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An equivalence lemma for a class of fuzzy implicit integro-differential equations

Masoumeh Zeinali* and Sedaghat Shahmorad

Abstract. In the current paper, an equivalence lemma is proposed which converts a second order implicit form of nonlinear fuzzy Volterra integro-differential equation to four different kinds of nonlinear integral equations based on kind of GH-differentiability of the first and second derivatives. Finally, some examples are given to illustrate the theoretical results.

Keywords: Implicit equation; Fuzzy integro-differential equation; equivalence lemma.

MSC(2010) 45JXX, 94DXX.

1 Introduction

In the crisp and fuzzy differential equations, converting a differential equation to an integral equation happens frequently, often, integral operator is preferred to differential operator due to its well-behavior and stability (see [7]). This strategy is more significant in fuzzy cases. Because, one encounters some serious problems such as the existence of H-differences, dealing with differential operator. In numerical and theoretical areas, there are several papers in which an equivalence lemma has been proved in order to convert the fuzzy differential equation into a fuzzy integral equation and viceversa. Investigating on the existence and uniqueness results for the fuzzy initial value problem

$$y'(t) = f(t, y(t)), y(a) = y_0 \quad (1.1)$$

under the Hukuhara differentiability, Kaleva [8] presented an equivalence lemma which converts that main problem into the following integral equation

$$y(t) = y_0 + \int_a^t f(s, y(s))ds.$$

After introducing strongly generalized differentiability by Bede et al. in [4], the equivalence lemma presented by Kaleva was generalized to include different kinds of differentiability for the possible solutions. Based on the new equivalence lemma, problem (1.1) is equivalent to the integral equation

$$y(t) = y_0 + \int_a^t f(s, y(s))ds$$

or

$$y(t) = y_0 \ominus (-1) \int_a^t f(s, y(s))ds$$

depending on the strongly generalized differentiability considered of type (i) or (ii), respectively. In [2], a second order nonlinear fuzzy differential equation with initial conditions has been converted to four different forms of integral equations depending on the kind of differentiability of the solution and its derivative. Considering the topics of fuzzy integro-differential equation (FIDE), Alikhani et al.

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[1] constructed an equivalence lemma which converts the FIDE to a corresponding integral equation. They presented two different kinds of solutions entitled “proper solution” and “mixed solution”. The term “proper solution” refers to the solutions which are (i) or (ii)-differentiable in the whole domain. When the kind of differentiability of the solution is changed within the domain (from (i) to (ii) or inverse), the term “mixed solution” is used.

In the above mentioned papers, a fuzzy initial value problem is converted to a fuzzy Volterra integral equation. Obviously, for a boundary value problem, the corresponding integral equation should be of Fredholm type. In [9, 10], the authors proved an equivalence lemma which converts a fuzzy boundary value problem to a fuzzy Fredholm integral equation with Green’s kernel. Some years later, Bede contradicted this equivalence by giving a counterexample. He also proved that a large number of fuzzy boundary value problems don’t have any “proper solution” [3]. After that some researchers tried to solve this problem by the use of switching points.

In the present paper, we consider the following implicit form of second order fuzzy integro-differential equation

$$y''(t) = f(t, y(t), y'(t)) + \int_0^t k(t, \tau, y(\tau), y'(\tau), y''(\tau)) d\tau, \quad t \in I = [0, b]$$

along with the initial conditions $y(0) = y_0$ and $y'(0) = v_0$. This equation is the general form of the equation of particle motion in a viscous fluid in the field of fluid mechanics. For this problem, four types of solutions are introduced based on the kind of differentiability. Then, for each solution, the given problem is converted to an integral equation using appropriately designed equivalence lemma. The proposed equivalence lemma, could be very useful for study of existence and uniqueness results and also for numerical solution of this problem.

The next parts of this paper are organized as follows: After introducing some basic definitions and theorems in Section 2, in Section 3, we propose our main results. The paper will be finished by giving an example.

2 Preliminaries

In this section, we state definitions and theorems that will be used throughout the next sections.

Definition 2.1 (see [6].) *A fuzzy number is a fuzzy convex and normal fuzzy subset in $\mathbb{R}$ with upper semicontinuous membership function and compact support. The set of all fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$.*

For $0 \leq r \leq 1$, the r -cut of fuzzy number u is defined as

$$[u]^r = \begin{cases} \{x \in \mathbb{R} : u(x) \geq r\}, & 0 < r \leq 1, \\ \overline{\{x \in \mathbb{R} : u(x) > 0\}}, & r = 0, \end{cases}$$

where $u(x)$ is the membership function of u . Then, it is clear that u is a fuzzy number if and only if it’s r -cuts are closed and bounded intervals and $[u]^1 \neq \emptyset$ (see [5]).

Definition 2.2 (See [4].) *Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$. The sum, H -difference and scalar product are defined levelwise as follows:*

$$\begin{aligned} [u + v]^r &= [u]^r + [v]^r, \\ [\lambda \cdot u]^r &= \lambda [u]^r, \\ [u \ominus v]^r &= [u_-^r - v_-^r, u_+^r - v_+^r], \end{aligned}$$

where $[u]^r = [u_-^r, u_+^r]$ and $[v]^r = [v_-^r, v_+^r]$.

The H-difference has the following properties:

$$\begin{aligned} A \ominus (B \ominus C) &= A + C \ominus B \\ A \ominus (B + C) &= A \ominus B \ominus C \\ (-1)(A \ominus B) &= (-A) \ominus (-B) \end{aligned}$$

The space $\mathbb{R}_{\mathcal{F}}$ can be metricized by the metric $D(u, v) = \sup_{r \in [0, 1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$, where $[u]^r = [u_-^r, u_+^r]$ and $[v]^r = [v_-^r, v_+^r]$ are r -cuts of $u, v \in \mathbb{R}_{\mathcal{F}}$. It is well known that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space with the following properties:

- (i) $D(u + v, u + w) = D(v, w)$,
- (ii) $D(k.u, k.v) = |k|D(u, v)$,
- (iii) $D(u + v, w + e) \leq D(u, w) + D(v, e)$.

Definition 2.3 (see [11]) Let $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$. The function f is called Riemann integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any partition $\Delta_n : a = x_0 < x_1 < \dots < x_n = b$ of $[a, b]$ with $\max_{1 \leq i \leq n} (x_i - x_{i-1}) < \delta$ and for any $\psi_i \in [x_{i-1}, x_i], \forall i = 1, \dots, n$, we have

$$D\left(\sum_{i=1}^n (x_i - x_{i-1}) \cdot f(\psi_i), A\right) \leq \epsilon$$

for some $A \in \mathbb{R}_{\mathcal{F}}$. Then, A is called Riemann integral of f and it is denoted by $A = \int_a^b f(t)dt$.

Definition 2.4 (see [4].) The function $f : (a, b) \rightarrow \mathbb{R}_{\mathcal{F}}$ is called strongly generalized differentiable at $x_0 \in (a, b)$ if there exists $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that

(i)
$$\lim_{h \searrow 0} D\left(\frac{f(x_0 + h) \ominus f(x_0)}{h}, f'(x_0)\right) = \lim_{h \searrow 0} D\left(\frac{f(x_0) \ominus f(x_0 - h)}{h}, f'(x_0)\right) = 0,$$

provided that H-differences $f(x_0 + h) \ominus f(x_0)$ and $f(x_0) \ominus f(x_0 - h)$ exist for sufficiently small $h > 0$, or

(ii)
$$\lim_{h \searrow 0} D\left(\frac{f(x_0) \ominus f(x_0 + h)}{-h}, f'(x_0)\right) = \lim_{h \searrow 0} D\left(\frac{f(x_0 - h) \ominus f(x_0)}{(-h)}, f'(x_0)\right) = 0,$$

provided that H-differences $f(x_0) \ominus f(x_0 + h)$ and $f(x_0 - h) \ominus f(x_0)$ exist for sufficiently small $h > 0$, or

(iii)
$$\lim_{h \searrow 0} D\left(\frac{f(x_0 + h) \ominus f(x_0)}{h}, f'(x_0)\right) = \lim_{h \searrow 0} D\left(\frac{f(x_0 - h) \ominus f(x_0)}{-h}, f'(x_0)\right) = 0,$$

provided that H-differences $f(x_0 + h) \ominus f(x_0)$ and $f(x_0 - h) \ominus f(x_0)$ exist for sufficiently small $h > 0$, or

(iv)
$$\lim_{h \searrow 0} D\left(\frac{f(x_0) \ominus f(x_0 + h)}{-h}, f'(x_0)\right) = \lim_{h \searrow 0} D\left(\frac{f(x_0) \ominus f(x_0 - h)}{h}, f'(x_0)\right) = 0,$$

provided that H-differences $f(x_0) \ominus f(x_0 + h)$ and $f(x_0) \ominus f(x_0 - h)$ exist for sufficiently small $h > 0$.

Lemma 2.5 (see [4]) Let $t_0 \in \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ be continuous. Then, the fuzzy boundary value problem $y' = f(t, y), y(t_0) = y_0 \in \mathbb{R}_{\mathcal{F}}$ is equivalent to one of the integral equations:

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s))ds, \forall t \in [t_0, t_1]$$

if y is considered to be (i)-differentiable, or

$$y_0 = y(t) + (-1) \int_{t_0}^t f(s, y(s)) ds, \forall t \in [t_0, t_1]$$

if y is considered to be (ii)-differentiable on some interval $(t_0, t_1) \subset \mathbb{R}$.

Lemma 2.6 (see [11]) Let $g : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be a continuous. Then

(a) The fuzzy function $G(t) = \int_a^t g(s) ds$ is (i)-GH-differentiable and we have $G'(t) = g(t)$.

(b) The fuzzy function $G(t) = \int_t^b g(s) ds$ is (ii)-GH-differentiable and we have $G'(t) = -g(t)$.

3 Problem description

In the present work, we will investigate the implicit form of the following second order fuzzy integro-differential equation with initial conditions:

$$y''(t) = f(t, y(t), y'(t)) + \int_0^t k(t, \tau, y(\tau), y'(\tau), y''(\tau)) d\tau, \quad t \in I = [0, b], \quad (3.1)$$

$$y(0) = y_0 \quad (3.2)$$

$$y'(0) = v_0 \quad (3.3)$$

in which $f \in C([0, b] \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$, $k \in C(G \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$, $h_0, v_0 \in \mathbb{R}_{\mathcal{F}}$ and

$$G = \{(t, \tau) | t \in [0, b], \tau \in [0, t]\}.$$

Let us denote by $C^2([0, b], \mathbb{R}_{\mathcal{F}})$ the space of fuzzy function with continuous second derivative (in a GH-derivative sense). In the remaining part of this paper, the fuzzy initial value problem (3.1)-(3.3) will be called FIVP. In the following definition, we introduce four different form of the solution for FIVP based on the kind of first and second derivative of the solution.

Definition 3.1 A solution for the FIVP is a fuzzy function $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$ satisfying the equations (3.1)-(3.3). Refereing to the different kind of GH-differentiability, various solutions can be considered for the FIVP as follows:

- A fuzzy solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$ is called **(i)-(i)-solution**, if y and y' are (i)-GH-differentiable.
- A fuzzy solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$ is called **(i)-(ii)-solution**, if y is (i)-GH-differentiable and y' is (ii)-GH-differentiable.
- A fuzzy solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$ is called **(ii)-(i)-solution**, if y is (ii)-GH-differentiable and y' is (i)-GH-differentiable.
- A fuzzy solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$ is called **(ii)-(ii)-solution**, if y and y' are (ii)-GH-differentiable.

3.1 The equivalence Lemma

In order to facilitate the investigation of FIVP, we convert the FIVP to an equivalent nonlinear integral equation in this section. To this end, we will prove a lemma which states FIVP has a unique solution if and only if the equivalent integral equation has a unique solution and also shows the relation between the solutions of FIVP and the equivalent integral equation.

Lemma 3.2 Let $g : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be continuous and $v_0, y_0 \in \mathbb{R}_{\mathcal{F}}$. Then

(a) The fuzzy function $G(t) = \int_0^t (t-s)g(s)ds$ is (i)-GH-differentiable and we have $G'(t) = \int_0^t g(s)ds$.

(b) If the H-differences

$$\begin{aligned} & y_0 + tv_0 \ominus (-1) \int_0^t (t-s)g(s)ds, \\ & v_0 \ominus (-1) \int_0^t g(s)ds, \\ & hv_0 \ominus (-h) \int_0^t g(s)ds \ominus (-1) \int_t^{t+h} (t+h-s)g(s)ds \end{aligned}$$

exist, then, the fuzzy function $G(t) = y_0 + tv_0 \ominus (-1) \int_0^t (t-s)g(s)ds$ is (i)-GH-differentiable and we have $G'(t) = v_0 \ominus (-1) \int_0^t g(s)ds$.

(c) If the H-difference $y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)g(s)ds$ exists, then, the fuzzy function $G(t) = y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)g(s)ds$ is (ii)-GH-differentiable and we have $G'(t) = v_0 + \int_0^t g(s)ds$.

(d) If the H-differences

$$\begin{aligned} & y_0 \ominus (-t)v_0 + \int_0^t (t-s)g(s)ds, \\ & v_0 \ominus (-1) \int_0^t g(s)ds, \\ & (-h)v_0 \ominus h \int_0^t g(\tau)d\tau \ominus \int_t^{t+h} (t+h-\tau)g(\tau)d\tau \end{aligned}$$

exist, then, the fuzzy function $G(t) = y_0 \ominus (-t)v_0 + \int_0^t (t-s)g(s)ds$ is (ii)-GH-differentiable and we have $G'(t) = v_0 \ominus (-1) \int_0^t g(s)ds$.

Proof. (a) For $h > 0$ we have

$$\begin{aligned} G(t+h) &= \int_0^{t+h} (t+h-s)g(s)ds = \int_0^t (t+h-s)g(s)ds + \int_t^{t+h} (t+h-s)g(s)ds \\ &= \int_0^t (t-s)g(s)ds + h \int_0^t g(s)ds + \int_t^{t+h} (t+h-s)g(s)ds \\ &= G(t) + h \int_0^t g(s)ds + \int_t^{t+h} (t+h-s)g(s)ds. \end{aligned}$$

Since $h \int_0^t g(s)ds + \int_t^{t+h} (t+h-s)g(s)ds \in \mathbb{R}_{\mathcal{F}}$, H-difference $G(t+h) \ominus G(t)$ exists. Furthermore,

$$\begin{aligned} \lim_{h \searrow 0} D \left(\frac{G(t+h) \ominus G(t)}{h}, \int_0^t g(s)ds \right) &= \lim_{h \searrow 0} D \left(\frac{h \int_0^t g(s)ds + \int_t^{t+h} (t+h-s)g(s)ds}{h}, \int_0^t g(s)ds \right) \\ &= \lim_{h \searrow 0} D \left(\frac{\int_t^{t+h} (t+h-s)g(s)ds}{h}, 0 \right) = 0. \end{aligned}$$

Hence, G is (i)-GH-differentiable and $G'(t) = \int_0^t g(s)ds$.

(b) For $h > 0$, we have

$$\begin{aligned}
 G(t+h) &= y_0 + (t+h)v_0 \ominus (-1) \int_0^{t+h} (t+h-s)g(s)ds \\
 &= y_0 + tv_0 + hv_0 \ominus (-1) \left(\int_0^t (t-s)g(s)ds + h \int_0^t g(s)ds + \int_t^{t+h} (t+h-s)g(s)ds \right) \\
 &= y_0 + tv_0 + hv_0 \ominus (-1) \int_0^t (t-s)g(s)ds \ominus (-h) \int_0^t g(s)ds \ominus (-1) \int_t^{t+h} (t+h-s)g(s)ds \\
 &= G(t) + hv_0 \ominus (-h) \int_0^t g(s)ds \ominus (-1) \int_t^{t+h} (t+h-s)g(s)ds
 \end{aligned}$$

Therefore, H-difference $G(t+h) \ominus G(t)$ exists and

$$\begin{aligned}
 \lim_{h \searrow 0} \frac{G(t+h) \ominus G(t)}{h} &= \lim_{h \searrow 0} \frac{hv_0 \ominus (-h) \int_0^t g(s)ds \ominus (-1) \int_t^{t+h} (t+h-s)g(s)ds}{h} \\
 &= v_0 \ominus (-1) \int_0^t g(s)ds.
 \end{aligned}$$

Thus, G is (i)-GH-differentiable and we have $G'(t) = v_0 \ominus (-1) \int_0^t g(s)ds$.

(c) For $h > 0$, we have

$$\begin{aligned}
 G(t+h) + (-h)v_0 + (-h) \int_0^t g(s)ds + (-1) \int_t^{t+h} (t+h-s)g(s)ds &= y_0 \ominus (-1)(t+h)v_0 \\
 \ominus (-1) \int_0^{t+h} (t+h-s)g(s)ds + (-h)v_0 + (-h) \int_0^t g(s)ds + (-1) \int_t^{t+h} (t+h-s)g(s)ds \\
 &= y_0 \ominus (-t)v_0 \ominus (-h)v_0 \ominus (-1) \left[\int_0^t (t-s)g(s)ds + h \int_0^t g(s)ds + \int_t^{t+h} (t+h-s)g(s)ds \right] \\
 &+ (-h)v_0 + (-h) \int_0^t g(s)ds + (-1) \int_t^{t+h} (t+h-s)g(s)ds \\
 &= y_0 \ominus (-t)v_0 \ominus (-h)v_0 \ominus (-1) \int_0^t (t-s)g(s)ds \ominus (-1)h \int_0^t g(s)ds \ominus (-1) \int_t^{t+h} (t+h-s)g(s)ds \\
 &+ (-h)v_0 + (-h) \int_0^t g(s)ds + (-1) \int_t^{t+h} (t+h-s)g(s)ds \\
 &= y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)g(s)ds = G(t).
 \end{aligned}$$

Hence, H-difference $G(t) \ominus G(t+h)$ exists and

$$\begin{aligned}
 \lim_{h \searrow 0} \frac{G(t) \ominus G(t+h)}{-h} &= \lim_{h \searrow 0} \frac{(-h)v_0 + (-h) \int_0^t g(s)ds + (-1) \int_t^{t+h} (t+h-s)g(s)ds}{-h} \\
 &= \lim_{h \searrow 0} \frac{(-h)v_0 + (-h) \int_0^t g(s)ds + (-1) \int_t^{t+h} (t+h-s)g(s)ds}{-h} \\
 &= v_0 + \int_0^t g(s)ds
 \end{aligned}$$

which means that G is (ii)-GH-differentiable and $G'(t) = v_0 + \int_0^t g(s)ds$.

(d) For $h > 0$, we have

$$\begin{aligned}
 & G(t+h) + (-h)v_0 \ominus h \int_0^t g(\tau) d\tau \ominus \int_t^{t+h} (t+h-\tau)g(\tau) d\tau \\
 &= y_0 \ominus (-t-h)v_0 + \int_0^{t+h} (t+h-\tau)g(\tau) d\tau + (-h)v_0 \ominus h \int_0^t g(\tau) d\tau \ominus \int_t^{t+h} (t+h-\tau)g(\tau) d\tau \\
 &= y_0 \ominus (-t)v_0 \ominus (-h)v_0 + \left(\int_0^t (t-\tau)g(\tau) d\tau + h \int_0^t g(\tau) d\tau + \int_t^{t+h} (t+h-\tau)g(\tau) d\tau \right) \\
 &+ (-h)v_0 \ominus h \int_0^t g(\tau) d\tau \ominus \int_t^{t+h} (t+h-\tau)g(\tau) d\tau \\
 &= y_0 \ominus (-t)v_0 + \int_0^t (t-\tau)g(\tau) d\tau = G(t).
 \end{aligned}$$

So, H-difference $G(t) \ominus G(t+h)$ exists and

$$\begin{aligned}
 \lim_{h \searrow 0} \frac{G(t) \ominus G(t+h)}{-h} &= \lim_{h \searrow 0} \frac{(-h)v_0 \ominus h \int_0^t g(\tau) d\tau \ominus \int_t^{t+h} (t+h-\tau)g(\tau) d\tau}{-h} \\
 &= \lim_{h \searrow 0} \frac{(-h)v_0}{-h} \ominus \frac{h \int_0^t g(\tau) d\tau}{-h} \ominus \frac{\int_t^{t+h} (t+h-\tau)g(\tau) d\tau}{-h} \\
 &= v_0 \ominus (-1) \int_0^t g(\tau) d\tau.
 \end{aligned}$$

Hence, G is (ii)-GH-differentiable and $G'(t) = v_0 \ominus (-1) \int_0^t g(\tau) d\tau$.

In the following lemma we reduce the FIVP to an equivalent integral equation (EIE) in order to facilitate the investigation of existence and uniqueness results.

Lemma 3.3 (Equivalence lemma) **(a)** *The FIVP has a unique (i)-(i)-solution y if and only if the integral equation*

$$\begin{aligned}
 u(t) &= f\left(t, y_0 + tv_0 + \int_0^t (t-s)u(s)ds, v_0 + \int_0^t u(s)ds\right) \\
 &+ \int_0^t k\left(t, \tau, y_0 + v_0\tau + \int_0^\tau (\tau-s)u(s)ds, v_0 + \int_0^\tau u(s)ds, u(\tau)\right) d\tau
 \end{aligned} \quad (3.4)$$

has a unique continuous solution u , where

$$y(t) = y_0 + tv_0 + \int_0^t (t-s)u(s)ds.$$

(b) *The FIVP has a unique (i)-(ii)-solution y if and only if the integral equation*

$$\begin{aligned}
 u(t) &= f\left(t, y_0 + tv_0 \ominus (-1) \int_0^t (t-s)u(s)ds, v_0 \ominus (-1) \int_0^t u(s)ds\right) \\
 &+ \int_0^t k\left(t, \tau, y_0 + v_0\tau \ominus (-1) \int_0^\tau (\tau-s)u(s)ds, v_0 \ominus (-1) \int_0^\tau u(s)ds, u(\tau)\right) d\tau
 \end{aligned} \quad (3.5)$$

has a unique continuous solution u , where

$$y(t) = y_0 + tv_0 \ominus (-1) \int_0^t (t-s)u(s)ds.$$

(c) The FIVP has a unique (ii)-(i)-solution y if and only if the integral equation

$$\begin{aligned} u(t) = & f\left(t, y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)u(s)ds, v_0 + \int_0^t u(s)ds\right) \\ & + \int_0^t k\left(t, \tau, y_0 \ominus (-\tau)v_0 \ominus (-1) \int_0^\tau (\tau-s)u(s)ds, v_0 + \int_0^\tau u(s)ds, u(\tau)\right) d\tau \end{aligned} \quad (3.6)$$

has a unique continuous solution u , where

$$y(t) = y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)u(s)ds.$$

(d) The FIVP has a unique (ii)-(ii)-solution y if and only if the integral equation

$$\begin{aligned} u(t) = & f\left(t, y_0 \ominus (-t)v_0 + \int_0^t (t-s)u(s)ds, v_0 \ominus (-1) \int_0^t u(s)ds\right) \\ & + \int_0^t k\left(t, \tau, y_0 \ominus (-\tau)v_0 + \int_0^\tau (\tau-s)u(s)ds, v_0 \ominus (-1) \int_0^\tau u(s)ds, u(\tau)\right) d\tau \end{aligned} \quad (3.7)$$

has a unique continuous solution u , where

$$y(t) = y_0 \ominus (-t)v_0 + \int_0^t (t-s)u(s)ds.$$

Proof (a) First, we show the equivalence of the existence of solutions. Suppose that the FIVP has a (i)-(i)-solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$. Let

$$u(t) := y''(t). \quad (3.8)$$

By using Lemma 2.5 and taking into account y' is (i)-GH-differentiable, the equality (3.8) along with the initial condition $y'(0) = v_0$ is equivalent to

$$y'(t) = v_0 + \int_0^t u(s)ds. \quad (3.9)$$

Reusing Lemma 2.5 and taking into account that y is (i)-GH-differentiable, the equality (3.9) along with the initial condition $y(0) = y_0$ is equivalent to

$$y(t) = y_0 + \int_0^t \left(v_0 + \int_0^\tau u(s)ds\right) d\tau = y_0 + tv_0 + \int_0^t (t-s)u(s)ds \quad (3.10)$$

By inserting (3.8), (3.9) and (3.10) in (3.1), we have

$$\begin{aligned} u(t) = & f\left(t, y_0 + tv_0 + \int_0^t (t-s)u(s)ds, v_0 + \int_0^t u(s)ds\right) \\ & + \int_0^t k\left(t, \tau, y_0 + v_0\tau + \int_0^\tau (\tau-s)u(s)ds, v_0 + \int_0^\tau u(s)ds, u(\tau)\right) d\tau \end{aligned}$$

which means the integral equation (3.4) has a continuous solution u . To prove the inverse, suppose that (3.4) has a continuous solution u and let

$$y(t) := y_0 + tv_0 + \int_0^t (t-s)u(s)ds. \quad (3.11)$$

It is enough to prove that $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$, y is (i)-GH-differentiable, y' is (i)-GH-differentiable and y satisfies (3.1), (3.2) and (3.3). From Lemma 3.2, y is (i)-GH-differentiable and

$$y'(t) = v_0 + \int_0^t u(s)ds \quad (3.12)$$

and by Lemma 2.6, $y'(t)$ is (i)-GH-differentiable and

$$y''(t) = u(t). \quad (3.13)$$

Replacing (3.11), (3.12) and (3.13) in (3.4), yields

$$y''(t) = f(t, y(t), y'(t)) + \int_0^t k(t, \tau, y(\tau), y'(\tau), y''(\tau)) d\tau.$$

Clearly (3.11) and (3.12) imply $y(0) = y_0$ and $y'(0) = v_0$.

Now, we show the equivalence of the uniqueness of solutions. Suppose that the FIVP has a unique (i)-(i)-solution y and the equation (3.4) has two different solutions u_1 and u_2 . By (3.11), the functions $y_1(t) = y_0 + tv_0 + \int_0^t (t-s)u_1(s)ds$ and $y_2(t) = y_0 + tv_0 + \int_0^t (t-s)u_2(s)ds$ are solutions of FIVP. Uniqueness of the solution of FIVP implies that $y_1(t) = y_2(t)$, hence $y'_1(t) = y'_2(t)$ and $y''_1(t) = y''_2(t)$ which means $u_1(t) = u_2(t)$ based on Lemmas 2.6 and 3.2. Conversely, let (3.4) has a unique continuous solution u and FIVP has two different solutions y_1 and y_2 . By (3.8), $u_1(t) = y''_1(t)$ and $u_2(t) = y''_2(t)$ are the solutions of (3.4). Since this equation has a unique solution, we have $u_1 = u_2$. Taking twice integral from both sides of this equality and considering the fact that y_1, y_2, y'_1 and y'_2 are (i)-GH-differentiable, we have

$$y_0 + tv_0 + \int_0^t (t-s)u_1(s)ds = y_0 + tv_0 + \int_0^t (t-s)u_2(s)ds \Rightarrow y_1(t) = y_2(t).$$

This completes the proof of part (a).

(b) Suppose that FIVP has a (i)-(ii)-solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$. Let

$$u(t) := y''(t). \quad (3.14)$$

We claim that u is a solution for integral equation (3.5). In order to prove this claim, from Lemma 2.5, (ii)-GH-differentiability of y' , the equality (3.14) and the initial condition (3.3), we conclude

$$y'(t) = v_0 \ominus (-1) \int_0^t u(s)ds. \quad (3.15)$$

By the same way, the (i)-GH-differentiable of y , the equality (3.15) and the initial condition (3.2) imply

$$\begin{aligned} y(t) &= y_0 + \int_0^t v_0 \ominus (-1) \int_0^s u(z)dz \\ &= y_0 + tv_0 \ominus (-1) \int_0^t (t-s)u(s)ds. \end{aligned} \quad (3.16)$$

Then we substitute (3.14), (3.15) and (3.16) in (3.1) to get

$$\begin{aligned} u(t) &= f\left(t, y_0 + tv_0 \ominus (-1) \int_0^t (t-s)u(s)ds, v_0 \ominus (-1) \int_0^t u(s)ds\right) \\ &\quad + \int_0^t k\left(t, \tau, y_0 + v_0\tau \ominus (-1) \int_0^\tau (\tau-s)u(s)ds, v_0 \ominus (-1) \int_0^\tau u(s)ds, u(\tau)\right) d\tau \end{aligned}$$

which means that the integral equation (3.5) has a continuous solution u . Conversely, suppose that the integral equation (3.5) has a continuous solution. To prove the existence of a (i)-(ii)-solution of FIVP, we define

$$y(t) = y_0 + tv_0 \ominus (-1) \int_0^t (t-s)u(s)ds. \quad (3.17)$$

Then, we will show that $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$, y is (i)-GH-differentiable, y' is (ii)-GH-differentiable and y satisfies (3.1), (3.2) and (3.3). By using part (b) of Lemma 3.2, y is (i)-GH-differentiable and

$$y'(t) = v_0 \ominus (-1) \int_0^t u(s) ds \quad (3.18)$$

From Lemma 2.5, y' is (ii)-GH-differentiable and we have

$$y''(t) = u(t) \quad (3.19)$$

Obviously, (3.17) and (3.18) imply $y(0) = y_0$ and $y'(0) = v_0$ respectively. Replacing (3.17), (3.18) and (3.19) in (3.5) implies

$$y''(t) = f(t, y(t), y'(t)) + \int_0^t k(t, s, y(s), y'(s), y''(s)) ds$$

which means that FIVP has a (i)-(ii)-solution.

Proof of the equivalence of the uniqueness of the solutions of FIVP and (3.4) is similar to case (a).

(c) Suppose that FIVP has a (ii)-(i)-solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$. Let

$$u(t) := y''(t). \quad (3.20)$$

Then, using Lemma (2.6) and taking into account y' is (i)-GH-differentiable, we have

$$y'(t) = v_0 + \int_0^t u(s) ds. \quad (3.21)$$

In a similar way, Lemma (2.6) and (ii)-GH-differentiability of y yield

$$\begin{aligned} y(t) &= y_0 \ominus (-1) \int_0^t v_0 + \int_0^s u(\tau) d\tau ds \\ &= y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)u(s) ds. \end{aligned} \quad (3.22)$$

Replacing (3.20), (3.21) and (3.22) in (3.1) imply

$$\begin{aligned} u(t) &= f\left(t, y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)u(s) ds, v_0 + \int_0^t u(s) ds\right) \\ &\quad + \int_0^t k\left(t, \tau, y_0 \ominus (-\tau)v_0 \ominus (-1) \int_0^\tau (\tau-s)u(s) ds, v_0 + \int_0^\tau u(s) ds, u(\tau)\right) d\tau \end{aligned}$$

which means that (3.6) has a continuous solution u .

Conversely, suppose that (3.6) has a continuous solution u . To prove the existence of a (ii)-(i)-solution for FIVP, let us define

$$y(t) = y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)u(s) ds. \quad (3.23)$$

Then, it is sufficient to show $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$, y is (ii)-GH-differentiable, y' is (i)-GH-differentiable and y satisfies (3.1), (3.2) and (3.3). From Lemma (3.2) part (c), we can conclude y is (ii)-GH-differentiable and

$$y'(t) = v_0 + \int_0^t u(s) ds. \quad (3.24)$$

(i)-GH-differentiability of y' can be concluded from Lemma (2.6) such that

$$y''(t) = u(t). \quad (3.25)$$

Replacing (3.23), (3.24) and (3.25) in (3.6) leads to

$$y''(t) = f(t, y(t), y'(t)) + \int_0^t k(t, s, y(s), y'(s), y''(s)) ds$$

Obviously (3.23) and (3.24) imply $y(0) = y_0$ and $y'(0) = v_0$, respectively.

(d) Suppose that FIVP has a (ii)-(ii)-solution $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$. Let

$$u(t) := y''(t). \quad (3.26)$$

Then, using Lemma (2.6) and the fact that y' is (ii)-GH-differentiable we have

$$y'(t) = v_0 \ominus (-1) \int_0^t u(s) ds. \quad (3.27)$$

Now, we use Lemma (2.6) and (ii)-GH-differentiability of y to obtain

$$\begin{aligned} y(t) &= y_0 \ominus (-1) \int_0^t (v_0 \ominus (-1) \int_0^s u(\tau) d\tau) ds \\ &= y_0 \ominus (-1) \left(t v_0 \ominus (-1) \int_0^t (t-s) u(s) ds \right) \\ &= y_0 \ominus \left((-t) v_0 \ominus \int_0^t (t-s) u(s) ds \right) \\ &= y_0 \ominus (-t) v_0 + \int_0^t (t-s) u(s) ds. \end{aligned} \quad (3.28)$$

Inserting (3.26), (3.27) and (3.28) in (3.1) imply

$$\begin{aligned} u(t) &= f\left(t, y_0 \ominus (-t) v_0 + \int_0^t (t-s) u(s) ds, v_0 \ominus (-1) \int_0^t u(s) ds\right) \\ &\quad + \int_0^t k\left(t, \tau, y_0 \ominus (-\tau) v_0 + \int_0^\tau (\tau-s) u(s) ds, v_0 \ominus (-1) \int_0^\tau u(s) ds, u(\tau)\right) d\tau \end{aligned}$$

which means that (3.7) has a continuous solution u .

Conversely, suppose that (3.7) has a continuous solution u . To prove existence of a (ii)-(ii)-solution for FIVP, let us define

$$y(t) = y_0 \ominus (-t) v_0 + \int_0^t (t-s) u(s) ds. \quad (3.29)$$

It is enough to show $y \in C^2([0, b], \mathbb{R}_{\mathcal{F}})$, y and y' are (ii)-GH-differentiable and y satisfies (3.1), (3.2) and (3.3). By part (d) of Lemma (3.2), y is (ii)-GH-differentiable and so

$$y'(t) = v_0 \ominus (-1) \int_0^t u(s) ds. \quad (3.30)$$

From Lemma (2.6) y' is (ii)-GH-differentiable and we have

$$y''(t) = u(t). \quad (3.31)$$

By replacing (3.29), (3.30) and (3.31) in (3.7) we have

$$y''(t) = f(t, y(t), y'(t)) + \int_0^t k(t, s, y(s), y'(s), y''(s)) ds$$

Obviously (3.29) and (3.30) imply $y(0) = y_0$ and $y'(0) = v_0$, respectively.

3.2 The advantage of EIE to FIVP

In crisp mathematics, converting a differential equation to an integral equation is common due to the stability of the integral operator in comparison with the derivative operator (see [7]). This is more important and more useful in fuzzy cases. Because, fuzzy derivative have various difficulties particularly due to the existence of H-differences. Also, in numerical solution, it is more important since the numerical derivative techniques are not satisfied for fuzzy derivative with the same quality as crisp case.

As it is seen in the theorems of this paper, in contrast with the crisp one, converting a differential equation to an integral equation is not trivial in fuzzy case.

The proposed equivalence lemma could be useful in the study of existence and uniqueness results for the integro-differential equation under study as well as in the construction of a numerical method to solve it. For example, in our result of the derivations, the FIVP (3.1)-(3.3) is equivalent by EIE (3.4) (for the (i)-(i)-solution case). To study the existence and uniqueness of solution for this case, one can introduce the following operator

$$T(u) = f\left(t, y_0 + tv_0 + \int_0^t (t-s)u(s)ds, v_0 + \int_0^t u(s)ds\right) + \int_0^t k\left(t, \tau, y_0 + v_0\tau + \int_0^\tau (\tau-s)u(s)ds, v_0 + \int_0^\tau u(s)ds, u(\tau)\right)d\tau.$$

It is clear that the fixed point of T is a solution of (3.4) and viceversa. Therefore, by using a suitable fixed point theorem, the existence and uniqueness results can be proved. Furthermore, to find an approximate solution for the problem (3.1)-(3.3) in the (i)-(i)-solution case, it is enough to use an accurate quadrature rule for EIE (3.4) as follows:

$$u_i = f\left(t_i, y_0 + t_i v_0 + \sum_{j=0}^i w_j(t_i - t_j)u_j, v_0 + \sum_{j=0}^i u_j\right) + \sum_{j=0}^i w_j \left(k\left(t_i, t_j, y_0 + v_0 t_j + \sum_{l=0}^j w_l(t_j - t_l)u_l, v_0 + \sum_{l=0}^j w_l u_l, u_j\right) \right).$$

As a future research, we propose the study of existence and uniqueness results for FIVP (3.1)-(3.3) as well as its numerical solution by using the current equivalence lemma.

Example 3.4 Here, an example is given for each of four forms of FIVP. Applying the equivalence lemma, it is shown that each type of FIVP is equivalent to an integral equation. Assume that $A = (0, 1, 2)$.

- **(i)-(i)-solution:** $y(t) = A.e^t$ is an (i)-(i)-solution for

$$\begin{cases} y''(t) = y(t) \ominus A.(te^t - e^t + 1) + \int_0^t \tau y''(\tau)d\tau, & t \in [0, 1], \\ y(0) = A, y'(0) = A. \end{cases}$$

Here, we have

$$\begin{cases} f(t, y(t), y'(t)) := y(t) \ominus A.(te^t - e^t + 1), \\ k(t, \tau, y(\tau), y'(\tau)) := \tau y''(\tau), \\ y_0 := A, v_0 = A. \end{cases}$$

By using part (a) of Lemma 3.3, the equivalent integral equation is

$$\begin{aligned} u(t) &= y_0 + tv_0 + \int_0^t (t-s)u(s)ds \ominus A(te^t - e^t + 1) + \int_0^t \tau u(\tau)d\tau \\ &= A + A.t \ominus A.(te^t - e^t + 1) + \int_0^t tu(s)ds. \end{aligned} \quad (3.32)$$

From equality (3.8), $u(t) = y''(t) = A.e^t$ should be solution of (3.32). This is simply confirmed by direct substitution.

- **(i)-(ii)-solution:** $y(t) = A.\sin t$ is an (i)-(ii)-solution for

$$\begin{cases} y''(t) = -y(t) \ominus \frac{1}{4}A^2(1 - \cos(2t)) + \int_0^t (-1)y'(\tau)y''(\tau)d\tau, & t \in [0, \frac{\pi}{2}], \\ y(0) = 0, y'(0) = A. \end{cases}$$

For this initial value problem, we have

$$\begin{cases} f(t, y(t), y'(t)) := -y(t) \ominus \frac{1}{4}A^2(1 - \cos(2t)), \\ k(t, \tau, y(\tau), y'(\tau), y''(\tau)) := (-1)y'(\tau)y''(\tau), \\ y_0 := 0, v_0 = A. \end{cases}$$

Considering part (b) of Lemma 3.3, the equivalent integral equation reads

$$\begin{aligned} u(t) &= -(y_0 + tv_0 \ominus (-1) \int_0^t (t-s)u(s)ds) \ominus \frac{1}{4}A^2(1 - \cos(2t)) + \int_0^t (-1)(v_0 \ominus (-1) \int_0^\tau u(s)ds)u(\tau)d\tau \\ &= -(At \ominus (-1) \int_0^t (t-s)u(s)ds) \ominus \frac{1}{4}A^2(1 - \cos(2t)) + \int_0^t (-1)\left(A \ominus (-1) \int_0^\tau u(s)ds\right)u(\tau)d\tau \end{aligned} \quad (3.33)$$

A direct substitution shows that $u(t) = y''(t) = \sin(t)(-2, -1, 0)$ is a solution of (3.33) which is in line with the equality (3.14).

- **(ii)-(i)-solution:** $y(t) = A.\cos t$ is an (ii)-(i)-solution for

$$\begin{cases} y''(t) = -y(t) \ominus \frac{1}{4}A^2 \sin(2t) + \int_0^t (y''(\tau)^2 - y'(\tau)^2)d\tau, & t \in [0, \frac{\pi}{2}], \\ y(0) = A, y'(0) = 0. \end{cases}$$

Here, we have

$$\begin{cases} f(t, y(t), y'(t)) := -y(t) \ominus \frac{1}{4}A^2 \sin(2t), \\ k(t, \tau, y(\tau), y'(\tau), y''(\tau)) := y''(\tau)^2 - y'(\tau)^2, \\ y_0 = A, v_0 = 0. \end{cases}$$

By using part (c) of Lemma 3.3, the equivalent integral equation takes the form

$$\begin{aligned} u(t) &= -(y_0 \ominus (-t)v_0 \ominus (-1) \int_0^t (t-s)u(s)ds) \ominus \frac{1}{4}A^2 \sin(2t) + \int_0^t \left(u^2(\tau) - (v_0 + \int_0^\tau u(s)ds)^2\right)d\tau \\ &= (-1)\left(A \ominus (-1) \int_0^t (t-s)u(s)ds\right) \ominus \frac{1}{4}A^2 \sin(2t) + \int_0^t \left(u(\tau)^2 - \left(\int_0^\tau u(s)ds\right)^2\right)d\tau \end{aligned} \quad (3.34)$$

According to (3.20), it is easy to see that $u(t) = y''(t) = \cos(t)(-2, -1, 0)$ is a solution of (3.34).

- **(ii)-(ii)-solution:** $y(t) = e^{-t}.A$ is an (ii)-(ii)-solution for

$$\begin{cases} y''(t) = \frac{1}{4}y(t) + (\frac{7}{4}e^{-t} - 1 + te^{-t})A + \int_0^t \tau y''(\tau)d\tau, & t \in [0, 1], \\ y(0) = A, y'(0) = -A. \end{cases}$$

Similar to previous examples, we have

$$\begin{cases} f(t, y(t), y'(t)) := \frac{1}{4}y(t) + (\frac{7}{4}e^{-t} - 1 + te^{-t})A, \\ k(t, \tau, y(\tau), y'(\tau), y''(\tau)) := \tau y''(\tau), \\ y_0 = A, v_0 = -A. \end{cases}$$

By using part (d) of Lemma 3.3, we obtain the equivalent integral equation

$$\begin{aligned} u(t) &= \frac{1}{4} \left(y_0 \ominus (-t)v_0 + \int_0^t (t-s)u(s)ds \right) + \left(\frac{7}{4}e^{-t} - 1 + te^{-t} \right) A + \int_0^t \tau u(\tau) d\tau \\ &= \frac{1}{4} \left(A \ominus At + \int_0^t (t-s)u(s)ds \right) + \left(\frac{7}{4}e^{-t} - 1 + te^{-t} \right) A + \int_0^t \tau u(\tau) d\tau \end{aligned} \quad (3.35)$$

It is straightforward to use (3.26) for showing that $u(t) = y''(t) = A.e^{-t}$ is a solution of (3.35).

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