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A dynamical communication system on a network

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ABSTRACT

A dynamical system is introduced and investigated. The system contains N vertices. The vertices send messages at discrete time instants according to a given rule. A conflict of two vertices takes place if the vertices try to send messages to each other at the same instant. Each vertex sends a message to another vertex at every step if no conflict takes place. In case of a conflict, only one of the two competing vertices sends a message. Deterministic and stochastic conflict resolution rules are considered. We investigate the average number of messages sent by a vertex per a time unit, called the productivity of this vertex, the total productivity of the system and other characteristics. The productivity of vertices depends on the initial state of the system, and the criterion of efficiency is the expected average productivity of vertices provided all possible initial states of the system are equiprobable. An ergodic version of the system is also considered in which any particle moves with approximately equal to 1 probability provided there is no conflict.

1. Model of a network

Consider a network, which contains N vertices. Each vertex can send a message to another vertex during a discrete time. The vertices are connected in accordance with a symmetrical *communication matrix*

$$C = \left(c_{ij}\right).$$

Each element equals 0 if no message is sent from the vertex i to the vertex j, $c_{ij} = 1$ if such transmission is possible at a discrete time period. Suppose

$$c_{ij} = c_{ji}, \quad i \neq j,\ i = 1, \dots, N,\ j = 1, \dots, N.$$

2. Dynamical communication system

2.1. General rules

The vertex i, $i = 1, \dots, N$, can send a signal to vertices $j = 1, \dots, N$ at discrete steps alternately. The vertex i can send a signal i to the vertex j if $c_{ij} = 1$ (there is a communication channel between the vertices i and j), and the vertex j does not try to send a signal to the vertex i during the same time period.

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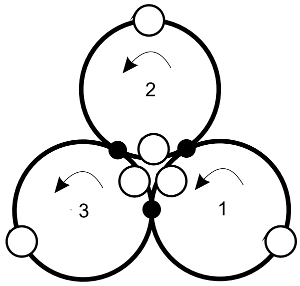

Fig. 1. A necklace with movement in one direction, $N = 3$, type 3(2).

If $c_{ij} = 1$, the vertex i tries to send a signal to the vertex j, and the vertex j tries to send a signal to the vertex i during the same time interval, then *a competition of the vertices i and j takes place, and only the vertex, winning the competition, sends a signal during this time interval.*

If there is a competition of vertices i and j, the vertex i wins the competition with probability p_{ij},

$$p_{ij} + p_{ji} = 1, \quad i \neq j,\ i, j = 1, \ldots, N.$$

The vertex, which is winning a competition, comes to the next state. The state of the loser does not change. The loser sends a message at the next step, i.e., the message is delayed.

We say that *the vertex i is in the state j if the vertex i sends a signal to the vertex* j, $i, j = 1, \ldots, N$.

Suppose $\vec{S} = (S_1, \ldots, S_N)$ is the vector such that S_i is the state of the vertex i, $i = 1, \ldots, N$ at the current time. *The vector* $\vec{S}$ *is called the state of the network.*

Now we give the following definition. We say that the system is in the state of *synergy* if there are no same coordinates with the same indexes in the vectors

$$S(T) = (S_1(T), \ldots, S_N(T))$$

and

$$S(T+1) = (S_1(T+1), \ldots, S_N(T+1))$$

since a time T_{syn}.

Consider two approaches. They are the *global synchronization* and the *local synchronization.*

2.2. Total synchronization of vertices behavior

The period of the message sending process is equal to the number of vertices. Rows of the communication matrix are processed simultaneously. So all elements of the matrix C are processed including zeros. If $c_{ij} = 0$, then no signal is sent from the vertex i to the vertex $j + 1$, and the vertex i comes to the state $j + 1$, i.e., to the state such that the vertex can send a signal to the vertex $j + 1$. If $j = N$, then a transition to the vertex 1 takes place.

2.3. Local chronometer of a signal source

Suppose every vertex of a dynamical system has its own chronometer, i.e., the vertex i passes the vertex j if $c_{ij} = 0$, and sends signal to the vertex j if $c_{ij} = 1$, and each vertex chooses alternately all states such that $c_{ij} = 1$. We describe the rules of competition below. Local indexes of vertices states can be also introduced.

2.4. Transport analogy

The considered problem is a generalization of the following transport construction, [1–3]. Each vertex can be interpreted as a contour, and there are *arranged cells* on the contour. A *single particle* occupies a cell, and can move according to given rules at discrete time units. There is a *node* between two neighboring cells. The node is *the point of a junction of two contours. An element of the communication matrix equals* 1, *if this element corresponds to the junction.* Such constructions have been studied in the literature [1].

The simplest example is the case of $N = 3$, Fig. 1. In this figure, black points correspond to nodes, and the small rings correspond to cells. A possible approach to numerate cells is to assign to each cell the index equal to the index of the common node running in the direction of movement.

We have the following communication matrix:

$$C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We say that the rule is *egalitarian* if the competition matrix has the form

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}.$$

2.5. Topological characterization of the network

Now we introduce the class of regular networks and give the following definitions. *A network is called regular* if the number of non-zero elements is the same for each row (for each column because of symmetry). This number is called the *index of the regular network.* So a network of index 2 is called *necklace.* A network of index 3 is called *honeycombs,* and a network of index 4 is called *chainmail,* [1–8]. If a network is not regular, *the type of the network* is defined with a vector. The values of coordinates of this vector are numbers of vertices. The coordinates are arranged such that these numbers decrease. We give examples below.

3. Competition of particles

In [1] a rule, called *egalitarian,* has been introduced. A symmetrical matrix P corresponds to this rule.

For example, in the case of a necklace, Fig. 3, if the rule is *egalitarian* the competition matrix has the form

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}. \tag{1}$$

If for any pair $(i, j), i \neq j$, such that $c_{ij} = c_{ji} = 1$ one of the elements p_{ij} and p_{ji} of the matrix P equals 1, and therefore the other element equals 0, then we say that the system works under the *priority rule.* If $c_{ij} = c_{ji} = 0$, then we suppose $p_{ij} = p_{ji} = 0$.

A vertex i is called father if $p_{ij} = 1$ for any j such that $c_{ij} = 1$. In this case, the other vertices are called *mothers.*

We can suppose that the matrix P is a function of time. For example, the following matrix corresponds to the priority, alternating during time

$$P(T) = \begin{pmatrix} 0 & 0.5(1+(-1)^T) & 0.5(1+(-1)^T) \\ 0.5(1-(-1)^T) & 0 & 0.5(1+(-1)^T) \\ 0.5(1-(-1)^T) & 0.5(1-(-1)^T) & 0 \end{pmatrix}. \tag{2}$$

4. Communication systems with four vertices

4.1. Necklace, $N = 4$, [5]

Consider the case of a necklace, $N = 4$, Fig. 2.

Let us numerate vertices from the left to the right. In Fig. 2, the nodes are small black points. A vacant cell is a large white circle. An occupied cell is a large black circle. States of the system are coded in accordance with the next vertex number. There is an alternative method of numerating: the state of vertex equals 2, if corresponding particle has upper position, and 1 otherwise.

We have the following communication matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

For example, in the case of the *father–mother* rule, we have the following competition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

4.2. Simplex, $N = 4$

Consider the full graph, $N = 4$. Each vertex of this graph is connected with every vertex, Fig. 3.

We have the following communication matrix:

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{5}$$

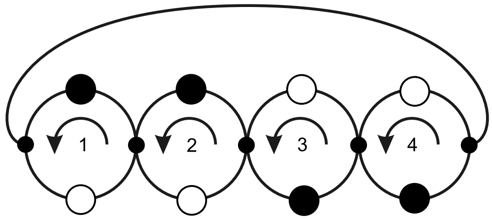

Fig. 2. A necklace with movement in one direction, $N = 4$, type 4(2), $S = (4, 1, 4, 1)$.

5. Necklaces, chainmails, and simplexes, $N = 9$

In [1,2,4,8] a periodic two-dimensional regular network of index 4, that is a chainmail on torus, has been investigated thoughtfully. The first non-trivial dimension, in the case of which the neighbors of rings are not duplicated, is a square of the dimension 3×3 containing 9 rings. Consider this minimal dimension of a network, for which there exists a periodic transport analogue that is a chainmail on a torus.

5.1. A necklace

If

$$c_{i,i+1} = c_{i+1,i} = 1,$$

$i = 1, \ldots, N$, (the addition is meant modulo N),

$$c_{ij} = 0, \quad |i - j| \neq 1,$$

i.e., only the neighboring vertices are joined, we have a necklace, Fig. 4. In case of $N = 9$ we have the following communication matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

5.2. Chainmail on a torus

The local coordinates have been introduced in [2] for more convenient representation of the system states as a matrix with a given set of elements from 1 until 4, Fig. 5.

Let us numerate vertices from the left to the right in the rows, and downwards, Fig. 6.

If $N = 9$, we have the communication matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{6}$$

6. Characteristics of a dynamical communication system on a network

Let us introduce a measure on the set of initial states $S(0)$ of the system. *By default, we assume that this measure is uniform.*

Now we introduce the concept of *productivity of vertices*. In accordance with rules, introduced above, at each step, the state of the vertex does not change or the vertex comes to the next state. Denote by $S_i(T, S(0))$ the number of changes of the vertex i state during the time interval $[0, T)$ if $S(0)$ is the initial state.

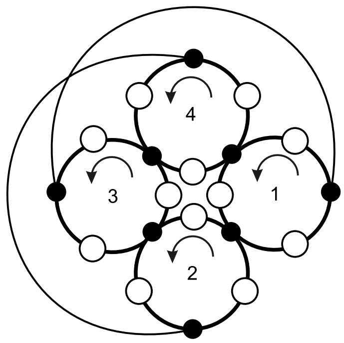

Fig. 3. A simplex, type 4(3).

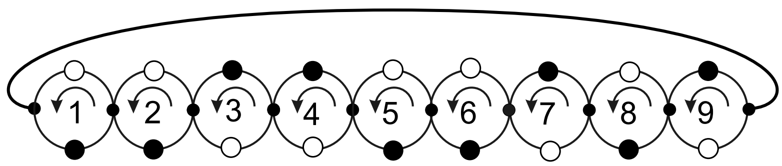

Fig. 4. Necklace, $N = 9$, $9(2)$, $S = (9, 3, 2, 3, 6, 7, 6, 9, 8)$.

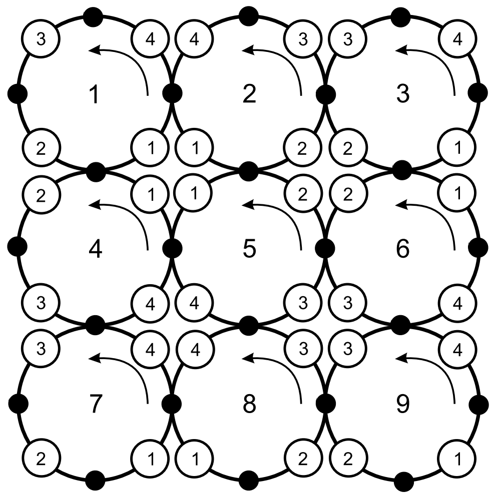

Fig. 5. A chainmail on a torus, type 9(4) with the local coordinates of cells.

We give the following definition. The limit

$$v_i(S(0)) = \lim_{T\to\infty} S_i(T, S(0))/T$$

is called *the productivity of the vertex* i, $i = 1, \ldots, N$.

We say that

$$v(T, S(0)) = \frac{1}{N}\sum_{i=1}^{N} v_i(T, S(0))$$

is *the average productivity of the network during the time* T.

We say that

$$v(S(0)) = \frac{1}{N}\sum_{i=1}^{N} v_i(S(0))$$

is *the average productivity of the network.*

We say that *a vertex is in the state of synergy beginning from an instant* T^* if from this instant this vertex changes its state at each step. We say that *the system is at the state of synergy* if since this instant each vertex of the network is in the state of synergy.

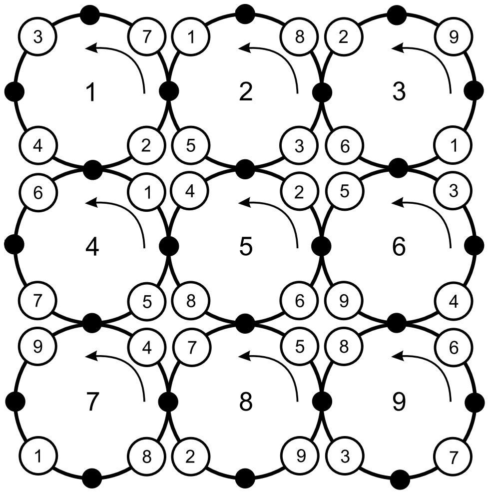

Fig. 6. A chainmail on a torus, type 9(4) with global coordinates of cells.

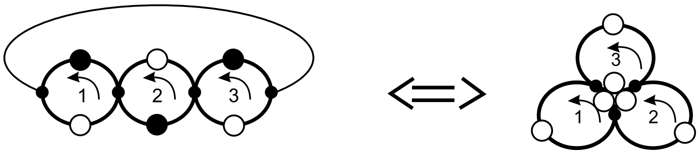

Fig. 7. Necklace and simplex are the same, $N = 3$, $S = (3, 3, 2)$.

If the system comes to the state of synergy, then $v_i = 1$ for any $i = 1, \ldots, N$, and $v = 1$.

The problem is to investigate characteristics of the system, first of all, the productivity of vertices and to find conditions of coming to the state of synergy determined by the structure, the competitions rules, etc.

7. General propositions

Denote by N_i the number of "ones" in the ith row of the communication matrix, $1 \leq i \leq N$.

Theorem 1. *Suppose $c_{ij} = 1$ for some i and j, i.e., vertices i and j are connected, $1 \leq i, j \leq N$. If the numbers N_i and N_j are coprime, then the state of the synergy is impossible.*

Proof. The number of possible states of the pair of vertices i and j is equal to N_iN_j. If the state of synergy were possible, then any state of this pair were repeated each N_iN_j steps. Since the numbers N_i and N_j are coprime, we have that no state can be repeated earlier than after N_iN_j steps. Hence the pair of vertices attends, during the period, all its possible states, and, in particular, the state when the vertex i sends a message to the vertex j, and the vertex j sends a message to the vertex i, but it is impossible if the system is in the state of synergy. □

Theorem 2. *Suppose $c_{ij} = 1$ for some i and j, i.e., the vertices i and j can be competing, $1 \leq i, j \leq N$. Suppose there exists a pair of vertices (i, j) such that the numbers (N_i, N_j) are coprime. Denote by K the least common multiple of the numbers N_i, N_j. Then, $v \leq \left(1 - \frac{1}{NK}\right)$.*

Proof. Since the numbers N_i and N_j are coprime, we have that, for any initial state, after no more than K steps, either the vertices i and j are competing or one of the vertices i and j and another vertex are competing. Therefore each K steps one of the vertices cannot send a message. □

8. Networks with three vertices

Consider quantitative characteristics of networks with three vertices, when the structures *of the necklace and the simplex are the same.*

8.1. Stochastic choice of priority

Consider the case of $N = 3$, Fig. 7.

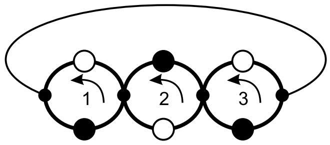

Fig. 8. The state $S3$.

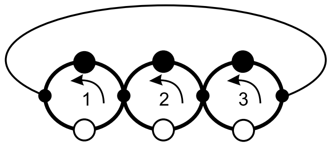

Fig. 9. The state $S8$.

In this case there are 8 possible states of the network, Figs. 8 and 9,

$$S1 = (2, 3, 1), \quad S2 = (2, 3, 2), \quad S3 = (2, 1, 1), \quad S4 = (2, 1, 2),$$
$$S5 = (3, 3, 1), \quad S6 = (3, 3, 2), \quad S7 = (3, 1, 1), \quad S8 = (3, 1, 2).$$

The communication matrix is

$$C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Theorem 3. *Suppose $N = 3$ and $0 < p_{ij} < 1$ for any $i, j = 1, 2, 3$, $i \neq j$. Then the system reaches to the state of synergy after time interval with a finite expectation. There exist initial states such that for any T the network, with a positive probability, does not reach the state of synergy before the time instant T.*

Proof. If $S1$ or $S8$ is the initial state, then the system comes by turns to each of these two states, i.e., we have a sequence of transitions

$$S1 \to S8 \to S1 \cdots,$$

and all vertices change their state at each step, i.e., the network is at the state of synergy.

In the state $S2$, a competition of the vertices 2 and 3 takes place. If the vertex 2 wins the competition, then the network comes to the state $S8$, and therefore the network comes to the state of synergy. If the vertex 3 wins the competition, then the network comes to the state $S5$.

In the state $S3$, a competition of the vertices 1 and 2 takes place. If the node 1 wins the competition, then the network comes to the state $S8$, and therefore the network comes to the state of synergy. If the vertex 2 wins the competition, then the network comes to the state $S2$.

In the state $S4$, a competition of the vertices 1 and 2 takes place. If the node 2 wins the competition, then the network comes to the state $S1$, and therefore the network comes to the state of synergy.

In the state $S5$, a competition of the vertices 2 and 3 takes place. If the vertex 3 wins the competition, then the network comes to the state $S8$, and therefore the network comes to the state of synergy. If the vertex 1 wins the competition, then the network comes to the state $S3$.

Therefore each of the states $S2$, $S3$, $S5$ can be repeated with a positive probability

$$S2 \to S5 \to S3 \to S2 \cdots.$$

On the other hand, at any step, the sequence of such transitions can break off, and the system comes to the state of synergy. Thus, if one of the states $S2$, $S3$, $S5$ is initial, then the system comes to the state of synergy, but the time to reach the state of synergy can be arbitrarily large. Namely, for any k the state of synergy the system can be not at the state of synergy at the time instant k.

In the state $S6$, a competition of vertices 2 and 3 takes place. If the vertex 3 wins the competition, then the network comes to the state $S1$, and therefore the network comes to the state of synergy.

A competition of vertices 1 and 3 takes place in the state $S7$. If the vertex 1 wins the competition, then the network comes to the state $S1$, and therefore the network comes to the state of synergy.

Thus, for any initial state $S2$, $S3$, $S4$, $S5$, $S6$, $S7$, the system can come to the state of synergy at once, but the time to reach the state of synergy can be arbitrarily large. □

Theorem 4. *Suppose $N = 3$ and the rule is egalitarian. If all permissible initial states are equiprobable, then, with probability 0.25, the initial state is a state of synergy, and, with probability 0.75, the number of steps k to reach the state of synergy is distributed geometrically, $p_k = \frac{1}{2^k}$, $k = 1, 2, \ldots$.*

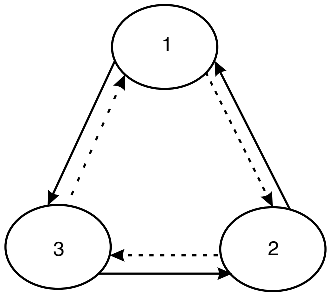

Fig. 10. Scheme of the priority.

Proof. We can see (proof of Theorem 3) that two of eight possible states, namely S_1 and S_8 are states of synergy. Therefore, the system has the state of synergy with probability 0.25, beginning from the initial time.

We can see also that, if the system is in one of the 6 states $S1, S2, S4, S5, S7, S8$, then this system, with probability 1/2, comes to the state of synergy at once. Hence, in this case, the probability that the network comes to the state of synergy for k steps is equal to $\frac{1}{2^k}$, $k = 1, 2, \ldots$ The probability of one of these states is initial equals 0.75. □

8.2. Deterministic priority

Suppose $p_{12} = p_{23} = p_{31} = 0$, $p_{21} = p_{32} = p_{13} = 1$. We call this rule *right-priority*, Fig. 10.
In this case the priority matrix is

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Theorem 5. *Suppose $N = 3$, and the rule is right-priority. Then the states $S1$ and $S8$ are states of synergy. If one of these two states is initial, then the sequence of transitions*

$$S1 \to S8 \to S1 \cdots$$

is repeated.

If one of the states $S1, S4, S7$ is initial, then the system comes to the state of synergy at the next step.

If one of the states $S2, S3, S5$ is initial, then the sequence of the transitions

$$S2 \to S5 \to S3 \to S2 \cdots$$

is repeated, and, during the period, equal to 3 steps, each vertex changes its state 2 times, i.e., the productivity of each vertex equals 2/3.

Proof. Theorem 5 is proved by exhaustion. □

9. Tetrahedron, $N = 4$

Consider the case of $N = 4$, Fig. 11, $0 < p_{ij} < 1$, $i, j = 1, 2, 3, 4$, $i \neq j$.
The number of possible states equals 81. These states are

$S1 = (2, 1, 1, 1)$, $S2 = (2, 1, 1, 2)$, $S3 = (2, 1, 1, 3)$,
$S4 = (2, 1, 2, 1)$, $S5 = (2, 1, 2, 2)$, $S6 = (2, 1, 2, 3)$,
$S7 = (2, 1, 4, 1)$, $S8 = (2, 1, 4, 2)$, $S9 = (2, 1, 4, 3)$,
$S10 = (2, 3, 1, 1)$, $S11 = (2, 3, 1, 2)$, $S12 = (2, 3, 1, 3)$,
$S13 = (2, 3, 2, 1)$, $S14 = (2, 3, 2, 2)$, $S15 = (2, 3, 2, 3)$,
$S16 = (2, 3, 4, 1)$, $S17 = (2, 3, 4, 2)$, $S18 = (2, 3, 4, 3)$,
$S19 = (2, 4, 1, 1)$, $S20 = (2, 4, 1, 2)$, $S21 = (2, 4, 1, 3)$,
$S22 = (2, 4, 2, 1)$, $S23 = (2, 4, 2, 2)$, $S24 = (2, 4, 2, 3)$,
$S25 = (2, 4, 4, 1)$, $S26 = (2, 4, 4, 2)$, $S27 = (2, 4, 4, 3)$,
$S28 = (3, 1, 1, 1)$, $S29 = (3, 1, 1, 2)$, $S30 = (3, 1, 1, 3)$,
$S31 = (3, 1, 2, 1)$, $S32 = (3, 1, 2, 2)$, $S33 = (3, 1, 2, 3)$,

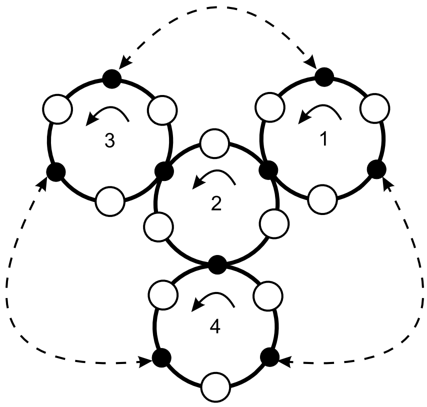

Fig. 11. Simplex, $N = 4$.

$$
\begin{array}{lll}
S34 = (3, 1, 4, 1), & S35 = (3, 1, 4, 2), & S36 = (3, 1, 4, 3), \\
S37 = (3, 3, 1, 1), & S38 = (3, 3, 1, 2), & S39 = (3, 3, 1, 3), \\
S40 = (3, 3, 2, 1), & S41 = (3, 3, 2, 2), & S42 = (3, 3, 2, 3), \\
S43 = (3, 3, 4, 1), & S44 = (3, 3, 4, 2), & S45 = (3, 3, 4, 3), \\
S46 = (3, 4, 1, 1), & S47 = (3, 4, 1, 2), & S48 = (3, 4, 1, 3), \\
S49 = (3, 4, 2, 1), & S50 = (3, 4, 2, 2), & S51 = (3, 4, 2, 3), \\
S52 = (3, 4, 4, 1), & S53 = (3, 4, 4, 2), & S54 = (3, 4, 4, 3), \\
S55 = (4, 1, 1, 1), & S56 = (4, 1, 1, 2), & S57 = (4, 1, 1, 3), \\
S58 = (4, 1, 2, 1), & S59 = (4, 1, 2, 2), & S60 = (4, 1, 2, 3), \\
S61 = (4, 1, 4, 1), & S62 = (4, 1, 4, 2), & S63 = (4, 1, 4, 3), \\
S64 = (4, 3, 1, 1), & S65 = (4, 3, 1, 2), & S66 = (4, 3, 1, 3), \\
S67 = (4, 3, 2, 1), & S68 = (4, 3, 2, 2), & S69 = (4, 3, 2, 3), \\
S70 = (4, 3, 4, 1), & S71 = (4, 3, 4, 2), & S72 = (4, 3, 4, 3), \\
S73 = (4, 4, 1, 1), & S74 = (4, 4, 1, 2), & S75 = (4, 4, 1, 3), \\
S76 = (4, 4, 2, 1), & S77 = (4, 4, 2, 2), & S78 = (4, 4, 2, 3), \\
S79 = (4, 4, 4, 1), & S80 = (4, 4, 4, 2), & S81 = (4, 4, 4, 3).
\end{array}
$$

Theorem 6. *Suppose $N = 4$ and $0 < p_{ij} < 1$ for any $i, j = 1, 2, 3, 4$, $i \neq j$. Then the system comes to the state of synergy after time interval with a finite expectation. There exist initial states such that for any T the network, with a positive probability, does not come to the state of synergy before the time T.*

Proof. The states $S12, S21, S22, S24, S31, S34, S35, S49, S62, S65, S66, S71$ are states of synergy.

If one of these states is initial, then one of the following sequences is repeated

$$
\begin{aligned}
&S12 \to S49 \to S62 \to S12 \ldots, S21 \to S31 \to S71 \to S21 \ldots, \\
&S22 \to S35 \to S66 \to S22 \ldots, S24 \to S34 \to S65 \to S24.
\end{aligned}
$$

From the states $S1, \ldots, S27$, i.e., from the states such that the vertex 1 is at the state 2, the system comes to the state of synergy after a finite number of steps, with positive probability, e.g., after one of the following sequences of transitions

$$
\begin{aligned}
&S1 \to S14 \to S51 \to S61 \to S65, S2 \to S15 \to S49, \\
&S3 \to S13 \to S50 \to S62, S4 \to S17 \to S48 \to S31, \\
&S5 \to S36 \to S66, S6 \to S34, \\
&S7 \to S29 \to S66, S8 \to S12, \\
&S9 \to S12, S10 \to S50 \to S62, \\
&S11 \to S51 \to S61 \to S65, S13 \to S50 \to S62, \\
&S14 \to S51 \to S61 \to S65, S15 \to S49, \\
&S16 \to S47 \to S56 \to S15 \to S49, S17 \to S48 \to S31,
\end{aligned}
$$

$$S18 \to S46 \to S56 \to S15 \to S49, S19 \to S32 \to S72 \to S21,$$
$$S20 \to S33 \to S70 \to S19 \to 32 \to S72 \to S21, S23 \to S35,$$
$$S25 \to S29 \to S66, S26 \to S48 \to S31,$$
$$S27 \to S34,$$

(the states $S12, S21, S22, S24$ are themselves states of synergy).

From the other states the network comes to one of the states $S1, \ldots, S27$ after a finite number of states, with positive probability.

Thus, from any initial state, the network comes to the state of synergy after a time with a finite expectation.

However there exist initial states such that the time interval to come to the state of synergy can be arbitrarily long. For example, the following sequence of transitions can be repeated

$$S8 \to S30 \to S64 \to S77 \to S1. \quad \square$$

10. Optimization of the competition matrix *P*

Suppose C is the communication matrix, and the matrix $P = p_{ij}$ is the competition matrix, $p_{ij} + p_{ji} = 1\ \forall i, j \neq j$. Suppose that all possible states are equiprobable. Denote by V the expectation of the average productivity of network vertices.

We can consider the problem *to find, from the class of possible matrix P, maximizing the value of V, if the communication matrix C is given.*

10.1. Pendulum rule of competition

The competition matrix consists of zeros and ones. However the matrix depends on time with the period $T = 2$

$$P(1) = P(3) = P(5) \ldots, P(2) = P(4) = P(6) \ldots, P^*(1) = P(2),$$

where P^* is the matrix transposed to P. We have a deterministic analogue of the egalitarian rule The problem is to optimize the initial state of the competition matrix.

10.2. Stepwise optimization of priority

At each step, an own priority matrix., i.e., a competition matrix of a special type is given. In accordance with this matrix the priority is given to particles such that, after transitions of these particles, we get a better state. The concept of a better state is defined depending on the aim. In our problem, the main aim is to reach the maximum possible productivity of system, in particular, to reach the state of synergy certainly.

10.3. Necklace

Consider a necklace. *Suppose the rule is right-priority.* In [5] it has been proved that, if all initial states are equiprobable, then, for any N, the expected average productivity of vertices equals $7/8$. *If the rule is egalitarian,* then, after a time interval with a finite expectation, the system comes to the state of synergy, i.e., in this case the expected average productivity of vertices is equal to 1, i.e., the egalitarian rule is more efficient. Consider the case of the *father–mother* rule. In accordance with this rule, the vertex with an even index always wins competitions. In [5] it is proved, that, for any N, the expected average probability of vertices equals $7/8$ as in the case of the right-priority rule, i.e., the efficiency of the father–mother rule and right-priority rule is the same.

10.4. Guaranteed coming to the state of synergy on a chainmail

Suppose the rule is egalitarian. To prove that the state of synergy is reached after a fixed time, we can assign priorities at each step. This construction is equivalent to the problem considered here.

11. System productivity in the case of randomization of the communication

11.1. Randomization of the communication

In Sections 1–11, we used as the efficiency criterion *the expectation of the average productivity provided all possible initial states are equiprobable.*

Consider another approach to define the concept of the efficiency criterion. If a system can be represented with an ergodic Markov chain [9], and the steady probabilities of the chain do not depend on the initial state, then *we can consider the value of an index of the efficiency of system work at the steady state.* The dynamical system, considered in Sections 1–11, is not ergodic but there is a large class of communication matrices such that the system is ergodic if *the probability of the realization of each attempt does not equal* 1 *but can be arbitrarily close to* 1. The sufficient condition for the system to be ergodic is that the system can come from any state to any other state. An equivalent condition of ergodicity is that a finite power of the transition probabilities matrix with no zero element exists. Therefore we can assign to the original system an auxiliary system with probability of realization of an attempt less than 1, and use, as the criterion of efficiency the average productivity of the auxiliary system as the probability that the attempt is realized tends to 1. It is equivalent to *the criterion of efficiency which is defined as the expected of the average productivity of the original system nodes, under the condition that the probabilities of initial states and the steady state probabilities of auxiliary system, in which the probability of the realization of an attempt is close to* 1, *are the same.*

Suppose, even in the case of no competition, the transition is realized not with the probability 1 but with probability $q\,(0 < q < 1)$. Though in the original system the productivity of the system does not depend on the initial state, the behavior of the new system can be represented with an ergodic Markov chain such that steady probabilities of its state depend on the initial state, and therefore the average productivity of vertices of the system. Suppose $V(q)$ is the average probability of the system vertices provided the probability of the realization of any attempt equals q. We assume that the limit

$$V = \lim_{q=1} V(q)$$

is the efficiency criterion.

11.2. Stochastic version of necklace, $N = 3$

Consider a necklace in the case $N = 3$. Suppose $q = 1$. There are 8 possible states of the system

$$\begin{aligned} &S1 = (2, 3, 1), \quad S2 = (2, 3, 2), \quad S3 = (2, 1, 1), \quad S4 = (2, 1, 2),\\ &S5 = (3, 3, 1), \quad S6 = (3, 3, 2), \quad S7 = (3, 1, 1), \quad S8 = (3, 1, 2). \end{aligned}$$

Consider the right-priority rule, i.e., the priority matrix is

$$P = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix}.$$

Theorem 7. *Suppose $N = 3$, $q = 1 - \varepsilon$, and the rule is right-priority. There is a single communicating class of aperiodic states. Suppose $p_i(\varepsilon)$ is the steady probability of the state S_i, $i = 1, \dots, 8$. There exist the steady probabilities of all states of the Markov chain, and*

$$\lim_{\varepsilon\to 0} p_1(\varepsilon) = \lim_{\varepsilon\to 0} p_8(\varepsilon) = \frac{1}{2}, \qquad \lim_{\varepsilon\to 0} p_i(\varepsilon) = 0, \quad 2 \le i \le 7.$$

Proof. The state space of the chain is a single communicating class of aperiodic states. Indeed, it is possible to come at once, from any state to the state S_1, and from the state S_1 to any other state. The system can stay at any state with positive probability. Hence the system can get to any state from either for two or three steps, and therefore all states are aperiodic. If the number of states of a Markov chain is finite, and the space of the chain is single communicating class of aperiodic states, then this chain is ergodic, i.e., there are positive probabilities of all steady states of the chain.

Denote by $p_i(\varepsilon)$ the steady state S_i, $i = 1, 2, \dots, 8$. The transition matrix is

$$\begin{pmatrix}
\varepsilon^3 & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^3\\
(1-\varepsilon)\varepsilon & \varepsilon^2 & 0 & 0 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & 0 & 0\\
(1-\varepsilon)\varepsilon & (1-\varepsilon)^2 & \varepsilon^2 & (1-\varepsilon)\varepsilon & 0 & 0 & 0 & 0\\
(1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & (1-\varepsilon)\varepsilon & \varepsilon^2 & 0 & 0 & 0 & 0\\
(1-\varepsilon)\varepsilon & 0 & (1-\varepsilon)^2 & 0 & \varepsilon^2 & 0 & (1-\varepsilon)\varepsilon & 0\\
(1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & 0 & 0 & (1-\varepsilon)\varepsilon & \varepsilon^2 & 0 & 0\\
(1-\varepsilon)^2 & 0 & (1-\varepsilon)\varepsilon & 0 & (1-\varepsilon)\varepsilon & 0 & \varepsilon^2 & 0\\
(1-\varepsilon)^3 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)\varepsilon^2 & \varepsilon^3
\end{pmatrix}.$$

The system for steady state probabilities has the form

$$p_1 = p_1\varepsilon^3 + p_2(1-\varepsilon)\varepsilon + p_3(1-\varepsilon)\varepsilon + p_4(1-\varepsilon)^2 + p_5(1-\varepsilon)\varepsilon + p_6(1-\varepsilon)^2 + p_7(1-\varepsilon)^2 + p_8(1-\varepsilon)^3, \tag{7}$$

$$p_2 = p_1(1-\varepsilon)^2\varepsilon + p_2\varepsilon^2 + p_3(1-\varepsilon)^2 + p_4(1-\varepsilon)\varepsilon + p_6(1-\varepsilon)\varepsilon + p_8(1-\varepsilon)^2\varepsilon, \tag{8}$$

$$p_3 = p_1(1-\varepsilon)\varepsilon^2 + p_3\varepsilon^2 + p_4(1-\varepsilon)\varepsilon + p_5(1-\varepsilon)^2 + p_7(1-\varepsilon)\varepsilon + p_8(1-\varepsilon)^2\varepsilon, \tag{9}$$

$$p_4 = p_1(1-\varepsilon)^2\varepsilon + p_3(1-\varepsilon)\varepsilon + p_4\varepsilon^2 + p_8(1-\varepsilon)\varepsilon^2, \tag{10}$$

$$p_5 = p_1(1-\varepsilon)\varepsilon^2 + p_2(1-\varepsilon)^2 + p_6(1-\varepsilon)\varepsilon + p_7(1-\varepsilon)\varepsilon + p_8(1-\varepsilon)^2\varepsilon, \tag{11}$$

$$p_6 = p_1(1-\varepsilon)^2\varepsilon + p_2(1-\varepsilon)\varepsilon + p_6\varepsilon^2 + p_8(1-\varepsilon)\varepsilon^2, \tag{12}$$
$$p_7 = p_1(1-\varepsilon)^2\varepsilon + p_4(1-\varepsilon)\varepsilon + p_7\varepsilon^2 + p_8(1-\varepsilon)\varepsilon^2, \tag{13}$$
$$p_8 = p_1(1-\varepsilon)^3 + p_8\varepsilon^3, \tag{14}$$
$$p_1 + p_2 + \cdots + \cdots + p_8 = 1. \tag{15}$$

From Eqs. (7)–(15) it follows that

$$p_1 = p_4 + p_6 + p_7 + p_8 + o\left(\sqrt{\varepsilon}\right), \tag{16}$$
$$p_2 = p_3 + o\left(\sqrt{\varepsilon}\right), \tag{17}$$
$$p_3 = p_5 + o\left(\sqrt{\varepsilon}\right), \tag{18}$$
$$p_4 = o\left(\sqrt{\varepsilon}\right), \tag{19}$$
$$p_5 = p_2 + o\left(\sqrt{\varepsilon}\right), \tag{20}$$
$$p_6 = o\left(\sqrt{\varepsilon}\right), \tag{21}$$
$$p_7 = o\left(\sqrt{\varepsilon}\right), \tag{22}$$
$$p_8 = p_1 + o\left(\sqrt{\varepsilon}\right) \tag{23}$$

as $\varepsilon \to 0$.

From (15)–(23) it follows that

$$p_i = \frac{1}{2} + o\left(\sqrt{\varepsilon}\right), \quad i = 1, 8,$$
$$p_i = o\left(\sqrt{\varepsilon}\right), \quad 1 < i < 8.$$

This proves Theorem 7. □

Suppose $\varepsilon = 0$. The transition matrix has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, from (7)–(15) it follows that $p_1 = p_4 + p_6 + p_7 + p_8$, $p_2 = p_3$, $p_3 = p_5$, $p_4 = 0$, $p_5 = p_2$, $p_6 = 0$, $p_7 = 0$, $p_8 = p_1$, $p_1 + p_2 + \cdots + \cdots + p_8 = 1$. The determinant of this system equals 0.

The solution set of this system can be described by the equations

$$(p_1, \ldots, p_8) = C_1(1, 0, 0, 0, 0, 0, 0, 1) + C_2(0, 1, 1, 0, 1, 0, 0, 0),$$
$$2C_1 + 3C_2 = 1.$$

If $\varepsilon = 0$, then the states S_1 and S_8 compose a communicating class (an orbit). If the system is at this class, then the system sends a message at each step, i.e., the system is at the state of synergy. From the states S_2, S_3 and S_5, the system comes at once to the state S_1, i.e., the synergy takes place. If the system is at this class, then the average productivity of nodes is equal to $2/3$. If all initial states are equiprobable, then the probability of the system at one of these states is equal to $3/8$. Thus, if all states are equiprobable, the average productivity of nodes during time is

$$v = \frac{5}{8} \cdot 1 + \frac{3}{8} \cdot \frac{2}{3} = \frac{7}{8}.$$

If ε is a small positive number, then the probability of the system comes at once from the set of states $\{S4, S6, S7\}$ to the set of states $\{S1, S8\}$ is close to 1. The probability that the system comes at once from the set of states $\{S1, S8\}$ to the set of states $\{S4, S6, S7\}$ is an infinitely small of order ε. The probability that the system comes at once from the set of states $\{S1, S8\}$ to the set of states $\{S2, S3, S5\}$ is an infinitely small of order ε^2. With a probability infinitely small of order ε, the system comes from the set of states $\{S2, S3, S5\}$ to the set of states $\{S4, S6, S7\}$, and, at the next step, with probability close to 1, the system comes to the set of states $\{S1, S8\}$.

From the state $S1$, with probability close to 1, the system comes to the state S8. From the state $S8$, with probability close to 1, the system comes to the state S1. From the above we conclude the following. With steady probability close to 1, the system is in the set of states $\{S1, S8\}$.

Suppose, at the state Si, $k(i)$ vertices send messages. The average productivity $v(\varepsilon)$ of vertices is equal to the average number of nodes, sending messages at present step, related to N (this relation equals the steady probability that a given node sends a message at the present step)

$$v(\varepsilon) = \frac{1-\varepsilon}{N}\sum_{i=1}^{N} p_i(\varepsilon)k(i).$$

Theorem 8. *Suppose $N = 3$, and the rule is right-priority. Then,*

$$\lim_{\varepsilon\to 0} v(\varepsilon) = 1.$$

Proof. Theorem 8 follows from Theorem 7, and from that, at states $S1$ and $S8$, every particle can move. □

11.3. N-necklace with the right-priority rule

Consider a necklace in the case of an arbitrary N. The rule is right-priority. Suppose the realization of each attempt equals $1-\varepsilon$.

There are $K = 2^N$ possible states of the system. Let us take each state to the vector $a(S) = (a_1(S), a_2(S), \dots, a_N(S))$, where $a_j(S) = 0$ if in the state S the node j sends a message to the node $j+1$ (the addition is meant modulo N), and $a_j = 1$ if at this state the node j sends a message to the node $j-1$. Let the index of the state S be $a_1a_2\dots a_N$ (the binary system is used). Denote by S_i the state with the index $i = 0, \dots, K-1$.

Hypothesis 1. Suppose $q = 1-\varepsilon$, $\varepsilon > 0$, and the rule is right-priority. All states of the Markov chain compose a single communicating class.

There exist positive steady probabilities of all states, and

$$\lim_{\varepsilon\to 0} p_0(\varepsilon) = \frac{1}{2}, \qquad \lim_{\varepsilon\to 0} p_{K-1}(\varepsilon) = \frac{1}{2},$$
$$\lim_{\varepsilon\to 0} p_i(\varepsilon) = 0, \quad i \neq 0, K-1$$

where $p_i(\varepsilon)$ is the steady probability of the state S_i, which is a function of ε.

Hypothesis 2. Suppose the rule is right-priority. Then, for any N,

$$\lim_{\varepsilon\to 0} v(\varepsilon) = 1.$$

If the system is in the state S_0 or S_{K-1}, then all particles move. Therefore, if Hypothesis 1 is true, Hypothesis 2 is true too.

12. Necklace in the case of father–mother rule, $N \geq 4$

Suppose N is an even number, $N \geq 4$, and the rule is father–mother. It was said in Section 11.3 that, if $\varepsilon = 0$, the expectation of the average productivity of nodes equals $7/8$ both for the right-priority and father–mother rule. Suppose $\varepsilon > 0$, the probability of the realization of any attempt equals $q = 1-\varepsilon$, and the rule is father–mother. We shall prove (Theorem 9) that the expectation of the average productivity of nodes tends to $7/8$ as $\varepsilon \to 0$, though in the case of the right-priority rule the average productivity of nodes tends to 1 as $\varepsilon \to 0$.

Consider the case of the father–mother rule. Let N be an arbitrary even number. Suppose each node with an even number always wins the competition. Let N be an arbitrary even number. The work of each priority node (father node) does not depend on the other nodes. Hence the average probability of each priority node during time is equal to $1-\varepsilon$. The work of each priority node (mother node) depends on only two neighboring father nodes. To find the productivity of mother node, it is sufficient to consider the behavior of three nodes. These nodes are a mother node and two neighboring father nodes. We consider three nodes 2, 3 and 4. There are 8 states of the set of these three nodes

$$S1 = (3,4,1), \quad S2 = (3,4,3), \quad S3 = (3,2,1), \quad S4 = (3,2,3),$$
$$S5 = (1,4,1), \quad S6 = (1,4,3), \quad S7 = (1,2,1), \quad S8 = (1,2,3).$$

Theorem 9. *Suppose $n \geq 4$, and the case of the father–mother rule takes place. Let the probability of the realization of any attempt equals $1-\varepsilon$, $\varepsilon > 0$, and $v(\varepsilon)$ is the average productivity of nodes of networks. Then*

$$\lim_{\varepsilon\to 0} v(\varepsilon) = \frac{7}{8}.$$

Proof. Consider the Markov chain corresponding to the set of three contours. The space of this chain is a single communicating class of aperiodic states. Indeed, the system can come, from any state to any other state through the state $S1$ or through the state $S8$, both for two or three steps. If the state space of a Markov chain is finite, and this space is a single communicating class of aperiodic states, then this chain is ergodic, i.e., there exist non-zero state probabilities.

The transition matrix is

$$\begin{pmatrix} \varepsilon^3 & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^3 \\ (1-\varepsilon)\varepsilon & \varepsilon^2 & 0 & 0 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon^2 & (1-\varepsilon)\varepsilon & 0 & 0 & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ 0 & 0 & (1-\varepsilon)\varepsilon & \varepsilon^2 & 0 & 0 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \\ (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^3 & \varepsilon^3 & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & 0 & 0 & (1-\varepsilon)\varepsilon & \varepsilon^2 & 0 & 0 \\ (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^3 & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^3 & (1-\varepsilon)\varepsilon^2 \\ (1-\varepsilon)^3 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)^2\varepsilon & (1-\varepsilon)\varepsilon^2 & (1-\varepsilon)\varepsilon^2 & \varepsilon^3 \end{pmatrix}.$$

Denote by $p_i(\varepsilon)$ the steady state probabilities of the state Si, $i = 1, 2, \ldots, 8$. The system of equations for steady probabilities of states is

$$p_1 = p_1\varepsilon^3 + p_2(1-\varepsilon)\varepsilon + p_5(1-\varepsilon)\varepsilon^2 + p_6(1-\varepsilon)^2 + p_7(1-\varepsilon)^2\varepsilon + p_8(1-\varepsilon)^3, \tag{24}$$

$$p_2 = p_1(1-\varepsilon)\varepsilon^2 + p_2\varepsilon^2 + p_5(1-\varepsilon)^2\varepsilon + p_6(1-\varepsilon)\varepsilon + p_7(1-\varepsilon)^3 + p_8(1-\varepsilon)^2\varepsilon, \tag{25}$$

$$p_3 = p_1(1-\varepsilon)\varepsilon^2 + p_3\varepsilon^2 + p_4(1-\varepsilon)\varepsilon + p_5(1-\varepsilon)^2\varepsilon + p_7(1-\varepsilon)\varepsilon^2 + p_8(1-\varepsilon)^2\varepsilon \tag{26}$$

$$p_4 = p_1(1-\varepsilon)^2\varepsilon + p_3(1-\varepsilon)\varepsilon + p_4\varepsilon^2 + p_5(1-\varepsilon)^3 + p_7(1-\varepsilon)^2\varepsilon + p_8(1-\varepsilon)\varepsilon^2, \tag{27}$$

$$p_5 = p_1(1-\varepsilon)\varepsilon^2 + p_2(1-\varepsilon)^2 + p_5\varepsilon^3 + p_6(1-\varepsilon)\varepsilon + p_7(1-\varepsilon)\varepsilon^2 + p_8(1-\varepsilon)^2\varepsilon, \tag{28}$$

$$p_6 = p_1(1-\varepsilon)^2\varepsilon + p_2(1-\varepsilon)\varepsilon + p_5(1-\varepsilon)\varepsilon^2 + p_6\varepsilon^2 + p_7(1-\varepsilon)^2\varepsilon + p_8(1-\varepsilon)\varepsilon^2, \tag{29}$$

$$p_7 = p_1(1-\varepsilon)^2\varepsilon + p_3(1-\varepsilon)\varepsilon + p_4(1-\varepsilon)^2 + p_5(1-\varepsilon)\varepsilon^2 + p_7(1-\varepsilon)^3 + p_8(1-\varepsilon)\varepsilon^2, \tag{30}$$

$$p_8 = p_1(1-\varepsilon)^3 + p_3(1-\varepsilon)^2 + p_4(1-\varepsilon)\varepsilon + p_5(1-\varepsilon)^2\varepsilon + p_7(1-\varepsilon)\varepsilon^2 + p_8\varepsilon^3, \tag{31}$$

$$p_1 + \cdots + p_8 = 1. \tag{32}$$

From (24)–(32) it follows that ($\varepsilon \to 0$)

$$p_1 = p_6 + p_8 + o\left(\sqrt{\varepsilon}\right), \tag{33}$$

$$p_2 = p_7 + o\left(\sqrt{\varepsilon}\right), \tag{34}$$

$$p_3 = o\left(\sqrt{\varepsilon}\right), \tag{35}$$

$$p_4 = p_5 + o\left(\sqrt{\varepsilon}\right), \tag{36}$$

$$p_5 = p_2 + o\left(\sqrt{\varepsilon}\right), \tag{37}$$

$$p_6 = o\left(\sqrt{\varepsilon}\right), \tag{38}$$

$$p_7 = p_4 + o\left(\sqrt{\varepsilon}\right), \tag{39}$$

$$p_8 = p_1 + p_3 + o\left(\sqrt{\varepsilon}\right). \tag{40}$$

From (33), (35), (38) it follows that

$$p_1 = p_8 + o\left(\sqrt{\varepsilon}\right). \tag{41}$$

Since the behavior of each father vertex does not depend on the movement of the other vertices, we see that

$$p_1(\varepsilon) + p_3(\varepsilon) = p_2(\varepsilon) + p_4(\varepsilon) + = p_5(\varepsilon) + p_7(\varepsilon) = p_6(\varepsilon) + p_8(\varepsilon). \tag{42}$$

From (34)–(39), (41) and (42) it follows that

$$\lim_{\varepsilon\to 0} p_1(\varepsilon) = \frac{1}{4}, \qquad \lim_{\varepsilon\to 0} p_2(\varepsilon) = \frac{1}{8},$$

$$\lim_{\varepsilon\to 0} p_3(\varepsilon) = 0, \qquad \lim_{\varepsilon\to 0} p_4(\varepsilon) = \frac{1}{8},$$

$$\lim_{\varepsilon\to 0} p_5(\varepsilon) = \frac{1}{8}, \qquad \lim_{\varepsilon\to 0} p_6(\varepsilon) = 0,$$

$$\lim_{\varepsilon\to 0} p_7(\varepsilon) = \frac{1}{8}, \qquad \lim_{\varepsilon\to 0} p_8(\varepsilon) = \frac{1}{4}.$$

The mother vertex sends a message at states $S1$, $S5$, $S7$, and $S8$, and does not send a message at states $S2$, $S3$, $S4$, $S6$. From this it follows that the average productivity of mother vertices is equal to

$$\lim_{\varepsilon\to 0}(p_1(\varepsilon)+p_5(\varepsilon)+p_7(\varepsilon)+p_8(\varepsilon))=\frac{3}{4}.$$

From this, since the number of fathers equals the number of mothers, we obtain Theorem 9. □

Thus the efficiency of the deterministic system under the right-priority rule is higher in the sense of the current criterion than under the father–mother rule (the value of this criterion equals 1 under the right-priority rule, and equals 7/8 under father–mother rule). Nevertheless the efficiency of two the rules is the same in the sense of the second criteria (the expected average productivity given all the initial states are equiprobable) The value of this criterion is equal to 7/8 under both the right-priority and the father–mother rule.

13. Conclusion

1. A problem of a system of points movement on the networks of a special kind is formulated exactly in this paper.
2. It has been shown that the model can be interpreted as a transmission of messages, as well as a traffic of low flow densities.
3. The presence of the synergy effect, which is always classified as the phase of free flow in physical concepts, has been found. The matter is presented on the exact mathematical language in contrast to physicists and specialists in simulation.
4. Different rules of the behavior of the particles system on a network have been considered. These rules give different results.
5. The considered problems are far reaching generalizations of results, obtained by M. Blank, about a system of points on a circle. The problem, considered by M. Blank, has been formulated earlier by K. Nagel, M. Schreckenberg et al.

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