a Sobolev space. Such functional contains several terms such that each one of them is controlled by a suitable parameter, for example, in [6] we study an approximation problem of surfaces introducing some fairness terms, which allows us to obtain a pleasing shape, meanwhile in [5] we add to such fairness terms some tangent ones in order to observe numerically and graphically the influence of each constraint for the curve case. Likewise, the authors in [8] present a smoothing method for fitting parametric surfaces from sets of data points and tangent planes. In addition to papers [3–6,8] the corresponding original curves or surfaces that are approximated do not present any discontinuity, in order to overcome this restriction, we allow in this work to introduce such discontinuity and hence to extend the results of previous papers to cover a wide set of data type.

We assume that a given differentiable function f in a subset Ω' of an open set $\Omega \subset \mathbb{R}^p$ with values in $\mathbb{R}^n$, $1 \leqslant p < n \leqslant 3$, or its first partial derivatives presents discontinuity over a finite subset of points of Ω. The problem is to construct a function σ that approximates f in the given set of points and whose tangent spaces associated to both σ and f will be close to each other.

To achieve that, firstly using the work of Arcangéli, Manzanilla and Torrens [1], we assume a set of conditions about Ω' that allows to model the contingent discontinuities of f. Secondly, we present a method of smoothness which results from adapting the theory of the smoothing variational splines over an open bounded set [3].

Among the most relevant differences of this paper respect to [9] we can indicate the following ones: we include the tangent data and we prove in Section 5 a convergence result of our approximation method.

This paper is organized as follows. In Section 2, we briefly recall some preliminary notations. Section 3 is devoted to the concept of discontinuity set. Next, we state the problem of smoothing variational splines over Ω' with tangent conditions in Section 4. The convergence of the method is established in Section 5. Finally, Section 6 provides some numerical and graphical examples.

2. Notations

Let n, m and p belonging to $\mathbb{N}^*$, we denote by $\langle \cdot \rangle_{\mathbb{R}^n}$, and $\langle \cdot\,,\,\cdot \rangle_{\mathbb{R}^n}$, respectively, the Euclidean norm and the inner product in $\mathbb{R}^n$. For all $E \subset \mathbb{R}^p$, we denote by $\overline{E}$, δE and card E, respectively, the adherence, the bounded and the cardinal of E.

Moreover, we denote by $\mathbb{R}^{N,n}$ the space of real matrices with N rows and n columns, equipped with the inner product

$$\langle A, B \rangle_{N,n} = \sum_{i=1}^{N} \sum_{j=1}^{n} a_{ij} b_{ij}$$

and the corresponding norm

$$\langle A \rangle_{N,n} = \langle A, A \rangle_{N,n}^{1/2}.$$

For all $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p$, we write $|\alpha| = \sum_{i=1}^p \alpha_i$ and we indicate by ∂^α the operator of partial derivative

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}.$$

Let ω be a nonempty open bounded set of $\mathbb{R}^p$ and we denote by $H^m(\omega; \mathbb{R}^n)$ the usual Sobolev space of (classes of) functions u belonging to $L^2(\omega; \mathbb{R}^n)$, together with all their partial derivatives $\partial^\alpha u$—in the distribution sense—of order $|\alpha| \leqslant m$. This space is equipped with the inner product of order ℓ

$$(u, v)_{\ell,\omega,\mathbb{R}^n} = \left(\sum_{|\alpha|=\ell} \int_\omega \langle \partial^\alpha u(x), \partial^\alpha v(x) \rangle_{\mathbb{R}^n} \right)^{1/2}, \quad \ell = 0, \ldots, m,$$

the corresponding semi-norms of order ℓ

$$|u|_{\ell,\omega,\mathbb{R}^n} = (u, u)^{1/2}_{\ell,\omega,\mathbb{R}^n}, \quad \ell = 0, \ldots, m,$$

the norm

$$\|u\|_{m,\omega,\mathbb{R}^n} = \left(\sum_{\ell=0}^m |u|^2_{\ell,\omega,\mathbb{R}^n} \right)^{1/2}$$

and the corresponding inner product

$$((u, v))_{m,\omega,\mathbb{R}^n} = \sum_{\ell=0}^m (u, v)_{\ell,\omega,\mathbb{R}^n}.$$

Finally, given a function $f : \omega \to \mathbb{R}^n$, we denote by $\operatorname{Im} Df(x)$ the image of the differential of f at the point $x \in \omega$, when this exists, i.e., the linear subspace generated by $\{\partial^\alpha f(x) : |\alpha| = 1\}$. Furthermore, if $1 \leqslant p < n \leqslant 3$, we can consider f as the parameterization of a curve ($p = 1$) or a surface ($p = 2$) and, if f is differentiable at $x \in \omega$, the space $\operatorname{Im} Df(x)$ is called the tangent space of f at x, sometimes when $p = 2$ it is written by $T_x(f) = \operatorname{span}\langle D_1 f(x), D_2 f(x) \rangle$, where $D_1 f$ and $D_2 f$ denote the first partial derivatives of f.

3. The set of discontinuities

The first step in developing this work is to have an adequate characterization over a set of discontinuity. Let us introduce the following definition due to Arcangéli, Manzanilla and Torrens [1].

Definition 3.1. Let Ω be a bounded open connected set of $\mathbb{R}^p$ with Lipschitz boundary and let F be a nonempty subset of $\overline{\Omega}$ such that, there exists a finite family $\{R_1, \ldots, R_I\}$ of open connected subsets of Ω with Lipschitz boundary, verifying the following conditions:

(i) for all $i, j = 1, \ldots, I,\ i \neq j,\ R_i \cap R_j = \emptyset$;
(ii) $\bigcup_{i=1}^I \overline{R}_i = \overline{\Omega}$;

(iii) $F \subset \delta R$, where $\bigcup_{i=1}^{I} R_i = R$;
(iv) F is contained in the interior of $\delta\Omega$ (equipped of the induced topology by $\mathbb{R}^p$) of $F \cap \delta\Omega$;
(v) the interior in δR of $\overline{F} \cap \Omega$ is contained in F;
(vi) $\overline{F} \cap \delta\Omega$ is contained in F.

It is said that the family $\{R_1, \ldots, R_I\}$ represents F in Ω and we write $\Omega' = \Omega \backslash \overline{F}$.

We denote by $C_F^k(\Omega'; \mathbb{R}^n)$ the space of functions $\varphi \in C^k(\Omega'; \mathbb{R}^n)$ such that

$$\forall i = 1, \ldots, I, \quad \varphi|_{R_i} \in C^k(\overline{R}_i; \mathbb{R}^n).$$

Such space is equipped by the norm

$$\|\varphi\|_{C_F^k(\Omega';\mathbb{R}^n)} = \max_{1 \leqslant i \leqslant I} \|\varphi|_{R_i}\|_{C^k(\overline{R}_i;\mathbb{R}^n)}. \tag{3.1}$$

Let us remember some useful properties for some spaces of functions defined over Ω' and whose proofs are proved in [1].

Theorem 3.1. *The space $C_F^k(\Omega'; \mathbb{R}^n)$ is a Banach space for the norm defined in* (3.1). *Moreover, such space and the norm defined in* (3.1) *are independent on the choose of the family $\{R_1, \ldots, R_I\}$ that presents F on Ω.*

Theorem 3.2. *For all $m, k \in \mathbb{N}$ such that $m > p/2 + k$:*

$$H^m(\Omega'; \mathbb{R}^n) \overset{c}{\subset} C_F^k(\Omega'; \mathbb{R}^n),$$

where $\overset{c}{\subset}$ designs the compact injection.

Theorem 3.3. *For all $l, l' \in \mathbb{N}$, with $l > l'$:*

$$H^l(\Omega'; \mathbb{R}^n) \overset{c}{\subset} H^{l'}(\Omega'; \mathbb{R}^n).$$

Theorem 3.4. *For all $m, k \in \mathbb{N}$ such that $m > p/2 + k$, the subspace $H^m(\Omega'; \mathbb{R}^n) \cap C^k(\overline{\Omega}; \mathbb{R}^n)$ is closed in $H^m(\Omega'; \mathbb{R}^n)$.*

4. Variational spline over Ω' with tangent conditions

We conserve the notations about Ω, F, Ω' and R_i for $i = 1, \ldots, I$ introduced in Section 3 and we suppose that

$$m > \frac{p}{2} + 1. \tag{4.1}$$

Let Υ_0 be a curve or surface parametrized by a function $f \in H^m(\Omega'; \mathbb{R}^n)$.

Let A_1 and A_2 be two ordered finite subsets of, respectively, N_1 and N_2 distinct points of $\overline{\Omega}$ and, for any $a \in A_1$, let us consider the linear form defined on $C^0_F(\Omega'; \mathbb{R}^n)$ by

$$\phi_a v = \begin{cases} v(a) & \text{if } a \in A_1 \backslash F, \\ v|_{R_i}(a) & \text{if } a \in A_1 \cap R_i \cap F, \ 1 \leqslant i \leqslant I. \end{cases} \tag{4.2}$$

Moreover, for any $a \in A_2$ let Π_a be the operator defined on $C^1_F(\Omega'; \mathbb{R}^n)$ by

$$\Pi_a v = \begin{cases} \left(P_{S_a^\perp}\left(\frac{\partial v}{\partial x_j}(a)\right)\right)_{1 \leqslant j \leqslant p} & \text{if } a \in A_2 \backslash F, \ 1 \leqslant i \leqslant I, \\ \left(P_{S_a^\perp}\left(\frac{\partial v|_{R_i}}{\partial x_j}(a)\right)\right)_{1 \leqslant j \leqslant p} & \text{if } a \in A_2 \cap R_i \cap F, \ 1 \leqslant i \leqslant I, \end{cases} \tag{4.3}$$

where for any $a \in A_2$, $P_{S_a^\perp}$ is the operator projection over $S_a^\perp$, being $S_a^\perp$ the orthogonal complement of the linear space $S_a = \operatorname{Im} Df(a)$.

Remark 4.1. The sets A_1 and A_2 are constituted by the parameter values of the points that are wished to be fitting and the points in those which are considered the tangent data. Likewise, the operators given, respectively, in (4.2) and (4.3) could be considered as the corresponding Lagrangean data and the tangent data, respectively.

Finally, let

$$Lv = (\phi_a v)_{a \in A_1} \quad \text{and} \quad \Pi v = (\Pi_a v)_{a \in A_2}$$

and we suppose that

$$\operatorname{Ker} L \cap \tilde{P}_{m-1}(\Omega'; \mathbb{R}^n) = \{0\}, \tag{4.4}$$

where $\tilde{P}_{m-1}(\Omega'; \mathbb{R}^n)$ designs the space of functions over Ω' into $\mathbb{R}^n$ that are polynomials of total degree $\leqslant m-1$ respect to the set of variables over each connected component of Ω'.

Now, given $\tau \geqslant 0$ and $\varepsilon > 0$ let $J_{\varepsilon\tau}$ be the functional defined on $H^m(\Omega'; \mathbb{R}^n)$ by

$$J_{\varepsilon\tau}(v) = \langle Lv - Lf \rangle^2_{N_1,n} + \tau \langle \Pi v \rangle^2_{N_2,pn} + \varepsilon |v|^2_{m,\Omega',\mathbb{R}^n}. \tag{4.5}$$

Remark 4.2. We observe that the functional $J_{\varepsilon\tau}(v)$ contains different terms which can be interpreted as follows:

- The first term, $\langle Lv - Lf \rangle^2_{N_1,n}$, indicates how well v approaches f in a least squares sense.
- The second term, $\langle \Pi v \rangle^2_{N_2,pn}$, indicates how well, for any point $a \in A_2$, the tangent spaces $\operatorname{Im} Df(a)$ and $\operatorname{Im} Dv(a)$ are really close.
- The last term, $|v|^2_{m,(a,b),\mathbb{R}^n}$, measures the degree of smoothness of v.

We note that the parameters τ and ε control the relative weights corresponding respectively, to the last two terms.

Now, we consider the following minimization problem: find an approximating curve or surface Υ of Υ_0 parametrized by a function $\sigma_{\varepsilon\tau}$ belonging to $H^m(\Omega'; \mathbb{R}^n)$ from the data $\{f(a) \mid a \in A_1\}$ and $\{S_a \mid a \in A_2\}$, such that $\sigma_{\varepsilon\tau}$ minimizes the functional $J_{\varepsilon\tau}$ on $H^m(\Omega'; \mathbb{R}^n)$, i.e., find $\sigma_{\varepsilon\tau}$ such that

$$\begin{aligned}&\sigma_{\varepsilon\tau} \in H^m(\Omega'; \mathbb{R}^n),\\ &\forall v \in H^m(\Omega'; \mathbb{R}^n), \quad J_{\varepsilon\tau}(\sigma_{\varepsilon\tau}) \leqslant J_{\varepsilon\tau}(v).\end{aligned} \tag{4.6}$$

Theorem 4.1. *The problem* (4.6) *has a unique solution*, *called the smoothing variational spline with tangent conditions in* Ω' *relative to* A_1, A_2, Lf, τ *and* ε, *which is also the unique solution of the following variational problem: find* $\sigma_{\varepsilon\tau}$ *such that*

$$\begin{aligned}&\sigma_{\varepsilon\tau} \in H^m(\Omega'; \mathbb{R}^n),\\ &\forall v \in H^m(\Omega'; \mathbb{R}^n), \quad \langle L\sigma_{\varepsilon\tau}, Lv\rangle_{N_1,n} + \tau\langle \Pi\sigma_{\varepsilon\tau}, \Pi v\rangle_{N_2,pn} + \varepsilon(\sigma_{\varepsilon\tau}, v)_{m,\Omega',\mathbb{R}^n} = \langle Lf, Lv\rangle_{N_1,n}.\end{aligned}$$

Proof. Taking into account (4.1), (4.4) and that the following norm:

$$v \longmapsto [[v]] = (\langle Lv\rangle^2_{N_1,n} + \tau\langle \Pi v\rangle^2_{N_2,pn} + \varepsilon|v|^2_{m,\Omega',\mathbb{R}^n})^{1/2}$$

is equivalent in $H^m(\Omega'; \mathbb{R}^n)$ to the norm $\|\cdot\|_{m,\Omega',\mathbb{R}^n}$ (cf. [1, Proposition 4.1]), one easily checks that the symmetric bilinear form $\widetilde{a} : H^m(\Omega'; \mathbb{R}^n) \times H^m(\Omega'; \mathbb{R}^n) \longrightarrow \mathbb{R}$ given by

$$\widetilde{a}(u, v) = \langle Lu, Lv\rangle_{N_1,n} + \tau\langle \Pi u, \Pi v\rangle_{N_2,pn} + \varepsilon(u, v)_{m,\Omega',\mathbb{R}^n}$$

is a continuous and $H^m(\Omega'; \mathbb{R}^n)$-elliptic. Likewise, the linear form

$$\varphi : v \in H^m(\Omega'; \mathbb{R}^n) \longmapsto \varphi(v) = \langle Lf, Lv\rangle_{N_1,n}$$

is continuous. The result is then a consequence of the Lax–Milgram Lemma (see [2]). □

5. Convergence

Under adequate hypotheses, we shall show that the *smoothing variational spline with tangent conditions* converges to *f*. To do this, we suppose the following are given:

- a set of real positive numbers $\mathscr{D}$ of which 0 is an accumulation point;
- for all $d \in \mathscr{D}$, two subsets A_1^d and A_2^d of, respectively, $N_1 = N_1(d)$ and $N_2 = N_2(d)$ distinct points of $\overline{\Omega}$;
- for all $d \in \mathscr{D}$ and any $a \in A_1^d$, let us consider the linear form defined on $C^0_F(\Omega'; \mathbb{R}^n)$ by

$$\phi^d_a v = \begin{cases} v(a) & \text{if } a \in A_1^d \backslash F,\\ v|_{R_i}(a) & \text{if } a \in A_1^d \cap R_i \cap F, \ 1 \leqslant i \leqslant I;\end{cases}$$

- for all $d \in \mathscr{D}$ and any $a \in A_2^d$, let Π_a^d be the operator defined in $C_F^1(\Omega'; \mathbb{R}^n)$ by

$$\Pi_a^d v = \begin{cases} \left(P_{S_a^\perp}\left(\frac{\partial v}{\partial x_j}(a)\right)\right)_{1 \leqslant j \leqslant p} & \text{if } a \in A_2^d \backslash F,\ 1 \leqslant i \leqslant I, \\ \left(P_{S_a^\perp}\left(\frac{\partial v|_{R_i}}{\partial x_j}(a)\right)\right)_{1 \leqslant j \leqslant p} & \text{if } a \in A_2^d \cap R_i \cap F,\ 1 \leqslant i \leqslant I, \end{cases}$$

where for any $a \in A_2^d$, $P_{S_a^\perp}$ is the operator projection over $S_a^\perp$, being $S_a^\perp$ the orthogonal complement of the linear space $S_a = \operatorname{Im} Df(a)$.

Finally, let

$$L^d v = (\phi_a^d v)_{a \in A_1^d} \quad \text{and} \quad \Pi^d v = (\Pi_a^d v)_{a \in A_2^d}.$$

We suppose that

$$\operatorname{Ker} L^d \cap \tilde{P}_{m-1}(\Omega'; \mathbb{R}^n) = \{0\} \tag{5.1}$$

and that

$$\sup_{x \in \Omega'} \min_{a \in A_1^d} \langle x - a \rangle_{\mathbb{R}^p} = d. \tag{5.2}$$

Now, for each $d \in \mathscr{D}$ let $\tau = \tau(d) \geqslant 0$, $\varepsilon = \varepsilon(d) > 0$ and let $J_{\varepsilon\tau}^d$ be the functional defined in $H^m(\Omega'; \mathbb{R}^n)$ as $J_{\varepsilon\tau}$ in (4.5), with L^d and Π^d instead of L and Π, respectively. Finally, let $\sigma_{\varepsilon\tau}^d$ be the *smoothing variational spline with tangent conditions* in Ω' relative to A_1^d, A_2^d, $L^d f$, τ and ε which is the minimum of $J_{\varepsilon\tau}^d$ in $H^m(\Omega'; \mathbb{R}^n)$.

In order to prove the convergence of $\sigma_{\varepsilon\tau}^d$ to *f*, under suitable hypotheses, we need the following results.

Proposition 5.1. *Let $B_0 = \{b_{01}, \ldots, b_{0\Delta}\}$ be a $\tilde{P}_{m-1}(\Omega'; \mathbb{R}^n)$-unisolvent subset of points of R. Then, there exists $\eta > 0$ such that if $\mathscr{B}_\eta$ designs the set of Δ-uplet $B = \{b_1, \ldots, b_\Delta\}$ of points of Ω' satisfying the condition*

$$\forall j = 1, \ldots, \Delta \quad \langle b_j - b_{0j} \rangle_{\mathbb{R}^p} < \eta,$$

the application $[[\cdot]]_{m,\Omega'}^B$ defined, for all $B \in \mathscr{B}_\eta$, by

$$[[v]]_{m,\Omega'}^B = \left(\sum_{j=1}^{\Delta} \langle v(b_j) \rangle_{\mathbb{R}^n}^2 + |v|_{m,\Omega',\mathbb{R}^n}^2 \right)^{1/2},$$

is a norm on $H^m(\Omega'; \mathbb{R}^n)$, uniformly equivalent over $\mathscr{B}_\eta$ to the usual norm of Sobolev $\| \cdot \|_{m,\Omega',\mathbb{R}^n}$.

Proof. It is analogous to the proof [1, Proposition 6.2]. □

Theorem 5.2. *We suppose that the hypotheses* (5.1) *and* (5.2) *hold*, *and that*

$$\varepsilon = \mathrm{o}(d^{-p}), \quad d \to 0. \tag{5.3}$$

Then, *one has*

$$\lim_{d \to 0} \|\sigma_{\varepsilon\tau}^d - f\|_{m,\Omega',\mathbb{R}^n} = 0. \tag{5.4}$$

Proof. (1) For all $d \in \mathscr{D}$, from Theorem 4.1 we have that

$$\langle L^d \sigma_{\varepsilon\tau}^d, L^d(\sigma_{\varepsilon\tau}^d - f)\rangle_{N_1,n} + \tau\langle \Pi^d \sigma_{\varepsilon\tau}, \Pi^d(\sigma_{\varepsilon\tau}^d - f)\rangle_{N_2,pn} + \varepsilon(\sigma_{\varepsilon\tau}, \sigma_{\varepsilon\tau}^d - f)_{m,\Omega',\mathbb{R}^n} = \langle L^d f, L^d(\sigma_{\varepsilon\tau}^d - f)\rangle_{N_1,n},$$

which means, for all $d \in \mathscr{D}$, that

$$\langle L^d(\sigma_{\varepsilon\tau}^d - f)\rangle_{N_1,n}^2 + \tau\langle \Pi^d(\sigma_{\varepsilon\tau}^d - f)\rangle_{N_2,pn}^2 + \varepsilon|\sigma_{\varepsilon\tau}|_{m,\Omega',\mathbb{R}^n}^2 = \varepsilon(\sigma_{\varepsilon\tau}^d, f)_{m,\Omega',\mathbb{R}^n},$$

because $\Pi^d f = 0$. We deduce that

$$\forall\, d \in \mathscr{D}, \quad |\sigma_{\varepsilon\tau}^d|_{m,\Omega',\mathbb{R}^n} \leqslant |f|_{m,\Omega',\mathbb{R}^n}, \tag{5.5}$$

and that

$$\forall\, d \in \mathscr{D}, \quad \langle L^d(\sigma_{\varepsilon\tau}^d - f)\rangle_{N_1,n} \leqslant \varepsilon^{1/2}|f|_{m,\Omega',\mathbb{R}^n}. \tag{5.6}$$

Let $B_0 = \{b_{01}, \ldots, b_{0\Delta}\}$ be a $\tilde{P}_{m-1}(\Omega')$-unisolvent subset of points of R and let η be the constant of Proposition 5.1. Obviously, there exists $\eta' \in (0, \eta]$ such that

$$\forall\, j = 1, \ldots, \Delta, \quad \overline{B}(b_{0j}, \eta') \subset \overline{R}.$$

From (5.2),

$$\forall d \in \mathscr{D}, \quad d < \eta', \ \forall\, j = 1, \ldots, \Delta, \quad \overline{B}(b_{0j}, \eta' - d) \subset \bigcup_{a \in A_1^d \cap \overline{B}(b_{0j},\eta')} \overline{B}(a, d).$$

If $\mathscr{N}_j = \mathrm{card}(A_1^d \cap \overline{B}(b_{0j}, \eta'))$, it follows:

$$\forall d \in \mathscr{D}, \quad d < \eta', \ \forall\, j = 1, \ldots, \Delta, \quad (\eta' - d)^p \leqslant \mathscr{N}_j d^p,$$

and, for consequently, for any $d_0 \in (0, \eta')$,

$$\forall\, d \in \mathscr{D}, \quad d \leqslant d_0, \ \forall\, j = 1, \ldots, \Delta, \quad \mathscr{N}_j \geqslant (\eta' - d_0)^p d^{-p}. \tag{5.7}$$

Now, from (5.3) and (5.6), it derives that

$$\forall\, j = 1, \ldots, \Delta, \quad \sum_{a \in A_1^d \cap \overline{B}(b_{0j},\eta')} \langle(\sigma_{\varepsilon\tau}^d - f)(a)\rangle_{\mathbb{R}^n}^2 = \mathrm{o}(d^{-p}), \quad d \to 0. \tag{5.8}$$

If a_j^d is a point of $A_1^d \cap \overline{B}(b_{0j}, \eta')$ such that

$$\langle(\sigma_{\varepsilon\tau}^d - f)(a_j^d)\rangle_{\mathbb{R}^n} = \min_{a \in A_1^d \cap \overline{B}(b_{0j},\eta')} \langle(\sigma_{\varepsilon\tau}^d - f)(a)\rangle_{\mathbb{R}^n},$$

we deduce, from (5.7) and (5.8), that

$$\forall\, j=1,\ldots,\Delta, \quad \langle(\sigma^d_{\varepsilon\tau}-f)(a^d_j)\rangle_{\mathbb{R}^n}=\mathrm{o}(1), \quad d\to 0. \tag{5.9}$$

We denote by B^d the set $\{a^d_1,\ldots,a^d_\Delta\}$. Applying Proposition 5.1 with $B=B^d$, for d sufficiently close to 0, it results from (5.5) and (5.9) that

$$\exists\, C>0, \quad \exists\,\alpha>0, \quad \forall\, d\in\mathscr{D}, \quad d\leqslant\alpha, \quad \|\sigma^d_{\varepsilon\tau}\|_{m,\Omega',\mathbb{R}^n}\leqslant C,$$

it means that the family $(\sigma^d_{\varepsilon\tau})_{d\in\mathscr{D},d\leqslant\alpha}$ is bounded in $H^m(\Omega';\mathbb{R}^n)$. Then, there exists a sequence $(\sigma^{d_l}_{\varepsilon_l\tau_l})_{l\in\mathbb{N}}$, extracted from such family, with $\lim_{l\to+\infty} d_l=0$, $\varepsilon_l=\varepsilon(d_l)$ and $\tau_l=\tau(d_l)$, and an element $f^*\in H^m(\Omega';\mathbb{R}^n)$ such that

$$f^* \text{ converges weakly to } \sigma^{d_l}_{\varepsilon_l\tau_l} \quad \text{in } H^m(\Omega';\mathbb{R}^n) \text{ when } l\to+\infty. \tag{5.10}$$

(2) Let us see that $f^*=f$. We shall argue by contradiction. Suppose that $f^*\neq f$. For the continuous injection of $H^m(\Omega';\mathbb{R}^n)$ in $C^0(\overline{\Omega'};\mathbb{R}^n)$ (cf. Theorem 3.2), there exist $\gamma>0$ and a nonempty open $\omega\subset\Omega'$ such that

$$\forall x\in\omega, \quad \langle f^*(x)-f(x)\rangle_{\mathbb{R}^n}>\gamma.$$

Moreover, such injection is compact. Thus, from (5.10) it follows that

$$\exists\, l_0\in\mathbb{N}, \quad \forall\, l\geqslant l_0, \quad \forall\, x\in\omega, \quad \langle\sigma^{d_l}_{\varepsilon_l\tau_l}(x)-f^*(x)\rangle_{\mathbb{R}^n}\leqslant\frac{\gamma}{2}.$$

Hence,

$$\forall\, l\geqslant l_0, \quad \forall x\in\omega, \ \langle\sigma^{d_l}_{\varepsilon_l\tau_l}(x)-f(x)\rangle_{\mathbb{R}^n}\geqslant\langle f^*(x)-f(x)\rangle_{\mathbb{R}^n}-\langle\sigma^{d_l}_{\varepsilon_l\tau_l}(x)-f^*(x)\rangle_{\mathbb{R}^n}>\frac{\gamma}{2}. \tag{5.11}$$

Now, by reasoning as the point (1) it would be proved, for l sufficiently large, that there exists a point $b^{d_l}\in A^{d_l}_1\cap\omega$ such that

$$\langle\sigma^{d_l}_{\varepsilon_l\tau_l}(b^{d_l})-f(b^{d_l})\rangle_{\mathbb{R}^n}=\mathrm{o}(1), \quad l\to+\infty,$$

which contradicts to (5.11). Then, $f^*=f$.

(3) From (5.10) and taking into account that $f^*=f$ and $H^m(\Omega';\mathbb{R}^n)$ is compactly injected in $H^{m-1}(\Omega';\mathbb{R}^n)$ (cf. Theorem 3.3) we have

$$f=\lim_{l\to+\infty}\sigma^{d_l}_{\varepsilon_l\tau_l} \quad \text{in } H^{m-1}(\Omega';\mathbb{R}^n). \tag{5.12}$$

Consequently,

$$\lim_{l\to+\infty}((\sigma^{d_l}_{\varepsilon_l\tau_l},f))_{m-1,\Omega',\mathbb{R}^n}=\|f\|^2_{m-1,\Omega',\mathbb{R}^n}.$$

Using again (5.10) and that $f=f^*$, we obtain

$$\lim_{l\to+\infty}(\sigma^{d_l}_{\varepsilon_l\tau_l},f)_{m,\Omega',\mathbb{R}^n}=\lim_{l\to+\infty}(((\sigma^{d_l}_{\varepsilon_l\tau_l},f))_{m,\Omega',\mathbb{R}^n}-((\sigma^{d_l}_{\varepsilon_l\tau_l},f))_{m-1,\Omega',\mathbb{R}^n})=|f|^2_{m,\Omega',\mathbb{R}^n}. \tag{5.13}$$

Since,

$$\forall\, l\in\mathbb{N}, \quad |\sigma^{d_l}_{\varepsilon_l\tau_l}-f|^2_{m,\Omega',\mathbb{R}^n}=|\sigma^{d_l}_{\varepsilon_l\tau_l}|^2_{m,\Omega',\mathbb{R}^n}+|f|^2_{m,\Omega',\mathbb{R}^n}-2(\sigma^{d_l}_{\varepsilon_l\tau_l},f)_{m,\Omega',\mathbb{R}^n},$$

we deduce, from (5.5) and (5.13), that

$$\lim_{l\to+\infty} |\sigma^{d_l}_{\varepsilon_l\tau_l} - f|_{m,\Omega',\mathbb{R}^n} = 0,$$

which, together with (5.12), implies that

$$\lim_{l\to+\infty} \|\sigma^{d_l}_{\varepsilon_l\tau_l} - f\|_{m,\Omega',\mathbb{R}^n} = 0.$$

(4) Finally, we shall argue by contradiction. Suppose that (5.4) does not hold. Then, there exist a real number $\mu > 0$ and three sequences $(d_{l'})_{l'\in\mathbb{N}}$, $(\varepsilon_{l'})_{l'\in\mathbb{N}}$ and $(\tau_{l'})_{l'\in\mathbb{N}}$, with $\lim_{l'\to+\infty} d_{l'} = 0$, $\varepsilon = \varepsilon(d_{l'})$ and $\tau = \tau(d_{l'})$, such that

$$\forall\, l' \in \mathbb{N}, \quad \|\sigma^{d_{l'}}_{\varepsilon_{l'}\tau_{l'}} - f\|_{m,\Omega',\mathbb{R}^n} \geqslant \mu. \tag{5.14}$$

Now well, the sequence $(\sigma^{d_{l'}}_{\varepsilon_{l'}\tau_{l'}})_{l'\in\mathbb{N}}$ is bounded in $H^m(\Omega'; \mathbb{R}^n)$. Then, the reasoning of points (1)–(3) shows that there exists a subsequence convergent to f, which produces a contradiction with (5.14). In short, (5.4) holds. □

6. Numerical and graphical examples

To test the smoothing method presented in Section 4, we discretize problem (4.6) in a suitable parametric finite element space. Let us consider a set of real positive numbers $\mathscr{H}$ of which 0 is an accumulation point and we suppose that (0, 0) is an accumulation point of $\mathscr{D} \times \mathscr{H}$. Problem (4.6) has been numerically solved using the finite element method. The approach is to replace the Sobolev space $H^m(\Omega', \mathbb{R}^n)$ by the parametric finite dimensional space V_h constructed on a partition $\mathscr{T}_h$ of $\overline{\Omega'}$ such that for all $K \in \mathscr{T}_h$, $K \subset \overline{R_i}$ for certain $i = 1, \ldots, I$.

For $(d, h) \in \mathscr{D} \times \mathscr{H}$, let $\sigma^{dh}_{\varepsilon\tau} \in V_h$ be the unique solution of the discrete problem relative to A^d_1, A^d_2, $L^d f$, ε and τ which is obtained by minimizing the functional $J^d_{\varepsilon\tau}$ in V_h. Let $\{v_1, \ldots, v_Z\}$ be a basis of V_h, then $\sigma^{dh}_{\varepsilon\tau} = \sum_{i=1}^{Z} \beta_i v_i$ where $\beta = (\beta_i)_{1\leqslant i\leqslant Z} \in \mathbb{R}^Z$ is the solution of a linear system of order Z whose coefficient matrix is symmetric positive definite and of band type.

Meanwhile, we have computed two types of error estimates, an estimate of the relative error in norm $|\cdot|_{0,\Omega',\mathbb{R}^n}$ and an estimate of the relative tangent error. The latest error estimate is the angle middle formed by the tangent vector subspaces of the approximated surface and the original one. Both error estimate are given through the expressions

$$E_{\mathrm{r}} = \left(\frac{\sum_{i=1}^{10\,000} \langle \sigma^{dh}_{\varepsilon\tau}(x_i) - f(x_i) \rangle_n^2}{\sum_{i=1}^{10\,000} \langle f(x_i) \rangle_n^2} \right)^{1/2}, \quad E_{\mathrm{t}} = \frac{\sum_{i=1}^{10\,000} \mathrm{Ang}(T_{x_i}(\sigma^{dh}_{\varepsilon\tau}), T_{x_i}(f))}{10\,000},$$

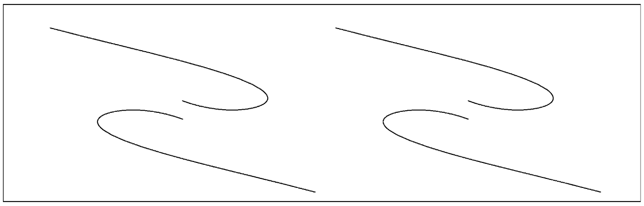

Fig. 1. Section 6.1. From left to right: the graphs of the original curve and its approximation one parameterized by $\sigma_{\varepsilon\tau}^{dh}$ with $N_1 = 61$, $N_2 = 201$, $\varepsilon = 10^{-4}$, $\tau = 10^{-6}$, $E_{\mathrm{r}} = 0.00127869$, $E_{\mathrm{t}} = 0.00388529$.

where $\{x_i\}_{1 \leqslant i \leqslant 10\,000}$ is a set of 10 000 arbitrary points of $\overline{\Omega}$ and $\mathrm{Ang}(T_{x_i}(\sigma_{\varepsilon\tau}^{dh}),\ T_{x_i}(f))$ is the angle that form $T_{x_i}(\sigma_{\varepsilon\tau}^{dh})$, tangent space of $\sigma_{\varepsilon\tau}^{dh}$ at x_i, and $T_{x_i}(f)$, tangent space of f at x_i.

6.1. The curve case

To test our smoothing method for the curve case, we consider the curve Υ_0 parameterized by the function

$$f = (f_1, f_2) : [-6, 6] \to \mathbb{R}^2,$$

where

$$f_1(u) = u \cos\left(\frac{u^2}{5\pi}\right),$$

$$f_2(u) = 2\,\mathrm{sign}(u)\,\frac{(|u| - 1.5)^2}{10},$$

The corresponding graphic is shown in Fig. 1 (left side). The following set of data have been taken:

- $\overline{\Omega} = [-6, 6]$;
- $F = \{0\}$;
- $\Omega' = \Omega \backslash \overline{F}$.

In this case, for $m = n = 2$, problem (4.6) has been numerically solved using the parametric finite element space V_h constructed on a partition $\mathscr{T}_h$ made up by six equal intervals from the generic finite element of Hermite of class C^1.

Fig. 1 shows an approximation curve (right side) obtained for some given data.

6.2. The surface case

We consider the surface Υ_0 parameterized by the function

$$f = (f_1, f_2, f_3) : [-6, 6] \times [-6, 6] \to \mathbb{R}^3,$$

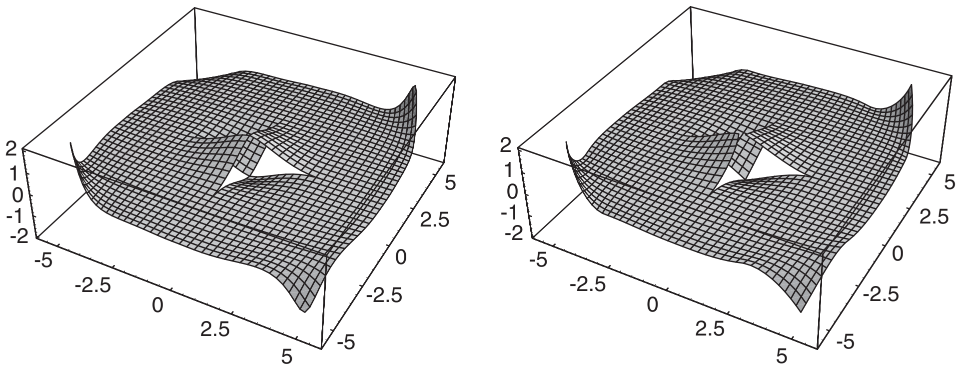

Fig. 2. Section 6.2. Surface Υ_0 and its approximation parameterized by $\sigma_{\varepsilon\tau}^{dh}$ with $N_1=250$, $N_2=0$, $\varepsilon=10^{-5}$, $\tau=0$, $E_{\mathrm{r}}=0.00454581$, $E_{\mathrm{t}}=0.0284015$, from left to right, respectively.

where

$$f_1(u,v)=u\,\cos\left(\frac{u^2+v^2}{35\pi}\right),$$
$$f_2(u,v)=v\,\cos\left(\frac{u^2+v^2}{35\pi}\right),$$
$$f_3(u,v)=\begin{cases} f_{3.1} & \text{if } |u|<3,\ |v|<3,\\ f_{3.2} & \text{if } |u|\geqslant 3 \text{ or} |v|\geqslant 3,\end{cases}$$

$$f_{3.1}=\operatorname{sign}(u)\operatorname{sign}(v)\sin\left(\pi\frac{(|u|-3)^2}{20}\right)\sin\left(\pi\frac{(|v|-3)^2}{20}\right),$$
$$f_{3.2}=\operatorname{sign}(u)\operatorname{sign}(v)\sin\left(\frac{\pi}{219\,952}u^2v^2(|u|-3)^2(|v|-3)^2\right).$$

The corresponding graphic is shown in Fig. 2 (left side). The following set of data have been taken:

- $\overline{\Omega}=[-6,6]\times[-6,6]$;
- $F=\{|x|\leqslant 3,\ y=0\}\cup\{x=0,\ |y|\leqslant 3\}$;
- $\Omega'=\Omega\backslash\overline{F}$.

In this example, for $m=3$ and $n=3$, the partition $\mathscr{T}_h$ is made up by 4×4 equal squares from the generic finite element of Bogner–Fox–Schmidt of class C^1.

The efficiency of our smoothing method presented in this paper is proved in this subsection as follows: graphically, the graphs of the original surface given in Fig. 2 and the approximation one given in Fig. 3 (right side) are similar. Furthermore, numerically in Fig. 3 the orders of the obtained relative errors are considerable.

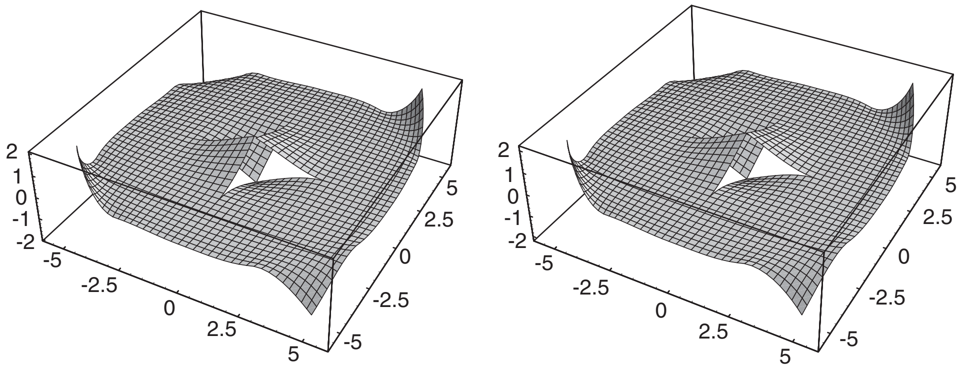

Fig. 3. Section 6.2. Two approximation surfaces of Υ_0 parameterized by $\sigma_{\varepsilon\tau}^{dh}$ with, respectively, $N_1=250$, $N_2=400$, $\varepsilon=10^{-5}$, $\tau=10^{-3}$, $E_{\mathrm{r}}=0.0044139$, $E_{\mathrm{t}}=0.019088$ and $N_1=900$, $N_2=400$, $\varepsilon=10^{-7}$, $\tau=10^{-3}$, $E_{\mathrm{r}}=0.00357$, $E_{\mathrm{t}}=0.0176559$, from left to right, respectively.

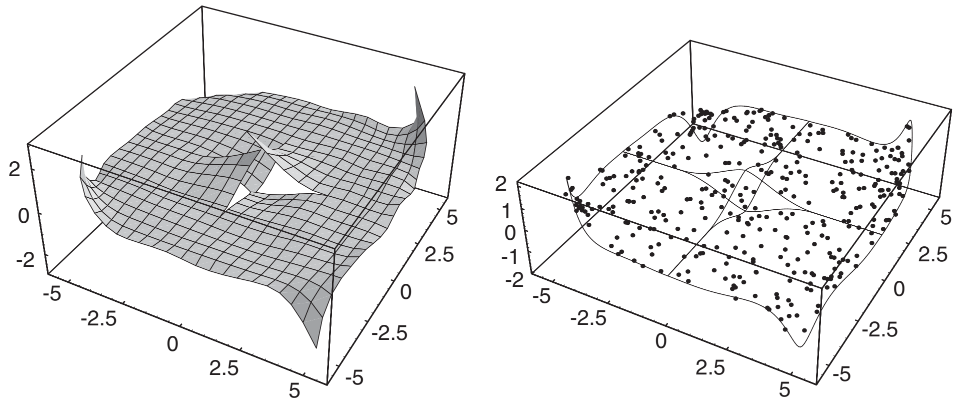

Fig. 4. Section 6.3. From left to right: the graph of an approximation surface of Υ_0 parameterized by $\sigma_{\varepsilon\tau}^{dh}$ with $N_1=900$, $N_2=0$, $\varepsilon=10^{-5}$, $\tau=0$, $E_{\mathrm{r}}=0.0202551$, $E_{\mathrm{t}}=0.0927968$, and the graph of the corresponding Lagrangean data points. Here the variance of v^d is $\eta=0.15$.

6.3. Using noisy data

We conserve the same notations of the previous subsection and we suppose now that the data are noisy, that is "white noise" hypothesis is satisfied. For each $h \in \mathscr{H}$ and $d \in \mathscr{D}$, let $v^d=(v_a^d)_{a\in A^d}$ be a vector "error" of $\mathbb{R}^{N_1}$.

By proceeding as the paper [10] for the noisy data, we introduce the hypothesis: for each $d \in \mathscr{D}$, v^d is a white noisy, i.e., v^d is a Gaussian vector of independent arbitrary variables of mean 0 and variance $\eta^2>0$ that are distributed of identical form.

Let $\sigma_{\varepsilon\tau}^{dh}$ be the solution of the discrete problem in V_h for $L^d f + v^d$ instead of $L^d f$. For a convergence result of $\sigma_{\varepsilon\tau}^{dh}$ to f, see [7] for an analogous way.

Figs. 4 and 5 show some approximation surfaces of the original surface Υ_0 from some Lagrangean noisy data.

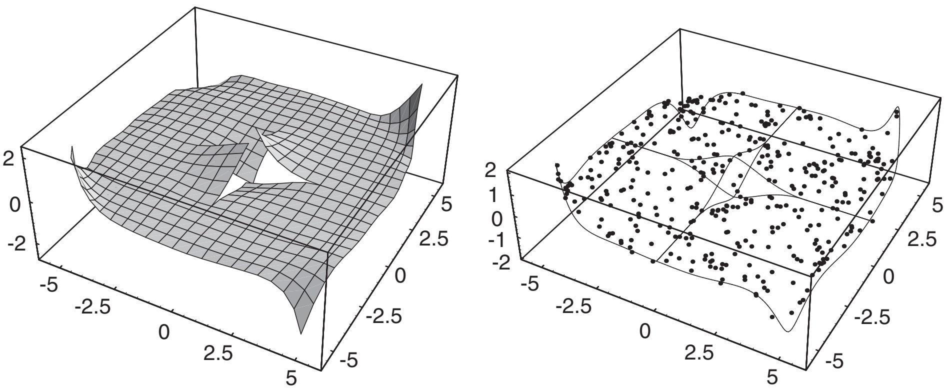

Fig. 5. Section 6.3. From left to right: the graph of an approximation surface of Υ_0 parameterized by $\sigma_{\varepsilon\tau}^{dh}$ with $N_1 = 900$, $N_2 = 0$, $\varepsilon = 10^{-5}$, $\tau = 0$, $E_\mathrm{r} = 0.0127351$, $E_\mathrm{t} = 0.0732392$ and the graph of the corresponding Lagrangean data points. In this case, the variance of ν^d is $\eta = 0.1$.

7. Conclusion

We conclude that our methodology is a well general approximation method of curves and surfaces for both exact and noisy data.

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