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Convexity-preserving approximation by univariate cubic splines

Xuli Han

School of Mathematics and Statistics, Central South University, Changsha, 410083, PR China

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ABSTRACT

A convexity-preserving approximation method is presented. Similar to the cubic spline interpolation, the given approximation function is univariate cubic spline with C^2 continuity. A very simple computing method is described. The approximation order and polynomial reproducing property of this convexity-preserving approximant are discussed. The computing method for interpolating a small quantity of data is given. Some numerical examples are given to show the effectivity.

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1. Introduction

We often required to generate a smooth function that approximates a prescribed set of data. An approximant is often needed to preserve certain geometric shape properties of the data such as monotonicity or convexity. At the same time, the approximant is wanted to be C^1 or C^2 continuous.

There are many methods for constructing shape-preserving interpolation. Some methods are appropriate for monotonicity-preserving, see [1–4]. Some methods are appropriate for convexity-preserving, see [5–9]. There are some methods available for the construction of a C^1 shape preserving interpolant, see [10,6,3,11]. On the other hand, it is a more difficult task to construct a C^2 shape preserving interpolant, see [12,13,7,14,9].

Polynomial spline methods have a common feature in that no additional knots need to be supplied, see [15–17,13,2,18]. In contrast, the papers [19,4,20,14,9,11] discussed the methods by adding one or two additional knots on the subinterval so that the monotonicity or convexity of the data is preserved.

Rational interpolant can produce satisfied shape-preserving interpolation. For C^2 continuity, the solution of the consistency equations is concerned, see [5,21,6,12,1,22]. In the paper [7], the given interpolant is convexity-preserving and C^2 continuous without solving a global system of equations.

Recently, in the application to computer-aided design, some methods for constructing shape-preserving interpolation have been presented, see [23–27].

For shape-preserving approximation, some error estimates have been discussed in [28–34]. As we know, the approximation by B-splines is shape-preserving, see [35], but the approximation order is not satisfied. In [33], a simple convexity-preserving algorithm is given by B-splines.

Polynomial spline functions are the most widely used. They are simple and easy to compute. However, classical spline interpolation methods usually yield solutions exhibiting undesirable inflections or oscillations. The aim of this paper is to present an effective convexity-preserving approximation method by univariate cubic splines.

The remainder of this paper is organized as follows. In Section 2, the piecewise expression of the convexity-preserving approximation is described. A very simple computing method for constructing the cubic splines is given. Approximation

E-mail address: xlhan@csu.edu.cn.

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properties of the presented cubic splines are discussed in Section 3. The computing method for interpolating a little inner data is presented in Section 4. Some numerical examples and conclusions are given in Sections 5 and 6 respectively.

2. Convexity-preserving cubic splines

Let $X = \{x_i : i = 1, 2, \ldots, n\}$ be a fixed partition of the interval $[a, b]$ by knots $a = x_1 < x_2 < \cdots < x_n = b$, and $D = \{f_i'' : i = 1, 2, \ldots, n\}$ be a set of n prescribed real numbers concerning the convexity of a given function $f(x)$ on X. If the data $f''(x_i)$ are given, we may set $f_i'' = f''(x_i),\ i = 1, 2, \ldots, n$. If only the data $f_i = f(x_i)$ are given, we may set

$$f_i'' = \frac{2}{h_{i-1} + h_i}(\Delta f_i - \Delta f_{i-1}) \tag{1}$$

for $i = 2, 3, \ldots, n-1$, and

$$f_1'' = f_2'', \qquad f_n'' = f_{n-1}'', \tag{2}$$

or natural boundary conditions

$$f_1'' = 0, \qquad f_n'' = 0, \tag{3}$$

where $h_i := x_{i+1} - x_i,\ \Delta f_i := [f(x_{i+1}) - f(x_i)]/h_i,\ i = 1, 2, \ldots, n-1$.

We will discuss the problem of convexity-preserving based on the given data f_i''. When all f_i'' have the same sign, based on the data $\{(x_i, f_i'') : i = 1, 2, \ldots, n\}$, we want to construct a concave approximation function (when $f_i'' \geq 0$) or a convex approximation function (when $f_i'' \leq 0$).

Definition. A convexity-preserving cubic spline on X is a real function S with properties:

(a) $S \in C^2[a, b]$, that is, S is twice continuously differentiable on $[a, b]$.
(b) S coincides on every subinterval $[x_i, x_{i+1}],\ i = 1, 2, \ldots, n-1$, with a polynomial of degree three.
(c) S satisfies $S''(x)f_i'' \geq 0$ on $[x_i, x_{i+1}]$ if $f_i''f_{i+1}'' \geq 0$.

In the classical construction of a C^2 cubic interpolating spline, the second derivatives at knots are unknown and spline values $S(x_i) = f(x_i)$. Here the order is reversed: the spline values are considered as unknown and second derivatives $S''(x_i) = f_i''$. Thus, convexity-preserving cubic spline can be constructed explicitly. Based on the given second derivatives at knots, a cubic spline is piecewise convexity-preserving if the condition (c) holds.

Based on (b), the second derivative of the spline $S(x)$ coincides with a linear function in each interval $[x_i, x_{i+1}], i = 1, 2, \ldots, n-1$, and for (c), these linear functions can be described in terms of the moments f_i'' of S:

$$S''(x) = (1-t)f_i'' + tf_{i+1}'', \quad x \in [x_i, x_{i+1}], \tag{4}$$

where $i = 1, 2, \ldots, n-1$, $t = (x - x_i)/h_i$. This expression ensures that $S(x)$ is convexity-preserving. A data set may not be strictly concave or convex. If the signs of f_i'' and f_{i+1}'' are different by (1), then from (4) we can know that the curve $S(x)$ has exactly one inflection in interval $[x_i, x_{i+1}]$.

By integration, denote the free parameters by $S(x_i) = y_i$, $i = 1, 2, \ldots, n$, from (4) we have

$$\begin{aligned} S(x) = {} & (1-t)^3 y_i + 3(1-t)^2 t\left[\frac{1}{3}(2y_i + y_{i+1}) - \frac{h_i^2}{18}(2f_i'' + f_{i+1}'')\right] \\ & + 3(1-t)t^2\left[\frac{1}{3}(y_i + 2y_{i+1}) - \frac{h_i^2}{18}(f_i'' + 2f_{i+1}'')\right] + t^3 y_{i+1} \end{aligned} \tag{5}$$

for $x \in [x_i, x_{i+1}]$, and then

$$S'(x) = (1-t)^2\left[\Delta y_i - \frac{h_i}{6}(2f_i'' + f_{i+1}'')\right] + 2(1-t)t\left[\Delta y_i + \frac{h_i}{6}(f_i'' - f_{i+1}'')\right] + t^2\left[\Delta y_i + \frac{h_i}{6}(f_i'' + 2f_{i+1}'')\right], \tag{6}$$

where $\Delta y_i = (y_{i+1} - y_i)/h_i$. Obviously,

$$S'(x_i^+) = \Delta y_i - \frac{h_i}{6}(2f_i'' + f_{i+1}''), \qquad S'(x_{i+1}^-) = \Delta y_i + \frac{h_i}{6}(f_i'' + 2f_{i+1}'').$$

We need compute all y_i to determine the cubic spline (5). Based on (a), the continuity conditions $S'(x_i^-) = s'(x_i^+)$ yield the linear system

$$h_i y_{i-1} - (h_{i-1} + h_i)y_i + h_{i-1}y_{i+1} = h_{i-1}h_i b_i, \quad i = 2, 3, \ldots, n-1, \tag{7}$$

where

$$b_i = \frac{1}{6}[h_{i-1}f_{i-1}'' + 2(h_{i-1} + h_i)f_i'' + h_i f_{i+1}''].$$

Naturally, we set boundary conditions

$$y_1 = f(x_1), \qquad y_n = f(x_n). \tag{8}$$

Thus the coefficient matrix of the linear system (7) is tridiagonal and irreducibly diagonally dominant (i.e., irreducible and weak row sum diagonally dominant). This means that the system has unique solutions for arbitrary right-hand side, and that consequently the convexity-preserving cubic spline (5) can be obtained uniquely.

By contraries, the solution of the cubic spline interpolation is to compute f_i'' by the linear system (7). Similar to the cubic spline interpolation problem, the solutions of the system (7) with (8) can be readily computed by using the finite recurrence relations for the LU-decomposition of the coefficient matrix, i.e., the Gaussian elimination algorithm. Since the coefficient matrix of the system (7) is particular, we can develop a very simple explicit formula.

Theorem 1. *The solutions of the system (7) with (8) can be given as follows:*

$$y_i = \frac{x_n - x_i}{x_n - x_1}\left[y_1 - \sum_{k=2}^{i-1}(x_k - x_1)b_k\right] + \frac{x_i - x_1}{x_n - x_1}\left[y_n - \sum_{k=i}^{n-1}(x_n - x_k)b_k\right] \tag{9}$$

for $i = 2, 3, \ldots, n-1$.

Proof. By (9), a straightforward computation gives that

$$\frac{y_{i+1} - y_i}{h_i} = \frac{1}{x_n - x_1}\left[y_n - y_1 + \sum_{k=2}^{i}(x_k - x_1)b_k - \sum_{k=i+1}^{n-1}(x_n - x_k)b_k\right],$$

and then

$$\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}} = b_i$$

for $i = 2, 3, \ldots, n-1$. This means that (9) coincides (7). □

For simplification, (9) can be written as

$$y_2 = y_1 + \frac{x_2 - x_1}{x_n - x_1}\left[y_n - y_1 - \sum_{k=2}^{n-1}(x_n - x_k)b_k\right],$$

$$y_i = y_{i-1} + h_{i-1}\left(\frac{y_2 - y_1}{h_1} + \sum_{k=2}^{i-1} b_k\right), \quad i = 3, 4, \ldots, n-1.$$

From these, we can present a simple algorithm by computing

$$a_1 = \frac{1}{x_n - x_1}\left[y_n - y_1 - \sum_{k=2}^{n-1}(x_n - x_k)b_k\right],$$

$$a_i = a_{i-1} + b_i, \quad i = 2, 3, \ldots, n-2,$$

$$y_i = y_{i-1} + h_{i-1}a_{i-1}, \quad i = 2, 3, \ldots, n-1.$$

3. Approximation properties of the cubic splines

We first show a property for the smoothness. The word smoothness is given a meaning in the following minimum property.

Theorem 2. *Let* $f(x) \in C^3[a, b]$. *If* $f_i'' = f''(x_i),\ i = 1, 2, \ldots, n$, *then*

$$\int_a^b [S'''(x)]^2 dx \le \int_a^b [f'''(x)]^2 dx. \tag{10}$$

Proof. Since

$$\int_a^b (f''')^2 dx = \int_a^b (S''')^2 dx + 2\int_a^b S'''(f''' - S''')dx + \int_a^b (f''' - S''')^2 dx,$$

we only need to show

$$\int_a^b S'''(f''' - S''')dx \ge 0.$$

Since S''' is constant on $[x_i, x_{i+1}]$ and $S''(x_i) = f_i''$, let $S'''(x_i) = c_i$, we have

$$\int_a^b S'''(f''' - S''')dx = \sum_{i=1}^{n-1} c_i \int_{x_i}^{x_{i+1}} (f''' - S''')dx = \sum_{i=1}^{n-1} c_i (f'' - S'')|_{x_i}^{x_{i+1}} = 0. \quad \square$$

Now we discuss the convergence of the presented cubic splines.

Theorem 3. *Let $f \in C^4[a, b]$. If $f_i'' = f''(x_i),\ i = 1, 2, \ldots, n$, then, for (9),*

$$|f(x_i) - y_i| \le \frac{n-1}{24(x_n - x_1)}(i-1)(n-i)h^5 M_4, \quad i = 2, 3, \ldots, n-1, \tag{11}$$

where

$$h := \max_{1\le k\le n-1}\{h_k\}, \qquad M_4 := \max_{a\le x\le b}|f^{(4)}(x)|.$$

Proof. Let $e_i = y_i - f(x_i)$,

$$\delta_i = b_i - \frac{1}{h_{i-1}}(f_{i-1} - f_i) - \frac{1}{h_i}(f_{i+1} - f_i).$$

From (7), we have

$$\frac{e_{i+1} - e_i}{h_i} - \frac{e_i - e_{i-1}}{h_{i-1}} = \delta_i.$$

According to Theorem 1, this implies that

$$e_i = \frac{x_n - x_i}{x_n - x_1}\left[e_1 - \sum_{k=2}^{i-1}(x_k - x_1)\delta_k\right] + \frac{x_i - x_1}{x_n - x_1}\left[e_n - \sum_{k=i}^{n-1}(x_n - x_k)\delta_k\right] \tag{12}$$

for $i = 2, 3, \ldots, n-1$.

By Taylor expansion, we have

$$\begin{aligned} b_i &= \frac{1}{2}(h_{i-1} + h_i)f''(x_i) + \frac{1}{6}(h_i^2 - h_{i-1}^2)f'''(x_i) \\ &\quad + \frac{h_{i-1}}{6}\int_{x_{i-1}}^{x_i}(u - x_{i-1})f^{(4)}(u)du + \frac{h_i}{6}\int_{x_i}^{x_{i+1}}(x_{i+1} - u)f^{(4)}(u)du, \\ \frac{1}{h_{i-1}}(f_{i-1} - f_i) + \frac{1}{h_i}(f_{i+1} - f_i) &= \frac{1}{2}(h_{i-1} + h_i)f''(x_i) + \frac{1}{6}(h_i^2 - h_{i-1}^2)f'''(x_i) \\ &\quad + \frac{1}{6h_{i-1}}\int_{x_{i-1}}^{x_i}(u - x_{i-1})^3 f^{(4)}(u)du + \frac{1}{6h_i}\int_{x_i}^{x_{i+1}}(x_{i+1} - u)^3 f^{(4)}(u)du, \end{aligned}$$

and then

$$\begin{aligned} \delta_i &= \frac{1}{6h_{i-1}}\int_{x_{i-1}}^{x_i}[h_{i-1}^2 - (u - x_{i-1})^2](u - x_{i-1})f^{(4)}(u)du \\ &\quad + \frac{1}{6h_i}\int_{x_i}^{x_{i+1}}[h_i^2 - (x_{i+1} - u)^2](x_{i+1} - u)f^{(4)}(u)du = \frac{1}{24}(h_{i-1}^3 + h_i^3)f^{(4)}(\xi) \end{aligned}$$

for some $\xi \in [x_{i-1}, x_{i+1}]$.

Since $e_1 = 0, e_n = 0$, we obtain

$$\begin{aligned} |e_i| &\le \frac{h^4}{12}M_4\left(\frac{x_n - x_i}{x_n - x_1}\sum_{k=1}^{i-2}k + \frac{x_i - x_1}{x_n - x_1}\sum_{k=1}^{n-i}k\right) \\ &= \frac{h^4}{24(x_n - x_1)}[(x_n - x_i)(i-1)(i-2) + (x_i - x_1)(n-i+1)(n-i)]M_4 \\ &\le \frac{n-1}{24(x_n - x_1)}(i-1)(n-i)h^5 M_4. \quad \square \end{aligned}$$

Theorem 4. *Let $f \in C^4[a, b]$. If $f_i'' = f''(x_i),\ i = 1, 2, \ldots, n$, then, for $x \in [x_i, x_{i+1}]$, $t = (x - x_i)/h_i$,*

$$|f(x) - S(x)| \le \frac{5}{384}h_i^4 M_{4,i} + \frac{(n-1)^3}{24(x_n - x_1)}h^5 M_4, \tag{13}$$

$$|f'(x) - S'(x)| \le \frac{19}{384}h_i^4 M_{4,i} + \frac{(n-1)(n-2)}{24(x_n - x_1)}h^4 M_4, \tag{14}$$

$$S''(x) - f''(x) = \frac{h_i^2}{2}(1-t)tf^{(4)}(\eta), \quad \eta \in [x_i, x_{i+1}], \tag{15}$$

where

$$M_{4,i} = \max_{x_i \le x \le x_{i+1}} |f^{(4)}(x)|.$$

Proof. Let $f_i = f(x_i),\ i = 1, 2, \ldots, n$,

$$\begin{aligned} P(x) \ =\ & (1-t)^3 f_i + 3(1-t)^2 t\left[\frac{1}{3}(2f_i + f_{i+1}) - \frac{h_i^2}{18}(2f_i'' + f_{i+1}'')\right] \\ & + 3(1-t)t^2\left[\frac{1}{3}(f_i + 2f_{i+1}) - \frac{h_i^2}{18}(f_i'' + 2f_{i+1}'')\right] + t^3 f_{i+1} \end{aligned}$$

for $x \in [x_i, x_{i+1}]$, $t = (x - x_i)/h_i$.

It is easy to show $P(x) = f(x)$ when $f(x) = 1, x, x^2$ respectively. When $f(x) = x^3$, since

$$\begin{aligned} & 2f_i + f_{i+1} - \frac{h_i^2}{6}(2f_i'' + f_{i+1}'') = 3x_i^2 x_{i+1}, \\ & f_i + 2f_{i+1} - \frac{h_i^2}{6}(f_i'' + 2f_{i+1}'') = 3x_i x_{i+1}^2, \end{aligned}$$

we have

$$\begin{aligned} P(x) \ =\ & (1-t)^3 x_i^3 + 3(1-t)^2 t x_i^2 x_{i+1} + 3(1-t)t^2 x_i x_{i+1}^2 + t^3 x_{i+1}^3 \\ =\ & [(1-t)x_i + t x_{i+1}]^3 = x^3 = f(x). \end{aligned}$$

Thus, by Taylor expansions, we have

$$\begin{aligned} P(x) \ =\ & f(x) + \frac{1}{6}(1-t)^3 \int_{x_i}^{x} (u - x_i)^3 f^{(4)}(u)du + 3(1-t)^2 t\left\{\frac{1}{9}\int_{x_i}^{x}[(u-x_i)^3 - h_i^2(u-x_i)]f^{(4)}(u)du\right. \\ & \left. + \frac{1}{18}\int_{x}^{x_{i+1}}[(x_{i+1}-u)^3 - h_i^2(x_{i+1}-u)]f^{(4)}(u)du\right\} \\ & + 3(1-t)t^2\left\{\frac{1}{18}\int_{x_i}^{x}[(u-x_i)^3 - h_i^2(u-x_i)]f^{(4)}(u)du + \frac{1}{9}\int_{x}^{x_{i+1}}[(x_{i+1}-u)^3 - h_i^2(x_{i+1}-u)]f^{(4)}(u)du\right\} \\ & + \frac{1}{6}t^3 \int_{x}^{x_{i+1}} (x_{i+1} - u)^3 f^{(4)}(u)du = f(x) + \frac{1}{6}\int_{x_i}^{x_{i+1}} A_i(u) f^{(4)}(u)du, \end{aligned}$$

where

$$A_i(u) = \begin{cases} (1-t)[(u-x_i)^3 - h_i^2(u-x_i)(2t-t^2)], & x_i \le u \le x, \\ t[(x_{i+1}-u)^3 - h_i^2(x_{i+1}-u)(1-t^2)], & x \le u \le x_{i+1}. \end{cases}$$

Since $A_i(u) \le 0$ and $A_i(u)$ is continuous on $[x_i, x_{i+1}]$, we obtain

$$\begin{aligned} P(x) \ =\ & f(x) + \frac{1}{6}f^{(4)}(\xi)\int_{x_i}^{x_{i+1}} A_i(u)du \\ =\ & f(x) - \frac{h_i^4}{24}(1-t)t[1 + (1-t)t]f^{(4)}(\xi), \end{aligned}$$

for some $\xi \in [x_i, x_{i+1}]$. Thus,

$$|f(x) - P(x)| \le \frac{5}{384}h_i^4 M_4.$$

Obviously,

$$S(x) - P(x) = (1-t)^3 e_i + (1-t)^2 t(2e_i + e_{i+1}) + (1-t)t^2(e_i + 2e_{i+1}) + t^3 e_{i+1},$$

and then

$$|S(x) - P(x)| \le \max\{|e_i|, |e_{i+1}|\} \le \frac{(n-1)^3}{24(x_n - x_1)} h^5 M_4.$$

Since $|f(x) - S(x)| \le |f(x) - P(x)| + |P(x) - S(x)|$, we obtain (13) immediately.

Direct computation gives that

$$P'(x) = (1-t)^2\left[\Delta f_i - \frac{h_i}{6}(2f_i'' + f_{i+1}'')\right] + 2(1-t)t\left[\Delta f_i + \frac{h_i}{6}(f_i'' - f_{i+1}'')\right] + t^2\left[\Delta f_i + \frac{h_i}{6}(f_i'' + 2f_{i+1}'')\right],$$

where

$$\begin{aligned}
\Delta f_i - \frac{h_i}{6}(2f_i'' + f_{i+1}'') &= f'(x) - h_i t f''(x) + \frac{1}{2}h_i^2 t^2 f'''(x) \\
&\quad - \frac{h_i^3}{24}(t^4 + 4t^2) f^{(4)}(\xi_1) + \frac{h_i^3}{24}[(1-t)^4 - 2(1-t)^2] f^{(4)}(\xi_2), \\
\Delta f_i + \frac{h_i}{6}(f_i'' - f_{i+1}'') &= f'(x) + \frac{h_i}{2}[(1-t) - t] f''(x) - \frac{1}{2}h_i^2(1-t)t f'''(x) \\
&\quad - \frac{h_i^3}{24}(t^4 - 2t^2) f^{(4)}(\xi_3) + \frac{h_i^3}{24}[(1-t)^4 - 2(1-t)^2] f^{(4)}(\xi_2), \\
\Delta f_i + \frac{h_i}{6}(f_i'' + 2f_{i+1}'') &= f'(x) + h_i(1-t) f''(x) + \frac{1}{2}h_i^2(1-t)^2 f'''(x) \\
&\quad - \frac{h_i^3}{24}(t^4 - 2t^2) f^{(4)}(\xi_3) + \frac{h_i^3}{24}[(1-t)^4 + 4(1-t)^2] f^{(4)}(\xi_4)
\end{aligned}$$

for some $\xi_1, \xi_3 \in [x_i, x]$, and $\xi_2, \xi_4 \in [x, x_{i+1}]$. From these we have

$$\begin{aligned}
|f'(x) - P'(x)| &\le \frac{h_i^3}{24} M_4 \{(1-t)^2[t^4 + 4t^2 + 2(1-t)^2 - (1-t)^4] + 2(1-t)t[2t^2 - t^4 + 2(1-t)^2 - (1-t)^4] \\
&\quad + t^2[2t^2 - t^4 + 4(1-t)^2 + (1-t)^4]\} \\
&= \frac{h_i^3}{24} M_4[1 + 4(1-t)^2 t^2 - 4(1-t)^3 t^3] \le \frac{19}{384} h_i^3 M_4.
\end{aligned}$$

Obviously,

$$P'(x) - S'(x) = \Delta f_i - \Delta y_i.$$

From (12) we have

$$\Delta f_i - \Delta y_i = \frac{1}{x_n - x_1}\left[\sum_{k=i+1}^{n-1}(x_n - x_k)\delta_k - \sum_{k=2}^{i}(x_k - x_1)\delta_k\right].$$

Thus, for $i = 2, 3, \ldots, n-2$, we obtain

$$|\Delta f_i - \Delta y_i| \le \frac{h^4}{12(x_n - x_1)} M_4 \left(\sum_{k=1}^{n-k-1} k + \sum_{k=1}^{k-1} k\right).$$

This inequality also holds for $i = 1, n-1$. Therefore, we obtain (14) immediately by $|f'(x) - S'(x)| \le |f'(x) - P'(x)| + |P'(x) - S'(x)|$.

For $x \in [x_i, x_{i+1}]$, $t = (x - x_i)/h_i$, by (5) we have

$$\begin{aligned}
S''(x) &= f''(x) + (1-t)\int_{x_i}^{x}(u - x_i) f^{(4)}(u)du + t\int_{x}^{x_{i+1}}(x_{i+1} - u) f^{(4)}(u)du \\
&= f''(x) + \frac{h_i^2}{2}(1-t)t f^{(4)}(\eta)
\end{aligned}$$

for some $\eta \in [x_i, x_{i+1}]$. This means (15) holds. □

With given second derivative values, (13) shows that the convexity-preserving cubic spline is cubic polynomial reproducing. Expressions (14) and (15) show that the derivative and the second derivative of the convexity-preserving cubic spline have satisfied approximation order.

In some practical situations, the data are only given by $f_i = f(x_i),\ i = 1, 2, \ldots, n$. In this case, we have the following result.

Theorem 5. *If the all data f_i'' are given by (1) and (2), then the convexity-preserving cubic splines are quadratic polynomial reproducing.*

Proof. For (1) and (2), let $f_i'' = f''(x_i) + \alpha_i$. Then, for $i = 2, 3, \ldots, n-1$, we have

$$\alpha_i = \frac{1}{h_{i-1}+h_i}\left[\frac{1}{h_i}\int_{x_i}^{x_{i+1}}(x_{i+1}-u)^2 f'''(u)du - \frac{1}{h_{i-1}}\int_{x_{i-1}}^{x_i}(u-x_{i-1})^2 f'''(u)du\right].$$

From this we obtain

$$|\alpha_i| \le \frac{h_{i-1}^2+h_i^2}{3(h_{i-1}+h_i)}M_3 \le \frac{1}{3}hM_3.$$

In the same way, since $\alpha_1 = f_2'' - f''(x_1)$ and $\alpha_n = f_{n-1}'' - f''(x_n)$ by (2), we have

$$\alpha_1 = \frac{1}{3}(2h_1+h_2)f'''(\xi), \quad |\alpha_1| \le hM_3,$$
$$\alpha_n = -\frac{1}{3}(h_{n-2}+2h_{n-1})f'''(\xi), \quad |\alpha_n| \le hM_3.$$

Therefore, $f_i'' = f''(x_i)$ for any quadratic polynomials. By (13), the result of the polynomial reproducing property follows. □

From above results, we can see that the given cubic splines have satisfied approximation property, since some shape-preserving interpolants found in the literature approach the piecewise linear interpolants (see [5,6,12]), and the approximation properties of some C^2 continuous convexity-preserving interpolants have not been discussed (see [5,21,15,6,13,19,9]). Although the cubic B-spline curves are convexity-preserving, they are not quadratic polynomial reproducing.

4. The method of interpolating a small quantity of data

The presented approximation scheme in Section 2 is a convexity-preserving rather than an interpolating scheme. Whether this convexity-preserving is desirable or not depends on one's point of view and on the data. Certainly if the data contain some noise then convexity-preserving is desirable. From Theorems 3 and 4 we can see that the approximation order of the obtained cubic splines may be 2 for the central data. The second order precision is the precision of the piecewise linear interpolants. In this section, we present a method of interpolating a small quantity of data.

By (9), a straightforward computation gives that

$$\begin{aligned} y_i = {} & \frac{x_n-x_i}{x_n-x_1}y_1 + \frac{x_i-x_1}{x_n-x_1}y_n - \frac{x_n-x_i}{6(x_n-x_1)}\left\{h_1^2 f_1'' + \sum_{k=2}^{i-1}(h_{k-1}+h_k)(x_{k-1}+x_k+x_{k+1}-3x_1)f_k''\right\} \\ & - \frac{(x_n-x_i)(x_i-x_1)}{6(x_n-x_1)}\left[\frac{x_{i-1}-x_1}{x_i-x_1}h_{i-1} + 2(h_{i-1}+h_i) + \frac{x_n-x_{i+1}}{x_n-x_i}h_i\right]f_i'' \\ & - \frac{x_i-x_1}{6(x_n-x_1)}\left\{\sum_{k=i+1}^{n-1}(h_{k-1}+h_k)(3x_n-x_{k-1}-x_k-x_{k+1})f_k'' + h_{n-1}^2 f_n''\right\} \end{aligned} \tag{16}$$

for $i = 2, 3, \ldots, n-1$.

For certain m $(1 < m < n)$, if the cubic spline is required to interpolate the given datum $f(x_m)$, then we can let $y_m = f(x_m)$ and compute f_m'' by using an expression of (16). From (16) we have

$$\begin{aligned} & \left[\frac{x_{m-1}-x_1}{x_m-x_1}h_{m-1} + 2(h_{m-1}+h_m) + \frac{x_n-x_{m+1}}{x_n-x_m}h_m\right]f_m'' \\ & = 6\left(\frac{f(x_n)-f(x_m)}{x_n-x_m} - \frac{f(x_m)-f(x_1)}{x_m-x_1}\right) - \frac{1}{x_m-x_1}\left\{h_1^2 f_1'' + \sum_{k=2}^{m-1}(h_{k-1}+h_k)(x_{k-1}+x_k+x_{k+1}-3x_1)f_k''\right\} \\ & \quad - \frac{1}{x_n-x_m}\left\{\sum_{k=m+1}^{n-1}(h_{k-1}+h_k)(3x_n-x_{k-1}-x_k-x_{k+1})f_k'' + h_{n-1}^2 f_n''\right\}, \end{aligned} \tag{17}$$

where $f(x_1), f(x_m), f(x_n), f_k''$ $(k = 1, 2, \ldots, n, k \ne m)$ are prescribed.

Therefore, with $f(x_1), f(x_m), f(x_n), f_k''$ $(k = 1, 2, \ldots, n, k \neq m)$, the method of interpolating the datum $f(x_m)$ is to compute f_m'' by (17) then compute all y_i according to the described method in Section 2. Thus, we have $y_m = f(x_m)$ then $S(x_m) = f(x_m)$. Obviously, the obtained cubic spline (5) is convexity-preserving for $x \in [x_i, x_{i+1}]$, $i = 1, 2, \ldots, n-1$, $i \neq m-1, m$. If the sign of the value f_m'' computed by (17) is the same as the sign of the prescribed value at $x = x_m$, then the cubic spline (5) is also convexity-preserving for $x \in [x_{m-1}, x_{m+1}]$. Thus, we can choose m based on the sign of f_m'' by using (17) repeatedly.

Of course, if several data are required to be interpolated, then the obtained cubic spline (5) may not be convexity-preserving. Now we describe the scheme for interpolating two inner data.

Let $1 < l < m < n$ and $f(x_1), f(x_l), f(x_m), f(x_n), f_k''$ $(k = 1, 2, \ldots, n,\ k \neq l, m)$ be prescribed, we may compute f_l'' and f_m'' by two equations of (16) with $y_l = f(x_l)$ and $y_m = f(x_m)$. In order to obtain simple equations for computing f_l'' and f_m'', we consider $\{x_i : i = 1, 2, \ldots, l\}$, $\{x_i : i = l, l+1, \ldots, m\}$ and $\{x_i : i = m, m+1, \ldots, n\}$ according to the described method in Section 2, respectively. Based on the continuity conditions $S'(x_l^-) = S'(x_l^+)$ and $S'(x_m^-) = S'(x_m^+)$, or directly based on (17), we obtain

$$\begin{aligned}
&\frac{1}{x_l - x_1}\left\{h_1^2 f_1'' + \sum_{k=2}^{l-1}(h_{k-1}+h_k)(x_{k-1}+x_k+x_{k+1}-3x_1)f_k''\right\} \\
&\quad + \left[\frac{x_{l-1}-x_1}{x_l-x_1}h_{l-1} + 2(h_{l-1}+h_l) + \frac{x_m - x_{l+1}}{x_m - x_l}h_l\right]f_l'' \\
&\quad + \frac{1}{x_m - x_l}\left\{\sum_{k=l+1}^{m-1}(h_{k-1}+h_k)(3x_m - x_{k-1} - x_k - x_{k+1})f_k'' + h_{m-1}^2 f_m''\right\} \\
&= 6\left(\frac{y_m - y_l}{x_m - x_l} - \frac{y_l - y_1}{x_l - x_1}\right), \qquad (18)\\
&\frac{1}{x_m - x_l}\left\{h_l^2 f_l'' + \sum_{k=l+1}^{m-1}(h_{k-1}+h_k)(x_{k-1}+x_k+x_{k+1}-3x_l)f_k''\right\} \\
&\quad + \left[\frac{x_{m-1}-x_l}{x_m-x_l}h_{m-1} + 2(h_{m-1}+h_m) + \frac{x_n - x_{m+1}}{x_n - x_m}h_m\right]f_m'' \\
&\quad + \frac{1}{x_n - x_m}\left\{\sum_{k=m+1}^{n-1}(h_{k-1}+h_k)(3x_n - x_{k-1} - x_k - x_{k+1})f_k'' + h_{n-1}^2 f_n''\right\} \\
&= 6\left(\frac{y_n - y_m}{x_n - x_m} - \frac{y_m - y_l}{x_m - x_l}\right). \qquad (19)
\end{aligned}$$

Obviously, from (18) and (19), with $y_l = f(x_l)$ and $y_m = f(x_m)$, we can obtain f_l'' and f_m''. If the signs of f_l'' and f_m'' computed by (18) and (19) are the same as the signs of the prescribed values at $x = x_l$ and $x = x_m$ respectively, then the obtained cubic spline is still convexity-preserving. We can choose l and m based on the signs of f_l'' and f_m'' by using (18) and (19) repeatedly.

5. Numerical examples

In this section we apply the method developed in the previous sections to numerical data. We first present the performance of the proposed method on two classical tests. On left of Fig. 1, the data may be found in [33]. On the right of Fig. 1, the data may be found in [33,35]. Near the peak, the fitting result is better than the presentation in [33]. In order to reproduce the peak better, a little more data are used in [33,35].

We next consider the sets of data taken from function $f(x) = \sqrt{0.1 + 4x - x^2}$, $0 \le x \le 4$. From this example, we can see that the presented cubic splines have satisfied convexity-preserving property and convergence.

Fig. 2 shows the curves for $n = 11$. Cubic spline interpolation curves with $f_1'' = f''(x_1)$ and $f_n'' = f''(x_n)$ are shown on the left of Fig. 2. Cubic spline approximation curves with (1) and (2) are shown on right of Fig. 2. Obviously, the cubic spline approximation curves have better shape-preserving property than the cubic spline interpolation curves. We also emphasize that the presented method is very simple.

Cubic spline approximation curves with (1), (2) and $n = 31$ are shown on the left of Fig. 3. Cubic spline approximation curves with $n = 51$ are shown on the right of Fig. 3. The solid lines show the curves with (1) and (2), and the dashed lines show the curves with all $f_i'' = f''(x_i)$. Since the influence of the data $f''(x_1)$ and $f''(x_n)$, the solid lines approximate to the data better than the dashed lines.

For not globally convex/concave data set, we consider a double-well function which concerns Duffing equations and potential functions, see [36–38]. We take data from function $u(x) = x^4 - 0.95x^2$, $-1 \le x \le 1$. Cubic spline approximation curves with (1), (3) and $n = 11$ are shown on the left of Fig. 4. The curves with (1), (3) and $n = 21$ are shown on the right of Fig. 4.

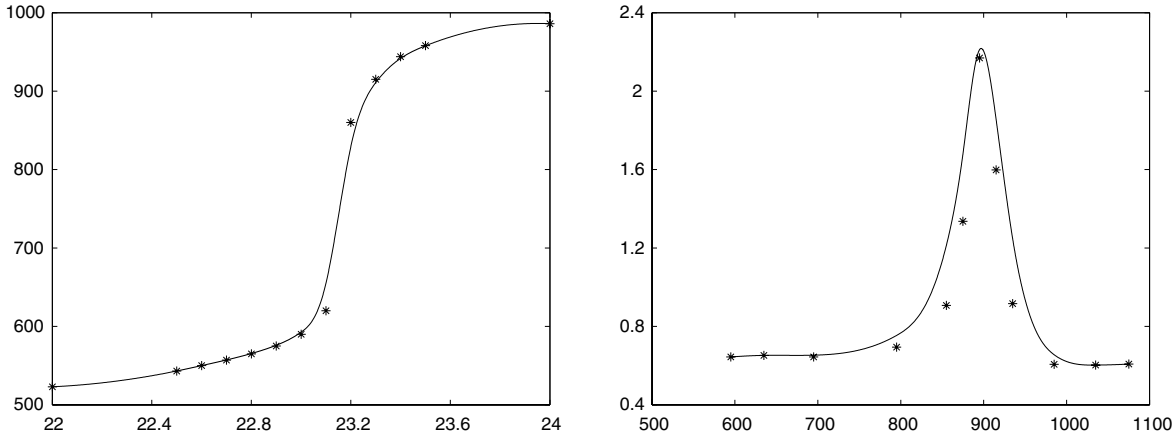


Fig. 1. Approximation on the classical test data set.

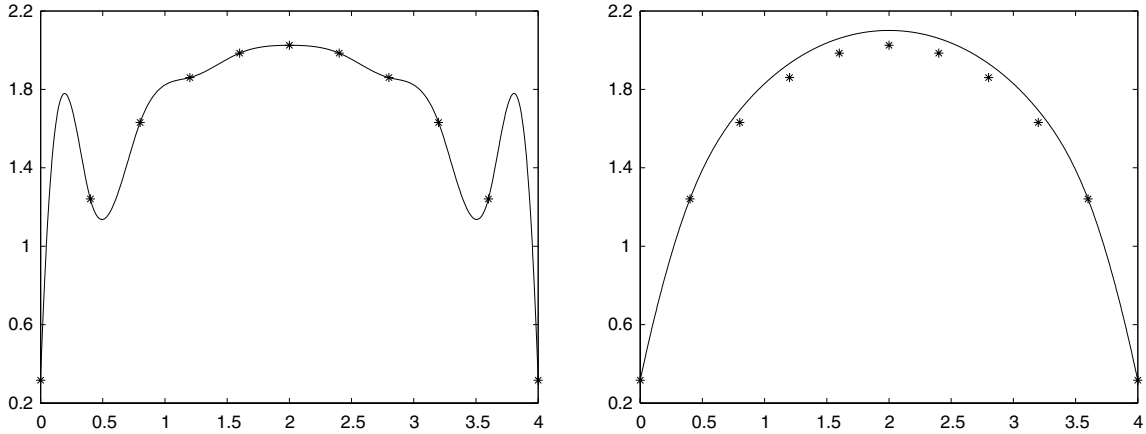


Fig. 2. Cubic spline curves for $f(x)$ with $n = 11$.

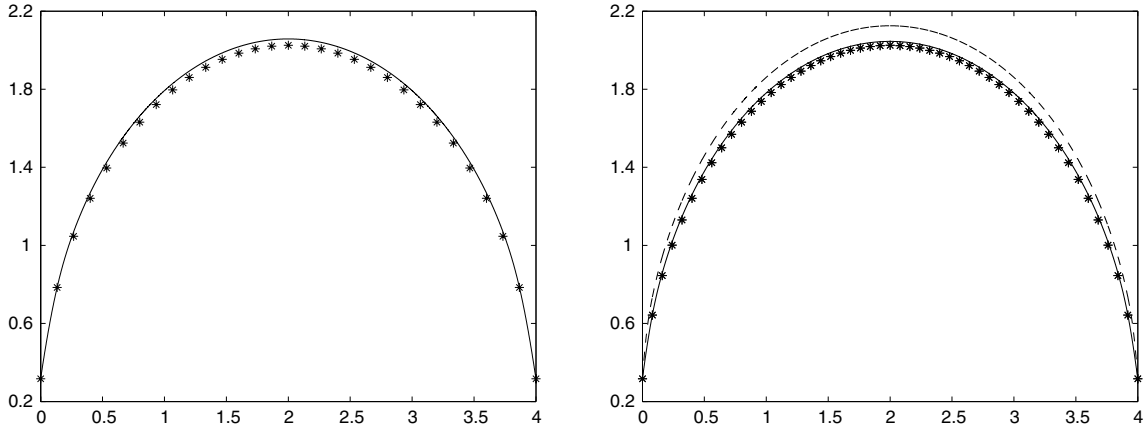


Fig. 3. Cubic spline approximation curves for $f(x)$ with $n = 31$ and $n = 51$.

Finally, we display the results by using (17)–(19). Since the approximation of the central data is not satisfied in Figs. 2 and 3, for $n = 11$, we let $y_6 = f(x_6)$ and compute f_6'' by (17), then compute other y_i by the method in Section 2. By (1) and (17), the values of f_6'' are -0.4988 and -0.2957 respectively. Their signs are the same, therefore the obtained cubic splines are still convexity-preserving. The curves are shown on the left of Fig. 5. For (18) and (19), we use the data from the example on the right of Fig. 1. By (1), the values of f_4'' and f_{10}'' are 3.81×10^{-5} and 1.22×10^{-4} respectively. By (18) and (19), setting

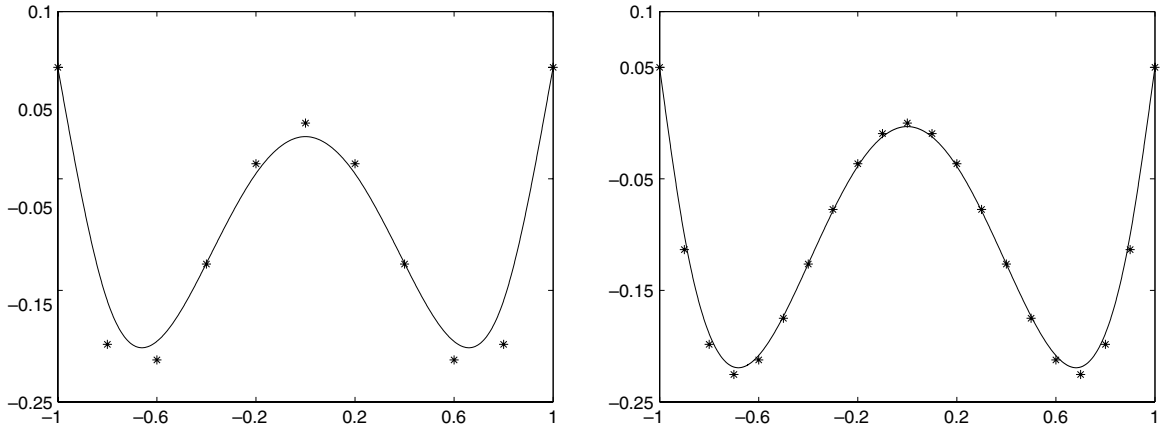


Fig. 4. Cubic spline approximation curves for $u(x)$ with $n = 11$ and $n = 21$.

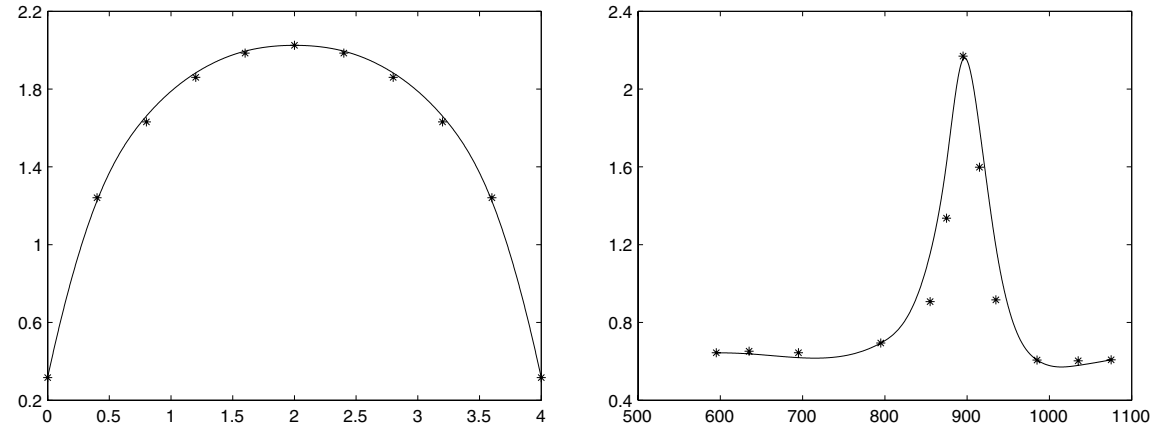


Fig. 5. Cubic spline curves for interpolating a small quantity of data.

$y_4 = f(x_4)$ and $y_{10} = f(x_{10})$, the values of f_4'' and f_{10}'' are 4.31×10^{-5} and 1.33×10^{-4} respectively. Therefore, the obtained cubic splines are still convexity-preserving. The curves are shown on the right of Fig. 5.

6. Conclusion

The presented approximation method in this paper provides a preferable solution of convexity-preserving approximation problem. The computing method for constructing the cubic splines is very simple. The convexity-preserving cubic splines have the same continuity as the cubic spline interpolants and the cubic B-spline curves, and they have better shape-preserving property than the cubic spline interpolants and better approximation order than the cubic B-spline curves. The derivative and the second derivative of the convexity-preserving cubic splines have satisfied approximation order. If the signs of f_i'' and f_{i+1}'' are different then the curve $S(x)$ has exactly one inflection in interval $[x_i, x_{i+1}]$. For some data, it is also feasible to obtain a convexity-preserving cubic spline which interpolates a small quantity of data.

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