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Restricted difference-based Liu estimator in partially linear model

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ABSTRACT

Partially linear model is useful in statistical model as a multivariate nonparametric fitting method. This paper deals with statistical inference for the partially linear model in the presence of multicollinearity. When some additional linear restrictions are assumed to hold, the corresponding restricted difference-based Liu estimator for the parametric component is constructed. The asymptotically properties of the proposed estimators are discussed. Finally, a simulation study is presented to explain the performance of the estimators.
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Contents

1. Introduction

Let us consider the following partially linear model

$$y_i = X_i'\beta + f(t_i) + \varepsilon_i, \quad i = 1, \ldots, n \tag{1}$$

with y_i denotes a scalar response, $X_i = (X_{i1}, \ldots, X_{ip})'$ denotes a $p \times 1$ independent vectors with a non-singular covariance matrix Σ_X, $\beta = (\beta_1, \ldots, \beta_p)'$ denotes a p-vector of unknown parameters, $f(\cdot)$ is the unknown function, the model error ε_i is an independent random error with zero mean and variance σ^2.

Rewrite model (1) in matrix notation as

$$y = X\beta + f(t) + \varepsilon \tag{2}$$

where $y = (y_1, \ldots, y_n)'$, $f(t) = (f(t_1), \ldots, f(t_n))'$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$ and $X = (X_1, \ldots, X_n)'$ is the $n \times p$ matrix.

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Partial linear models are more flexible than standard linear models since they have a parametric and a nonparametric component. They can be a suitable choice when one suspects that the response y linearly depends on X, but that it is nonlinearly related to X.

The condition number is a measure of the presence of multicollinearity. The condition number of the matrix X presents some information about the existence of multicollinearity, however it does not illustrate the structure of the linear dependency among the column vectors $X_1, X_2, \dots, X_n$. The best way of illustrating the existence and structure of multicollinearity is to see the eigenvalues of $X'X$. If $X'X$ is ill-conditioned with a large condition number a Liu regression estimator can be used to estimate β (see e.g. [1–7]). In this paper, we will examine a biased estimation techniques to be followed when the matrix $X'X$ appears to be ill-conditioned in the partial linear model. We suppose that the condition number of the parametric component is large explains that a biased estimation procedure is desirable.

In this paper, a restricted difference-based estimator is presented for the vector parameter β in the partially linear model when the linear nonstochastic constraint is assumed to hold. We also examine the properties of the proposed estimator.

The rest of the paper is organized as follows: the restricted difference-based Liu estimator is defined in Section 2 and the properties of the proposed estimator are discussed in Section 3. The performance of the new estimator is evaluated by a simulation study in Section 4 and some conclusions are given in Section 5.

2. Profile least-squares estimator

In this section we will propose the restricted difference-based Liu estimator in partially linear model.

2.1. Difference-based estimator

Let $d = (d_0, \dots, d_m)$ be a $m + 1$ vector, where m is the order of differencing and $d_0, \dots, d_m$ are differencing weights satisfying the conditions

$$\sum_{j=0}^{m} d_j = 0, \qquad \sum_{j=0}^{m} d_j^2 = 1. \tag{3}$$

Moreover, for $k = 1, \dots, m$ let $c_k = \sum_{i=1}^{m+1-k} d_i d_{i+k}$. Now, we denote the $(n-m) \times n$ differencing matrix D whose elements satisfy Eq. (3) as follows:

$$D = \begin{pmatrix} d_0 & d_1 & \cdots & d_m & 0 & 0 & \cdots & 0 \\ 0 & d_0 & d_1 & \cdots & d_m & 0 & \cdots & 0 \\ \cdots & \cdots & & & & & & \\ \cdots & \cdots & & & & & & \\ \cdots & \cdots & & & & & & \\ 0 & 0 & \cdots & d_1 & \cdots & d_m & 0 & 0 \\ 0 & 0 & \cdots & d_0 & d_1 & \cdots & d_m & 0 \\ 0 & 0 & \cdots & 0 & d_0 & d_1 & \cdots & d_m \end{pmatrix}. \tag{4}$$

This and related matrices are given, for example, in [8]. Then we can use the differencing matrix to model (2), and this leads to direct estimation of the parametric effect. In particular, take

$$Dy = DX\beta + Df(t) + D\epsilon. \tag{5}$$

Since the data have been reordered so that the $X's$ are close, the application of the differencing matrix D in model (3) can remove the nonparametric effect in large samples [8]. This ignores the presence of $Df(t)$. Thus, we may write Eq. (7) as

$$Dy \doteq DX\beta + D\varepsilon \tag{6}$$

or

$$\widetilde{y} \doteq \widetilde{X}\beta + \widetilde{\varepsilon} \tag{7}$$

where $\widetilde{y} = Dy$, $\widetilde{X} = DX$ and $\widetilde{\varepsilon} = D\varepsilon$.

For arbitrary differencing coefficients satisfying Eq. (6), Yatchew [9] defines a simple differencing estimator of the parameter β in a partial linear model

$$\hat{\beta} = (\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}'\widetilde{y}. \tag{8}$$

In order to account for the parameter β in Eq. (3), we propose the modified estimator of σ^2, defined as

$$\hat{\sigma}^2 = \frac{\widetilde{y}'(I-P)\widetilde{y}}{tr(D'(I-P)D)} \tag{9}$$

where P is the projection matrix and defined as

$$P = \widetilde{X}(\widetilde{X}'\widetilde{X})^{-1}\widetilde{X}'. \tag{10}$$

2.2. Restricted difference-based Liu estimator

In this subsection, we will propose the restricted difference-based Liu estimator when the matrix $\widetilde{X}'\widetilde{X}$ appears to be ill-conditioned.

In order to overcome this problem, one method is to consider biased estimator, such as, Duran et al. [4] proposed a difference-based Liu estimator which is defined as

$$\hat{\beta}(d) = (\widetilde{X}'\widetilde{X} + I)^{-1}(\widetilde{X}'\widetilde{y} + d\hat{\beta}), \quad 0 < d < 1. \tag{11}$$

Alternative method to combat the multicollinearity is to consider restrictions for the parametric components. As pointed by Rao et al. [10], some prior information may improve the efficiency of the estimator, now we consider the linear restriction

$$H\beta = h. \tag{12}$$

For a given $k \times p$ matrix H and a given $k \times 1$ known vector h. Combining the method of the Liu [1], models (2) and (7), using the Lagrange method, we can propose a restricted difference-based Liu estimator which is defined as

$$\hat{\beta}_H(d) = \hat{\beta}(d) - (\tilde{X}'\tilde{X} + I)^{-1}H'\left[H(\tilde{X}'\tilde{X} + I)^{-1}H'\right]^{-1}\left(H\hat{\beta}(d) - h\right). \tag{13}$$

The estimator $\hat{\beta}_H(d)$ is called restricted difference-based Liu estimator. When the errors are correlated, Akdeniz et al. [5] proposed this estimator (13) and they also discuss the small sample of this estimator (13). In this paper we mainly discuss the asymptotic properties of this estimator. When $d = 1$, then the restricted difference-based Liu estimator becomes the restricted difference-based estimator

$$\hat{\beta}_H = \hat{\beta} - (\tilde{X}'\tilde{X})^{-1}H'\left[H(\tilde{X}'\tilde{X})^{-1}H'\right]^{-1}\left(H\hat{\beta} - h\right). \tag{14}$$

In the next section we will give the asymptotic normality of the new estimator $\hat{\beta}_H(d)$.

3. Properties of the new estimator

In order to present the properties of the new estimator we firstly present some assumptions and lemmas.

Definition 1. Define the Lipschitz ball $\Lambda^\alpha(M)$ in the usual way

$$\begin{aligned}\Lambda^\alpha(M) = \{g:\ &\text{for all } 0 \le x, y \le 1,\ k = 0, \dots, \lfloor\alpha\rfloor - 1,\\ &|g^{(k)}(x)| \le M,\ \text{and } |g^{(\lfloor\alpha\rfloor)}(x) - g^{(\lfloor\alpha\rfloor)}(y)| \le M|X - Y|^{\alpha'}\}\end{aligned} \tag{15}$$

where $\lfloor\alpha\rfloor$ is the largest integer less than α and $\alpha' = \alpha - \lfloor\alpha\rfloor$.

Assumptions 1. In this paper, we always assume that $f \in \Lambda^\alpha(M)$.

Assumptions 2. Let $c_k = \sum_{i=1}^{m+1-k} d_i d_{i+k},\ k = 1, \dots, m$, then we have $c_k^2 = O(m^{-1})$ as $m \to \infty$.

Lemma 1. *Suppose that $\alpha > 0$, $m \to \infty$, $m/n \to 0$ and Assumptions 1–2 are satisfied, then the estimator $\hat{\beta}$ given in (8) is asymptotically normal, i.e.*

$$\sqrt{n}(\hat{\beta} - \beta) \longrightarrow^L N\left(0, \sigma^2\Sigma_X^{-1}\right) \tag{16}$$

where Σ_X^{-1} is the non-singular covariance matrix of $X'X$, $\longrightarrow^L$ denotes convergence in distribution.

Proof. See [11].

Theorem 1. *Suppose that $\alpha > 0$, $m \to \infty$, $m/n \to 0$ and Assumptions 1–2 are satisfied, then the estimator $\hat{\beta}(d)$ given in (11) is asymptotically normal, i.e.*

$$\sqrt{n}\left(\hat{\beta}(d) - \beta\right) \longrightarrow^L N\left(0, \sigma^2\Sigma_X^{-1}\right). \tag{17}$$

Proof. By (11), we obtain

$$\begin{aligned}\hat{\beta}(d) - \beta &= (\tilde{X}'\tilde{X} + I)^{-1}(\tilde{X}'\tilde{Y} + d\hat{\beta}) - \beta\\ &= (\tilde{X}'\tilde{X} + I)^{-1}(\tilde{X}'\tilde{X} + dI)(\hat{\beta} - \beta) + (d - 1)(\tilde{X}'\tilde{X} + I)^{-1}\beta.\end{aligned} \tag{18}$$

Then

$$\sqrt{n}\left(\hat{\beta}(d) - \beta\right) = (\tilde{X}'\tilde{X} + I)^{-1}(\tilde{X}'\tilde{X} + dI)\sqrt{n}(\hat{\beta} - \beta) + (d - 1)\sqrt{n}(\tilde{X}'\tilde{X} + I)^{-1}\beta. \tag{19}$$

It is easy to prove that

$$\frac{1}{n}(\tilde{X}'\tilde{X}+I)^{-1} \longrightarrow^{p} \Sigma_X^{-1} \tag{20}$$

$$\frac{1}{n}(\tilde{X}'\tilde{X}+dI) \longrightarrow^{p} \Sigma_X \tag{21}$$

$$(1-d)\sqrt{n}(\tilde{X}'\tilde{X}+I)\beta = O_p(n^{-1/2}) \tag{22}$$

where $\longrightarrow^{p}$ denotes convergence in probability. By Lemma 1 and (19)–(22), we have

$$(\tilde{X}'\tilde{X}+I)^{-1}(\tilde{X}'\tilde{X}+dI)\Sigma_X^{-1}(\tilde{X}'\tilde{X}+dI)(\tilde{X}'\tilde{X}+I)^{-1} = \Sigma_X^{-1}. \tag{23}$$

Then by the Slutsky theorem, (19) and (23), we obtain

$$\sqrt{n}\left(\hat{\beta}(d)-\beta\right) \longrightarrow^{L} N\left(0, \sigma^2\Sigma_X^{-1}\right). \tag{24}$$

Theorem 2. *Suppose that $\alpha > 0$, $m \to \infty$, $m/n \to 0$ and Assumptions 1–2 are satisfied, then the estimator $\hat{\beta}_H(d)$ given in (13) is asymptotically normal, i.e.*

$$\sqrt{n}\left(\hat{\beta}_H(d)-\beta\right) \longrightarrow^{L} N(0, \sigma^2\Omega) \tag{25}$$

where $\Omega = \Sigma_X^{-1} - \Sigma_X^{-1}H'\left[H\Sigma_X^{-1}H'\right]^{-1}H\Sigma_X^{-1}$.

Proof. By (13), we obtain

$$\begin{aligned}\hat{\beta}_H(d)-\beta &= \hat{\beta}(d) - (\tilde{X}'\tilde{X}+I)^{-1}H'\left[H(\tilde{X}'\tilde{X}+I)^{-1}H'\right]^{-1}\left(H\hat{\beta}(d)-h\right)-\beta \\ &= \left(\hat{\beta}(d)-\beta\right) - (\tilde{X}'\tilde{X}+I)^{-1}H'\left[H(\tilde{X}'\tilde{X}+I)^{-1}H'\right]^{-1}\left(H\hat{\beta}(d)-H\beta+H\beta-h\right) \\ &= \left[I-(\tilde{X}'\tilde{X}+I)^{-1}H'\left[H(\tilde{X}'\tilde{X}+I)^{-1}H'\right]^{-1}H\right]\left(\hat{\beta}(d)-\beta\right).\end{aligned} \tag{26}$$

By (24)–(26) we have

$$(\tilde{X}'\tilde{X}+I)^{-1}H'\left[H(\tilde{X}'\tilde{X}+I)^{-1}H'\right]^{-1}H \longrightarrow^{p} \Sigma_X^{-1}H'\left[H\Sigma_X^{-1}H'\right]^{-1}H. \tag{27}$$

Thus by Theorem 1 and (27), we have

$$\left[I-\Sigma_X^{-1}H'\left[H\Sigma_X^{-1}H'\right]^{-1}H\right]\Sigma_X^{-1}\left[I-\Sigma_X^{-1}H'\left[H\Sigma_X^{-1}H'\right]^{-1}H\right]' = \Omega \tag{28}$$

where $\Omega = \Sigma_X^{-1} - \Sigma_X^{-1}H'\left[H\Sigma_X^{-1}H'\right]^{-1}H\Sigma_X^{-1}$. Then by (26)–(28) and the Slutsky theorem, we obtain

$$\sqrt{n}\left(\hat{\beta}_H(d)-\beta\right) \longrightarrow^{L} N(0, \sigma^2\Omega). \tag{29}$$

Remark 1. Compared the asymptotic covariance matrices of $\hat{\beta}_H(d)$ and $\hat{\beta}(d)$, we note that: $\Sigma_X^{-1}-\Omega = \Sigma_X^{-1}H'\left[H\Sigma_X^{-1}H'\right]^{-1}H\Sigma_X^{-1}$ is a positive definite matrix. That is to say when the linear restrictions (12) are assumed to hold, the restricted difference-based Liu estimator $\hat{\beta}_H(d)$ is more efficient than the difference-based Liu estimator $\hat{\beta}(d)$.

4. Simulation study

In order to show the performance of the proposed estimator. Following McDonald and Galarneau [12], the explanatory variables are generated using the following device:

$$x_{ij} = (1-\gamma^2)z_{ij} + \gamma z_{i(p+1)}, \quad i = 1,\ldots,n,\ j = 1,\ldots,p$$

where z_{ij} and $z_{i(p+1)}$ present independent standard normal pseudo-random numbers and γ is specified so that the correlation between any two explanatory variables is given by γ^2.

And observations on the dependent variable are then produced by

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + f(t_i) + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2)$$

Table 1
The estimated SMSE of the estimators when $\gamma = 0.9$ and $m = 4$.

n	$\hat{\beta}$	$\hat{\beta}(d)$	$\hat{\beta}_H$	$\hat{\beta}_H(d)$
$n = 100$	0.0718	0.0712	0.0235	0.0233
$n = 200$	0.0444	0.0443	0.0138	0.0137
$n = 400$	0.0212	0.0211	0.0072	0.0071

Table 2
The estimated SMSE of the estimators when $\gamma = 0.99$ and $m = 4$.

n	$\hat{\beta}$	$\hat{\beta}(d)$	$\hat{\beta}_H$	$\hat{\beta}_H(d)$
$n = 100$	0.9138	0.8320	0.3477	0.3203
$n = 200$	0.4110	0.3923	0.1517	0.1443
$n = 400$	0.2012	0.1966	0.0635	0.0622

Table 3
The estimated SMSE of the estimators when $\gamma = 0.999$ and $m = 4$.

n	$\hat{\beta}$	$\hat{\beta}(d)$	$\hat{\beta}_H$	$\hat{\beta}_H(d)$
$n = 100$	8.1494	5.7412	2.4472	1.7466
$n = 200$	4.1384	3.2039	1.3038	1.0336
$n = 400$	2.0152	1.6964	0.6942	0.5789

where

$$f(t_i) = 1 + 4\left(e^{-550(t_i-0.2)^2} + e^{-200(t_i-0.5)^2} + e^{-950(t_i-0.8)^2}\right)$$

is called the Doppler function for $t_i = (i - 0.5)/n, i = 1, \ldots, n$.

In this paper we consider $n = 100, 200, 400$, $m = 4, 6, 25, 50$, $p = 4$, $\sigma^2 = 0.5$, $\gamma = 0.9, 0.99, 0.999$. And we consider the following linear restrictions

$$H = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{30}$$

and

$$h = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{31}$$

In this section, we use the method that proposed in this paper to estimate β, that is using differencing procedure to estimate β. For example, a fourth-order differencing coefficients, $d_0 = 0.8873$, $d_1 = -0.3099$, $d_2 = -0.2464$, $d_3 = -0.1901$, $d_4 = -0.1409$ in which case $m = 4$. Now we denote a $(n-4) \times n$ differencing matrix as

$$D = \begin{pmatrix} 0.8873 & -0.3099 & \cdots & -0.1409 & 0 & \cdots & 0 \\ 0 & 0.8873 & -0.3099 & \cdots & 0 & \cdots & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -0.1409 \end{pmatrix} \tag{32}$$

when $m = 6, 25, 50$, the differencing coefficients we can see [8]. The simulation is replicated 2000 times by generating new random numbers and the simulated scalar mean squared error (SMSE) values of the estimator are calculated respectively as follows

$$\mathrm{SMSE}(\tilde{\beta}_r) = \frac{\sum_{r=1}^{2000} (\tilde{\beta}_r - \beta)'(\tilde{\beta}_r - \beta)}{2000} \tag{33}$$

where $\tilde{\beta}_r$ is any estimator considered in the study in the ith replication.

From Tables 1 to 4, we summarize our findings as follows. As the sample size increases, the SMSE of all the estimators decreases. For all the cases, the SMSEs increase with the increase in γ. The difference-based estimator $\hat{\beta}$ performs the worst among all estimators in terms of SMSE, and the new estimator $\hat{\beta}_H(d)$ performs better than the $\hat{\beta}$, $\hat{\beta}(d)$ and $\hat{\beta}_H$.

Table 4
The estimated SMSE of the estimators when $\gamma = 0.999$ and $n = 400$.

n	$\hat{\beta}$	$\hat{\beta}(d)$	$\hat{\beta}_H$	$\hat{\beta}_H(d)$
$m = 6$	1.9755	1.6669	0.6679	0.5631
$m = 25$	2.0114	1.6330	0.6798	0.5572
$m = 50$	2.2151	1.7685	0.7624	0.6256

5. Conclusions

In this paper, we proposed a restricted difference-based Liu estimator when some additional linear restrictions are supposed to hold on the parametric component. And then we discuss the asymptotic properties of the new estimator.

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