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Some results on generalized Szász operators involving Sheffer polynomials

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In this paper we consider old and new operators of Szász type involving Sheffer polynomials. We present an asymptotic expansion formula for operators of Ismail type. Then, in order to improve the accuracy of the approximation of a function f in a fixed point, we apply a well-known extrapolation algorithm. We also introduce some new special sequences of Appell and Sheffer polynomials and construct new generalized Szász-type operators. By using classical techniques we investigate approximation properties and rate of convergence for these operators and compare the results with other existing operators. Finally, we present numerical examples which confirm the validity of the theoretical analysis and the effectiveness of the presented operators.

1. Introduction

In the approximation theory one of the fundamental problems is to approximate a function f by functions having better properties of integration, differentiation and efficient calculations than f. After the famous Theorems of Weierstrass and Korovkin [1], positive linear polynomial operators have been widely used for approximating regular functions.

In this paper we are dealing with the approximation of real functions f defined in the semi-infinite interval $[0, +\infty)$ which have a suitable rate of growth as $x \to \infty$. In this case, one of the well-known examples of sequences of positive linear operators is Szász operators, introduced by Szász in 1950 [2]:

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1}$$

S_n converges to f (as $n \to \infty$) at each point $t = x \geq 0$ where f is continuous. In [2] Szász investigated the detailed approximation properties of the operators (1). Particularly, if f is bounded in every finite interval and $f(x) = O\left(x^k\right)$ for some $k > 0$ as $x \to \infty$, then there hold:

– if f is continuous at a point ξ, then S_n converges uniformly to f at ξ;

– if f is differentiable at a point $\xi > 0$, then $n^{\frac{1}{2}} \left[S_n(f;\xi) - f(\xi)\right] \to 0, n \to \infty$.

Later, in 1969, Jakimovski and Leviatan [3] gave a generalization of Szász operators by means of Appell polynomials:

$$P_n(f;x) = \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{2}$$

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where $\{p_k\}_{k\in\mathbb{N}}$ is the Appell polynomial sequence [4] defined from the holomorphic function $A(t)$ in the disk $|z| < R$, $(R > 1)$ with $A(1) \neq 0$.

Under the assumption $p_k(x) \geq 0$, for $x \in [0, \infty)$ and $k \geq 0$, for each function f defined in $[0, \infty)$, Jakimovski and Leviatan [3] gave several approximation properties of these operators in view of Szász's method [2].

If $A(t) = 1$, $p_k(x) = \frac{x^k}{k!}$, then $P_n(f; x)$ reduces to the Szász operators (1).

Let $E = E[0, \infty)$ denote the class of all functions of exponential type defined on the semi-axis and such that $|f(t)| < ce^{\alpha t}$ $(t \geq 0)$ for some c and α finite positive constants and let $C_E[0, \infty) = C[0, \infty) \cap E$. Observe that, if $n > \alpha \log R$, the series in (2) is convergent. Hence the operators $P_n(f; x)$ are well-defined for all sufficiently great n.

In [3] the authors proved that, if $f \in C_E[0, \infty)$, then $P_n(f; x)$ converges to $f(x)$ as $n \to \infty$. The convergence is uniform in each compact subset $[0, a]$, $a > 0$.

In 1996 Ciupa [5] studied the rate of convergence of these operators.

If f admits derivatives of sufficiently high order at $x \geq 0$, a complete asymptotic expansion for $P_n(f; x)$ has been derived by Abel and Ivan [6]:

$$P_n(f; x) \sim f(x) + \sum_{k=1}^{\infty} c_k(f; x) n^{-k}, \tag{3}$$

where the coefficients $c_k(f; x)$ do not depend on n. From (3) it follows that for all $m \geq 1$

$$P_n(f; x) = f(x) + \sum_{k=1}^{m} h^k c_k(f; x) + o\left(h^m\right), \tag{4}$$

with $h = 1/n$.

Operators of Jakimovski and Leviatan type have been generalized by substituting the Appell sequence by a more general Sheffer sequence [7]. In particular, in [8] Ismail considered the polynomial operators

$$F_n(f; x) = \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{5}$$

where $\{p_k\}_{k\in\mathbb{N}}$ is the Sheffer polynomial sequence related to the analytic functions $A(t)$ and $H(t)$ with $A(0) \neq 0$, $H(0) = 0$ and $H'(0) \neq 0$. Moreover, in (5), $A(1) \neq 0$, $H(1) \neq 0$. Under the assumptions $H'(1) = 1$, $p_k(x) \geq 0$ for $x \geq 0$, Ismail showed that the same type of results obtained by Jakimovski and Leviatan are still valid for operators $F_n(f; x)$.

Observe that for $H(t) = t$, operators $F_n(f; x)$ reduce to operators $P_n(f; x)$.

Further generalizations of Szász-type operators have been studied in [9–12].

To the knowledge of the authors, an explicit asymptotic expansion for operators (5) does not exist in the literature. Therefore, in this paper, with the same technique used in [6], a complete asymptotic expansion formula for the Ismail-type operators $F_n(f; x)$ is given, provided f admits derivatives of sufficiently high order at $x \geq 0$. Asymptotic expansions of type (4) are very important, since a well-known extrapolation algorithm can be applied, in order to improve the approximation results. This algorithm provides new sequences of operators having a faster rate of convergence than $F_n(f; x)$.

The paper is structured as follows. In Section 2 necessary background and definitions are given; in Section 3 the asymptotic expansion for $F_n(f; x)$ is presented and Richardson extrapolation technique is applied for any fixed $x \in [0, +\infty)$. Then, in Section 4 some known and new special operators of Jakimovski and Leviatan type and of Ismail type are considered. Rates of convergence of the above operators are compared by using classical techniques. Numerical experiments are given in Section 5 which provide favorable comparisons with other existing operators and show the effectiveness of the considered extrapolation algorithm. Finally, conclusions are reported in Section 6.

2. Preliminaries

In this section we recall some well-known definitions and properties on Appell and Sheffer polynomials which will be useful hereafter. Moreover the estimation of the rate of convergence for $P_n(f, x)$ and $F_n(f, x)$ operators by means of modulus of continuity is examined.

Appell sequences [4,13–15] and Sheffer sequences [7,13,15–17] can be introduced in many equivalent ways. Here we use the method of generating functions.

If $A(t)$ is an invertible power series, the polynomial sequence $\{p_k\}_{k\in\mathbb{N}}$ defined by

$$A(t)e^{xt} = \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!} \tag{6}$$

is called *Appell polynomial sequence* for $A(t)$. If

$$A(z) = \sum_{i=0}^{\infty} a_i \frac{z^i}{i!}, \tag{7}$$

with $a_0 \neq 0$, it is easy to prove, from (6), that

$$p_k(x) = \sum_{\nu=0}^{k} \binom{k}{\nu} a_{k-\nu} x^{\nu}, \qquad k \geq 0. \tag{8}$$

Moreover [13], the sequence $\{p_k\}$ is positive $\forall x \geq 0$ if and only if

$$\forall i \geq 0, \qquad a_i \geq 0.$$

Observe that for $A(t) = 1$, we get $p_k(x) = x^k$, $k \geq 0$, that is, the sequence $\{p_k\}$ is the well-known monomial sequence.

In 1939 Sheffer [7] introduced the polynomial sequences, called *of type-zero*, which include Appell and binomial sequences.

Let $H(t)$ be a δ-series, that is

$$H(t) = \sum_{i=0}^{\infty} b_i \frac{t^i}{i!}, \qquad b_0 = 0,\ b_1 \neq 0,$$

and $A(t)$ an invertible power series as in (7).

Sheffer polynomial sequences are defined by the generating function $A(t)e^{xH(t)}$. That is, for the Sheffer sequence $\{s_k\}_{k\in\mathbb{N}}$ we get

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!}. \tag{9}$$

Observe that for $H(t) = t$ the sequence $\{s_k\}$ coincides with the Appell sequence $\{p_k\}$.

For $A(t) = 1$ the Sheffer sequence $\{s_k\}$ defined by

$$e^{xH(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!},$$

is called polynomial sequence *of binomial-type* (see [13,15,18] and references therein).

In this paper Sheffer polynomial sequences such that $s_k(x) \geq 0$, $\forall x \geq 0$ will be considered. For this kind of sequences the following proposition holds.

Proposition 1. *Let $s_k(x)$ be the Sheffer sequence defined in (9). If $a_i \geq 0$, $b_i \geq 0$, $i = 0, 1, \ldots$, then, $\forall k \in \mathbb{N}$, we get $s_k(x) \geq 0$ for $x \in [0, +\infty)$.*

Proof. The decomposition of $s_k(x)$ in the basic monomials is

$$s_k(x) = \sum_{i=0}^{k} s_{k,i} x^i.$$

In [13] it is proved that

$$s_{k,0} = a_k, \qquad s_{k,k} = a_0 b_1^k, \tag{10}$$

and, for $i = 0, \ldots, k-1$,

$$s_{k,i+1} = \frac{1}{k+1} \sum_{j=i}^{k} \binom{k}{j} b_{k-j}\, s_{j,i}. \tag{11}$$

From (10) and (11) the result follows.

Proposition 2. *If $s_k(x) \geq 0$ for $x \in [0, +\infty)$, then $a_k \geq 0$ and $b_1 > 0$.*

Proof. Since $s_k(0) = s_{k0}$ and [13] $s_{k0} = a_k$, we get $s_k(0) = a_k \geq 0$.

For $x \to +\infty$, $s_k(x) \to +\infty$. It follows that $s_{kk} > 0$. But $s_{kk} = a_0 b_1^k$, hence $b_1 > 0$.

In [13] the conjugate sequence of a polynomial sequence is defined. Sheffer polynomial sequences form a group with respect to the umbral composition [15]. It can be shown [13] that the conjugate sequences are the inverse elements in the group. That is, if $s_k(x)$ is a Sheffer sequence, the conjugate sequence $\hat{s}_k(x)$ satisfies

$$s_k\left(\hat{s}_k(x)\right) = \hat{s}_k\left(s_k(x)\right) = x^k, \qquad k \geq 0.$$

In [13] it is proved that if

$$A(t)\, e^{xH(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!},$$

then

$$\frac{1}{A\left(\overline{H}(t)\right)}e^{x\overline{H}(t)} = \sum_{k=0}^{\infty}\hat{s}_k(x)\frac{t^k}{k!},$$

where $\overline{H}(t)$ is the compositional inverse of $H(t)$ [15]:

$$H\left(\overline{H}(t)\right) = \overline{H}\left(H(t)\right) = t,$$

that is $H \circ \overline{H} = I$, where "$\circ$" is the umbral composition.

For the Appell sequence, since $H(t) = t$, we get

$$A(t)e^{xt} = \sum_{k=0}^{\infty}p_k(x)\frac{t^k}{k!}$$

and

$$\frac{1}{A(t)}e^{xt} = \sum_{k=0}^{\infty}\hat{p}_k(x)\frac{t^k}{k!},$$

where $\hat{p}_k(x)$ is the conjugate sequence of $p_k(x)$.

In order to consider some estimation of the error for operators $P_n(f;x)$ and $F_n(f;x)$, let

$$C_E^2[0,\infty) := \{g \in C_E[0,\infty) : g', g'' \in C_E[0,\infty)\}$$

with the norm $\|f\|_{C_E^2} = \|f\|_{C_E} + \|f'\|_{C_E} + \|f''\|_{C_E}$ [5,19].

Theorem 1 (*[5]*)**.** *If $f \in C_E^2[0,\infty)$, then*

$$|P_n(f;x) - f(x)| \le \lambda_n(x)\|f\|_{C_E^2}, \tag{12}$$

where $\lambda_n(x) = \frac{1}{n}\left(x + \frac{A''(1)+A'(1)}{A(1)}\right)$.

Now, let $\widetilde{C}[0,\infty)$ be the space of uniformly continuous functions defined on $[0,\infty)$ and let $\widetilde{C}_E[0,\infty) = \widetilde{C}[0,\infty) \cap E$.

The next theorem gives the rate of convergence of the sequence F_n by means of the modulus of continuity ω of f, very much used in the approximations by Szász-type operators (see [20–23] and the references therein).

Theorem 2 (*[24,25]*)**.** *Let $f \in \widetilde{C}_E[0,\infty)$, then F_n operators satisfy the following estimation*

$$|F_n(f;x) - f(x)| \le \beta_n(x)\,\omega\left(f, \frac{1}{\sqrt{n}}\right), \tag{13}$$

where $\beta_n(x) = 1 + \sqrt{(H''(1)+1)\,x + \frac{A'(1)+A''(1)}{nA(1)}}$.

Corollary 1. *If f is Lipschitzian with Lipschitz constant L, then*

$$|F_n(f;x) - f(x)| \le \frac{L}{\sqrt{n}}\left(1 + \sqrt{(H''(1)+1)\,x + \frac{A'(1)+A''(1)}{nA(1)}}\right).$$

3. Asymptotic expansion and extrapolation for operators F_n

For a fixed $x \in [0,\infty)$, let us denote by $K_x^{[q]}$, with $q \in \mathbb{N}$, the class of all functions $f : E \to \mathbb{R}$ which admit a derivative of order q at x.

Theorem 3. *Let $f \in K_x^{[2q]}$, with $x \ge 0$ and $q \in \mathbb{N}$. There exist coefficients $c_k(f;x)$ which depend on A, H, f, x and not on n such that the operators F_n defined in (5) possess the following complete asymptotic expansion*

$$F_n^{(l)}(f;x) = f^{(l)}(x) + \sum_{k=1}^{q}c_k(f;x)n^{-k} + o\left(n^{-q}\right), \qquad l = 0,\dots,q. \tag{14}$$

The theorem can be proved, after some straightforward calculations, by the same techniques used by Abel and Ivan in [6] for operators $P_n(f,x)$.

The asymptotic expansions (4) and (14) in powers of $h = 1/n$, $n > 0$, for a fixed $x > 0$, with coefficients independent on n, allow to apply the classical Richardson extrapolation process [26], in order to obtain faster convergent approximations. In [27,28] for the first time this idea has been applied in the approximation theory.

Let $f \in K_x^{[2q]}$ and let $\{n_i\}_{i\geq 0}$ be an increasing sequence of positive integers. We define a new sequence of polynomials $\left\{G_k^{(i)}\right\}_{k\in N}$ such that

$$\begin{cases} G_0^{(i)} := G_0^{(i)}(f, x) = F_{n_i}(f; x), & i = 0, \dots, q, \\ G_k^{(i)} := G_k^{(i)}(f, x) = \dfrac{h_{i+k} G_{k-1}^{(i)} - h_i G_{k-1}^{(i+1)}}{h_{i+k} - h_i}, & \begin{array}{l} k = 1, \dots, q, \\ i = 0, \dots, q-k, \end{array} \end{cases} \tag{15}$$

where $h_i = n_i^{-1}$.

The following classical result holds [26,28].

Theorem 4. *For i fixed positive number*

$$\lim_{h_i \to 0} G_k^{(i)} = f(x), \qquad k = 1, 2, \dots, q-1.$$

Moreover, the following representation of the error holds

$$G_k^{(i)} - f(x) = (-1)^k h_i\, h_{i+1} \cdots h_{i+k} \left(c_k(f; x) + O\left(h_i\right)\right).$$

Several common choices of the sequence n_i can be considered. For example:

(1) $n_i = \rho^i$, $\rho \geq 2$ positive integer (Romberg sequence)
(2) $n_i = b_i$ or $n_i = n\, b_i$, where b_i are the Bulirsch numbers $b_i = 2, 3, 4, 6, 8, 12, 16, \dots$
(3) $n_i = 2i$, double harmonic sequence (Deuflhard sequence)
(4) $n_i = n + i$, $n \in \mathbb{N}$
(5) $n_i = \frac{n}{m}(m + i)$, with m fixed constant as n/m is integer.

As we know, the numerical stability of extrapolation algorithm depends on the stability factor $M(q)$, which is the sum of the absolute value of extrapolation coefficients [29] $M(q) = \sum_{j=0}^{q} |C_{qj}|$ where C_{qj} are such that $G_q^{(0)} = \sum_{j=0}^{q} C_{qj} G_0^{(j)}$.

Particularly, the smaller $M(q)$, the more stable the extrapolation algorithm.

Let us denote by $M_k(q)$, $k = 1, \dots, 5$, the stability factor related to the kth considered sequence n_i in the above list.

By comparing the five analyzed sequences, we can observe that, at least for $q \leq 300$, the stability factor of Richardson extrapolation with $n_i = \rho^i$ is very small. In fact, for example, $M_1(q) < 8.26$ for $\rho = 2$, $M_1(q) < 1.97$ if $\rho = 4$, and $M_1(q) < 1.33$ when $\rho = 8$. While for $M_2(q)$ we have $M_2(q) < 221$. The stability of the other sequences is poor. In fact $M_4(q)$ and $M_5(q)$ increase rapidly with q for each choice of n and m. Nevertheless, we observe that the grid points of the refinement related to sequence (1) increases rapidly, especially for high values of ρ and, consequently, the amount of work increases.

4. Special cases: examples of operators

In this section, in order to provide the reader with a complete view of particular sequences of positive operators of Szász type, for the approximation of real regular functions on the semi infinite interval, we consider some known operators of Jakimovski and Leviatan type (Table 1) and of Ismail-type (Table 2). Then we present new operators of Jakimovski and Leviatan type and of Ismail type, respectively, and compare the rates of convergence.

4.1. New operators of Jakimovski and Leviatan type

In order to present new operators of Jakimovski and Leviatan type, we introduce new sequences of Appell polynomials which are positive for $x \geq 0$.

- The conjugate Bernoulli polynomials [13]:

$$p_k(x) = \frac{(x+1)^{k+1} - x^{k+1}}{(k+1)!}, \tag{16}$$

related to the invertible power series

$$A(t) = \frac{e^t - 1}{t}. \tag{17}$$

- The conjugate Euler polynomials [13]:

$$\widetilde{p}_k(x) = \frac{(x+1)^k + x^k}{2k!}, \tag{18}$$

Table 1
Operators of Jakimovski and Leviatan type.

	Sequence of polynomials	Operators
(1)	Gould–Hopper *d*-orthogonal polynomials [19,30] $g_k(x,h,d)=\sum_{s=0}^{\lfloor \frac{k}{d+1}\rfloor}\frac{k!}{s!(k-(d+1)s)!}h^s x^{k-(d+1)s},\quad h\geq 0$ $A(t)=e^{ht^{d+1}}$	$L_n^{GH}(f,d;x)=e^{-nx-h}\sum_{k=0}^{\infty}\frac{g_k(nx,h,d)}{k!}f\left(\frac{k}{n}\right)$
(2)	Hermite polynomials of variance ν [30] $H_k(x,\nu)=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}\left(-\frac{\nu}{2}\right)^i\frac{k!}{i!(k-2i)!}x^{k-2i},\quad \nu\leq 0$[a] $A(t)=e^{-\frac{\nu t^2}{2}}$	$L_n^{HP}(f,\nu;x)=e^{-nx+\frac{\nu}{2}}\sum_{k=0}^{\infty}\frac{H_k(nx,\nu)}{k!}f\left(\frac{k}{n}\right)$
(3)	Miller–Lee polynomials [30] $G_k(x,m)=\sum_{i=0}^{k}\frac{(m+1)_i}{i!(k-i)!}x^{k-i},\quad m>-1$ $(\alpha)_i$ is the Pochhammer's symbol $(\alpha)_0=1,(\alpha)_i=\alpha(\alpha+1)\cdots(\alpha+i-1),\quad i\geq 1$ $A(t)=\left(1-\frac{t}{2}\right)^{-m-1}$	$L_n^{ML}(f,m;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{G_k(2nx,m)}{2^{m+k+1}}f\left(\frac{k}{n}\right)$

[a] For $\nu=-\frac{1}{4}$ we get the Hermite–Appell conjugate sequences [13] with generating function $e^{xt+\frac{t^2}{4}}$.

Table 2
Operators of Ismail type.

	Sequence of polynomials	Operators
(i)	Modified Laguerre polynomials [31–34] $\widetilde{L}_k(x,\alpha)=\frac{L_k^{(\alpha)}\left(-\frac{x}{2}\right)}{2^k}$ where $L_k^{(\alpha)}(x)=\sum_{m=0}^{k}\frac{(\alpha+k)!}{(k-m)!(\alpha+m)!m!}(-x)^m$ is the Laguerre polynomial, $\alpha>-1$ $A(t)=\left(1-\frac{t}{2}\right)^{-\alpha-1},\quad H(t)=\frac{t}{2(2-t)}$	$L_n^{LG}(f,\alpha;x)=e^{-\frac{nx}{2}}\sum_{k=0}^{\infty}\frac{\widetilde{L}_k(nx,\alpha)}{2^{\alpha+1}}f\left(\frac{k}{n}\right)$
(ii)	Modified 2-orthogonal Laguerre polynomials [25] $\hat{L}_k(x,\alpha)=\frac{1}{2^k}\sum_{m=0}^{k}\frac{1}{(k-m)!}L_m^{(\alpha)}\left(-\frac{x}{2}\right)$ $L_m^{(\alpha)}(x)$ the Laguerre polynomials, $\alpha>-1$ $A(t)=e^{\frac{t}{2}}\left(1-\frac{t}{2}\right)^{-\alpha-1},\quad H(t)=\frac{t}{2(2-t)}$	$L_n^{LG_2}(f,\alpha;x)=e^{-\frac{nx+1}{2}}\sum_{k=0}^{\infty}\frac{\hat{L}_k(nx,\alpha)}{2^{\alpha+1}}f\left(\frac{k}{n}\right)$
(iii)	Modified Charlier polynomials [31,32,34,35] $\widetilde{C}_k(x,a)=\frac{C_k^{(a)}((1-a)x)}{k!}$ where $C_k^{(a)}(x)=\sum_{r=0}^{k}\binom{k}{r}\left(\frac{1}{a}\right)^r(-x)_r,\quad x\leq 0$ is the Charlier polynomial, $a>0$ $A(t)=e^t,\quad H(t)=(1-a)\ln\left(1-\frac{t}{a}\right)$	$L_n^{MC}(f,a;x)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\widetilde{C}_k(nx;a)f\left(\frac{k}{n}\right)$
(iv)	Modified Reverse Bessel polynomials [24] $\widetilde{B}_k(x)=\frac{B_k\left(2\sqrt{2x}\right)}{4^k k!}$ where $B_k(x)=\sum_{j=1}^{k}\frac{(2k-j-1)!}{(j-1)!(k-j)!2^{k-j}}x^j$ is the Reverse Bessel polynomial $A(t)=1,\quad H(t)=2\sqrt{2}\left(1-\sqrt{1-\frac{t}{2}}\right)$	$L_n^{RB}(f,d;x)=e^{-2\left(\sqrt{2}-1\right)nx}\sum_{k=0}^{\infty}\widetilde{B}_k^{(a)}(nx;a)f\left(\frac{k}{n}\right)$
(v)	Modified Meixner polynomials [25] $\widetilde{M}_k(x;\gamma,c)=\frac{(\gamma)_k}{2^{\gamma+k}k!}M_k\left(\frac{(2c-1)x}{2(c-1)};\gamma,c\right)$ where $M_k(x;\gamma,c)=\sum_{r=0}^{k}\binom{k}{r}\left(\frac{1}{c}-1\right)^r\frac{(-x)_r}{(\gamma)_r}$ is the Meixner polynomial, $\gamma>0,\ \frac{1}{2}<c<1, x\geq 0$ $A(t)=\left(1-\frac{t}{2}\right)^{-\gamma},\quad H(t)=\frac{2c-1}{2(c-1)}\ln\left(\frac{2c-t}{c(2-t)}\right)$	$L_n^{MM}(f,\gamma,c;x)=\left(2-\frac{1}{c}\right)^{\frac{nx(1-2c)}{2(c-1)}}\sum_{k=0}^{\infty}\widetilde{M}_k(nx;\gamma,c)f\left(\frac{k}{n}\right)$

related to

$$A(t)=\frac{e^t+1}{2}.$$

In this case we obtain, respectively, the two operators $T_n(f;x)$ and $\widetilde{T}_n(f;x)$ defined as follows:

$$T_n(f;x)=\frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right), \tag{19}$$

$$\widetilde{T}_n(f;x)=\frac{2e^{-nx}}{e+1}\sum_{k=0}^{\infty}\widetilde{p}_k(nx)f\left(\frac{k}{n}\right). \tag{20}$$

For the rate of convergence of T_n and $\widetilde{T}_n$ operators, from Theorem 2 we get

$$|T_n(f;x)-f(x)| \le \left(x+\frac{1}{n}\right)\,\omega\left(f,\frac{1}{\sqrt{n}}\right),$$

and

$$|\widetilde{T}_n(f;x)-f(x)| \le \left(x+\frac{1}{n}\right)\,\omega\left(f,\frac{1}{\sqrt{n}}\right).$$

Moreover the following theorem holds.

Theorem 5. *If $f \in C_E^2[0,\infty)$, then*

$$|T_n(f;x)-f(x)| \le \overline{\lambda}_n(x)\|f\|_{C_E^2}, \tag{21}$$

where $\overline{\lambda}_n(x) = \frac{1}{2n}\left(x+\frac{1}{n}+\frac{2}{e-1}\right)$;

$$|\widetilde{T}_n(f;x)-f(x)| \le \widetilde{\lambda}_n(x)\|f\|_{C_E^2}, \tag{22}$$

where $\widetilde{\lambda}_n(x) = \frac{e}{n(e+1)} + \frac{x}{2n} + \frac{e}{n^2(e+1)}$.

Proof. The results follow by the same technique used by Ciupa in [5], taking into account that $T_n\,(s-x;x) = \frac{1}{n(e-1)}$ and $T_n\left((s-x)^2;x\right) = \frac{x}{n}+\frac{1}{n^2}$.

Remark 1. In the special case where $A(t)$ is defined as in (17), from the upper bound (12) $\lambda_n(x) = \frac{1}{n}(x+1)$ and $\forall n, x$, $\lambda_n(x) > \overline{\lambda}_n(x)$.

4.2. New operators of Ismail-type

In order to present new operators of Ismail type, we introduce new positive sequences of Sheffer polynomials satisfying $A(1) \neq 0$ and $H'(1) = 1$.

- The conjugate generalized Boole polynomials

$$b_k(x) = \sum_{j=0}^{k} s_{k,j}\,x^j,$$

where

$$s_{k,j} = \hat{b}_{k,j} + \frac{j+1}{2}\hat{b}_{k,j+1}$$

with $\hat{b}_{k,j}$ defined by

$$\hat{b}_{k,j} = \begin{cases} \hat{b}_{k,j} = \delta_{k,j} \\ \hat{b}_{k,0} = 0,\ \ \hat{b}_{k,1} = 1 \\ \hat{b}_{k,j} = \dfrac{1}{k}\displaystyle\sum_{i=1}^{k-j+1}\binom{k}{i}\hat{b}_{k-i,j-1}\,. \end{cases} \tag{23}$$

The generating function is

$$\frac{1+e^t}{2}e^{x(e^t-1)} = \sum_{k=0}^{\infty}\frac{b_k(x)}{k!}t^k. \tag{24}$$

Now, let be $q_k(x) = \frac{1}{k!}\,b_k\left(\frac{x}{e}\right)$. Then, by virtue of (24), $A(t) = \frac{1+e^t}{2}$ and $H(t) = \frac{e^t-1}{e}$, so that $A(1) \neq 0$ and $H'(1) = 1$.
- The conjugate Poisson–Charlier polynomials

$$c_k(x) = \sum_{j=0}^{k}\widetilde{s}_{k,j}\,x^j,$$

where

$$\widetilde{s}_{k,j} = \sum_{i=j}^{k}\binom{i}{j}\hat{b}_{k,i},$$

Table 3
Rate of convergence of some operators.

	Operators	Rate of convergence
(1)	$L_n^{GH}(f,d;x)$ [19,30]	$x+\frac{h(h+1)(1+d)^2}{n}$
(2)	$L_n^{HP}(f,\nu;x)$ [30]	$x+\frac{\nu(\nu-2)}{n}$
(3)	$L_n^{ML}(f,m;x)$ [30]	$x+\frac{(m+1)(m+3)}{n}$
(i)	$L_n^{LG}(f,\alpha;x)$ [31–34]	$3x+\frac{\alpha(\alpha+4)+3}{n}$
(ii)	$L_n^{LG_2}(f,\alpha;x)$ [25]	$3x+\frac{4\alpha(\alpha+5)+19}{n}$
(iii)	$L_n^{MC}(f,a;x)$ [31,32,34,35]	$ax+\frac{2}{n}$
(iv)	$L_n^{RB}(f,d;x)$ [24]	$\frac{3}{2}x+\frac{1}{n}$
(v)	$L_n^{MM}(f,\gamma,c;x)$ [25]	$\frac{1-4c}{1-2c}x+\frac{\gamma(\gamma+2)}{n}$
(a)	$T_n(f;x)$ [13]	$x+\frac{1}{n}$
(b)	$\widetilde{T}_n(f;x)$ [13]	$x+\frac{1}{n}$
(c)	$Q_n(f;x)$	$2x+\frac{2e}{n(e+1)}$
(d)	$\widetilde{Q}_n(f;x)$	$2x+\frac{2e}{n(e+1)}$

with $\hat{b}_{k,j}$ defined as in (23). The generating function is

$$e^{(e^t-1)}e^{x(e^t-1)}=\sum_{k=0}^{\infty}\frac{c_k(x)}{k!}t^k.$$

If $\widetilde{q}_k(x)=\frac{1}{k!}c_k\left(\frac{x}{e}\right)$, then $A(t)=e^{e^t+1}$ and $H(t)=\frac{e^t-1}{e}$ so that $A(1)\neq 0$ and $H'(1)=1$.

We get, respectively, the two operators $Q_n(f;x)$ and $\widetilde{Q}_n(f;x)$:

$$Q_n(f;x)=\frac{2e^{-nx\frac{e-1}{e}}}{1+e}\sum_{k=0}^{\infty}q_k(nx)f\left(\frac{k}{n}\right), \tag{25}$$

$$\widetilde{Q}_n(f;x)=\frac{e^{-nx(e-1)}}{e^{e-1}}\sum_{k=0}^{\infty}\widetilde{q}_k(nx)f\left(\frac{k}{n}\right). \tag{26}$$

For the rate of convergence of Q_n and $\widetilde{Q}_n$ operators, from Theorem 2 we have

$$|Q_n(f;x)-f(x)|\le\left(2x+\frac{2e}{n(e+1)}\right)\omega\left(f,\frac{1}{\sqrt{n}}\right),$$

and

$$|\widetilde{Q}_n(f;x)-f(x)|\le\left(2x+\frac{2e}{n(e+1)}\right)\omega\left(f,\frac{1}{\sqrt{n}}\right).$$

The following theorem can be proved analogously to Theorem 5.

Theorem 6. *If $f\in C_E^2[0,\infty)$, then*

$$|Q_n(f;x)-f(x)|\le\lambda_n(x)\|f\|_{C_E^2}, \tag{27}$$

where $\lambda_n(x)=\frac{1}{n}\left(x+\frac{e(1+n)}{n(1+e)}\right)$;

$$|\widetilde{Q}_n(f;x)-f(x)|\le\widetilde{\lambda}_n(x)\|f\|_{C_E^2}, \tag{28}$$

where $\widetilde{\lambda}_n(x)=\frac{1}{n}\left(x+\frac{e(2+e+2n)}{2n}\right)$.

4.3. Comparison of rate of convergence

In order to compare the rates of convergence of the operators considered in Tables 1 and 2, we give Table 3.

Observe that in the case of operators $T_n(f;x)$ and $\widetilde{T}_n(f;x)$ the asymptotic error constant is 1. For the other operators this does not always happen.

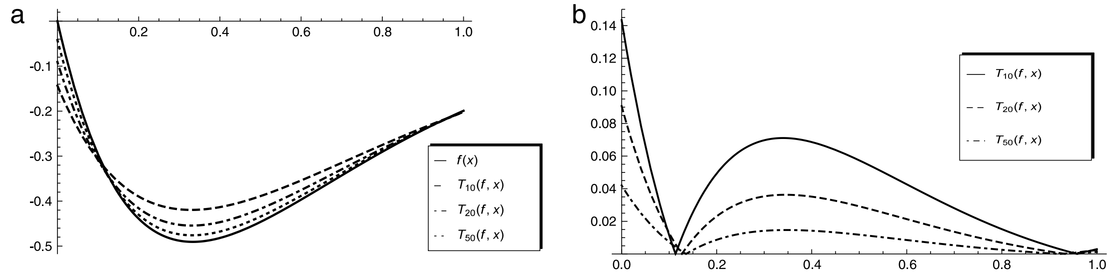

Fig. 1. (a) Convergence of $T_n(f;x)$ to $f(x)$ in [0, 1]. (b) Absolute errors in [0, 1].

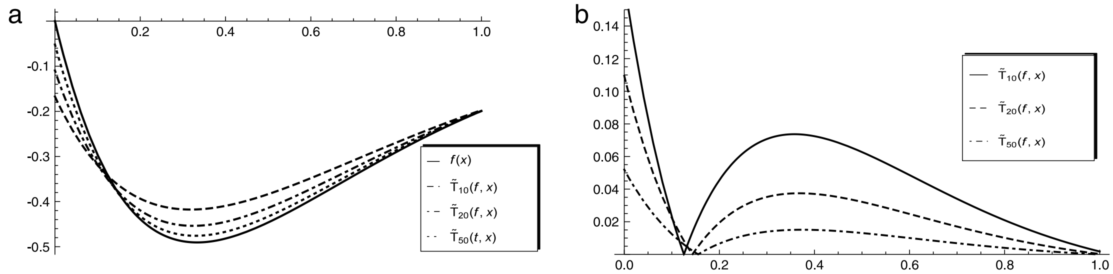

Fig. 2. (a) Convergence of $T_n(f;x)$ to $f(x)$ in [0, 1]. (b) Absolute errors in [0, 1].

Table 4
Error bound by using modulus of continuity—Example 1.

n	Err_{T_n}	$Err_{\widetilde{T}_n}$	Err_{Q_n}	$Err_{\widetilde{Q}_n}$	Error in [36]
10	0.33809900	0.34169700	0.40677900	0.4640070	0.89388445
10^3	0.03423310	0.03423710	0.04132160	0.04132902	0.84099669
10^5	0.003423750	0.003423750	0.00413282	0.00413288	0.38033509
10^7	0.000342374	0.000342374	0.00041328	0.00041329	0.13488244
10^9	0.0000342374	0.0000342374	0.00004132	0.00004132	0.04423507
10^{11}	3.42374e−06	3.42374e−06	4.1328e−06	4.1328e−06	0.01415055
10^{13}	3.42374e−07	3.42374e−07	4.1328e−07	4.1328e−07	0.00449115
10^{15}	3.42374e−08	3.42374e−08	4.1328e−08	4.1328e−08	0.00142186
10^{17}	3.42374e−09	3.42374e−09	4.1328e−09	4.1328e−09	4.497971e−04
10^{19}	3.42374e−10	3.42374e−10	4.1328e−10	4.1328e−10	1.422548e−04

5. Numerical examples

In the following examples we determine error estimations in the approximation by T_n, $\widetilde{T}_n$, Q_n and $\widetilde{Q}_n$ operators for some functions, by using modulus of continuity. We denote by Err_{W_n}, with $W_n = T_n, \widetilde{T}_n, Q_n, \widetilde{Q}_n$, the computed error bound, respectively, for operators T_n, $\widetilde{T}_n$, Q_n and $\widetilde{Q}_n$. We compare the obtained results to those achieved with other operators in the literature.

Then, we apply the extrapolation algorithm by considering several choices of the sequence n_i and compute the error

$$G_{W_n} = \max_{0\le x\le 1} |G_{q-1}^{(1)}(f;x) - f(x)| \,,$$

where W_n is as above and indicates the specific operators utilized for the computation of $G_0^{(i)}$ in (15).

All calculations have been carried out using Mathematica 10.0 and Matlab R2015b.

Example 1 (*[36]*). Consider the following function $f(x) = -4xe^{-3x}$, $x \in [0, 1]$.

Fig. 1(a) and (b) show, respectively, the plots of $f(x)$ and the approximating operators $T_n(f;x)$, and the plots of the absolute errors $|T_n(f;x) - f(x)|$ for $n = 10, 20, 50$ in the interval [0, 1].

Fig. 2(a) and (b) show, respectively, the plots of $f(x)$ and the approximating operators $\widetilde{T}_n(f;x)$, and the plots of the absolute errors $|\widetilde{T}_n(f;x) - f(x)|$ for $n = 10, 20, 50$ in [0, 1].

In Table 4 we give the computed error estimations Err_{W_n} for several values of n. They are compared with the ones in [36].

Table 5 shows the true errors

$$E_{W_n} = \max_{a\le x\le b} |W_n(f;x) - f(x)|.$$

Table 5
Maximum absolute error—Example 1.

n	E_{T_n}	$E_{\widetilde{T}_n}$	E_{Q_n}	$E_{\widetilde{Q}_n}$
10	0.1433940	0.1671720	0.1671716	0.3341707
50	0.0420539	0.0519629	0.0519629	0.1656723
100	0.0221170	0.0275517	0.0750055	0.0946478
10^3	0.00231595	0.0029067	0.4878836	0.4878836
10^5	0.00002328	0.0000294	2.9241e−05	1.0871e−04
10^7	2.3279e−07	2.9242e−07	2.9242e−07	1.0873e−06
10^9	4.6216e−08	4.5056e−08	2.9242e−09	1.0874e−08

Table 6
Extrapolation error—Example 1.

n_i	i	G_{T_n}	i	$G_{\widetilde{T}_n}$
2^i	1, 2, 3	8.959338e−05	1, 2, …, 4	9.525916e−01
	1, 2, …, 14	3.737659e−10	1, 2, …, 6	4.671524e−05
4^i	1, 2, 3	5.130503e−05	1, 2, 3	8.900901e−05
	1, 2, …, 7	2.280971e−10	1, 2, …, 5	8.676678e−10
$10+i$	1, 2, 3	8.360350e−05	1, 2, …, 4	7.251635e−05
	1, 2, …, 11	1.422075e−08	1, 2, …, 11	2.872629e−08
$2i$	1, 2, …, 5	9.515105e−05	1, 2, …, 5	3.024622e−05
	1, 2, …, 10	6.537760e−10	1, 2, …, 10	6.973742e−10

Table 7
Extrapolation error—Example 1.

n_i	i	G_{Q_n}	n_i	i	$G_{\widetilde{Q}_n}$
2^i	1, 2, …, 4	7.027857e−05	2^i	1, 2, …, 4	1.085701e−01
	1, 2, …, 7	5.005474e−07		1, 2, …, 6	2.862507e−03
$10+i$	1, 2, …, 5	8.404059e−05	$30+i$	1, 2, …, 5	3.554468e−04
	1, 2, …, 11	8.432076e−08		1, 2, …, 7	6.819058e−06
$2i$	1, 2, …, 7	9.049401e−05	$2i$	1, 2, …, 10	2.337181e−04
	1, 2, …, 12	7.990505e−10		1, 2, …, 12	7.432047e−06

Table 8
Error bound by using modulus of continuity—Example 2.

n	Err_{T_n}	$Err_{\widetilde{T}_n}$	Err_{Q_n}	$Err_{\widetilde{Q}_n}$	Error in [24]
10	0.644675	0.651536	0.406779	0.884751	0.70003463
10^2	0.200489	0.200718	0.130482	0.245873	0.22246336
10^3	0.0632613	0.0632686	0.0413216	0.0764873	0.07035257
10^4	0.0200005	0.0200007	0.0130689	0.0241467	0.02224744
10^5	0.00632457	0.00632458	0.00413282	0.00763456	0.00703526
10^6	0.00200000	0.00200000	0.00130691	0.00241422	0.00222474
10^7	0.00063246	0.00063246	0.000413282	0.000763442	0.00070352
10^8	0.00020000	0.00020000	0.000130691	0.000241421	0.00022247
10^9	0.00006324	0.00006324	0.000041328	0.000076344	0.00007035
10^{10}	0.00002000	0.00002000	0.000013069	0.000024142	0.00002224
10^{11}	6.3246e−06	6.3246e−06	4.13282e−06	7.63441e−06	7.03526e−06
10^{12}	2.0000e−06	2.0000e−06	1.30691e−06	2.41421e−06	2.22474e−06
10^{13}	6.3246e−07	6.3246e−07	4.13282e−07	7.63441e−06	7.03526e−07

Since our machines have not enough speed and power to compute the infinite series in (25) and (26), we had to approximate the series with finite sums in order to evaluate Q_n and $\widetilde{Q}_n$ in Table 5.

The absolute errors obtained by the extrapolation process, for several choices of the sequence n_i are displayed in Tables 6 and 7.

Example 2 ([24]). Consider now $f(x)=\frac{x}{\sqrt{1+x^4}}$, $x\in[0,1]$. The error estimations Err_{W_n} are displayed in Table 8 and compared with the estimations in [24].

Table 9 shows the true errors in the approximation by means of the considered operators.

The absolute errors obtained by extrapolation, for several choices of the sequence n_i are displayed in Tables 10 and 11.

Table 9
Maximum absolute error—Example 2.

n	E_{T_n}	$E_{\widetilde{T}_n}$	E_{Q_n}	$E_{\widetilde{Q}_n}$
10	0.0581096	0.0727902	0.0983694	0.2565774
20	0.0329087	0.0365364	0.0580220	0.1351967
50	0.0137139	0.0146206	0.0265477	0.0543578
100	0.00704207	0.00731051	0.0140616	0.0288311

Table 10
Extrapolation error—Example 2.

n_i	i	G_{T_n}	i	$G_{\widetilde{T}_n}$
2^i	$1, 2, \ldots, 4$	8.509047e−05	$1, 2, \ldots, 6$	4.966732e−05
	$1, 2, \ldots, 8$	1.596833e−07	$1, 2, \ldots, 9$	7.389635e−10
4^i	$1, 2, 3$	5.087599e−05	$1, 2, \ldots, 4$	1.893475e−05
	$1, 2, \ldots, 4$	2.032594e−05	$1, 2, \ldots, 5$	9.944473e−08
$10 + i$	$1, 2, \ldots, 4$	8.210283e−05	$1, 2, \ldots, 5$	8.727032e−05
	$1, 2, \ldots, 13$	2.289737e−07	$1, 2, \ldots, 9$	7.859487e−07
$2i$	$1, 2, \ldots, 8$	8.905467e−05	$1, 2, \ldots, 7$	8.941022e−05
	$1, 2, \ldots, 16$	3.743760e−08	$1, 2, \ldots, 21$	6.585855e−07

Table 11
Extrapolation error—Example 2.

n_i	i	G_{Q_n}	n_i	i	$G_{\widetilde{Q}_n}$
2^i	$1, 2, \ldots, 6$	9.200485e−04	2^i	$1, 2, \ldots, 4$	3.185449e−02
	$1, 2, \ldots, 6$	9.200485e−04		$1, 2, \ldots, 5$	8.468367e−03
4^i	$1, 2, 3$	3.980982e−05	4^i	$1, 2, 3$	7.766571e−03
	$1, 2, 3$	3.980982e−05		$1, 2, 3, 4$	9.91e−01
$10 + i$	$1, 2, \ldots, 4$	8.775534e−05	$30 + i$	$1, 2, \ldots, 4$	5.367207e−04
	$1, 2, \ldots, 7$	7.329339e−05		$1, 2, \ldots, 6$	4.545825e−05
$2i$	$1, 2, \ldots, 8$	8.217080e−05	$2i$	$1, 2, \ldots, 7$	7.201657e−03
	$1, 2, \ldots, 11$	5.341235e−06		$1, 2, \ldots, 16$	4.598149e−06

6. Conclusions

In this paper old and new operators of Jakimovski and Leviatan type and of Ismail type are introduced. The rate of convergence for these operators is examined by using classical techniques. An asymptotic expansion for operators of Ismail type is presented. Hence, Richardson extrapolation algorithm is applied, in order to accelerate the convergence of these operators. Numerical examples support theoretical results and show that high accuracy in the approximation by means of extrapolation can be achieved.

Observe that the numerical results are strongly influenced by the power of the software and computing tools which have been used. Probably, by the use of more powerful tools, the approximations would be more stringent. However, by comparing the absolute errors on the operators and those on the extrapolated operators, the effectiveness of the use of extrapolation in approximation with the considered operators is evident. In fact, from Tables 6, 7, 10 and 11 we can observe that stringent tolerances are achieved also for small values of n.

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