Journal of Computational and Applied Mathematics 229 (2009) 158–167

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New periodic and soliton solutions by application of Exp-function method for nonlinear evolution equations

A. Borhanifar [a,*], M.M. Kabir [b]

[a] Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran

[b] Department of Mechanical Engineering, University of Mohaghegh Ardabili, Ardabil, Iran

ARTICLE INFO

Article history:
Received 20 February 2008
Received in revised form 19 May 2008

Keywords:
Exp-function method
Kaup–Kupershmidt equation
$(2+1)$-dimensional Potential Kadomtsev–Petviashvili (PKP) equation
Traveling wave solution
Periodic solutions
Soliton

ABSTRACT

In this letter, the Kaup–Kupershmidt, $(2+1)$-dimensional Potential Kadomtsev–Petviashvili (shortly PKP) equations are presented and the Exp-function method is employed to compute an approximation to the solution of nonlinear differential equations governing the problem. It has been attempted to show the capabilities and wide-range applications of the Exp-function method. This method can be used as an alternative to obtain analytic and approximate solution of different types of differential equations applied in engineering mathematics.

1. Introduction

Nonlinear phenomena play important roles in applied mathematics, physics and also in engineering problems in which each parameter varies depending on different factors. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts that are not simply understood through common observations. Moreover, obtaining exact solutions for these problems is a great purpose that has been quite untouched. However, in recent years, analytical solutions [1,2] have been developed considerably to be used for nonlinear partial equations. Recently Ji-Huan He [3–14] introduced some new method such as the variation iteration method (VIM), homotopy perturbation method (HPM) and Exp-function method to solve these equations. The Exp-function method is very strong for solving high nonlinearity of nonlinear equations. Other authors such as Zhu [15,16] and Zhang [17] have been working in this field.

In this study, the Exp-function method is used to derive new solitary and periodic solutions for a form of fifth order nonlinear KdV (fKdV), namely, Kaup–kupershmidt, and also for potential kadomtsev–petviashvili (PKP) equations.

The fKdV equation, describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice and has wide applications in quantum mechanics, nonlinear optics, plasma physics and fluid dynamics.

Recently, several investigations on the solutions of the Kaup–Kupershmidt equation have been done. Wazwaz used the tanh method and the extended the tanh method for finding solitonary solutions of this equation [18,19]. M. Al-Mdallal and Syam applied the sine–cosine method to obtain solitonary and periodic Solutions of the generalized fifth order nonlinear equation [20], also, Chun used the Exp-function method for finding solutions of another form of the fKdV equation [22].

Some methods, such as, the Adomian decomposition method (ADM) [23], the tanh method [24] and improved tanh function method [25], were applied for solving the PKP equation.

* Corresponding author. Tel.: +98 451 5514703; fax: +98 451 5514701.
E-mail addresses: borhani@uma.ac.ir (A. Borhanifar), kabir.mehdi@gmail.com (M.M. Kabir).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2008.10.052

2. Basic idea of Exp-function method

We first consider a nonlinear equation of form

$$N(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \ldots) = 0 \tag{1}$$

where N, is a nonlinear function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformations. Introducing a complex variation η defined as

$$u = u(\eta), \quad \eta = kx + \omega t, \tag{2}$$

where k, ω are constants to be determined later. Then Eq. (1) reduces to the ODE:

$$N(u, \omega u', ku', k^2u'', \omega^2 u'', k\omega u'', \ldots) = 0. \tag{3}$$

And then solution of $u(\eta)$ is a form of

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} = \frac{a_c \exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{a_p \exp(p\eta) + \cdots + a_{-q}\exp(-q\eta)} \tag{4}$$

where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants.

3. Application of Exp-function method

3.1. The kaup–kupershmidt equation

To illustrate the basic idea of the Exp-function method, we first consider the Kaup–Kupershmidt equation [20,21] in the form

$$u_{xxxxx} + u_t + 45u_x u^2 - \frac{75}{2}u_{xx}u_x - 15uu_{xxx} = 0. \tag{5}$$

Introducing a complex variation η defined as Eq. (2), and then Eq. (5) becomes an ordinary differential equation, which is a form of

$$\omega u' + 45ku^2u' - \frac{75}{2}k^3u'u'' - 15k^3uu''' + k^5u''''' = 0. \tag{6}$$

In order to determine values of c and p, we balance the linear term of the highest order u''''' with the highest order nonlinear term $u'''u$ in Eq. (6), we have

$$u''''' = \frac{c_1 \exp((c+31p)\eta) + \cdots}{c_2 \exp(32p\eta) + \cdots}, \tag{7}$$

and

$$uu''' = \frac{c_3 \exp((c+7p)\eta) + \cdots}{c_4 \exp(8p\eta) + \cdots} \times \frac{\exp(c\eta)}{\exp(p\eta)} = \frac{c_3 \exp((2c+30p)\eta) + \cdots}{c_4 \exp(32p\eta) + \cdots}, \tag{8}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs. (7) and (8), we have

$$c + 31p = 2c + 30p, \tag{9}$$

which leads to the result

$$p = c. \tag{10}$$

Similarly to determine values of d and q,we balance the linear term of lowest order in Eq. (6)

$$u''''' = \frac{\cdots + d_1 \exp(-(d+31q)\eta)}{\cdots + d_2 \exp(-32q\eta)}, \tag{11}$$

and

$$u'''u = \frac{\cdots + d_3 \exp(-(d+7q))\eta}{\cdots + d_4 \exp(-8q\eta)} \times \frac{\exp(-d\eta)}{\exp(-q\eta)} = \frac{\cdots + d_3 \exp(-(2d+30q))\eta}{\cdots + d_4 \exp(-32q\eta)}, \tag{12}$$

where d_i are determine coefficients only for simplicity. Balancing lowest order of Exp-function in Eqs. (11) and (12), we have

$$30q + 2d = d + 31q, \tag{13}$$

which leads to the result

$$q = d. \tag{14}$$

3.1.1. Case A: $p = c = 1, q = d = 1$

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (4) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{15}$$

Substituting Eq. (15) in to Eq. (6) and by the help of Maple, we have

$$\frac{1}{A}[c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta)] = 0, \tag{16}$$

where we have

$$A = (\exp(\eta) + b_0 + b_{-1} \exp(-\eta))^6 \tag{17}$$

and c_n are coefficients of $\exp(n\eta)$. Equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}, k$ and ω. Solving the system of algebraic equations with the aid of Maple, we obtain:

Case 1.

$$\begin{aligned} &\omega = -11k^5, \quad b_{-1} = \frac{1}{4}b_0^2, \quad a_1 = \frac{2}{3}k^2, \quad k = k, \\ &a_{-1} = \frac{1}{6}k^2 b_0^2, \quad b_0 = b_0, \quad a_0 = -\frac{10}{3}k^2 b_0. \end{aligned} \tag{18}$$

Inserting Eq. (18) into (15), one admits to the generalized solitary wave solution of Eq. (5) as

$$\begin{aligned} u(x,t) &= \frac{\frac{2}{3}k^2 \exp(kx - 11k^5 t) - \frac{10}{3}k^2 b_0 + \frac{1}{6}k^2 b_0^2 \exp(-(kx - 11k^5 t))}{\exp(kx - 11k^5 t) + b_0 + \frac{1}{4}b_0^2 \exp(-(kx - 11k^5 t))} \\ &= \frac{2}{3}k^2 - \frac{16 b_0 k^2}{4\exp(kx - 11k^5 t) + 4b_0 + b_0^2 \exp(-(kx - 11k^5 t))}. \end{aligned} \tag{19}$$

In case k and ω are imaginary numbers, the obtained solitary solution (19) reduces to the periodic solution. We write $k = \mathrm{i}K$ and using the transformation

$$\begin{aligned} \exp(kx - 11k^5 t) &= \cos(Kx - 11K^5 t) + \mathrm{i}\sin(Kx - 11K^5 t), \\ \exp(kx - 11k^5 t) &= \cos(Kx - 11K^5 t) - \mathrm{i}\sin(Kx - 11K^5 t). \end{aligned} \tag{20}$$

Substituting Eq. (20) into (19) results in a periodic solution

$$u = -\frac{2}{3}K^2 - \frac{16 b_0 K^2}{(4 + b_0^2)\cos(Kx - 11K^5 t) + 4b_0 + (4 - b_0^2)\mathrm{i}\sin(Kx - 11K^5 t)}. \tag{21}$$

If we search for a periodic solution or compact-like solution, the imaginary part in Eq. (21) must be zero, that requires that

$$4 - b_0^2 = 0. \tag{22}$$

From Eq. (22) we obtain

$$b_0 = \pm 2. \tag{23}$$

Substituting $b_0 = 2$ into Eq. (21) results

$$u(x,t) = \frac{32K^2}{8\cos(Kx - 11K^5 t) + 8} - \frac{2}{3}K^2 = \frac{2K^2}{\cos^2\left(\frac{K}{2}x - \frac{11}{2}K^5 t\right)} - \frac{2}{3}K^2. \tag{24}$$

And substituting $b_0 = -2$ into Eq. (21) results

$$u(x,t) = \frac{-32K^2}{8\cos(Kx - 11K^5 t) - 8} - \frac{2}{3}K^2 = \frac{2K^2}{\sin^2\left(\frac{K}{2}x - \frac{11}{2}K^5 t\right)} - \frac{2}{3}K^2, \tag{25}$$

where as $k = \mathrm{i}K$, we write $K = -\mathrm{i}k$ and using the transformation into Eqs. (24) and (25)

$$u(x,t) = -\frac{2k^2}{\cos^2\left(\mathrm{i}\left(\frac{1}{2}kx - \frac{11}{2}k^5 t\right)\right)} + \frac{2}{3}k^2 = -2k^2 \operatorname{sech}^2\left(\frac{k}{2}x - \frac{11}{2}k^5 t\right) + \frac{2}{3}k^2 \tag{26}$$

$$u(x,t) = -\frac{2k^2}{\sin^2\left(\mathrm{i}\left(\frac{1}{2}kx - \frac{11}{2}k^5 t\right)\right)} + \frac{2}{3}k^2 = 2k^2 \operatorname{csch}^2\left(\frac{k}{2}x - \frac{11}{2}k^5 t\right) + \frac{2}{3}k^2. \tag{27}$$

If we choose $k = 4\mu$, our solution, Eq. (27), turns out to be Wazwaz's solution as expressed in Eq. (59) into [18]

Case 2.

$$\begin{aligned}&\omega=-\frac{1}{16}k^5, \qquad b_{-1}=\frac{1}{4}b_0^2, \qquad a_1=\frac{1}{12}k^2, \qquad k=k,\\ &a_{-1}=\frac{1}{48}k^2b_0^2, \qquad b_0=b_0, \qquad a_0=-\frac{15}{12}k^2b_0.\end{aligned} \tag{28}$$

Inserting Eq. (28) into (15), one admits to the generalized solitary wave solution of Eq. (5) as follows:

$$\begin{aligned}u(x,t)&=\frac{\frac{1}{12}k^2\exp\left(kx-\frac{1}{16}k^5t\right)-\frac{5}{12}k^2b_0+\frac{1}{48}k^2b_0^2\exp\left(-\left(kx-\frac{1}{16}k^5t\right)\right)}{\exp\left(kx-\frac{1}{16}k^5t\right)+b_0+\frac{1}{4}b_0^2\exp\left(-\left(kx-\frac{1}{16}k^5t\right)\right)}\\ &=\frac{1}{12}k^2-\frac{2b_0k^2}{4\exp\left(kx-\frac{1}{16}k^5t\right)+4b_0+b_0^2\exp\left(-\left(kx-\frac{1}{16}k^5t\right)\right)}.\end{aligned} \tag{29}$$

In case k and ω are imaginary numbers, the solitary solution obtained (29) reduces to the periodic solution. We write $k=\mathrm{i}K$ and using the transformation

$$\begin{aligned}\exp\left(kx-\frac{1}{16}k^5t\right)&=\cos\left(Kx-\frac{1}{16}K^5t\right)+\mathrm{i}\sin\left(Kx-\frac{1}{16}K^5t\right),\\ \exp\left(kx-\frac{1}{16}k^5t\right)&=\cos\left(Kx-\frac{1}{16}K^5t\right)-\mathrm{i}\sin\left(Kx-\frac{1}{16}K^5t\right).\end{aligned} \tag{30}$$

Substituting Eq. (30) into (29) results in a periodic solution

$$u=-\frac{1}{12}K^2+\frac{2b_0K^2}{(4+b_0^2)\cos\left(Kx-\frac{1}{16}K^5t\right)+4b_0+(4-b_0^2)\mathrm{i}\sin\left(Kx-\frac{1}{16}K^5t\right)}. \tag{31}$$

If we search for a periodic solution or compact-like solution, the imaginary part in Eq. (31) must be zero, Substituting $b_0=2$ into Eq. (31) results

$$u(x,t)=\frac{4K^2}{8\cos\left(Kx-\frac{1}{16}K^5t\right)+8}-\frac{1}{12}K^2=\frac{K^2}{4\cos^2\left(\frac{K}{2}x-\frac{1}{32}K^5t\right)}-\frac{1}{12}K^2, \tag{32}$$

and substituting $b_0=-2$ into Eq. (31) results

$$u(x,t)=\frac{-4K^2}{8\cos\left(Kx-\frac{1}{16}K^5t\right)-8}-\frac{1}{12}K^2=\frac{K^2}{4\sin^2\left(\frac{K}{2}x-\frac{1}{32}K^5t\right)}-\frac{1}{12}K^2, \tag{33}$$

where as $k=\mathrm{i}K$, we write $K=-\mathrm{i}k$ and using the transformation into Eqs. (32) and (33)

$$u(x,t)=-\frac{k^2}{4\cos^2\left(\mathrm{i}\left(\frac{1}{2}kx-\frac{1}{32}k^5t\right)\right)}+\frac{1}{12}k^2=-\frac{1}{4}k^2\,\mathrm{sech}^2\left(\frac{k}{2}x-\frac{1}{32}k^5t\right)+\frac{1}{12}k^2. \tag{34}$$

$$u(x,t)=-\frac{k^2}{4\sin^2\left(\mathrm{i}\left(\frac{1}{2}kx-\frac{1}{32}k^5t\right)\right)}+\frac{1}{12}k^2=\frac{1}{4}k^2\mathrm{csch}^2\left(\frac{k}{2}x-\frac{1}{32}k^5t\right)+\frac{1}{12}k^2. \tag{35}$$

If we choose $k=4\mu$, our solution, Eq. (35), turns out to be wazwaz's solution as expressed in Eq. (58) into [18]

3.1.2. *Case B:* $p=c=2$, $q=d=2$

Since the values of c and d can be freely chosen, we can $p=c=2$ and $q=d=2$, the trial function, Eq. (4) becomes

$$u(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)}{b_2\exp(2\eta)+b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)+b_{-2}\exp(-2\eta)}. \tag{36}$$

There are some free parameters in Eq. (36), we set $b_2=1$, $b_1=b_{-1}=0$ for simplicity, the trial function, Eq. (36) is simplified as follows:

$$u(\eta)=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)}{\exp(2\eta)+b_0+b_{-2}\exp(-2\eta)}. \tag{37}$$

By the same manipulation as illustrated above, we have the following sets of solutions:

Case 1.

$$\begin{aligned} &\omega = -k^5, \qquad b_{-2} = \frac{9}{100}\frac{a_0^2}{k^4}, \qquad a_{-2} = \frac{3}{100}\frac{a_0^2}{k^2}, \qquad a_1 = 0, \qquad k = k, \\ &a_{-1} = 0, \qquad a_0 = a_0 \qquad a_2 = \frac{k^2}{3}, \qquad b_0 = -\frac{3}{5}\frac{a_0}{k^2}. \end{aligned} \tag{38}$$

Substituting Eq. (38) into (37), we get the generalized solitary wave solution of Eq. (5) as follows:

$$u = \frac{\frac{k^2}{3}\exp(2(kx-k^5t)) + a_0 + \frac{3}{100}\frac{a_0^2}{k^2}\exp(-2(kx-k^5t))}{\exp(2(kx-k^5t)) - \frac{3}{5}\frac{a_0}{k^2} + \frac{9}{100}\frac{a_0^2}{k^4}\exp(-2(kx-k^5t))}. \tag{39}$$

If we set $a_0 = -\frac{10}{3}k^2$, then Eq. (39), can be easily converted to

$$u = -\frac{2}{3}k^2 + k^2\tanh^2(kx-k^5t). \tag{40}$$

If we set $a_0 = \frac{10}{3}k^2$, then Eq. (39), can be easily converted to

$$u = -\frac{2}{3}k^2 + k^2\coth^2(kx-k^5t). \tag{41}$$

Comparing our results, Eqs. (40) and (41), with Wazwaz's results in (83), (85) into [19], it can be seen that the results are same. Here k acts as the role of μ in [19].

As illustrated in the previous cases, the obtained solitary solutions can be converted into periodic solutions or compact-like solutions if k is chosen as an imaginary number. For case ($p = c = 2, q = d = 2$), we only discuss the solution given by Eq. (39).

If $k = \mathrm{i}K$, then it becomes

$$u = \frac{-\left(\frac{K^2}{3} + \frac{3}{100}\frac{a_0^2}{K^2}\right)\cos(2(Kx-K^5t)) + a_0 + \mathrm{i}\left(\frac{-K^2}{3} + \frac{3}{100}\frac{a_0^2}{K^2}\right)\sin(2(Kx-K^5t))}{\left(1 + \frac{9}{100}\frac{a_0^2}{K^4}\right)\cos(2(Kx-K^5t)) + \frac{3}{5}\frac{a_0}{K^2} + \mathrm{i}\left(1 - \frac{9}{100}\frac{a_0^2}{K^4}\right)\sin(2(Kx-K^5t))}. \tag{42}$$

Elimination of the imaginary parts requires that

$$a_0 = \pm\frac{10}{3}K^2. \tag{43}$$

We, therefore, obtain from Eq. (42) the periodic solutions

$$u = K^2\sec^2(Kx-K^5t) - \frac{K^2}{3}, \tag{44}$$

$$u = K^2\csc^2(Kx-K^5t) - \frac{K^2}{3}. \tag{45}$$

Case 2.

$$\begin{aligned} &\omega = -176k^5, \qquad b_{-2} = \frac{b_0^2}{4}, \qquad a_{-2} = \frac{2}{3}b_0^2k^2, \qquad a_1 = 0, \qquad k = k, \\ &a_{-1} = 0, \qquad b_0 = b_0, \qquad a_0 = -\frac{40}{3}k^2b_0 \qquad a_2 = \frac{8}{3}k^2. \end{aligned} \tag{46}$$

By using Eq. (46) into (37), we obtain

$$u = \frac{\frac{8}{3}k^2\exp(2\eta) - \frac{40}{3}k^2b_0 + \frac{2}{3}k^2b_0^2\exp(-2\eta)}{\exp(2\eta) + b_0 + \frac{b_0^2}{4}\exp(-2\eta)} \tag{47}$$

where $\eta = kx - 176k^5t$.

If we choose $b_0 = \pm 2$, then Eq. (47), can be easily converted to

$$u = -\frac{16}{3}k^2 + 8k^2\tanh^2(kx-176k^5t), \tag{48}$$

$$u = -\frac{16}{3}k^2 + 8k^2\coth^2(kx-176k^5t). \tag{49}$$

Comparing our results, Eqs. (48) and (49), with Wazwaz's solutions in (84), (86) into [19], it can be seen that the results are same.

Case 3.

$$\begin{aligned}&\omega=-k^5,\qquad b_{-2}=\frac{b_0^2}{4},\qquad a_{-2}=\frac{1}{12}b_0^2k^2,\qquad a_1=0,\qquad k=k,\\&a_{-1}=0,\qquad b_0=b_0,\qquad a_0=-\frac{5}{3}k^2b_0\qquad a_2=\frac{k^2}{3}.\end{aligned}\tag{50}$$

By using Eq. (50) into (37), we have

$$u=\frac{\frac{k^2}{3}\exp(2(kx-k^5t))-\frac{5}{3}k^2b_0+\frac{1}{12}b_0^2k^2\exp(-2(kx-k^5t))}{\exp(2(kx-k^5t))+b_0+\frac{b_0^2}{4}\exp(-2(kx-k^5t))}.\tag{51}$$

If we set $b_0=\pm 2$, we can obtain solutions (40) and (41).

Case 4.

$$\begin{aligned}&\omega=-\frac{1}{16}k^5,\qquad b_{-2}=\frac{a_1^4}{k^8},\qquad a_{-2}=\frac{1}{12}\frac{a_1^4}{k^6},\qquad a_1=a_1,\qquad k=k,\\&a_{-1}=\frac{a_1^3}{k^4},\qquad b_0=\frac{-2a_1^2}{k^4},\qquad a_0=\frac{11}{6}\frac{a_1^2}{k^2}\qquad a_2=\frac{1}{12}k^2.\end{aligned}\tag{52}$$

Substituting Eq. (52) into (37), we obtain

$$u=\frac{\frac{k^2}{12}\exp(2\eta)+a_1\exp(\eta)+\frac{11}{6}\frac{a_1^2}{k^2}+\frac{a_1^3}{k^4}\exp(-\eta)+\frac{1}{12}\frac{a_1^4}{k^6}\exp(-2\eta)}{\exp(2\eta)-\frac{2a_1^2}{k^4}+\frac{a_1^4}{k^8}\exp(-2\eta)}\tag{53}$$

where $\eta=kx-\frac{k^5}{16}t$.

Case 5.

$$\begin{aligned}&\omega=-11k^5,\qquad b_{-2}=\frac{1}{4096}\frac{a_1^4}{k^8},\qquad a_{-2}=\frac{1}{6144}\frac{a_1^4}{k^6},\qquad a_1=a_1,\qquad k=k,\\&a_{-1}=\frac{1}{64}\frac{a_1^3}{k^4},\qquad b_0=\frac{-1}{32}\frac{a_1^2}{k^4},\qquad a_0=\frac{11}{48}\frac{a_1^2}{k^2}\qquad a_2=\frac{2}{3}k^2.\end{aligned}\tag{54}$$

Substituting Eq. (54) into (37), we have

$$u=\frac{\frac{2}{3}k^2\exp(2\eta)+a_1\exp(\eta)+\frac{11}{48}\frac{a_1^2}{k^2}+\frac{1}{64}\frac{a_1^3}{k^4}\exp(-\eta)+\frac{1}{6144}\frac{a_1^4}{k^6}\exp(-2\eta)}{\exp(2\eta)-\frac{1}{32}\frac{a_1^2}{k^4}+\frac{1}{4096}\frac{a_1^4}{k^8}\exp(-2\eta)}.\tag{55}$$

3.2. $(2+1)$-*dimensional Potential Kadomtsev–Petviashvili (PKP) equation*

Now, let us consider the PKP equation [23,25] in the form

$$\frac{1}{4}u_{xxxx}+\frac{3}{2}u_xu_{xx}+\frac{3}{4}u_{yy}+u_{xt}=0.\tag{56}$$

Using the transformation

$$\eta=kx+ly+\omega t,\qquad u=u(\eta),\tag{57}$$

then Eq. (56) becomes an ordinary differential equation, which is a form of

$$\frac{1}{4}k^4u''''+\frac{3}{2}k^3u'u''+\left(\frac{3}{4}l^2+k\omega\right)u''=0.\tag{58}$$

In order to determine values of c and p we balance the linear term of the highest order u'''' with the highest order nonlinear term $u'u''$ in Eq. (58), we have

$$u''''=\frac{c_1\exp((c+15p)\eta)+\cdots}{c_2\exp(16p\eta)+\cdots},\tag{59}$$

and

$$u'u'' = \frac{c_3 \exp((c+3p)\eta) + \cdots}{c_4 \exp(4p\eta) + \cdots} \times \frac{\exp((c+p)\eta)}{\exp(2p\eta)} = \frac{c_3 \exp((2c+14p)\eta) + \cdots}{c_4 \exp(16p\eta) + \cdots}, \tag{60}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp function in Eqs. (59) and (60), we have

$$15p + c = 2c + 14p, \tag{61}$$

which leads to the result

$$p = c. \tag{62}$$

Similarly to determine values of d and q, we balance the linear term of lowest order in Eq. (58)

$$u'''' = \frac{\cdots + d_1 \exp(-(d+15q)\eta)}{\cdots + d_2 \exp(-16q\eta)}, \tag{63}$$

and

$$u'u'' = \frac{\cdots + d_3 \exp(-(d+3q))\eta}{\cdots + d_4 \exp(-4q\eta)} \times \frac{\exp(-(d+q)\eta)}{\exp(-2q\eta)} = \frac{\cdots + d_3 \exp(-(2d+14q))\eta}{\cdots + d_4 \exp((-16q)\eta)} \tag{64}$$

where d_i are determine coefficients only for simplicity. Balancing lowest order of Exp-function in Eqs. (63) and (64), we have

$$15q + d = 2d + 14q, \tag{65}$$

which leads to the result

$$q = d. \tag{66}$$

3.2.1. *Case A:* $p = c = 1, q = d = 1$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (4) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \tag{67}$$

substituting Eq. (67) in to Eq. (58) and by the help of Maple, we have

$$\begin{aligned} &\frac{1}{A}[c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) \\ &\quad + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta)] = 0, \end{aligned} \tag{68}$$

where we have

$$A = (\exp(\eta) + b_0 + b_{-1} \exp(-\eta))^5 \tag{69}$$

and c_n are coefficients of $\exp(n\eta)$. Equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}, k, l$ and ω. Solving the system of algebraic equations with the aid of Maple, we obtain:

$$\begin{aligned} &\omega = -\frac{3l^2 + k^4}{4k}, \qquad a_1 = a_1, \\ &a_{-1} = -\frac{4a_1 b_0^2 k^2 - 4a_0 b_0 k^2 - 4a_1^2 b_0^2 k + 6a_1 a_0 b_0 k + a_1 a_0^2 + a_1^3 b_0^2 - 2a_1^2 a_0 b_0 - 2a_0^2 k}{4k^2}, \\ &b_0 = b_0, \qquad a_0 = a_0, \qquad b_{-1} = -\frac{-2ka_1 b_0^2 + 2ka_0 b_0 + a_0^2 + a_1^2 b_0^2 - 2a_1 a_0 b_0}{4k^2}, \qquad k = k, \qquad l = l. \end{aligned} \tag{70}$$

Inserting Eq. (70) into (67) admits to the generalized solitary wave solution of Eq. (56) as follows:

$$\begin{aligned} u &= \frac{a_1 \exp\left(kx + ly - \frac{k^4+3l^2}{4k} t\right) + a_0}{\exp\left(kx + ly - \frac{k^4+3l^2}{4k} t\right) + b_0} \\ &\quad \times \frac{-\frac{1}{4k^2}(4a_1 b_0^2 k^2 - 4a_0 b_0 k^2 - 4a_1^2 b_0^2 k + 6a_1 a_0 b_0 k + a_1 a_0^2 + a_1^3 b_0^2 - 2a_0 b_0 a_1^2 - 2ka_0^2) \exp\left(-\left(kx + ly - \frac{k^4+3l^2}{4k} t\right)\right)}{+\frac{2a_1 b_0^2 k - 2a_0 b_0 k - a_0^2 - a_1^2 b_0^2 + 2a_1 a_0 b_0}{4k^2} \exp\left(-\left(kx + ly - \frac{k^4+3l^2}{4k} t\right)\right)}, \\ &= a_1 - \frac{4k^2 a_1 b_0 - 4k^2 a_0 + 2k(2a_1 b_0^2 k - 2a_0 b_0 k - a_1^2 b_0^2 + 2a_1 a_0 b_0 - a_0^2) \exp\left(-\left(kx + ly - \frac{k^4+3l^2}{4k} t\right)\right)}{4k^2 \exp\left(kx + ly - \frac{k^4+3l^2}{4k} t\right) + 4b_0 k^2 + (2ka_1 b_0^2 - 2ka_0 b_0 - a_0^2 - a_1^2 b_0^2 + 2a_1 a_0 b_0) \exp\left(-\left(kx + ly - \frac{k^4+3l^2}{4k} t\right)\right)}. \end{aligned} \tag{71}$$

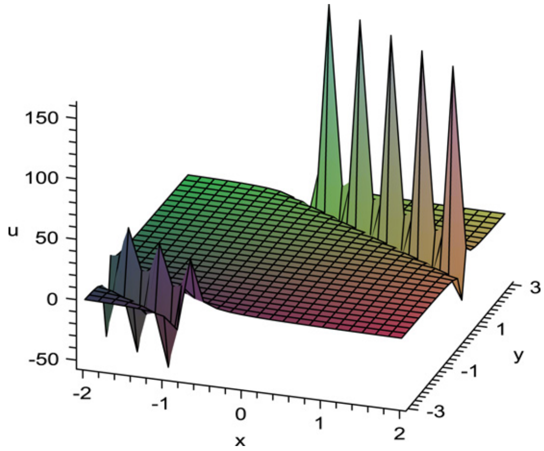

Fig. 1. One of periodic solutions (74).

In case k, l and ω are imaginary numbers, the obtained solitary solution (71) reduces to the periodic solution. We write $k = \mathrm{i}K$, $l = \mathrm{i}L$ and using the transformation

$$\begin{aligned}\exp\left(kx + ly - \frac{3l^2 + k^4}{4k}t\right) &= \cos\left(Kx + Ly + \frac{K^4 - 3L^2}{4K}t\right) + \mathrm{i}\sin\left(Kx + Ly + \frac{K^4 - 3L^2}{4K}t\right),\\ \exp\left(-\left(kx + ly - \frac{3l^2 + k^4}{4k}t\right)\right) &= \cos\left(Kx + Ly + \frac{K^4 - 3L^2}{4K}t\right) - \mathrm{i}\sin\left(Kx + Ly + \frac{K^4 - 3L^2}{4K}t\right).\end{aligned} \tag{72}$$

Substituting Eq. (72) into (71) results in a periodic solution

$$\begin{aligned}u = a_1 &+ \frac{2K[-2a_0K + 2a_1b_0K + (2Ka_1b_0^2 - 2Ka_0b_0)\cos(\eta) + (a_1^2b_0^2 + a_0^2 - 2a_1a_0b_0)\sin(\eta)}{(-4K^2 - a_0^2 - a_1^2b_0^2 + 2a_1a_0b_0)\cos(\eta) - 4b_0K^2 + (2Ka_1b_0^2 - 2Ka_0b_0)\sin(\eta)}\\ &\times \frac{+(a_1^2b_0^2 + a_0^2 - 2a_1a_0b_0)\mathrm{i}\cos(\eta) + (-2Ka_1b_0^2 + 2Ka_0b_0)\mathrm{i}\sin(\eta)]}{+(2Ka_1b_0^2 - 2Ka_0b_0)\mathrm{i}\cos(\eta) + (-4K^2 + a_0^2 + a_1^2b_0^2 - 2a_1a_0b_0)\mathrm{i}\sin(\eta)},\end{aligned} \tag{73}$$

where in this case $\eta = Kx + Ly + \frac{K^4-3L^2}{4K}t$ and a_0, a_1 and b_0 are free parameters. If we set $b_0 = 0$, $a_0 = \pm 2K$ in Eq. (73), becomes

$$u(x, y, t) = a_1 - K\mathrm{i} \pm K\sec\left(Kx + Ly + \frac{K^4 - 3L^2}{4K}t\right) - K\tan\left(Kx + Ly + \frac{K^4 - 3L^2}{4K}t\right). \tag{74}$$

where as $k = \mathrm{i}K$, $l = \mathrm{i}L$, we write $K = -\mathrm{i}k$, $L = -\mathrm{i}l$ and with Substituting into (74) we obtain

$$u(x, y, t) = (a_1 - k) \mp \mathrm{i}k\,\mathrm{sech}\left(kx + ly - \frac{k^4 + 3l^2}{4k}t\right) + k\tanh\left(kx + ly - \frac{k^4 + 3l^2}{4k}t\right). \tag{75}$$

To compare our results, Eqs. (74) and (75),with Inan and kaya's solutions in Eq. (14), (15) into [25], we set $a_1 = i$, $K = 1$ and $L = \beta$ in Eq. (74) and also, we set $a_1 = k$, $k = 1$ and $l = \beta$ in Eq. (75), it can be seen that the results are the same.

One of periodic solutions (74), is shown at $a_1 = i$, $K = L = 1$ and $t = 2$ (see Fig. 1).

3.2.2. Case B: $p = c = 2$, $q = d = 2$

As mentioned above the values of c and d can be freely chosen, we set $p = c = 2$ and $d = q = 2$, then trial function, Eq. (4), reduces to Eq. (37).

By using Eq. (37) with (58), we determine three cases for coefficients, namely,

Case 1.

$$\begin{aligned}&\omega = -\frac{1}{4}\frac{3l^2 + 4k^4}{k}, \qquad b_{-2} = \frac{1}{16}\frac{b_0^2(-a_2^2 + 4ka_2) + a_0b_0(2a_2 - 4k) - a_0^2}{k^2}, \qquad l = l,\\ &a_1 = 0, \qquad k = k, \qquad a_{-1} = 0,\\ &b_0 = b_0, \qquad a_0 = a_0, \qquad a_2 = a_2,\\ &a_{-2} = \frac{-1}{16}\frac{a_0b_0(-16k^2 - 2a_2^2 + 12ka_2) + b_0^2(16a_2k^2 + a_2^3 - 8ka_2^2) + a_0^2(a_2 - 4k)}{k^2}.\end{aligned} \tag{76}$$

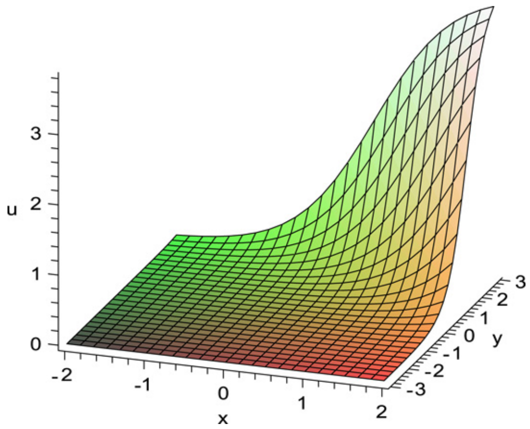

Fig. 2. Traveling wave solution (78).

Substituting Eq. (76) into (37), we have:

$$u=\frac{a_2\exp(2\eta)+a_0+\left(\frac{-1}{16}\frac{a_0b_0(-16k^2-2a_2^2+12ka_2)+b_0^2(16a_2k^2+a_2^3-8ka_2^2)+a_0^2(a_2-4k)}{k^2}\right)\exp(-2\eta)}{\exp(2\eta)+b_0+\left(\frac{1}{16}\frac{b_0^2(-a_2^2+4ka_2)+a_0b_0(2a_2-4k)-a_0^2}{k^2}\right)\exp(-2\eta)} \tag{77}$$

where $\eta=kx+ly-\frac{1}{4}\frac{3l^2+4k^4}{k}t$.

If we set $k=1$, $b_0=0$, $a_2=4$ and $a_0=\pm4$ in Eq. (77) we obtain:

$$u=2\coth(2\eta)-2csch(2\eta)+2=2\tanh(\eta)+2, \tag{78}$$

$$u=2\coth(2\eta)+2csch(2\eta)+2=2\coth(\eta)+2. \tag{79}$$

Comparing our results, Eqs. (78) and (79), with Inan and kaya's results in [25], it can be seen that the results are same. Here l acts as the role of β in [25].

The traveling wave solution (78), is shown at $k=l=1$ and $t=2$ (see Fig. 2).

Case 2.

$$\begin{aligned}&\omega=-\frac{1}{4}\frac{3l^2+k^4}{k},\qquad b_{-2}=\frac{-1}{16}\frac{a_1^2(a_1^2+4b_0k^2)}{k^4},\qquad l=l,\qquad a_1=a_1,\\&k=k,\qquad a_{-1}=\frac{1}{4}\frac{a_1(a_1^2+4b_0k^2)}{k^2},\\&b_0=b_0,\qquad a_0=\frac{1}{2}\frac{2ka_2b_0+a_1^2}{k},\qquad a_2=a_2,\qquad a_{-2}=\frac{1}{16}\frac{a_1^2(8k^3b_0-a_2a_1^2-4a_2b_0k^2+2a_1^2k)}{k^4}.\end{aligned} \tag{80}$$

By using Eq. (80) into Eq. (37), we have

$$u=\frac{a_2\exp(2\eta)+a_1\exp(\eta)+\frac{1}{2}\frac{2ka_2b_0+a_1^2}{k}+\left(\frac{1}{4}\frac{a_1(a_1^2+4b_0k^2)}{k^2}\right)\exp(-\eta)+\left(\frac{1}{16}\frac{a_1^2(8k^3b_0-a_2a_1^2-4a_2b_0k^2+2a_1^2k)}{k^4}\right)\exp(-2\eta)}{\exp(2\eta)+b_0+\left(\frac{-1}{16}\frac{a_1^2(a_1^2+4b_0k^2)}{k^4}\right)\exp(-2\eta)} \tag{81}$$

where $\eta=kx+ly-\frac{1}{4}\frac{3l^2+k^4}{k}t$.

4. Conclusions

In this Letter, the Exp-function method was used for finding solutions of the kaup–kupershmidt equation and the (2+1)-dimensional Potential Kadomtsev–Petviashvili (PKP) equation. It can be concluded that the Exp-function method is a very powerful and efficient technique for finding exact solutions for wide classes of problems. The Exp-function method has many merits and many more advantages than exact solutions. Calculations in the Exp-function method are simple and straightforward. The reliability of the method and the reduction in the size of computational domain give this method a wide applicability. The results show that the Exp-function method is a powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in engineering.

Acknowledgements

The authors wishes to thank an anonymous referee for valuable ideas that allowed us to improve the article.

References

[1] R.L. Burden, J.D. Faires, Numerical Analysis, PWS Publishing Company, Boston, 1993.
[2] D. Kaya, S.M. El-Sayed, On a generalized fifth order KdV equations, Physics Letters A 310 (2003) 44–51.
[3] J.H. He, Asymptotology by homotopy perturbation method, Applied Mathematics and Computation 156 (2006) 591–596.
[4] J.H. He, Variational iteration method for autonomous ordinary differential systems, Applied Mathematics and Computation 114 (2005) 115–123.
[5] J.H. He, Comparison of homotopy perturbation method and homotopy analysis method, Applied Mathematics and Computation 156 (2004) 527–539.
[6] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, International Journal of Non-Linear Mechanics 35 (2000) 37–43.
[7] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons & Fractals 26 (2005) 695–700.
[8] J.H. He, Homotopy perturbation method: A new nonlinear analytical technique, Applied Mathematics and Computation 156 (2004) 591–596.
[9] H.X. Wu, J.H. He, Solitary solutions periodic solutions and compacton-like solutions using EXP-function method, Computers & Mathematics with Applications (2007) (in press).
[10] J.H. He, X.H. Wu, Exp-function method for nonlinear wave equations, Chaos, Solitons & Fractals 30 (2006) 700–708.
[11] Ji-Huan He, Li-Na Zhang, Generalized solitary solution and compacton-like solution of the Jaulent–Miodek equations using the Exp-function method, Physics Letters A (2007) (in press).
[12] J.H. He, Non-Perturbative method for strongly nonlinear problems. Berlin: Dissertation. De-Verlag im Internet GmbH. 2006.
[13] H.X. Wu, J.H. He, Exp-function method and its application to nonlinear equations, Chaos, Solitons & Fractals (2007) (in press).
[14] Ji-Huan He, M.A. Abdou, New periodic solutions for nonlinear evolution equations using Exp-function method, Chaos, Solitons & Fractals 34 (2007) 1421–1429.
[15] S.D. Zhu, Exp-function method for the hybrid-lattice system, International Journal of Nonlinear Science and Numerical Solution 8 (3) (2007) 461–464.
[16] S.D. Zhu, Exp-function method for the discrete mKdV lattice, International Journal of Nonlinear Science and Numerical Solution 8 (3) (2007) 465–468.
[17] S. Zhang, Application of Exp-function method to high-dimensional nonlinear evolution equation, Chaos, Solitons & Fractals (2006) (in press).
[18] Abdul-majid Waz waz, The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, Applied Mathematics and Computation 184 (2007) 1002.
[19] Abundant solitons solutions for several forms of the fifth-order KdV equations by using the tanh method, Applied Mathematics and Computaion 182 (2006) 283.
[20] M. Qasem, Al-Mdallal, Muhammad I. Syam, Sine–Cosine method for finding the soliton Solutions of the generalized fifth-order nonlinear equation, Chaos, Solitons & Fractals 33 (2007) 1610–1617.
[21] A. Parker, On soliton solutions of the Kaup–Kupershmidt equation. I. Direct bilinearisation and solitary wave, Physica D 137 (2000) 25–33.
[22] Changbum Chun, Solitons and periodic solutions for the fifth-order KdV equation with the Exp-function method, Physics Letters A (in press).
[23] Dogan Kaya, Salah M. El-Sayed, Numerical soliton-like solutions of the potential Kadomtsev–Petviashvili equation by the decomposition method, Physics Letters A 320 (2003) 192–199.
[24] M. Senthilvelan, Applied Mathematics and Computation 123 (2001) 381.
[25] Ibrahim E. Inan, Dogan Kaya, Some exact solutions to the potential Kadomtsev petviashvili equation and to a system of shallow water wave equations, Physics Letters. A 355 (2006) 314.