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Ioannis K. Argyros, Alicia Cordero, Á. Alberto Magreñán, Juan R. Torregrosa

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Third-degree anomalies of Traub's method☆

Ioannis K. Argyros[a], Alicia Cordero[b,*], Á. Alberto Magreñán[c], Juan R. Torregrosa[b]

[a] Department of Mathematics Science, Cameron University, Lawton OK 73505, USA
[b] Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, València, Spain
[c] Universidad Internacional de La Rioja, C/Gran Vía 41, 26005, Logroño (La Rioja), Spain

Abstract

Traub's method is a tough competitor of Newton's scheme for solving nonlinear equations as well as nonlinear systems. Due to its third-order convergence and its low computational cost, it is a good procedure to be applied on complicated multidimensional problems. In order to better understand its behavior, the stability of the method is analyzed on cubic polynomials, showing the existence of very small regions with unstable behavior. Finally, the performance of the method on cubic matrix equations arising in control theory is presented, showing a good performance.

Keywords: Nonlinear equations, Traub's iterative method, basin of attraction, cubic polynomial, parameter plane, stability, matrix equations.

1. Motivation

In many branches of Science and Technology it is necessary to solve different kinds of nonlinear equations or systems $F(x) = 0$, where $F : X \to Y$, being X and Y Banach spaces. The best known iterative scheme is Newton's method

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \quad k = 0, 1, \ldots$$

but Traub's scheme increases the order of convergence of Newton's one, without a complex iterative formula

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - [F'(x^{(k)})]^{-1}F(y^{(k)}), \quad k = 0, 1, \ldots \end{aligned} \tag{1}$$

where $F'(x)$ denotes the Fréchet derivative of F. This scheme can be successfully used, with third-order convergence, on nonlinear problems.

In Control Theory (in the calculation of the logarithm of a matrix or in the computation of sector function), nuclear magnetic resonance, lattice quantum chromo-dynamics and other areas of applications, matrix equations such as $X^p - A = 0$ where the p-th root of a matrix A must be calculated, can appear (see, for example, [1], [2] and [3]). Most of the known algorithms are useless for their numerical instability, unless A is very well conditioned. So, in order to adapt only the best iterative methods for solving this kind of nonlinear problems, we wonder about their behavior on these polynomials, as many of them can be adapted to solve matrix equations holding the order of convergence but it is necessary to know about their stability properties.

In the last years, the use of tools from Complex Dynamics has allowed the researchers in this area of Numerical Analysis to deep in the understanding of the stability of iterative schemes (see, for example, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). The analysis, in these terms, of the rational function R associated to the iterative procedure applied on quadratic polynomials, gives us valuable information about its role on the convergence's dependence on initial estimations, the size and shape of convergence regions and even on a possible convergence to fixed points that are not solution of the problems to be solved or to different attracting or even superattracting cycles. Moreover, if a parametric family is studied under

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*Corresponding author

Email addresses: ioannisa@cameron.edu (Ioannis K. Argyros), acordero@mat.upv.es (Alicia Cordero), alberto.magrenan@unir.net (Á. Alberto Magreñán), jrtorre@mat.upv.es (Juan R. Torregrosa)

this point of view, the most stable elements of the class can be chosen, by means of an appropriated use of the parameter plane.

In this paper, we analyze the dynamics of the rational operator associated to Traub's method on cubic polynomials. Stable and pathological behaviors are obtained depending on the polynomial.

1.1. Dynamical Concepts

In this section, we recall some concepts of complex dynamics that we use in this paper. These concepts can be completed in [14]. So, we need that nonlinear function f is defined on Riemann sphere $\hat{\mathbb{C}}$, as ∞ becomes one more point to be taken into account.

Let us assume that a fixed point iteration function acts on an arbitrary polynomial $p(z)$; that yields a rational function, that will be denoted by R. So, given any rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, \, R(z_0), \, R^2(z_0), ..., R^n(z_0), ...\}.$$

Then, we analyze the phase plane of the map R by classifying the starting points from the asymptotical behavior of their orbits. A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_0) = z_0$ is satisfied. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$.

Moreover, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$. The fixed points different from those associated with the roots of the polynomial $p(z)$ are called *strange fixed points*.

A point z_0 is a *critical point* of the rational map R if R fails to be injective in any neighborhood of z_0. Indeed, if a critical point is different from those associated with the roots of the polynomial $p(z)$, it is called *free critical point*. Indeed, any superattracting fixed point is a critical point (let us remark that, if the iterative method has order of convergence at least two, the roots of $p(z)$ are superattracting fixed points).

The basin of attraction of an attractor α is defined as the set of points that, used as initial estimation, converge to α:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} \, : \, R^n(z_0) \to \alpha, \; n \to \infty\}.$$

The Fatou set of the rational function R, $\mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point or periodic orbit). Its complement in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That means that the basin of attraction of any fixed or periodic point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The following classical result is a key fact to be used in the definition and interpretation of parameter planes. In it, the concept of immediate basin of attraction is introduced, that is, the connected component of the basin of attraction that includes the attracting fixed point.

Theorem 1 ([15, 16]). *Let R be a rational function. The immediate basin of attraction of an attracting fixed or periodic point holds, at least, a critical point.*

The conjugacy classes are extremely useful because they allow us to get general results by using simple functions. Let f and g be functions defined and with image at Riemann sphere. An analytic conjugation between f and g is a diffeomorfism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $h \circ f = g \circ h$.

The following results assure us that, if our aim is to analyze the stability of Traub's method on cubic polynomials, it is enough to study its behavior on $p(z) = (z-1)(z-r)(z+1)$, as the dynamics are equivalent, that is, a conjugacy preserves fixed and periodic points as well as their character and basins of attraction.

Theorem 2 (Scaling Theorem [17]). *Let $f(z)$ be an analytic function, and let $T(z) = \alpha z + \gamma$, with $\alpha \neq 0$, be an affine map. If $g(z) = (f \circ A)(z)$, then $(T \circ R_g \circ T^{-1})(z) = R_f(z)$, that is, R_f is affine conjugated to R_g by T, where R_f and R_g denote the fixed point operator of Traub's method on f and g, respectively.*

Theorem 3 ([18]). *Let $q(z)$ be any cubic polynomial with simple roots. Then, it can be parametrized by means of an affine map to $p(z) = (z-1)(z-r)(z+1)$, $r \in \mathbb{C}$. This map induces a conjugacy between $R_q(z)$ and $R_p(z)$.*

By using conjugacy classes on quadratic and cubic polynomials, it can be proved that second-order Newton's method is globally convergent for quadratic polynomials (there is no free critical point, so no more basins of attraction than those of the roots are possible, by Theorem 1), and its behavior on cubic polynomials was analyzed by Curry et al. in [19], founding attracting periodic orbits, being unstable in small regions of the complex plane. On the other hand, third-order Traub's scheme is not globally convergent for quadratic polynomials but it is quite stable (see [9], [17] or [20], for instance).

The rest of the paper is organized as follows: Section 2 is devoted to analyze the dynamics of Traub's method on cubic polynomials. The calculus of the fixed points and their stability as well as the critical points are presented. Moreover, the parameter planes of the independent free critical points are obtained and some dynamical planes, corresponding to stable or unstable behavior, are shown. In Section 3 we adapt Traub's method for solving matrix equations and test it on several problems with different sizes of the matrices. The paper finishes with some conclusions and the references used in it.

2. Dynamics of Traub's scheme on cubic polynomials

In the following, we apply the fixed point operator associated to Traub's scheme (1) on polynomial $p(z) = (z-1)(z-r)(z+1)$. Then, the following rational operator appears, depending on both $z \in \hat{\mathbb{C}}$ and $r \in \mathbb{C}$,

$$R_p(z,r) = \frac{4(25r^2-9)z^7 + 20r(1-2r^2)z^6 + 2r(19-24r^2)z^4 + 12r^2(r^2-5)z^3 + 12r(2r^2-1)z^2}{(2rz-3z^2+1)^4} + \frac{2r^2(4-r^2)z + 6(r^4+8r^2+1)z^5 - 111rz^8 + r + 46z^9}{(2rz-3z^2+1)^4}. \quad (2)$$

2.1. Fixed and critical points

Let us remark that the rational function $R_p(z,r)$ is simpler for specific values of parameter r that make numerator denominator of $R_p(z,r)$ have common roots. In fact, they are double roots of $p(z)$:

- If $r = 1$, the fixed point operator is $R_p(z,1) = \dfrac{1+10z+46z^2+80z^3+73z^4+46z^5}{(1+3z)^4}$.
- For $r = -1$, the resulting rational function is $R_p(z,-1) = \dfrac{-1+10z-46z^2+80z^3-73z^4+46z^5}{(1-3z)^4}$.

Respect to the fixed points of operator $R_p(z,r)$, the third-order of convergence of the original iterative scheme induces the superattracting character of the roots, as fixed points. However, there exist some strange fixed points of $R_p(z,r)$, whose character is analyzed in the following results. Firstly, we analyze the role of the infinity as a fixed point: $z = \infty$ corresponds to the divergence of the iterative method and it can be checked that it is a fixed point of $R_p(z,r)$. In fact, its character is repulsive as it happens in Newton's scheme.

Lemma 1. *The strange fixed points of $R_p(z,r)$ are the roots of polynomial $s(z) = -1+(-8r+2r^3)z+(11-22r^2)z^2+(54r-10r^3)z^3+(-37+46r^2)z^4-70rz^5+35z^6$, that will be denoted by $s_i(r)$, $i = 1,2,\ldots,6$. Then, there exist six different strange fixed points except in cases $r = \pm 1$, when the operator is simpler and there exist only three strange fixed points.*

- *$s_1(r)$ and $s_6(r)$ are repulsive for all $r \in \mathbb{C}$*
- *$s_i(r)$, $i = 2,3,4,5$ can be attractive (even superattractive) in different small areas of the complex plane.*

Moreover,

- *$s_2(r)$ is superattracting for $r = -3.06574$, $r = 2.93636$ and $r = 0.0161684$*
- *$s_5(r)$ is superattracting for $r = 3.06574$, $r = -2.93636$ and $r = -0.0161684$.*

In fact, the regions of the complex plane where some of these strange fixed points become attractive are showed in Figures 1 and 2. In Figure 1, the stability function of $s_2(r)$, $St2(r) = |R_p'(s_2(r),r)|$, is plotted in those regions of $\mathbb{C}$ where it takes values lower than one, that is, where $s_2(r)$ is attractive or superattractive. The behavior of $s_5(r)$ is similar but symmetric respect the imaginary axis.

The stability of strange fixed points $s_3(r)$ and $s_4(r)$ is complementary, as it can be seen at Figure 2. In it, we show as the loci where $St3(r) = |R_p'(s_3(r),r)| < 1$ and $St4(r) = |R_p'(s_4(r),r)| < 1$ are complementary subsets of small cardioids. When a value of r is taken in these regions, only one of them ($s_3(r)$ or $s_4(r)$) will be attracting, existing a bifurcation curve (the intersection between dark and clear regions in Figure 2) in each case defined by those values of r where the stability of both points change simultaneously.

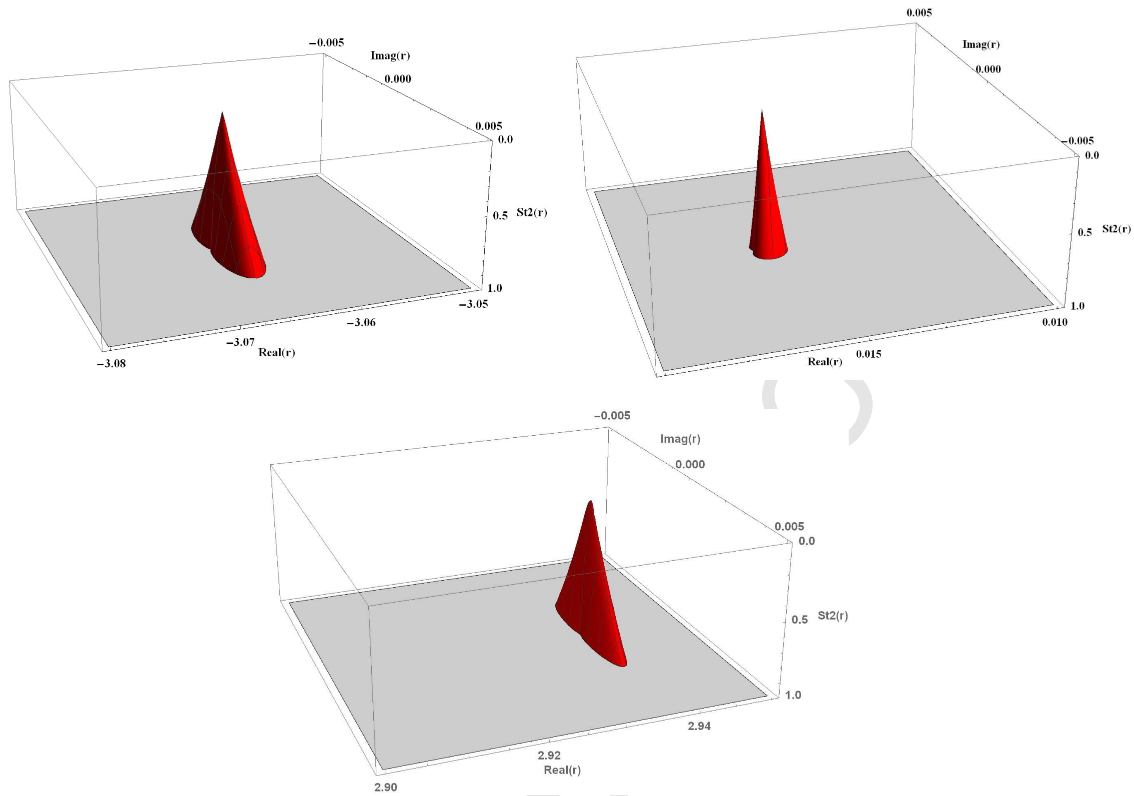


Figure 1: Stability regions of $s_2(r)$ in different areas

Nevertheless, the convergence of the iterative schemes can lead us to other elements, such as periodic orbits, that can also be attractive. In order to detect this kind of behavior, the analysis of the critical points is necessary, by using the classical result of Julia and Fatou (see the Section 1), that establishes the existence of at least one critical point at each immediate component of any basin of attraction.

Critical points are obtained by solving the equation $R_p'(z,r) = 0$. Due to the third order of convergence of the method, the roots of $p(z)$ are also critical points; so, the zeros of $R_p'(z,r)$ are $z = \pm 1$, $z = r$, and also the free critical points, that are described in the following result.

Lemma 2. *The free critical points of $R_p(z,r)$ are*

$$\begin{aligned}
c_1(r) &= \frac{r}{3},\\
c_2(r) &= \frac{1}{69}\left(\sqrt[3]{23}A(r) + \frac{5\dot{2}3^{2/3}\left(r^2+3\right)}{A(r)} + 23r\right),\\
c_3(r) &= \frac{1}{138}\left((-2.84387 + 4.92572i)A(r) + \frac{10\dot{2}3^{2/3}\left(r^2+3\right)}{A(r)} + 46r\right),\\
c_4(r) &= \frac{1}{138}\left(-2\sqrt[3]{-23}A(r) + \frac{10\dot{2}3^{2/3}\left(r^2+3\right)}{A(r)} + 46r\right).
\end{aligned}$$

where $A(r) = \sqrt[3]{-92r^3 + 3\sqrt{69}\sqrt{9r^6 - 287r^4 + 979r^2 - 125} + 828r}$, except in the following cases:

i) If $r = 0$, $c_1(0) = c_4(0) = 0$, so there exist only two different free critical points.

ii) For $r = 3$, $c_1(3) = c_4(3) = 1$, so there are two different free critical points.

iii) When $r = -3$, $c_1(-3) = c_4(-3) = -1$, and there are also two different free critical points.

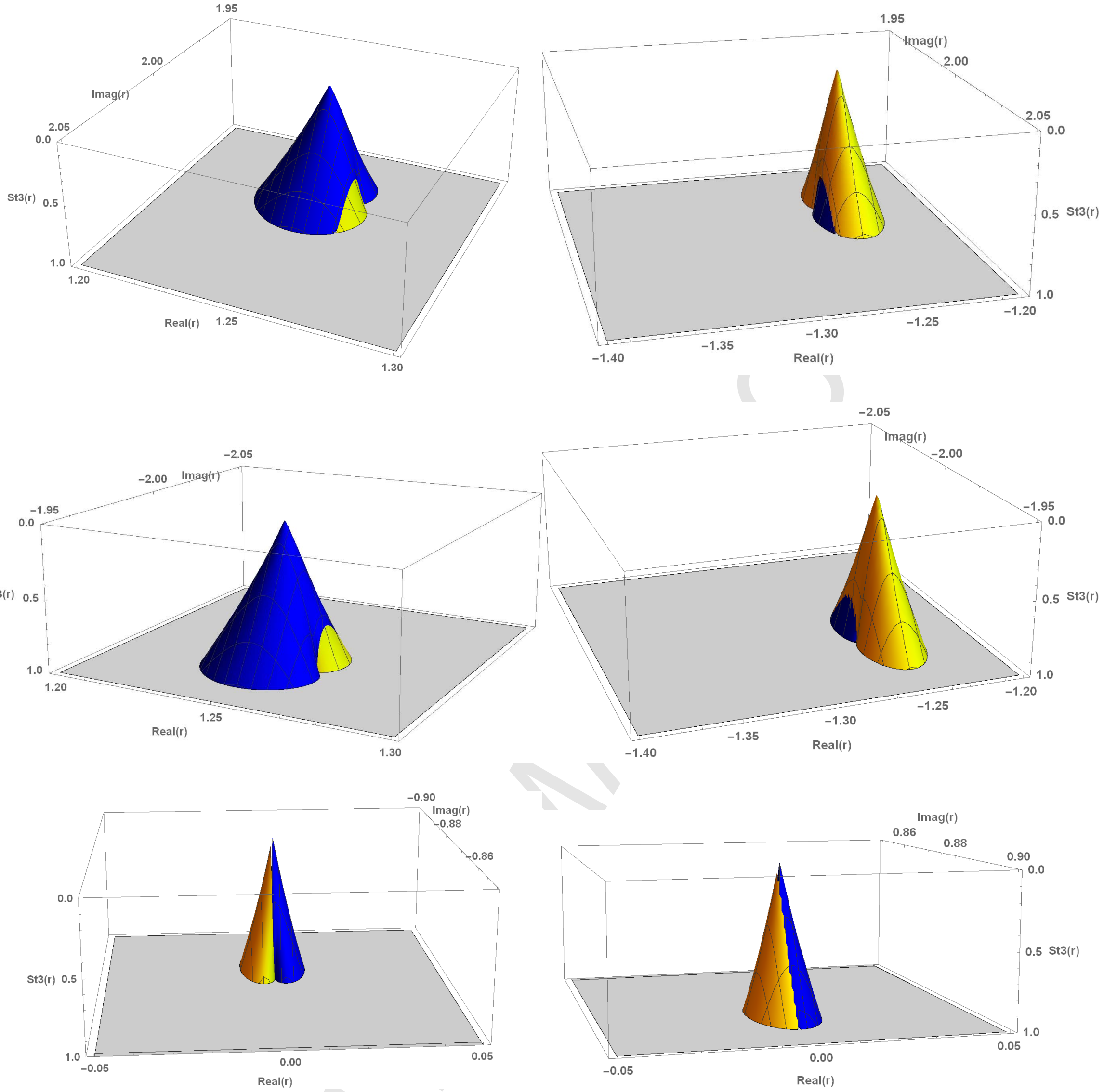


Figure 2: Stability regions of $s_3(r)$ and $s_4(r)$ in different areas

iv) *If $r = 1$, $c_2(1) = 1$ and moreover the rational operator is simpler as $r = 1$ is a double root. In this case, there are two different free critical points.*

v) *In a similar way, when $r = -1$, $c_3(1) = -1$ and also the rational operator is simpler ($r = -1$ is a double root). Then there are two different free critical points.*

vi) *For $r \approx 5.29388$, $r \approx -1.93156$ or $r \approx -0.364462$, $c_2(r) = c_4(r)$ and there are three different free critical points.*

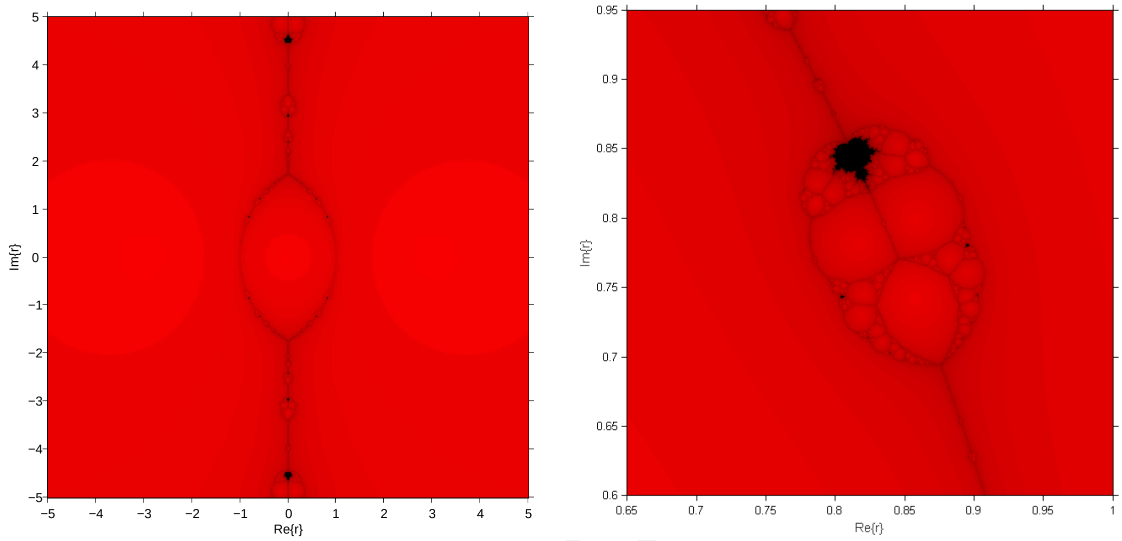
vii) *Finally, for $r \approx -5.29388$, $r \approx 1.93156$ or $r \approx 0.364462$, $c_3(r) = c_4(r)$ and the number of different free critical points is three.*

Let us note that, as lower is the number of free critical points, more stable is the behavior of the method. In fact, the rational operator associated to Newton's method on $p(z)$ has an only free critical point $c_1(r) = \frac{r}{3}$. It is also interesting to notice that the values of r where the strange fixed points are superattracting coincide with those where they coincide with any critical point:

- $s_2(r) = c_3(r)$ if $r \approx -3.06574$, $r \approx 2.93636$ and $r \approx 0.0161684$.

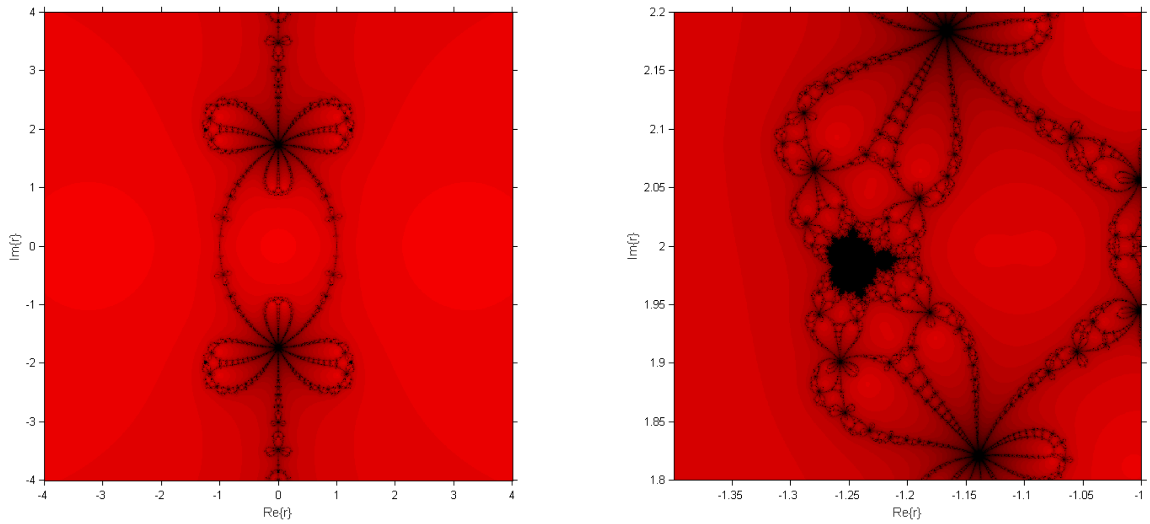
- $s_3(r) = c_1(r)$ for $r \approx 1.25171 \pm 1.9841i$ and $r \approx \pm 0.88115i$.
- $s_4(r) = c_1(r)$ if $r \approx -1.25171 \pm 1.9841i$.
- $s_5(r) = c_2(r)$ when $r \approx 3.06574$, $r \approx -2.93636$ and $r \approx -0.0161684$.

Moreover, in case of Traub's scheme, two critical points satisfy the relation $c_2(r) = -c_3(-r)$, so at most three critical points are independent and only $c_1(r)$, $c_2(r)$ and $c_4(r)$ will be taken into account to obtain parameter planes.



(a) Critical point $c_1(r) = \frac{r}{3}$ (b) $c_1(r) = \frac{r}{3}$, a detail

Figure 3: Parameter plane associated to Newton's method on $p(z)$

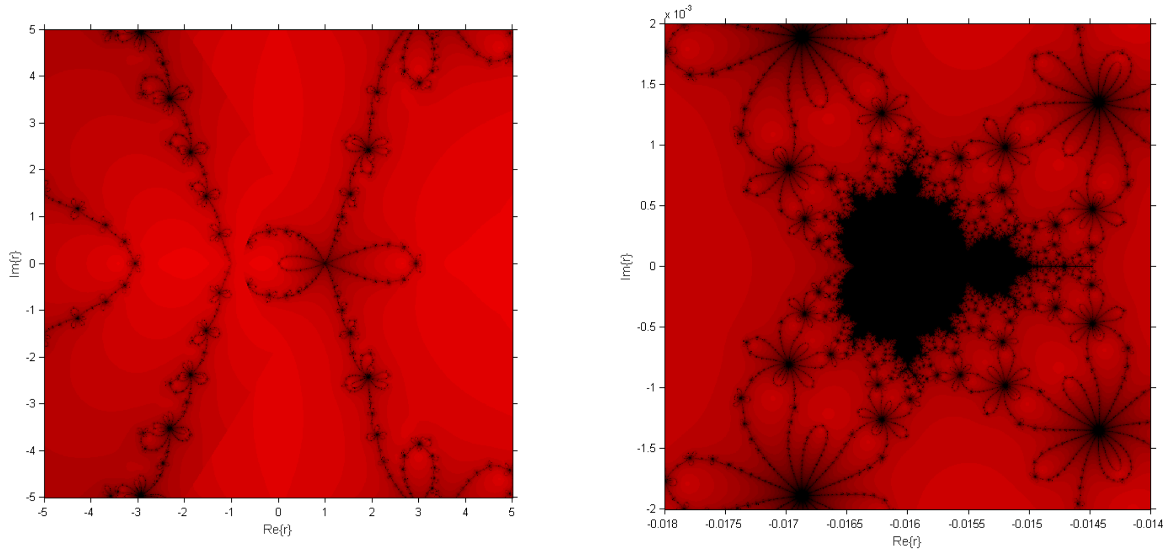


(a) Critical point $c_1(r)$ (b) $c_1(r)$, a detail

Figure 4: Parameter plane associated to Traub's method for the critical point $c_1(r)$

2.2. Parameter and dynamical planes

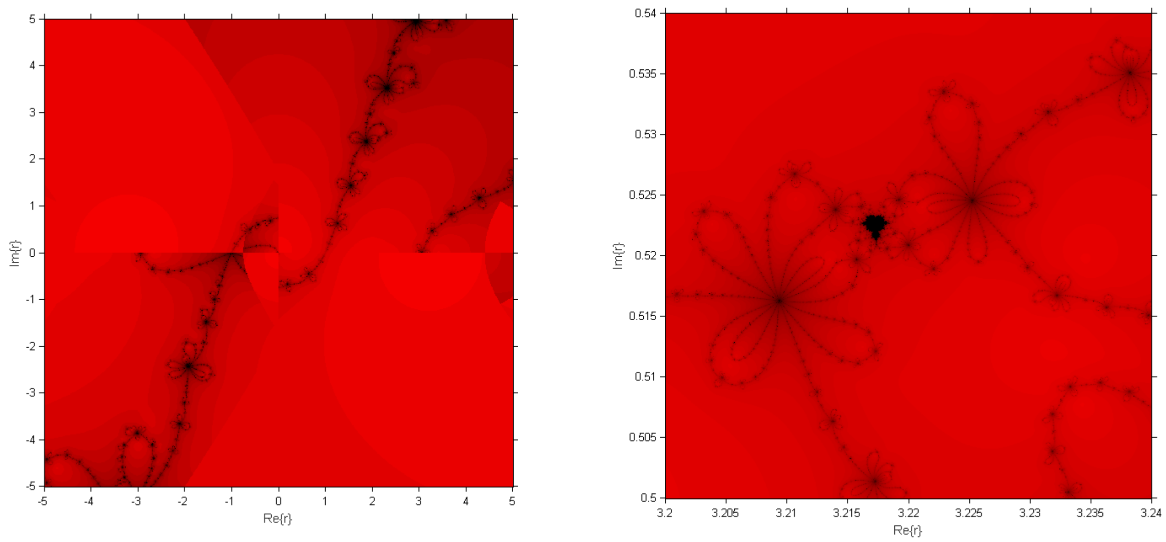
The dynamical behavior of rational operator (2) is globally observed in the different parameter planes and depends on parameter r. These parameter planes are obtained by applying the operator on an independent free critical point as



(a) Critical point $c_2(r)$

(b) $c_2(r)$, a detail

Figure 5: Parameter plane associated to Traub's method for the critical point $c_2(r)$



(a) Critical point $c_4(r)$

(b) $c_4(r)$, a detail

Figure 6: Parameter plane associated to Traub's method for the critical point $c_4(r)$

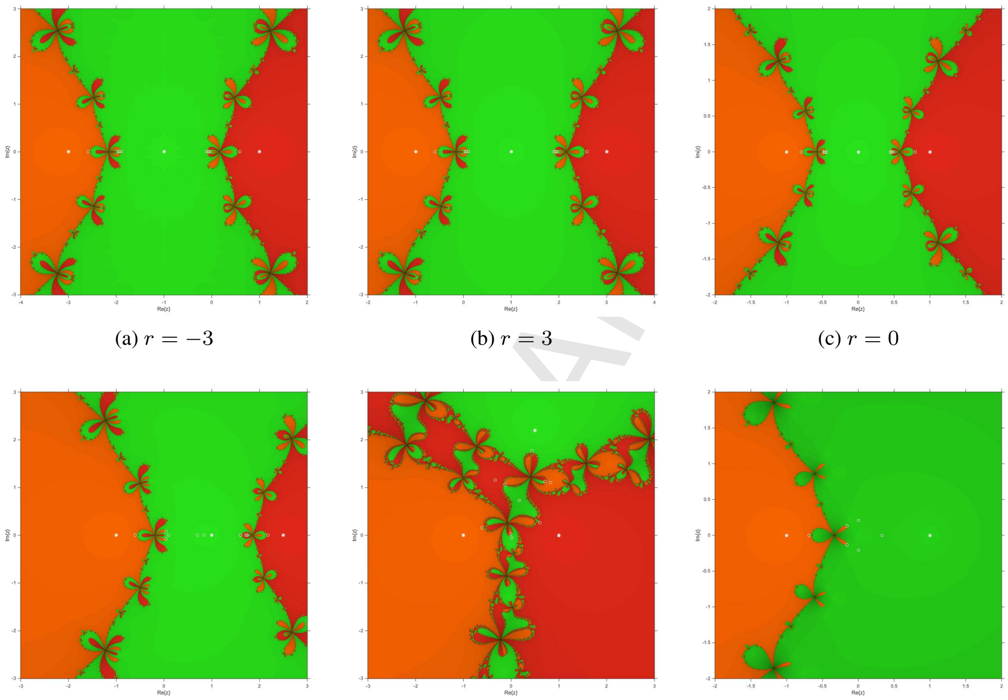
initial estimation and coloring the point of the plane corresponding to the value of r depending on whether this critical point goes to the basin of attraction of any of the roots of $p(z)$ (red color) or to another unknown basin of attraction (black color). Moreover, each point of the parameter plane is associated with a complex value of r, that is, with an specific cubic polynomial. Every value of parameter r belonging to the same connected component of the parameter space gives rise to subsets of polynomials with equivalent dynamical behavior. We have used the code presented in [6] with a mesh of 2000×2000 points, a maximum number of iterations of 500 and a tolerance of 10^{-3}.

Those values of r appearing in red in all parameter planes are the most stable ones, as the method always converge to one of the roots. On the contrary, if a point (a value of r) appears in black in at least one parameter plane, then it means that, for this polynomial, there exists at least one basin of attraction different from the basins of the roots. Let us remark that Figures 4, 5 and 6 show wide red regions of stable behavior and pathological conduct is detected only in tiny areas.

Once a value of parameter r is chosen, in order to analyze the behavior of the method in this case, the associated dynamical plane is obtained. In it, the dependence of the method from initial estimations is observed, as the basins of attraction of any attracting fixed point is drawn in different colors.

As in case of parameter planes, these dynamical planes has been generated by using the routines appearing in [6]. The dynamical plane associated to a value of the parameter r, that is, corresponding to a particular cubic polynomial, is generated by using each point of the complex plane as initial estimation (we have used a mesh of 800×800 points). We paint in the same color the points whose orbit converges to the same fixed point (with a tolerance of 10^{-3}) (all fixed points appear marked as a white circle in the figures and with a star if they are attracting; critical points are marked with a square). For interpretation of the references to color in the text, the reader is referred to the web version of this article.

In Figure 7, some dynamical planes corresponding to some ”red points” in the parameter planes (that is, values of r where the behavior of the rational function is stable) appear. It deserves a special mention the case $r = 1$, where one of the roots has multiplicity two and there exist only two wide basins on attraction.



(a) $r = -3$ (b) $r = 3$ (c) $r = 0$

(d) $r = 2.5$ (e) $r = 0.5 + 2.2i$ (f) $r = 1$

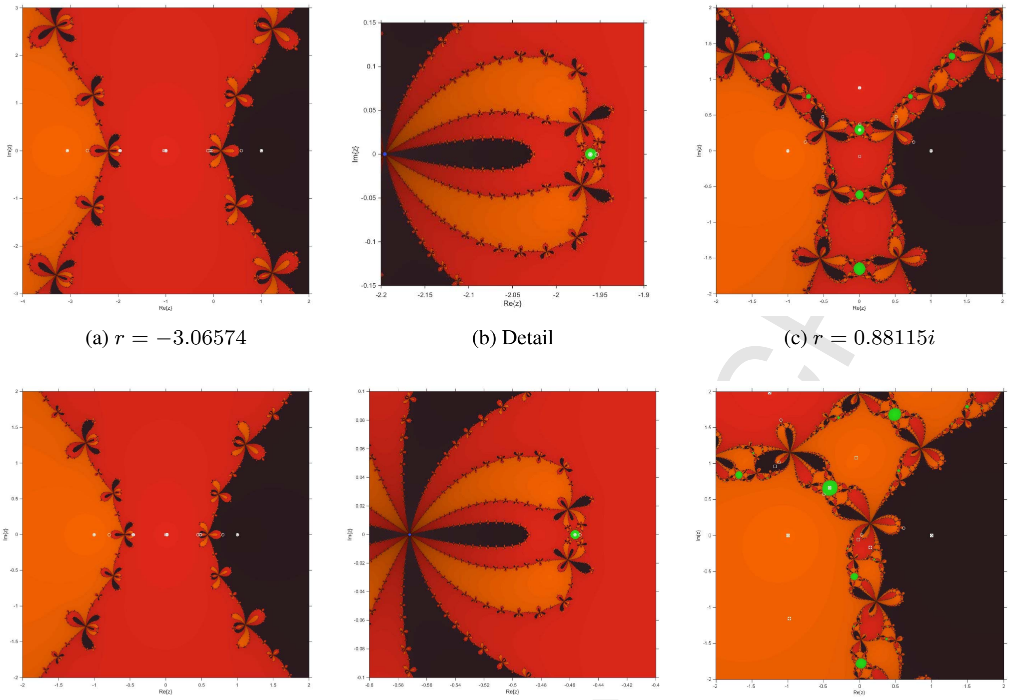
Figure 7: Dynamical planes associated to values of r in red regions of the parameter planes

Nevertheless, for those points appearing in black of any of the parameter planes, unstable behavior of any kind appear: if they belong to any of the cardioids defined by the stability function of an strange fixed point, then the method can fail and converge to any of these fixed points that are not solutions of the problem. On the other hand, there are also loci of attracting periodic orbits in the bulbs surrounding the cardioid of the Mandelbrot-type sets. Some particular cases showing this kind of behavior can be observed in Figure 8. In particular, these values of r have been selected because in all of them one strange fixed point is superattracting. In spite of this, their basins of attraction are very small and frequently it is necessary to get closer in order to see them (Figure 8(b) is a detail of Figure 8(a) and Figure 8(e) is a detail of Figure 8(d)).

As a result, it can be concluded that the behavior of Traub’s method on cubic polynomials is very stable, and the existing anomalies are extremely rare in practice. In the following section, we will test its performance on a singular cubic problem.

3. Application on matrix cubic equations

Given a complex matrix A of size $n \times n$, let us consider the differentiable operator $F : \mathbb{C}^{n\times n} \rightarrow \mathbb{C}^{n\times n}$ such that $F(X) = X^3 - A$. The zeros of this function are the cubic roots of matrix A. In this section, we are interested in the



(a) $r = -3.06574$ (b) Detail (c) $r = 0.88115i$

(d) $r = 0.0161684$ (e) Detail (f) $r = -1.25 + 1.98i$

Figure 8: Dynamical planes associated to values of r in black regions of the parameter planes

numerical computation of the principal cubic root of A, that is, in the unique solution X of $X^3 - A = 0$, for a matrix A having no nonpositive real eigenvalues, such that the eigenvalues of X have argument less in modulus than $\pi/3$.

We are going to adapt Traub's method for obtaining the principal cubic root of A. Let us remark that the Fréchet derivative of F is an operator

$$F' : \mathbb{C}^{n\times n} \to \mathcal{L}(\mathbb{C}^{n\times n}),$$

where $\mathcal{L}(\mathbb{C}^{n\times n})$ denotes the space of linear operators defined in $\mathbb{C}^{n\times n}$, such that

$$F'(X)(H) = XHX + HX^2 + X^2H.$$

If we take an initial estimation X_0, Newton's scheme generates a sequence of iterates $\{Y_k\}_{k\geq 0}$ defined by

$$Y_k = X_k + H_k, \text{ such that } X_kH_kX_k + H_kX_k^2 + X_k^2H_k = A - X_k^3.$$

If we assume the commutativity relation $X_kH_k = H_kX_k$, the previous equation is $3X_k^2H_k = A - X_k^3$. Then,

$$H_k = \frac{1}{3}(X_k^{-2}A - X_k)$$

and therefore the iterative expression of Newton's method is

$$Y_k = X_k + \frac{1}{3}(X_k^{-2}A - X_k) = \frac{1}{3}(2X_k + X_k^{-2}A),$$

where X_k^{-2} denotes the square of the inverse of X_k.

By applying the standard local convergence theorem, it can be proved the quadratic convergence of Newton's method to the principal cubic root of A, when matrix A is nonsingular and the initial guess X_0 commutes with A.

For obtaining the sequence of iterates $\{X_k\}_{k\geq 0}$ of Traub's scheme, the second step of the method is made

$$X_{k+1} = Y_k + H_k, \text{ such that } X_kH_kX_k + H_kX_k^2 + X_k^2H_k = A - Y_k^3.$$

If we assume again that X_k and H_k commute, we obtain $H_k = \frac{1}{3}(X_k^{-2}A - X_k^{-2}Y_k^3)$. Therefore, the iterative formula of Traub's method is

$$X_{k+1} = Y_k + \frac{1}{3}(X_k^{-2}A - X_k^{-2}Y_k^3) = \frac{1}{3}(2X_k + 2X_k^{-2}A - X_k^{-2}Y_k^3).$$

Although it is not the object of this work, it can be proved that Traub's method converges cubically to the desired root, when matrix A is nonsingular and the initial approximation X_0 commutes with A.

Now, we apply this iterative expression for obtaining the principal cubic root of A, for different sizes of the matrix.

Example 1. *Let us consider the nonsingular and no diagonalizable matrix*

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

We apply Newton' and Traub's methods for obtaining the principal cubic root of A. We work in Matlab, in variable precision arithmetics, with 50 significant digits, and a stopping criterion using the 2-norm, $\|X_k^3 - A\| < 10^{-40}$. The computer specifications are Intel(R) Core(TM), i5-2500, CPU 3.30 GHz, with 16 GB of RAM. In Table 1 we show the number of iterations for different initial guess, as well as the error in the last iteration and the approximated computational order of convergence (ACOC), according to

$$p \approx ACOC = \frac{\ln(\|X_{k+1} - X_k\|/\|X_k - X_{k-1}\|)}{\ln(\|X_k - X_{k-1}\|/\|X_{k-1} - X_{k-2}\|)}.$$

Let us remark that the value of ACOC that is presented in the different tables is the last coordinate of vector ACOC when the variation between its values is small. We denote by I_3 the identity matrix of size 3×3 and by $\|A\|$ the 2-norm of matrix A.

Newton's method				Traub's method			
X_0	Iter	ACOC	Error	X_0	Iter	ACOC	Error
I_3	7	2.0000	1.58e-47	I_3	10	2.9999	1.88e-56
$4I_3$	7	2.0436	8.91e-50	$4I_3$	5	3.1352	1.36e-57
$10I_3$	9	2.0302	1.29e-57	$10I_3$	6	3.1259	1.32e-57
A	7	2.0000	1.58e-47	A	5	2.9997	1.31e-57
$A/\|A\|$	38	1.0006	6.3e-41	$A/\|A\|$	26	1.0013	9.45e-41
$A/\|A\|^2$	11	2.0283	1.31e-57	$A/\|A\|^2$	11	3.2653	2.99e-45

Table 1: Numerical results for A by using variable precision arithmetics

In Table 1 we observe the stability of both methods and the best results of Traub's method compared to Newton's scheme. The results that appear in this table confirm the theoretical results, except when we choose as an initial guess $X_0 = A/\|A\|$. In this case, both methods have only linear convergence.

Example 2. *Let us consider the following tridiagonal matrix B, of arbitrary size $n \times n$, related to the discretization of partial differential equations*

$$B = \begin{pmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & 1-2\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & 1-2\lambda \end{pmatrix},$$

where $\lambda = 2 \cdot 10^{-3}$.

We apply again Newton' and Traub's methods for approximating the principal cubic root of B, for different values of n. In this case, we work in double precision arithmetics, with 32 significant digits, and the stopping criterion $\|X_k^3 - B\| <$

Newton's method				Traub's method			
n	X_0	Iter	Error	n	X_0	Iter	Error
20	I_n	3	3.33e-16	20	I_n	2	6.66e-16
50	$2I_n$	6	7.10e-16	50	$2I_n$	4	7.05e-15
50	$10I_n$	10	1.43e-15	50	$10I_n$	7	4.44e-16
100	$2I_n$	6	7.10e-16	100	$2I_n$	4	7.06e-15
100	$10I_n$	10	1.43e-15	100	$10I_n$	7	4.44e-16
500	$2I_n$	6	7.86e-16	500	$2I_n$	4	7.05e-15
500	$10I_n$	10	1.48e-15	500	$10I_n$	7	4.44e-16
1000	$2I_n$	6	7.86e-16	1000	$2I_n$	4	7.05e-15
1000	$10I_n$	10	1.48e-15	1000	$10I_n$	7	4.44e-16

Table 2: Numerical results for B by using double precision arithmetics and different values of n

10^{-12}. In Table 2 we show the number of iterations for different initial guess and sizes of matrix A and the error in the last iteration.

From the results of Table 2 we would like to highlight that the number of iterations and the error in the last iteration keep on, with independence of the size of matrix A, both Newton's method and Traub's method. The difference in the number of iterations among both methods is due to the difference in the order of convergence.

4. Conclusions

In this paper we have shown that third-order Traub's scheme has a very stable behavior on cubic polynomials. The only anomalies appear as very small basins of attracting strange fixed points, inside the cardioids of Mandelbrot-type sets found in the parameter planes. Moreover, existing bulbs around Mandelbrot-type sets are the loci of attracting periodic orbits of different periods. As, except in these small regions of the parameter planes, the behavior of Traub's method on cubic polynomials is extremely stable, the numerical results obtained by applying Traub's scheme for approximating the cubic roots of complex matrices are satisfactory.

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