Journal of Computational and Applied Mathematics 236 (2012) 3728–3740

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Conservative finite difference schemes for the Degasperis–Procesi equation

Yuto Miyatake *, Takayasu Matsuo

Department of Mathematical Informatics, Graduate School of Information Science and Technology, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo, 113-8656, Japan

ARTICLE INFO

Article history:
Received 10 September 2010
Received in revised form 1 September 2011

Keywords:
Degasperis–Procesi equation
Conservation
Discrete variational derivative method

ABSTRACT

We consider the numerical integration of the Degasperis–Procesi equation, which was recently introduced as a completely integrable shallow water equation. For the equation, we propose nonlinear and linear finite difference schemes that preserve two invariants associated with the bi-Hamiltonian form of the equation at the same time. We also prove the unique solvability of the schemes, and show some numerical examples.

1. Introduction

In this paper,[1] we consider numerical methods for the Degasperis–Procesi equation:

$$u_t - u_{xxt} = uu_{xxx} + 3u_x u_{xx} - 4uu_x, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

where the subscript t (or x, respectively) denotes the differentiation with respect to time variable t (or x). This equation was found in a study by Degasperis and Procesi [2], where they considered asymptotic integrability of a family of the third order dispersive nonlinear equations:

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x.$$

The parameters α, γ, c_0, c_1, c_2, c_3 are constants, and Degasperis and Procesi concluded that under only three special choices of them the equation becomes integrable; namely, the KdV equation: $u_t + uu_x + u_{xxx} = 0$, the Camassa–Holm (CH) equation:

$$u_t - u_{xxt} = uu_{xxx} + 2u_x u_{xx} - 3uu_x, \tag{2}$$

and the DP equation (1). As for the KdV and the CH, the structure of solutions has been studied in detail so far, and also considerable effort has been devoted to their numerical treatment. In contrast to this, there seems to have been relatively few studies on the DP, and much remains to be investigated both mathematically and numerically. In view of this background, this paper is intended to provide a good numerical scheme for the equation.

At this point, some readers might feel that due to the apparent similarity between the DP and CH (actually they only differ in two coefficients), we can naturally extend the knowledge on the CH to the DP, and nothing special is left to be investigated separately. In fact, the DP is quite similar to the CH in the following senses. They both were discovered in the context of integrable systems (CH [3], DP [2]), and can be viewed as models of shallow water waves (CH [4,5], DP [6,7]). They both have bi-Hamiltonian structures, are completely-integrable, and thus have infinitely many conservation laws. They have peakon (peaked soliton) solutions (CH [4], DP [2,8]).

* Corresponding author.
E-mail address: yuto_miyatake@mist.i.u-tokyo.ac.jp (Y. Miyatake).

[1] This is an augmented English version of [1] (in Japanese) with extra numerical results and discussions.

0377-0427/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2011.09.004

Nevertheless, the DP and CH are in some aspects truly different. From the perspective of PDE theory, the solutions of the CH basically belong to $H^1(\mathbb{R})$ (the first order Sobolev space), while the DP can develop shock solutions which should be understood as entropy solutions [9–11] (see also [12,13] for smoother solutions). Furthermore there are differences in invariants and bi-Hamiltonian structures. The CH has the following two invariants:

$$\tilde{H}_2 = -\frac{u^3 + u{u_x}^2}{2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \tilde{H}_2 \mathrm{d}x = 0, \qquad \tilde{H}_1 = -\frac{u^2 + {u_x}^2}{2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \tilde{H}_1 \mathrm{d}x = 0,$$

which define the associated bi-Hamiltonian structure:

$$m_t = (\partial_x - {\partial_x}^3)\frac{\delta \tilde{H}_2}{\delta m} = (m\partial_x + \partial_x m)\frac{\delta \tilde{H}_1}{\delta m}, \quad m = (1 - \partial_x^2)u. \tag{3}$$

(The symbol ∂_x denotes $\partial/\partial x$.) Note that, as usual in this research field, we promise that the "operator" $(m\partial_x + \partial_x m)$ operates to a function f in such a way that $(m\partial_x + \partial_x m)f = mf_x + \partial_x(mf)$. For the DP, the first two invariants are

$$H_{-1} = -\frac{1}{6}u^3, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} H_{-1}\mathrm{d}x = 0,$$
$$H_0 = -\frac{9}{2}(u - u_{xx}), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} H_0\mathrm{d}x = 0,$$

and the bi-Hamiltonian structure is

$$m_t = \mathcal{B}_0\frac{\delta H_{-1}}{\delta m} = \mathcal{B}_1\frac{\delta H_0}{\delta m},$$

where $\mathcal{B}_0$, $\mathcal{B}_1$ are the skew-symmetric operators:

$$\mathcal{B}_0 = \partial_x(1 - {\partial_x}^2)(4 - {\partial_x}^2), \qquad \mathcal{B}_1 = m^{\frac{2}{3}}\partial_x m^{\frac{1}{3}}(\partial_x - {\partial_x}^3)^{-1} m^{\frac{1}{3}}\partial_x m^{\frac{2}{3}}.$$

As can be easily seen, the invariants and the bi-Hamiltonian structures are substantially different. Considering all the differences above, now researchers believe that the equations are essentially different, and there is no simple transformation that casts the DP into the CH and vice versa (see the introductions of [13,14]).

The mathematical differences also make the numerical treatment of the DP inevitably different. Before precisely stating this point, we first like to recall a fact that for conservative PDEs such as the CH and DP, numerical methods preserving the invariants are often advantageous, and in the last two decades much effort has been devoted in this topic to finally find out several general frameworks. For example, Furihata and Mori [15] and Celledoni et al. [16] proposed frameworks in finite-difference context (see also related references in [17]), and Matsuo [18] devised a Galerkin framework. With these frameworks, some conservative schemes for the CH have been already proposed based on the bi-Hamiltonian structure: $\tilde{H}_1$-preserving finite difference schemes [19–21] and a Galerkin scheme [22] based on the second form of (3); and $\tilde{H}_2$-preserving finite difference schemes [20,21] and a Galerkin scheme [23] based on the first form. Obviously similar study is expected for the DP equation, but to the best of the authors' knowledge, none has been done yet. Main reason of this may be simply because the DP is quite new, but our attention should also go to the fact that, as mentioned above, the bi-Hamiltonian structures are very different, which causes several difficulties in the DP case. The first difficulty is caused by the complex skew-symmetric operators $\mathcal{B}_0$ and $\mathcal{B}_1$; they are much more complicated than those in the CH (see (3)), and due to this, the general Galerkin framework [18] cannot be applied. Thus, whether a conservative Galerkin scheme for the DP can be constructed or not remains an open question, as of writing this paper (we will come back to this point in the end of this paper). In contrast, the finite-difference frameworks can be applied, but here arises a second difficulty that, even in the finite-difference framework, the operator $\mathcal{B}_1$ is too cumbersome, and it is quite unlikely that the resulting H_0-preserving scheme is numerically efficient in practice. This is in sharp contrast to the CH case where both of the bi-Hamiltonian representations were fully utilized for efficient $\tilde{H}_2$-and $\tilde{H}_1$-preserving schemes.

Taking these backgrounds into account, in this paper we propose conservative finite-difference schemes based on the first form of the bi-Hamiltonian structure. More precisely, we propose two finite-difference schemes. The first scheme is a nonlinear scheme derived with the general framework [15] (which is now called the "discrete variational derivative method"), and the second is its linearized version based on a linearization technique [24]. Both schemes conserve H_{-1} (or its approximation), and furthermore, it turns out that fortunately they also conserve H_0 at the same time. This means we can preserve H_0 without the second form of the bi-Hamiltonian structure; this happens thanks to the fact that H_0 is a linear invariant. This is an interesting property in that, as far as the authors know, this is the first scheme that preserves both invariants associated with a bi-Hamiltonian structure. We also prove the unique existence of the solutions of the proposed schemes, and show several numerical examples.

Finally, we like to briefly mention the existing numerical schemes. Operator splitting schemes were devised in [25,26], and a particle method based on multi-shockpeakon solutions was exploited in [27]. These numerical methods were mainly intended to capture shock solutions, and does not conserve H_{-1} (actually H_{-1} becomes less important in such a situation; it is no longer conserved when the solution develops a shock [11]). In this sense, these schemes and our conservative schemes (which mainly focus on smoother solutions) seem to complement each other. We will mention this point again in the end of this paper.

In view of numerical computation, hereafter let us choose the periodic boundary condition:

$$u(x+L,t)=u(x,t),\quad t\geq 0,\ x\in\mathbb{R}. \tag{4}$$

We denote the periodic first-order Sobolev space by $H^1(\mathbb{S})$ where $\mathbb{S}$ is the torus of the length L. Global existence of the smooth solutions in $H^1(\mathbb{S})$ was proved in [12]. It is straightforward to confirm that the invariants mentioned above are also invariants under the periodic boundary condition.

This paper is organized as follows. In Section 2 notation and useful lemmas are introduced. In Section 3 the proposed schemes are presented, and its properties are discussed. In Section 4 some numerical results are provided. Concluding remarks and comments are given in Section 5.

2. Preliminaries

In this section, we prepare notation and useful lemmas. The numerical solution is denoted by $U_k^{(n)}\simeq u(k\Delta x,n\Delta t)$, where N is the number of the spatial grids (i.e., $\Delta x=L/N$), and Δt is the time mesh size. We often write this as a vector: $\boldsymbol{U}^{(n)}=(U_0^{(n)},U_1^{(n)},\dots,U_{N-1}^{(n)})^\top$. In order to treat the periodic boundary condition (4), we consider $\{U_k^{(n)}\}_{k=-\infty}^{\infty}$, an infinitely long vector, and then its N-dimensional restriction by the discrete periodic boundary condition $U_k^{(n)}=U_{k \bmod N}^{(n)}$ ($\forall k\in\mathbb{Z}$). We denote the latter space by $\mathbb{R}^{(N)}$, and define its inner product by $(\boldsymbol{a},\boldsymbol{b})=\sum_{k=0}^{N-1}a_kb_k\Delta x$. The norm $\|\cdot\|$ is defined accordingly. They are the natural discretization of $L^2(\mathbb{S})$. For $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^{(N)}$, we define the operator $*:\mathbb{R}^{(N)}\times\mathbb{R}^{(N)}\to\mathbb{R}^{(N)}$ by $\boldsymbol{a}*\boldsymbol{b}:=(a_0b_0,a_1b_1,\dots,a_{N-1}b_{N-1})^\top$. It is easy to see $\|\boldsymbol{a}*\boldsymbol{b}\|\leq\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|/\sqrt{\Delta x}$.

The standard central difference operators that approximate ∂_x,∂_x^2 are denoted by $\delta_k^{\langle 1\rangle},\delta_k^{\langle 2\rangle}$ respectively:

$$\delta_k^{\langle 1\rangle}U_k^{(n)}:=\frac{U_{k+1}^{(n)}-U_{k-1}^{(n)}}{2\Delta x},\qquad \delta_k^{\langle 2\rangle}U_k^{(n)}:=\frac{U_{k+1}^{(n)}-2U_k^{(n)}+U_{k-1}^{(n)}}{\Delta x^2}.$$

We often write them in matrix form; for example, $D^{\langle 1\rangle}\boldsymbol{a}:=(\delta_0^{\langle 1\rangle}a_0,\delta_1^{\langle 1\rangle}a_1,\dots,\delta_{N-1}^{\langle 1\rangle}a_{N-1})^\top$ for $\boldsymbol{a}\in\mathbb{R}^{(N)}$.

The following estimates are useful. Although they are well known, we here show a rough proof for the readers' convenience. We note that $(1-\partial_x{}^2)^{-1}$ has a meaning as $(1-\partial_x{}^2)^{-1}:L^2(\mathbb{S})\to H^2(\mathbb{S})$ [28].

Lemma 2.1. *For $D^{\langle 1\rangle},D^{\langle 2\rangle}:\mathbb{R}^{(N)}\to\mathbb{R}^{(N)}$,*

$$\|D^{\langle 1\rangle}\|\leq\frac{1}{\Delta x},\qquad \|D^{\langle 2\rangle}\|\leq\frac{4}{\Delta x^2},\qquad \left\|\left(I-D^{\langle 2\rangle}\right)^{-1}\right\|=1$$

hold, where I is the identity mapping, $I\boldsymbol{a}=\boldsymbol{a}$.

Proof. We first prove the statement for $\|D^{\langle 1\rangle}\|$. Since for every $\boldsymbol{v}\in\mathbb{R}^{(N)}$,

$$\|D_k^{(1)}\boldsymbol{v}\|^2=\sum_{k=0}^{N-1}\left(\frac{v_{k+1}-v_{k-1}}{2\Delta x}\right)^2\Delta x\leq\frac{1}{(\Delta x)^2}\sum_{k=0}^{N-1}\frac{v_{k+1}^2+v_{k-1}^2}{2}\Delta x=\frac{1}{(\Delta x)^2}\|\boldsymbol{v}\|^2$$

holds, we get $\|D^{\langle 1\rangle}\|\leq 1/\Delta x$. The statements for $D^{\langle 2\rangle}$ and $\left(I-D^{\langle 2\rangle}\right)^{-1}$ can be shown by eigenvalue arguments. Let us introduce a cyclic matrix B whose subdiagonal and superdiagonal elements are 1 (note that $B_{1,N}=B_{N,1}=1$ by periodicity). Then the eigenvalues of the matrix are $2\cos(2j\pi/N)$ ($j=0,\dots,N-1$). This immediately implies $\|D^{\langle 2\rangle}\|\leq 4/\Delta x^2$, since $D^{\langle 2\rangle}=(B-2I)/\Delta x^2$ (note that we defined the norm by the two-norm). Next, the eigenvalues of matrix $I-D^{\langle 2\rangle}=I-(B-2I)/\Delta x^2$ are

$$1-\frac{2}{(\Delta x)^2}\left\{\cos\left(\frac{2j\pi}{N}\right)-1\right\},\quad j=0,1,\dots,N-1,$$

thus $\left(I-D^{\langle 2\rangle}\right)^{-1}$ exists because all eigenvalues of $I-D^{\langle 2\rangle}$ are greater than or equal to 1 (independent of Δx). Then $\left\|\left(I-D^{\langle 2\rangle}\right)^{-1}\right\|=1$ holds because all eigenvalues of $\left(I-D^{\langle 2\rangle}\right)^{-1}$ is less than or equal to 1. □

As for the difference operators, the following summation-by-parts formulas hold [29]:

$$\sum_{k=0}^{N-1}U_k\left(\delta_k^{\langle 1\rangle}V_k\right)\Delta x+\sum_{k=0}^{N-1}\left(\delta_k^{\langle 1\rangle}U_k\right)V_k\Delta x=\left[\frac{U_kV_{k-1}+U_{k-1}V_k}{2}\right]_{k=0}^{N}, \tag{5}$$

$$\sum_{k=0}^{N-1}U_k\left(\delta_k^{\langle 2\rangle}V_k\right)\Delta x-\sum_{k=0}^{N-1}\left(\delta_k^{\langle 2\rangle}U_k\right)V_k\Delta x=\left[U_k\left(\delta_k^{\langle 1\rangle}V_k\right)-\left(\delta_k^{\langle 1\rangle}U_k\right)V_k\right]_{k=0}^{N}, \tag{6}$$

where $[U_k]_{k=0}^N=U_N-U_0$. Under the discrete periodic boundary condition, the boundary terms are automatically canceled.

Next we prepare a useful lemma on the discrete operators. In this paper, we use the discrete versions of ∂_x, $(1-\partial_x{}^2)^{-1}\partial_x(4-\partial_x{}^2)$, $(4-\partial_x{}^2)$, and $(1-\partial_x{}^2)^{-1}$. Obviously they are skew-symmetric or symmetric, and the following lemma shows that difference operators inherit the symmetries.

Lemma 2.2. *With respect to the inner product of $\mathbb{R}^{(N)}$, $\delta_k^{\langle 1\rangle}$, $(1-\delta_k^{\langle 2\rangle})^{-1}\delta_k^{\langle 1\rangle}(4-\delta_k^{\langle 2\rangle})$ are skew-symmetric, and $(4-\delta_k^{\langle 2\rangle})$, $(1-\delta_k^{\langle 2\rangle})^{-1}$ are symmetric operators.*

Proof. The skew-symmetry of $\delta_k^{\langle 1\rangle}$ and symmetry of $\delta_k^{\langle 2\rangle}$ are obvious form (5) and (6). Accordingly $(4-\delta_k^{\langle 2\rangle})$ and $(1-\delta_k^{\langle 2\rangle})^{-1}$ are symmetric. Combining these symmetries, we see that $(1-\delta_k^{\langle 2\rangle})^{-1}\delta_k^{\langle 1\rangle}(4-\delta_k^{\langle 2\rangle})$ is skew-symmetric. □

3. H_{-1}, H_0-conserving schemes

In this section we propose two finite difference schemes (nonlinear and linear) that conserve both H_{-1} and H_0. For comparison, we also show the standard Crank–Nicolson scheme.

3.1. The first form expressed in u

As noted above, we consider the first form of the bi-Hamiltonian structure. Although it is possible to directly apply the framework [15] to the first form (in the variable m), it is more convenient to rewrite it with u, in view of the fact that the initial data is given by $u(x,0)$ and the final solution is demanded also in the form of $u(x,t)$. We then see

$$\left(1-\partial_x{}^2\right)u_t=\partial_x\left(4-\partial_x{}^2\right)\frac{\delta H_{-1}}{\delta u}. \tag{7}$$

This still keeps the H_{-1} conservation law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H_{-1}\mathrm{d}x=\int_0^L\frac{\partial H_{-1}}{\partial u}u_t\mathrm{d}x=\int_0^L\frac{\delta H_{-1}}{\delta u}\left\{(1-\partial_x{}^2)^{-1}\partial_x(4-\partial_x{}^2)\frac{\delta H_{-1}}{\delta u}\right\}\mathrm{d}x=0.$$

The first equality is just the chain rule. The second equality follows from (7). The third equality is from the skew-symmetry of $(1-\partial_x{}^2)^{-1}\partial_x(4-\partial_x{}^2)$. Also the H_0 conservation follows from (7):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H_0\mathrm{d}x=-\frac{9}{2}\int_0^L\partial_x(4-\partial_x{}^2)\frac{\delta H_{-1}}{\delta u}\mathrm{d}x=-\frac{9}{2}\left[(4-\partial_x{}^2)\frac{\delta H_{-1}}{\delta u}\right]_0^L=0.$$

The final equality is from the periodic boundary condition.

Remark 1. We here like to make additional comments on the H_0 conservation. Above we showed it based on the u expression (7), but the property itself is more easily seen by the original m expression. Actually, if we rewrite it to $m_t-((1-\partial_x{}^2)(4-\partial_x{}^2)\delta H_{-1}/\delta m)_x=0$, it is in the so-called "conservation law" form, and the conservation of $\int m\,\mathrm{d}x$ is immediate. This implies that, although the second form of the DP states an important mathematical fact that the invariant H_0 can generate the equation, it is not so relevant for the (proof of the) preservation of H_0. This happens since H_0 is a linear invariant. Below we show that based on the same principle, the H_{-1} conservative schemes based on the discrete version of the first form fortunately conserve H_0 as well.

3.2. A H_{-1}, H_0-conserving nonlinear scheme

Now we present a nonlinear scheme preserving both H_{-1} and H_0. We define a discrete version of $H_{-1}=-u^3/6$, and accordingly a "discrete variational derivative" that approximate $\delta H_{-1}/\delta u=-u^2/2$ as

$$(H_{-1})_k^{(n)}:=-\frac{(U_k^{(n)})^3}{6},\qquad\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}:=-\frac{(U_k^{(n+1)})^2+U_k^{(n+1)}U_k^{(n)}+(U_k^{(n)})^2}{6}. \tag{8}$$

Due to the restriction of space, the detail of the concept of "discrete variational derivative," and the derivation of its concrete form from the discrete Hamiltonian $(H_{-1})_k^{(n)}$ is skipped; interested readers may refer to the general framework: the "discrete variational derivative method" [15]. For the purpose of this paper, it is sufficient to note the key equality:

$$\sum_{k=0}^{N-1}(H_{-1})_k^{(n+1)}\Delta x-\sum_{k=0}^{N-1}(H_{-1})_k^{(n)}\Delta x=\sum_{k=0}^{N-1}\left\{\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}(U_k^{(n+1)}-U_k^{(n)})\right\}\Delta x, \tag{9}$$

which can be easily checked with the concrete forms in (8). We also define the discrete H_0 as

$$(H_0)_k^{(n)}:=-\frac{9}{2}(1-\delta_k^{\langle 2\rangle})U_k^{(n)}.$$

Now we are in a position to define the nonlinear finite difference scheme.

Scheme 1 (H_{-1}, H_0-*Conserving Nonlinear Finite Difference Scheme*). *We define the initial approximate solution by* $U_k^{(0)} = u(0, k\Delta x)$ $(k = 0, \ldots, N-1)$. *Then for* $n = 1, 2, \ldots,$

$$\left(1-\delta_k^{\langle 2\rangle}\right)\frac{U_k^{(n+1)}-U_k^{(n)}}{\Delta t} = \delta_k^{\langle 1\rangle}\left(4-\delta_k^{\langle 2\rangle}\right)\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}, \quad k = 0,\ldots,N-1. \tag{10}$$

Obviously (10) corresponds to (7). Numerical solutions by Scheme 1 conserve both $(H_{-1})_k^{(n)}$ and $(H_0)_k^{(n)}$ under the discrete periodic boundary condition.

Theorem 3.1 (Scheme 1: *Conservation Laws*). *Under the discrete periodic boundary condition, the numerical solution by Scheme 1 conserves both* H_{-1} *and* H_0*:*

$$\sum_{k=0}^{N-1}(H_{-1})_k^{(n)}\Delta x = \sum_{k=0}^{N-1}(H_{-1})_k^{(0)}\Delta x, \quad n = 1, 2, \ldots,$$
$$\sum_{k=0}^{N-1}(H_0)_k^{(n)}\Delta x = \sum_{k=0}^{N-1}(H_0)_k^{(0)}\Delta x, \quad n = 1, 2, \ldots.$$

Proof. We first prove the discrete H_{-1} conservation law.

$$\begin{aligned}\frac{1}{\Delta t}\left(\sum_{k=0}^{N-1}(H_{-1})_k^{(n+1)}\Delta x - \sum_{k=0}^{N-1}(H_{-1})_k^{(n)}\Delta x\right) &= \sum_{k=0}^{N-1}\left\{\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}\frac{U_k^{(n+1)}-U_k^{(n)}}{\Delta t}\right\}\Delta x\\ &= \sum_{k=0}^{N-1}\left\{\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}\right\}\\ &\quad\times\left\{\left(1-\delta_k^{\langle 2\rangle}\right)^{-1}\delta_k^{\langle 1\rangle}\left(4-\delta_k^{\langle 2\rangle}\right)\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}\right\}\Delta x\\ &= 0.\end{aligned}$$

The first equality is from (9), the second is from (10), and the third is from the skew-symmetry of the operator $\left(1-\delta_k^{\langle 2\rangle}\right)^{-1}\delta_k^{\langle 1\rangle}\left(4-\delta_k^{\langle 2\rangle}\right)$ (Lemma 2.1). Next we prove the discrete H_0 conservation law.

$$\begin{aligned}\frac{1}{\Delta t}\left(\sum_{k=0}^{N-1}(H_0)_k^{(n+1)}\Delta x - \sum_{k=0}^{N-1}(H_0)_k^{(n)}\Delta x\right) &= -\frac{9}{2}\sum_{k=0}^{N-1}\left(1-\delta_k^{\langle 2\rangle}\right)\frac{U_k^{(n+1)}-U_k^{(n)}}{\Delta t}\Delta x\\ &= -\frac{9}{2}\sum_{k=0}^{N-1}\delta_k^{\langle 1\rangle}\left(4-\delta_k^{\langle 2\rangle}\right)\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}\Delta x\\ &= -\frac{9}{4}\left[\left(4-\delta_k^{\langle 2\rangle}\right)\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_k}+\left(4-\delta_k^{\langle 2\rangle}\right)\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)},\boldsymbol{U}^{(n)})_{k-1}}\right]_{k=0}^{N}\\ &= 0.\end{aligned}$$

The second equality is from (10). By substituting $\boldsymbol{V} = (1, 1, \ldots, 1)^\top$ into (5), we obtain the third equality. The final is from the discrete periodic boundary condition. □

Since Scheme 1 is nonlinear, it requires nonlinear solvers in each time step. The next theorem states that if we set time step Δt adequately based on $\boldsymbol{U}^{(n)}$, unique existence of the solution is guaranteed. Note that even if the time mesh size Δt is changed, Theorem 3.1 still holds.

Theorem 3.2 (*Unique Solvability of* Scheme 1). *Let* $\boldsymbol{U}^{(n)}$ *be given. If* Δt *satisfies*

$$\frac{2\Delta t}{3\Delta x^{3/2}} < \frac{2\sqrt{3}-3}{3\|\boldsymbol{U}^{(n)}\|}.$$

Scheme 1 has a unique numerical solution $\boldsymbol{U}^{(n+1)}$.

To prove this theorem, we make some preparations. First we transform (10) into

$$U_k^{(n+1)} = U_k^{(n)} - \frac{\Delta t}{6}\left(1-\delta_k^{\langle 2\rangle}\right)^{-1}\delta_k^{\langle 1\rangle}\left(4-\delta_k^{\langle 2\rangle}\right)\left\{\left(U_k^{(n+1)}\right)^2 + U_k^{(n+1)}U_k^{(n)} + \left(U_k^{(n)}\right)^2\right\}. \tag{11}$$

Then for $\boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^{(N)}$, we define $\phi_{\boldsymbol{a}} : \mathbb{R}^{(N)} \to \mathbb{R}^{(N)}$ as

$$\phi_{\boldsymbol{a}}(\boldsymbol{v}) := \boldsymbol{a} - \frac{\Delta t}{6}\left(I - D^{\langle 2\rangle}\right)^{-1} D^{\langle 1\rangle}\left(4 - D^{\langle 2\rangle}\right)(\boldsymbol{v} * \boldsymbol{v} + \boldsymbol{v} * \boldsymbol{a} + \boldsymbol{a} * \boldsymbol{a}). \tag{12}$$

If we substitute $\boldsymbol{a} = \boldsymbol{U}^{(n)}$, $\boldsymbol{v} = \boldsymbol{U}^{(n+1)}$ into (12), (12) is equivalent to (11). Therefore, in order to prove the unique solvability of Scheme 1, it is sufficient to prove the existence of a fixed point of the map $\phi_{\boldsymbol{a}}$.

Proof of Theorem 3.2. We first define a ball $\mathcal{K}_{\boldsymbol{a}} := \{\boldsymbol{v} \in \mathbb{R}^{(N)} \mid \|\boldsymbol{v}\| \le p r_{\boldsymbol{a}}\}$, where $r_{\boldsymbol{a}} := \|\boldsymbol{a}\|$, $p = 1 + \sqrt{3}$. The proof will be done in the following two steps. (i) $\phi_{\boldsymbol{a}}$ is a map from $\mathcal{K}_{\boldsymbol{a}}$ to $\mathcal{K}_{\boldsymbol{a}}$ if Δt is sufficiently small. (ii) Then $\phi_{\boldsymbol{a}}$ is a contraction mapping.

(i) We will use following inequality:

$$\left\|\left(I - D_k^{\langle 2\rangle}\right)^{-1}\left(4 - D_k^{\langle 2\rangle}\right)\right\| = \left\|I + 3\left(I - D_k^{\langle 2\rangle}\right)^{-1}\right\| \le \|I\| + 3\left\|\left(I - D_k^{\langle 2\rangle}\right)^{-1}\right\| \le 4.$$

Then for every $\boldsymbol{v} \in \mathcal{K}_{\boldsymbol{a}}$,

$$\begin{aligned}
\|\phi_{\boldsymbol{a}}(\boldsymbol{v})\| &\le \|\boldsymbol{a}\| + \frac{\Delta t}{6}\left\|\left(I - D_k^{\langle 2\rangle}\right)^{-1}\left(4 - D_k^{\langle 2\rangle}\right)\right\| \|D_k^{\langle 1\rangle}\| \left(\|\boldsymbol{v} * \boldsymbol{v}\| + \|\boldsymbol{v} * \boldsymbol{a}\| + \|\boldsymbol{a} * \boldsymbol{a}\|\right) \\
&\le \|\boldsymbol{a}\| + \frac{\Delta t}{6} \cdot 4 \cdot \frac{1}{\Delta x} \cdot \left(\frac{\|\boldsymbol{v}\|^2 + \|\boldsymbol{v}\|\,\|\boldsymbol{a}\| + \|\boldsymbol{a}\|^2}{\sqrt{\Delta x}}\right) \\
&\le r_{\boldsymbol{a}} + \frac{2\Delta t}{3\Delta x^{3/2}}(p^2 + p + 1) r_{\boldsymbol{a}}^2.
\end{aligned}$$

Thus if Δt satisfies

$$r_{\boldsymbol{a}} + \frac{2\Delta t}{3\Delta x^{3/2}}(p^2 + p + 1) r_{\boldsymbol{a}}^2 \le p r_{\boldsymbol{a}} \Leftrightarrow \frac{2\Delta t}{3\Delta x^{3/2}} \le \frac{p - 1}{r_{\boldsymbol{a}}(p^2 + p + 1)} = \frac{2\sqrt{3} - 3}{3 r_{\boldsymbol{a}}},$$

$\phi_{\boldsymbol{a}}$ is a map from $\mathcal{K}_{\boldsymbol{a}}$ to $\mathcal{K}_{\boldsymbol{a}}$.

(ii) If Δt satisfy the above condition, for every $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{K}_{\boldsymbol{a}}$,

$$\begin{aligned}
\|\phi_{\boldsymbol{a}}(\boldsymbol{v}_1) - \phi_{\boldsymbol{a}}(\boldsymbol{v}_2)\| &\le \frac{\Delta t}{6}\left\|\left(I - D_k^{\langle 2\rangle}\right)^{-1}\left(4 - D_k^{\langle 2\rangle}\right)\right\| \|D_k^{\langle 1\rangle}\| \left(\frac{\|\boldsymbol{v}_1 + \boldsymbol{v}_2\|\,\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| + \|\boldsymbol{a}\|\,\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|}{\sqrt{\Delta x}}\right) \\
&\le \frac{2\Delta t}{3\Delta x^{3/2}}(\|\boldsymbol{v}_1 + \boldsymbol{v}_2\| + \|\boldsymbol{a}\|)\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \\
&\le \frac{2\Delta t}{3\Delta x^{3/2}}(2p + 1) r_{\boldsymbol{a}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|
\end{aligned}$$

holds. Thus if Δt satisfies

$$\frac{2\Delta t}{3\Delta x^{3/2}}(2p + 1) r_{\boldsymbol{a}} < 1 \Leftrightarrow \frac{2\Delta t}{3\Delta x^{3/2}} < \frac{1}{(2p + 1) r_{\boldsymbol{a}}} = \frac{2\sqrt{3} - 3}{3\|\boldsymbol{U}^{(n)}\|},$$

$\phi_{\boldsymbol{a}}$ is a contraction mapping from $\mathcal{K}_{\boldsymbol{a}}$ to $\mathcal{K}_{\boldsymbol{a}}$, and hence $\phi_{\boldsymbol{a}}$ has a unique fixed point on $\mathcal{K}_{\boldsymbol{a}}$ by the contraction mapping theorem. □

3.3. A H_{-1}, H_0-conserving linear finite difference scheme

Next we propose a H_{-1}, H_0-conserving linear finite difference scheme. We define $(H_{-1})_k^{(n+\frac{1}{2})}$ and $\delta H_{-1}/\delta(\boldsymbol{U}_k^{(n+1)}, \boldsymbol{U}_k^{(n)}, \boldsymbol{U}_k^{(n-1)})$ that approximate $H_{-1} = -u^3/6$ and $\delta H_{-1}/\delta u = -u^2/2$ as

$$\begin{aligned}
(H_{-1})_k^{\left(n+\frac{1}{2}\right)} &:= -\frac{U_k^{(n+1)} U_k^{(n)} (U_k^{(n+1)} + U_k^{(n)})}{12}, \\
\frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} &:= -\frac{U_k^{(n)}(U_k^{(n+1)} + U_k^{(n)} + U_k^{(n-1)})}{6}.
\end{aligned}$$

Here, we utilized the linearization technique in [24] (again we skip the details of this). We can easily check the following key equality:

$$\sum_{k=0}^{N-1} (H_{-1})_k^{(n+\frac{1}{2})} \Delta x - \sum_{k=0}^{N-1} (H_{-1})_k^{\left(n-\frac{1}{2}\right)} \Delta x = \sum_{k=0}^{N-1} \left\{ \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} \frac{U_k^{(n+1)} - U_k^{(n-1)}}{2} \right\} \Delta x. \tag{13}$$

Using the above discrete variational derivative, we define the linear finite difference scheme as follows.

Scheme 2 (H_{-1}, H_0-*Conserving Linear Finite Difference Scheme*). *We define the initial approximate solution by* $U_k^{(0)} = u(0, k\Delta x)$ $(k = 0, \ldots, N-1)$, *and compute a starting value* $\boldsymbol{U}^{(1)}$ *by Scheme 1. Then, for* $n = 1, 2, \ldots,$

$$\left(1 - \delta_k^{\langle 2\rangle}\right) \frac{U_k^{(n+1)} - U_k^{(n-1)}}{2\Delta t} = \delta_k^{\langle 1\rangle} \left(4 - \delta_k^{\langle 2\rangle}\right) \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k}, \quad k = 0, \ldots, N-1. \tag{14}$$

Obviously (14) corresponds to (7). We note that since Scheme 2 is a multistep scheme, we need not only the initial value $\boldsymbol{U}^{(0)}$ but also a starting value $\boldsymbol{U}^{(1)}$. In Scheme 2 we set $\boldsymbol{U}^{(1)}$ by Scheme 1, but also other schemes can be used if Δt is chosen appropriately small such that the invariants are kept with enough accuracy. The numerical solution by Scheme 2 satisfies $(H_{-1})_k^{(n+\frac{1}{2})}$, $(H_0)_k^{(n)}$ conservation laws under the discrete periodic boundary condition.

Theorem 3.3 (Scheme 2*: Conservation Laws*). *Under the discrete periodic boundary condition, the numerical solution by Scheme 2 conserves* H_{-1} *and* H_0*:*

$$\sum_{k=0}^{N-1} (H_{-1})_k^{\left(n+\frac{1}{2}\right)} \Delta x = \sum_{k=0}^{N-1} (H_{-1})_k^{\left(\frac{1}{2}\right)} \Delta x, \quad n = 1, 2, \ldots,$$

$$\sum_{k=0}^{N-1} (H_0)_k^{(n)} \Delta x = \sum_{k=0}^{N-1} (H_0)_k^{(0)} \Delta x, \quad n = 1, 2, \ldots.$$

Proof. We first prove the discrete H_{-1} conservation law.

$$\begin{aligned}
\frac{1}{\Delta t}\left(\sum_{k=0}^{N-1} (H_{-1})_k^{\left(n+\frac{1}{2}\right)} \Delta x - \sum_{k=0}^{N-1} (H_{-1})_k^{\left(n-\frac{1}{2}\right)} \Delta x\right) &= \sum_{k=0}^{N-1} \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} \frac{U_k^{(n+1)} - U_k^{(n-1)}}{2\Delta t} \Delta x \\
&= \sum_{k=0}^{N-1} \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} \\
&\quad \times \left\{ \left(1 - \delta_k^{\langle 2\rangle}\right)^{-1} \delta_k^{\langle 1\rangle} \left(4 - \delta_k^{\langle 2\rangle}\right) \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} \right\} \Delta x \\
&= 0.
\end{aligned}$$

The first equality is from (13), the second is from (14), and the third is from the skew-symmetry of the operator $\left(1 - \delta_k^{\langle 2\rangle}\right)^{-1} \delta_k^{\langle 1\rangle} \left(4 - \delta_k^{\langle 2\rangle}\right)$ (Lemma 2.1). Next we prove the discrete H_0 conservation law. To this end, we note the following identity:

$$\begin{aligned}
\frac{1}{\Delta t}\left(\sum_{k=0}^{N-1} (H_0)_k^{(n+1)} \Delta x - \sum_{k=0}^{N-1} (H_0)_k^{(n-1)} \Delta x\right) &= -9 \sum_{k=0}^{N-1} \left(1 - \delta_k^{\langle 2\rangle}\right) \frac{U_k^{(n+1)} - U_k^{(n-1)}}{2\Delta t} \Delta x \\
&= -9 \sum_{k=0}^{N-1} \delta_k^{\langle 1\rangle} \left(4 - \delta_k^{\langle 2\rangle}\right) \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} \Delta x \\
&= -\frac{9}{2} \left[\left(4 - \delta_k^{\langle 2\rangle}\right) \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} \right. \\
&\quad \left. + \left(4 - \delta_k^{\langle 2\rangle}\right) \frac{\delta H_{-1}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_{k-1}} \right]_{k=0}^{N} \\
&= 0.
\end{aligned}$$

The second equality is from (14). By substituting $\boldsymbol{V} = (1, 1, \ldots, 1)^\top$ into (5), we obtain the third equality. The final is from the discrete periodic boundary condition. From the identity, $\sum_k (H_0)_k^{(n+2)} \Delta x = \sum_k (H_0)_k^{(n)} \Delta x$ $(n = 0, 1, 2, \ldots)$. Since in the scheme we demanded the first time step to be conservative: $\sum_k (H_0)_k^{(1)} \Delta x = \sum_k (H_0)_k^{(0)} \Delta x$, $\sum_k (H_0)_k^{(n)} \Delta x = \sum_k (H_0)_k^{(0)} \Delta x$ holds for every n. □

As for Scheme 2, the unique existence of the solution is guaranteed if the time mesh size is sufficiently small as the next theorem states.

Theorem 3.4 (*Unique Solvability of* Scheme 2). *For $n \geq 1$, let $\boldsymbol{U}^{(n-1)}, \boldsymbol{U}^{(n)}$ be given. Then if $\Delta t > 0$ is sufficiently small,* Scheme 2 *has a unique numerical solution $\boldsymbol{U}^{(n+1)}$. For example, a sufficient condition is*

$$\Delta t < \frac{3\Delta x^3}{4\|\boldsymbol{U}^{(n)}\|(1+\Delta x^2)}. \tag{15}$$

Proof. Scheme 2 can be rewritten as follows:

$$\left[\frac{I - D^{\langle 2\rangle}}{2} + \Delta t \left\{D^{\langle 1\rangle}(4I - D^{\langle 2\rangle})\frac{U^{(n)}}{6}\right\}\right]\boldsymbol{U}^{(n+1)} = \boldsymbol{F}(\boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})$$

where $U^{(n)} = \operatorname{diag}(U_0^{(n)}, \dots, U_{N-1}^{(n)})$, and $\boldsymbol{F}$ denotes the remaining terms with $\boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)}$. Let us consider the coefficient matrix of the left hand side. Since $I - D^{\langle 2\rangle}$ is nonsingular, the coefficient matrix should be nonsingular for sufficiently small Δt by the continuity of the determinant. The sufficient condition can be checked by noting that if

$$\left\|\left(\frac{I - D^{\langle 2\rangle}}{2}\right)^{-1}\right\| \cdot \left\|\Delta t \left\{D^{\langle 1\rangle}(4I - D^{\langle 2\rangle})\frac{U^{(n)}}{6}\right\}\right\| < 1, \tag{16}$$

then the coefficient matrix is nonsingular [30, Theorem 2.3.4]. With the estimates:

$$\left\|\left(\frac{I - D^{\langle 2\rangle}}{2}\right)^{-1}\right\| = 2\left\|\left(I - D^{\langle 2\rangle}\right)^{-1}\right\| = 2,$$

$$\begin{aligned}\left\|\Delta t \left\{D^{\langle 1\rangle}(4I - D^{\langle 2\rangle})\frac{U^{(n)}}{6}\right\}\right\| &\leq \frac{\Delta t}{6}\|D^{\langle 1\rangle}\| \cdot \left\|4I - D^{\langle 2\rangle}\right\| \cdot \|U^{(n)}\| \\ &\leq \frac{\Delta t}{6}\frac{1}{\Delta x}\left(4 + \frac{4}{\Delta x^2}\right)\|U^{(n)}\| \\ &= \frac{2\Delta t(1+\Delta x^2)\|U^{(n)}\|}{3\Delta x^3},\end{aligned}$$

we get (15). □

The conditions on Δt in Theorems 3.2 and 3.4 are sufficient, but necessary. We checked numerically that the schemes practically work well even if the parameters do not satisfy the conditions in the theorems.

3.4. Crank–Nicolson scheme

For comparison, we employ the standard Crank–Nicolson Scheme constructed based on the

$$\left(1 - \partial_x{}^2\right)u_t = \partial_x(-2u^2 + u_x{}^2 + uu_{xx}). \tag{17}$$

We can also easily check that the Crank–Nicolson scheme preserves H_0. This scheme is a nonlinear scheme, like as Scheme 1.

Scheme 3 (H_0 *Conservative Crank–Nicolson Scheme*). *We define the initial solution by $\boldsymbol{U}^{(0)}$ as $U_k^{(0)} = u(0, k\Delta x)$ $(k = 0, \dots, N-1)$. Then, for $n = 0, 1, \dots,$*

$$\begin{aligned}\left(1 - \delta_k^{\langle 2\rangle}\right)\left(\frac{U_k^{(n+1)} - U_k^{(n)}}{\Delta t}\right) = \delta_k^{\langle 1\rangle}&\left[-2\left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2}\right)^2 + \left\{\delta_k^{\langle 1\rangle}\left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2}\right)\right\}^2\right. \\ &\left. + \left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2}\right)\delta_k^{\langle 2\rangle}\left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2}\right)\right], \quad k = 0, \dots, N-1.\end{aligned}$$

The numerical solution by Scheme 3 conserves H_0 (but not H_{-1}).

Theorem 3.5 (Scheme 3*: Conservation Law*). *Under the discrete periodic boundary condition, the numerical solution by* Scheme 3 *conserves H_0.*

$$\sum_{k=0}^{N-1}(H_0)_k^{(n)}\Delta x = \sum_{k=0}^{N-1}(H_0)_k^{(0)}\Delta x, \quad n = 0, 1, \dots.$$

Proof. As noted in Remark 1, it is obvious from the summation-by-parts formula (6) and the periodic boundary condition. □

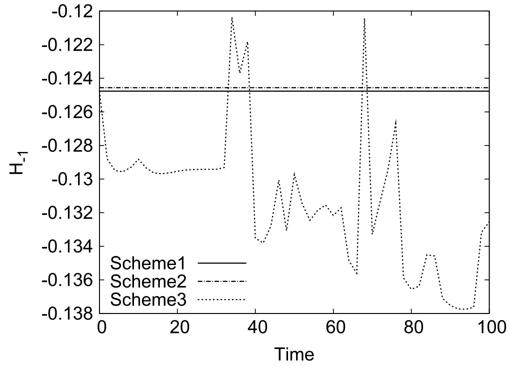

Fig. 1. Evolution of H_{-1}.

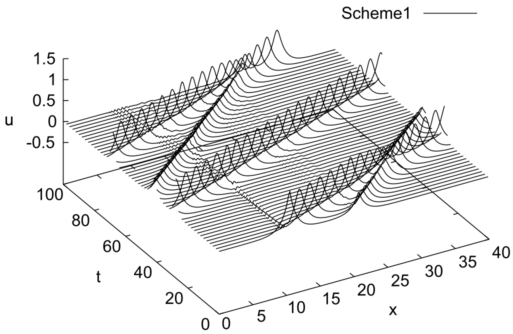

Fig. 2. The numerical solution obtained by Scheme 1 (H_{-1}, H_0-conserving nonlinear scheme) with $\Delta x = 40/2^8$ and $\Delta t = 0.05$.

4. Numerical examples

In this section, we test Schemes 1 and 2 numerically. For comparison, we also show the results by Scheme 3. Computation environment is CPU Xeon (3.00 GHz), 16 GB memory, Linux OS. We used MATLAB (R2007b), and nonlinear equations in Schemes 1 and 3 are solved by "fsolve" in MATLAB with tolerance $TolX = 10^{-16}$ and $TolFun = 10^{-16}$.

First we compare Schemes 1–3 for 2-peakon interaction. It is known that the DP has a solution called "multi-peakon" solutions [8]:

$$u(t,x) = \sum_{i=1}^{n} p_i(t)\mathrm{e}^{-|x-q_i(t)|}.$$

The parameters are set as follows: $n = 2$, $x \in [0, 40]$, $t \in [0, 100]$, $\Delta x = 40/2^8 = 0.15625$, $\Delta t = 0.05$, and the initial value is $u(0,x) = \exp(-|x-10.43|) + 0.5\exp(-|x-20.66|)$.

Fig. 1 shows the evolutions of H_{-1}. According to Theorems 3.1, 3.3 and 3.5, Schemes 1 and 2 conserve both H_{-1} and H_0, but Scheme 3 conserves only H_0. We confirmed that the three schemes preserve the value of H_0, 13.49730 up to at least 5 significant digits. On the other hand, Fig. 1 shows that Scheme 3 actually does not conserve H_{-1}, and the deviation (10% or more) is observed.

Figs. 2 and 3 show the numerical solutions by Schemes 1 and 3. We here omit the numerical solution by Scheme 2 because Schemes 1 and 2 produce visually identical results. We observe that, compared to the conservative Schemes 1 and 2, the numerical solution by Scheme 3 (Crank–Nicolson) is oscillating and unstable. Thus proposed schemes (Schemes 1 and 2) are better in the sense of qualitative behavior. We also note that Scheme 1 and the Crank–Nicolson scheme are both nonlinear schemes, and requires almost the same computational cost. This implies that Scheme 1 is preferred over the Crank–Nicolson scheme.

Next we compare Schemes 1 and 2 in detail. Here we set the space mesh size $\Delta x = 40/2^9 = 0.078125$, and tried three time mesh sizes: $\Delta t = 4/56 \simeq 0.109$, $\Delta t = 0.1$, and $\Delta t = 0.01$. Initial value was set to $u(0,x) = 1.5\exp(-|x-20.1|)$. Figs. 4 and 5 are the numerical solutions by Schemes 1 and 2 at $t = 50$. In case of $\Delta t = 0.01$, the numerical solution by Schemes 1 and 2 are both fine. But as Δt gets large, the results by Scheme 2 become slightly worse. In fact, when $\Delta t = 0.1$, unstable oscillations appear in the result of Scheme 2 in the whole spatial interval (Fig. 5). We also find tiny oscillation in Scheme 1 (Fig. 4), but that is limited to the initial peak position. (The small oscillation seems to be caused by the singular initial solution; this phenomena is often observed in other peakon simulations.) For $\Delta t = 0.109$, oscillations in Scheme 2 get

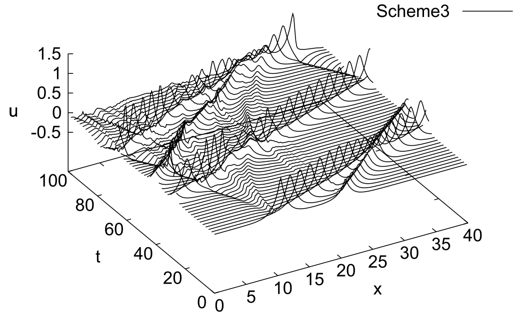

Fig. 3. The numerical solution obtained by Scheme 3 (H_0-conserving Crank–Nicolson scheme) with $\Delta x = 40/2^8$ and $\Delta t = 0.05$.

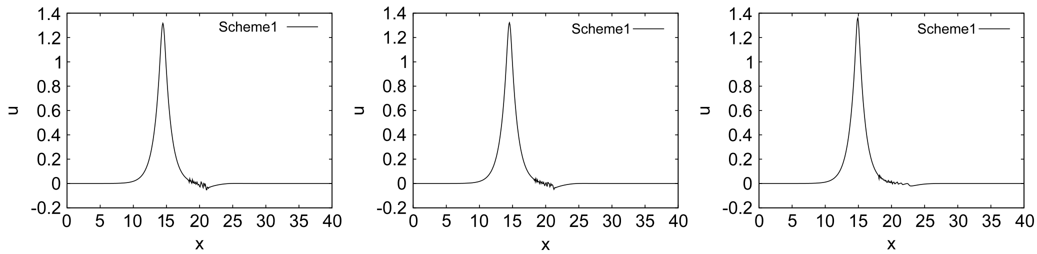

Fig. 4. The numerical solution of 1-peakon solution obtained by Scheme 1: (left) $\Delta t = 0.109$, (center) $\Delta t = 0.1$, (right) $\Delta t = 0.01$.

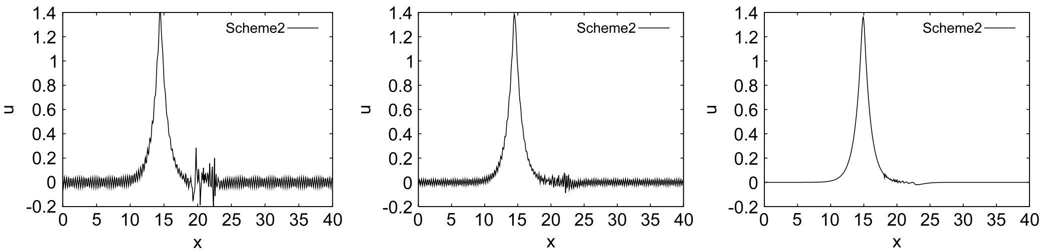

Fig. 5. The numerical solution of 1-peakon solution obtained by Scheme 2: (left) $\Delta t = 0.109$, (center) $\Delta t = 0.1$, (right) $\Delta t = 0.01$.

Table 1
Computation time in Schemes 1 and 2.

Δx	$40/2^5$ (s)	$40/2^6$ (s)	$40/2^7$ (s)	$40/2^8$ (s)	$40/2^9$ (s)	$40/2^{10}$ (s)
Scheme 1	0.14498	0.37947	1.18499	6.36891	67.4376	1144.05
Scheme 2	0.01262	0.02674	0.07658	0.42090	4.07037	64.9714

even worse. We observed that these oscillations grew bigger with time. Therefore we conclude that, from the perspective of numerical stability, Scheme 1 is the best among the three schemes.

Next we compare the computation time. Table 1 shows the computation times of Schemes 1 and 2 required to proceed for 20 time steps for various Δx. Here we set the time mesh size $\Delta t = 0.05$. Initial value was set to $u(0, x) = 1.5 \exp(-|x - 20.1|)$. In contrast to Scheme 2, Scheme 1 costs much more when the space mesh size Δx is small. The increase of computation time in Scheme 2 is much more moderate. Furthermore, the increase is mainly due to the first step where we used Scheme 1; this can be relaxed by replacing Scheme 1 with other cheap integrators.

The above two experiments suggest that there is a trade-off between stability and computation time, as is often observed. In the present case, then, how can we decide which one should be used? In order to answer this question, we carried out many numerical experiments under various conditions, and came to a conclusion that in most practical cases Scheme 2 is more efficient.

Let us demonstrate this fact in the following experiment. In this experiment, we first prepared two settings by which Scheme 2 could obtain reasonable soliton evolutions: in Fig. 6, one soliton solution was integrated for $0 \le t \le 40$ with $\Delta x = 40/2^8$, $\Delta t = 0.05$; in Fig. 7, it was integrated for longer period $0 \le t \le 100$ with the same meshes. Then we tested Scheme 1 with various Δx and Δt under the condition that the computation time was equal to or less than the corresponding one by Scheme 2. Let us first see Fig. 6. The results show that in all cases Scheme 1 yielded worse results. This in other words

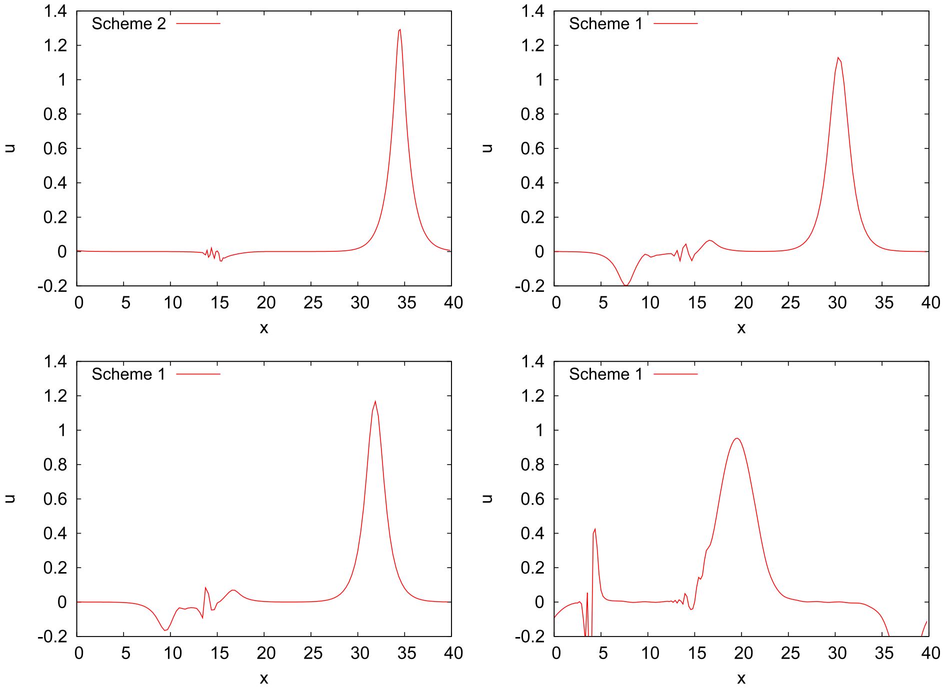

Fig. 6. The numerical solutions of 1-peakon solution at $t = 40$ obtained by: (left-top) Scheme 2, $\Delta x = 40/2^8$, $\Delta t = 0.05$, (right-top) Scheme 1, $\Delta x = 40/2^7$, $\Delta t = 0.5$, (left-bottom) Scheme 1, $\Delta x = 40/2^7$, $\Delta t = 0.33$, (right-bottom) Scheme 1, $\Delta x = 40/192$, $\Delta t = 2$.

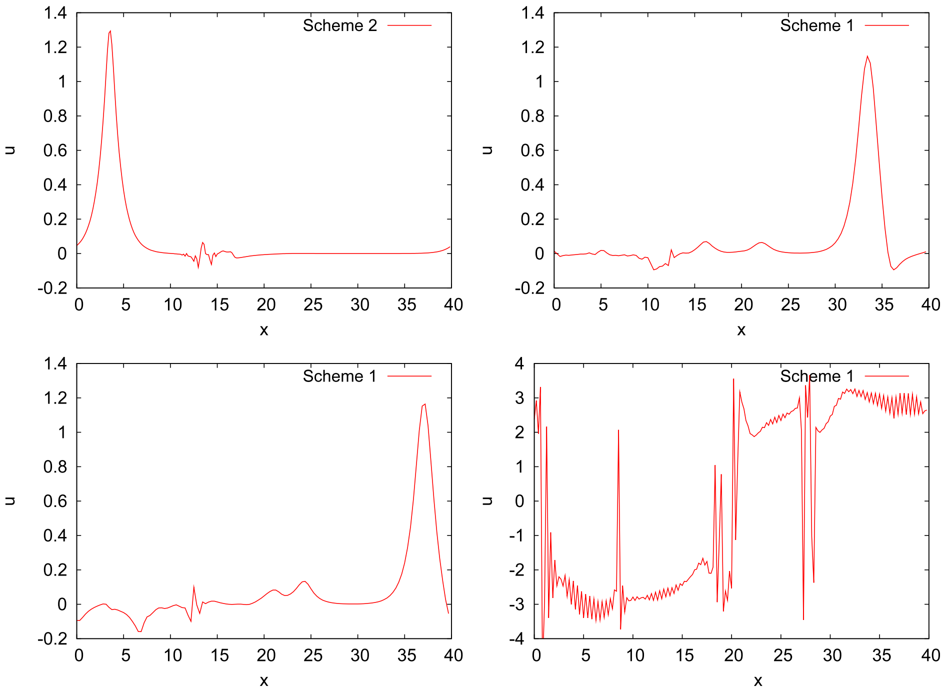

Fig. 7. The numerical solutions of 1-peakon solution at $t = 100$ obtained by: (left-top) Scheme 2, $\Delta x = 40/2^8$, $\Delta t = 0.05$, (right-top) Scheme 1, $\Delta x = 40/2^7$, $\Delta t = 0.5$, (left-bottom) Scheme 1, $\Delta x = 40/2^7$, $\Delta t = 0.33$, (right-bottom) Scheme 1, $\Delta x = 40/192$, $\Delta t = 2$.

means Scheme 1 is more expensive, for this problem, in order to obtain a solution of the same quality. Fig. 7 shows the same tendency in a longer integration. It should be noted, however, these results strongly depend on how we solve the nonlinear scheme and also on the computation environment.

5. Concluding remarks

We have proposed two finite difference schemes for the DP equation preserving both H_{-1} and H_0. We also proved the unique existence of the numerical solutions for the schemes. Numerical examples on 2-peakon solution indicated that proposed schemes are more stable than the standard Crank–Nicolson scheme. The last numerical experiment showed that Scheme 2 is better than Scheme 1 from a practical point of view.

Our future works include the followings.

- In this paper, we proved the unique solvability of the proposed schemes, but we did not mention the theoretical stability in terms of $\|\boldsymbol{U}^{(n)}\|_\infty$, nor the convergence of the schemes. This is partly because of the restriction of space, and more essentially due to the fact that the DP can develop discontinuous entropy solutions, and even in the original continuous DP equation, it is not easy to bound $\|u\|_\infty$. Now we are trying to establish such estimates for sufficiently smooth global solutions, and the result will be presented in the near future elsewhere.
- In this paper, we derived conservative schemes in the finite-difference context using the general framework (the discrete variational derivative method) [15]. However, it is difficult to do the same thing based on the Galerkin version [18]. As mentioned in the introduction, this is caused by the substantial difference in the bi-Hamiltonian structures between the CH and DP (it is much more complicated in the DP). Whether conservative Galerkin schemes can be constructed for the DP or not is an interesting research topic, and we are now working on this as well.
- In this paper, we focused on sufficiently smooth (at least continuous) solutions, for which H_{-1} and H_0 conservation laws hold without any problems. But generally for these integrable equations, conservation laws can become meaningless as the solution loses smoothness. In the CH case, the two invariants (out of infinitely many invariants) that derive the bi-Hamiltonian structure is proved to be conserved even for peakon solutions [31], but to the best of the author's knowledge, it is not yet known whether the other invariants are also kept or not. For the DP, H_{-1} and $\int u\,\mathrm{d}x$ were proved to make sense and in fact remain constant for peakon solutions, but other invariants are yet to be investigated (note that the latter is not H_0). Even worse, the DP has entropy solutions; for instance, when peakon and antipeakon solutions collide, the solution inevitably becomes an entropy solution, and in such a circumstance, invariants of the DP are no longer preserved [11]. Whether under such circumstances conservative schemes make any sense or not is still an open question. In this sense, in the DP case, there seems to be another trade-off between conservative schemes like Schemes 1 and 2 (which should work better for smooth solutions) and those focusing on entropy solutions (better for singular solutions). Whether we can construct a ''hybrid'' scheme that inherits both of the advantages or not is also an interesting point.

References

[1] Y. Miyatake, T. Matsuo, Conservative finite difference schemes for the Degasperis–Procesi equation, Trans. Japan Soc. Ind. Appl. Math. 20 (2010) 219–239 (in Japanese).
[2] A. Degasperis, M. Procesi, Asymptotic integrability, in: Symmetry and Perturbation Theory, World Scientific Publishing, River Edge, 1999, pp. 23–37.
[3] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their bäcklund transformations and hereditary symmetries, Physica D 4 (1981) 47–66.
[4] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
[5] R. Camassa, D.D. Holm, J. Hyman, A new integrable shallow water equation, Adv. Appl. Mech. 31 (1993) 1–33.
[6] R.S. Johnson, Camassa–Holm, Korteweg–de Vries and related models for water, J. Fluid Mech. 455 (2002) 63–82.
[7] A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, Arch. Ration. Mech. Anal. 192 (2009) 165–186.
[8] A. Degasperis, D.D. Holm, A.N.W. Hone, A new integrable equation with peakon solutions, Theoret. and Math. Phys. 133 (2002) 1463–1474.
[9] G.M. Coclite, K.H. Karlsen, On the well-posedness of the Degasperis–Procesi equation, J. Funct. Anal. 233 (2006) 60–91.
[10] G.M. Coclite, K.H. Karlsen, On the uniqueness of discontinuous solutions to the Degasperis–Procesi equation, J. Differential Equations 234 (2007) 142–160.
[11] H. Lundmark, Formation and dynamics of shock waves in the Degasperis–Procesi equation, J. Nonlinear Sci. 17 (2007) 169–198.
[12] Z. Yin, Global existence for a new periodic integrable equation, J. Math. Anal. Appl. 283 (2003) 129–139.
[13] Z. Yin, Global weak solution for a new periodic integrable equation with peakon solutions, J. Funct. Anal. 212 (2004) 182–194.
[14] J. Escher, Y. Liu, Z. Yin, Global weak solutions and blow-up structure for the Degasperis–Procesi equation, J. Funct. Anal. 241 (2006) 457–485.
[15] D. Furihara, D. Mori, General derivation of finite difference schemes by means of a discrete variation, Trans. Japan Soc. Ind. Appl. Math. 8 (1998) 317–340 (in Japanese).
[16] E. Celledoni, V. Grimm, R.I. McLachlan, D.I. McLaren, D.R.J. O'Neale, B. Owren, G.R.W. Quispel, Preserving Energy Resp. Dissipation in Numerical PDEs, Using the Average Vector Field Method, NTNU Preprint Series: Numerics No.7/2009. http://www.math.ntnu.no/preprint/numerics/2009/N7-2009.pdf.
[17] R.I. McLachlan, G.R.W. Quispel, N. Robidoux, Geometric integration using discrete gradients, Philos. Trans. R. Soc. Lond. A Math. Phys. Eng. Sci. 357 (1999) 1021–1045.
[18] T. Matsuo, Dissipative/conservative Galerkin method using discrete partial derivatives for nonlinear evolution equations, J. Comput. Appl. Math. 218 (2008) 506–521.
[19] A.A. Abdelgadir, Y. Yao, Y. Fu, P. Huang, A Difference Scheme for the Camassa–Holm Equation, in: Lecture Notes in Comput. Sci., vol. 4682, 2007, pp. 1287–1295.
[20] K. Takeya, Conservative finite difference schemes for the Camassa–Holm equation, Master's Thesis, Osaka University, 2007 (in Japanese).
[21] K. Takeya, D. Furihata, Conservative Finite Difference Schemes for the Camassa–Holm Equation (in preparation).
[22] T. Matsuo, A Hamiltonian-conserving Galerkin scheme for the Camassa–Holm equation, J. Comput. Appl. Math. 234 (2010) 1258–1266.
[23] T. Matsuo, H. Yamaguchi, An energy-conserving Galerkin scheme for a class of nonlinear dispersive equations, J. Comput. Phys. 228 (2009) 4346–4358.
[24] T. Matsuo, D. Furihata, Dissipative or conservative finite-difference schemes for complex-valued nonlinear partial differential equations, J. Comput. Phys. 171 (2001) 425–447.
[25] G.M. Coclite, K.H. Karlsen, N.H. Risebro, Numerical schemes for computing discontinuous solutions of the Degasperis–Procesi equation, IMA J. Numer. Anal. 28 (2008) 80–105.
[26] B.F. Feng, Y. Liu, An operator splitting method for the Degasperis–Procesi equation, J. Comput. Phys. 228 (2009) 7805–7820.

[27] H.A. Hoel, A numerical scheme using multi-shock peakons to compute solutions of the Degasperis–Procesi equation, Electron. J. Differential Equations 2007 (2007) 1–22.
[28] H. Brezis, Analyse Fonctionnelle, Masson, Paris, 1983.
[29] D. Furihata, Finite difference schemes for $\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{\alpha} \frac{\delta G}{\delta u}$ that inherit energy conservation or dissipation property, J. Comput. Phys. 156 (1999) 181–205.
[30] G.H. Golub, C.F. Van Loan, Matrix Computations, third ed., Johns Hopkins University Press, Baltimore, 1996.
[31] A. Constantin, L. Molinet, Global weak solutions for a shallow water equation, Comm. Math. Phys. 211 (2000) 45–61.