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Auxiliary point on the semilocal convergence of Newton's method

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ABSTRACT

We use an auxiliary point on the semilocal convergence of Newton's method when the majorant principle of Kantorovich is applied to operators with high order derivatives satisfying a center Lipschitz type condition, so that we extend the classical conditions of these types, that are centered at the starting point of Newton's method, to other points belonging to the domain of definition of the operator involved. This extension provides a modification of the domain of starting points for Newton's method which allows increasing the choice of starting points. We illustrate this study with nonlinear Fredholm integral equations.

1. Introduction

Using mathematical modeling, many problems from computational sciences, physics and other disciplines can be solved if they are brought into a form similar to equation $F(x) = 0$. To give sufficient generality to this problem, we consider that F is a nonlinear operator defined on a nonempty open convex subset Ω of a Banach space X with values in a Banach space Y. Usually, it is not possible to find a solution of $F(x) = 0$ in closed form and, therefore, it is necessary to apply iterative methods for solving it. For this, starting from one initial approximation x_0 of a solution x^* of the equation $F(x) = 0$, a sequence $\{x_n\}$ of approximations is constructed such that the sequence $\{\|x_n - x^*\|\}$ is strictly decreasing to zero.

Three types of studies can be done when we are interested in proving the convergence of the sequence $\{x_n\}$ given by an iterative method, that depend on the conditions required to the operator F and the starting point of the iterative method: local [1], semilocal [2] and global [3]. In this paper, we pay our attention to the semilocal convergence.

Three types of conditions are required to obtain a semilocal convergence result for an iterative method: conditions on the starting point x_0, conditions on the operator involved F and conditions that connect both types of the previous conditions. An important feature of the semilocal convergence results obtained for an iterative method is that conclusions about the existence and uniqueness of solution of the equation to solve are drawn from the theoretical significance of the result and the initial approximation of the iterative method used to solve the equation. This fact makes the choice of the starting points for iterative methods is a basic aspect in semilocal convergence studies. Moreover, the fact that the conditions required to the starting point and to the operator F are independent makes that we can choose the initial approximation inside a domain of starting points depending on the conditions that connect both types of conditions. In another case, if both types of conditions are connected, the domain of starting points can be significantly reduced and this is a problem.

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We then try to solve the last problem by introducing an auxiliary point, different from the starting point, that allows us to eliminate the connection between the conditions required to the starting points and to the operator F, and thus recover the domain of starting points.

We use the best known and most used iterative method in practice to solve equation $F(x) = 0$, that is Newton's method, whose algorithm is:

$$x_0 \in \Omega, \quad x_n = x_{n-1} - [F'(x_{n-1})]^{-1}F(x_{n-1}), \quad n \in \mathbb{N}. \tag{1}$$

The main aim of this paper is to obtain, under a center type condition for the operator F, a domain of starting points for Newton's method that, in the case of requiring the condition on the starting point x_0, such as we have indicated previously, this domain is then reduced to x_0. We consider that the operator F is sufficiently differentiable; namely, $F \in C^{(k)}(\Omega)$ with $k \geq 3$. In [4], we obtain a semilocal convergence result for Newton's method under the following conditions:

(A1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ for some $x_0 \in \Omega$, with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y to X; moreover, $\|F^{(i)}(x_0)\| \leq b_i$, with $i = 2, 3, \ldots, k$ and $k \geq 2$.

(A2) $\|F^{(k)}(x) - F^{(k)}(x_0)\| \leq \omega(\|x - x_0\|)$ for $x \in \Omega$, where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function such that $\omega(0) = 0$.

(A3) $g(\alpha) \leq 0$, where

$$g(t) = \int_0^t \int_0^{\theta_{k-1}} \cdots \int_0^{\theta_1} \omega(z)\,dz\,d\theta_1 \cdots d\theta_{k-1} + \sum_{i=2}^{k} \frac{b_i}{i!}\,t^i - \frac{t}{\beta} + \frac{\eta}{\beta}$$

and α is the unique positive solution of $g'(t) = 0$, and $B(x_0, t^*) \subset \Omega$, where t^* is the smallest positive solution of $g(t) = 0$.

It is obvious that a solution x^* of equation $F(x) = 0$ must satisfy the convergence conditions in a trivial way, so that it seems clear that we can choose, by continuity, points next to x^* as initial approximations for Newton's method. However, under conditions (A1)–(A3), the solution x^* can be only chosen as starting point for Newton's method if condition (A2) is satisfied at $x_0 = x^*$. In this case, the domain of starting points for Newton's method is just reduced to x_0 and Newton's sequence $\{x_n\}$ is convergent to the solution x^* of $F(x) = 0$. In another case, we cannot choose other starting points for Newton's method, since condition (A2), required to the operator F, is only given in x_0. To solve this problem, we use an auxiliary point $\widetilde{x}$, that can be different from x_0, so that (A2) is given at this point $\widetilde{x}$ instead of the point x_0 and, if condition (A3) is not satisfied, once x_0 is fixed, we can consider, as initial approximation for Newton's method, any point different from x_0 that satisfies (A3), expanding so the domain of starting points.

The paper is organized as follows. In Section 2, we introduce the majorant principle of Kantorovich and show how we can use it under Lipschitz type conditions for high order derivatives centered at a point different from the starting point of Newton's method. For this, we construct majorizing sequences from solving initial value problems, which is an approach different from that given by Kantorovich, that is based on interpolation fitting. In Section 3, we analyze the semilocal convergence of Newton's method and provide a domain of existence of solution. In Section 4, we show the uniqueness of solution and prove the quadratic convergence of Newton's method under conditions required in the paper. In Section 5, we illustrate the previous study with a nonlinear Fredholm integral equation. Finally, in Section 6, we analyze two particular cases of the main condition required previously, that are reduced to high order derivatives that are center Lipschitz or Höolder continuous at an auxiliary point. Both cases are illustrated with nonlinear Fredholm integral equations.

Throughout the paper we denote $\overline{B(x, \varrho)} = \{y \in X;\ \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in X;\ \|y - x\| < \varrho\}$.

2. Initial focus

The application of Newton's method to solve nonlinear equations has a long history and different techniques have been developed over the past years to analyze the semilocal convergence of the method. In this paper, we use the most famous, "the majorant principle" of Kantorovich [5], devised by Kantorovich at the beginning of the 50s of the last century. The idea of the principle is simple, since it allows us to translate the problem of solving $F(x) = 0$ to the problem of solving a scalar equation. For this, we look for a scalar sequence $\{t_n\}$ such that

$$\|x_n - x_{n-1}\| \leq t_n - t_{n-1}, \quad n \in \mathbb{N},$$

where

$$t_n = t_{n-1} - \frac{f(t_{n-1})}{f'(t_{n-1})},\ n \in \mathbb{N},\ \text{with } t_0 \text{ given},$$

and $f(t)$ is a suitable scalar function that we have to find.

Taking into account the aims indicated in the introduction, we then analyze the semilocal convergence of Newton's method by "the majorant principle" of Kantorovich under the following conditions:

(P1) There exists the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$; moreover, $\|F^{(i)}(x_0)\| \leq b_i$ with $i = 2, 3, \ldots, k-1$ and $k \geq 3$.
(P2) There exists $\widetilde{x} \in \Omega$ such that $\|x_0 - \widetilde{x}\| = \gamma$, where $x_0 \in \Omega$, and $\|F^{(k)}(\widetilde{x})\| \leq \delta$.
(P3) There exists a nondecreasing continuous function $\omega : [0, +\infty) \longrightarrow \mathbb{R}$ such that $\|F^{(k)}(x) - F^{(k)}(\widetilde{x})\| \leq \omega(\|x - \widetilde{x}\|)$ for $x \in \Omega$ and $\omega(0) = 0$.

On the one hand, it seems clear that Kantorovich solved a problem of interpolation fitting, from the conditions required to the operator F and the starting point x_0, to obtain the real function from which the majorizing sequences are built.

On the other hand, if we pay attention to the condition imposed to the operator F in (P3), it also seems clear that we cannot use the same procedure as Kantorovich to find the real function, since (P3) does not allow us to determine the class of functions where (P1)–(P2) can be applied. To solve this problem, we proceed differently, without interpolation fitting, and solve an initial value problem. Then, we look for a real function $f \in \mathcal{C}^j([\widetilde{t}, +\infty))$, with $\widetilde{t} \in \mathbb{R}_+$ and $j \geq k$, such that

$$\|F^{(k)}(x) - F^{(k)}(\widetilde{x})\| \leq f^{(k)}(t) - f^{(k)}(\widetilde{t}) \text{ with } \|x - \widetilde{x}\| \leq t - \widetilde{t}, x \in \Omega \text{ and } t \in [\widetilde{t}, +\infty).$$

So, from (P3) and the last, it follows

$$\|F^{(k)}(x) - F^{(k)}(\widetilde{x})\| \leq \omega(\|x - \widetilde{x}\|) \leq \omega(t - \widetilde{t}) = f^{(k)}(t) - f^{(k)}(\widetilde{t})$$

if $\|x - \widetilde{x}\| \leq t - \widetilde{t}$, since ω is a nondecreasing continuous function, and then

$$f^{(k)}(t) = f^{(k)}(\widetilde{t}) + \omega(t - \widetilde{t}).$$

In addition, if we consider $t_0 = \widetilde{t} + \gamma$, $f^{(k)}(\widetilde{t}) = \delta$ and take into account (P1)–(P2), we can solve the initial value problem

$$\begin{cases} y^{(k)}(t) = \delta + \omega(t - t_0 + \gamma), \\ y(t_0) = \dfrac{\eta}{\beta}, \quad y'(t_0) = -\dfrac{1}{\beta}, \\ y''(t_0) = b_2, \quad y'''(t_0) = b_3, \quad \ldots, \quad y^{(k-1)}(t_0) = b_{k-1}, \end{cases}$$

to find the real function $f(t)$, since we can choose, from (P1), $-\frac{1}{f'(t_0)} = \beta$, $-\frac{f(t_0)}{f'(t_0)} = \eta$ and $f^{(i)}(t_0) = b_i$, for $i = 2, 3, \ldots, k-1$. So, next result is given.

Theorem 1. *Suppose that the function $\omega(t - t_0 + \gamma)$ is continuous in $[t_0, +\infty)$. Then, for any nonnegative real numbers γ, δ, $\beta \neq 0$, η, $b_2, b_3, \ldots, b_{k-1}$, the last initial value problem has a unique solution $\widehat{\psi}(t) \in \mathcal{C}^j([\widetilde{t}, +\infty))$, with $j \geq k \geq 3$, which is given by*

$$\widehat{\psi}(t) = \int_{t_0}^{t} \int_{t_0}^{\theta_{k-1}} \cdots \int_{t_0}^{\theta_1} \omega(s - t_0 + \gamma)\, ds\, d\theta_1 \cdots d\theta_{k-1} + \frac{\delta}{k!}(t - t_0)^k + \sum_{i=2}^{k-1} \frac{b_i}{i!}(t - t_0)^i - \frac{t - t_0}{\beta} + \frac{\eta}{\beta}.$$

Remark 2. Observe that $t_0 \geq 0$, but we can choose $t_0 = 0$, since function $\widehat{\psi}(t)$ is such that $\widehat{\psi}(t + t_0) = \psi(t)$, with

$$\psi(t) = \int_{0}^{t} \int_{0}^{\theta_{k-1}} \cdots \int_{0}^{\theta_1} \omega(s + \gamma)\, ds\, d\theta_1 \cdots d\theta_{k-1} + \frac{\delta}{k!} t^k + \sum_{i=2}^{k-1} \frac{b_i}{i!} t^i - \frac{t}{\beta} + \frac{\eta}{\beta}, \tag{2}$$

and the sequence $\{t_n = N_{\widehat{\psi}}(t_{n-1})\}_{n \in \mathbb{N}}$, for any $t_0 > 0$, satisfies $t_n = N_{\widehat{\psi}}(t_{n-1}) = t_0 + N_{\psi}(s_{n-1})$, $n \in \mathbb{N}$, where $s_n = N_{\psi}(s_{n-1})$ with $s_0 = 0$, since we have, for $t_0 \geq 0$ and $s_0 = 0$,

$$t_0 + s_n = t_0 + N_{\psi}(s_{n-1}) = t_0 + s_{n-1} - \frac{\psi(s_{n-1})}{\psi'(s_{n-1})} = t_0 + s_{n-1} - \frac{\widehat{\psi}(s_{n-1} + t_0)}{\widehat{\psi}'(s_{n-1} + t_0)} = t_{n-1} - \frac{\widehat{\psi}(t_{n-1})}{\widehat{\psi}'(t_{n-1})} = t_n = N_{\widehat{\psi}}(t_{n-1})$$

for all $n \in \mathbb{N}$. Therefore, the real sequences $\{t_n\}$ and $\{s_n\}$ given by Newton's method when they are constructed from $\widehat{\psi}(t)$ and $\psi(t)$, respectively, can be obtained, one from the other, by translation. Besides, $t_n - t_{n-1} = s_n - s_{n-1}$, for all $n \in \mathbb{N}$, and all the results obtained previously are independent of the value $t_0 \geq 0$, so that we choose $t_0 = 0$ because, in practice, it is the most favorable situation.

3. Majorizing sequence and semilocal convergence

Once we have found the scalar function $\psi(t)$ to construct the majorizing sequence $\{t_n = N_{\psi}(t_{n-1})\}_{n \in \mathbb{N}}$, we have to guarantee that $\psi(t)$ has at least one zero greater than $t_0 = 0$, so that the sequence converges to this zero. So, we give the next result.

Theorem 3. *Let ψ and ω be the two functions defined in (2) and (P3), respectively.*

(a) *If there exists a solution $\alpha > 0$ of the equation $\psi'(t) = 0$, then α is the unique minimum of $\psi(t)$ in $[0, +\infty)$ and $\psi(t)$ is nonincreasing in $[0, \alpha)$.*
(b) *If $\psi(\alpha) \leq 0$, then equation $\psi(t) = 0$ has at least one solution t^* in $[0, +\infty)$. In addition, if t^* is the smallest root of $\psi(t) = 0$ in the interval $[0, +\infty)$, then $t^* \in (0, \alpha]$.*

Proof. First, from (P3), it follows that $\psi^{(k)}(t) \geq 0$ in $[\widetilde{t}, +\infty)$ and $\psi^{(k-1)}(t)$ is nondecreasing in $[\widetilde{t}, +\infty)$. In addition, as $\psi^{(k-1)}(0) = b_{k-1} > 0$, then $\psi^{(k-1)}(t) \geq 0$ in $[0, +\infty)$. Repeating the previous reasoning until the second derivative, we obtain $\psi''(t) \geq 0$ in $[0, +\infty)$. As a consequence, $\psi(t)$ is convex in $[0, +\infty)$, α is the unique minimum of $\psi(t)$ in $[0, +\infty)$ and $\psi'(t)$ is nondecreasing in $[0, +\infty)$.

Moreover, as $\psi'(0) = -\frac{1}{\beta} < 0$, then $\psi'(t) \leq 0$ in $[0, \alpha)$ and $\psi(t)$ is nonincreasing in $[0, \alpha)$.

Second, if $\psi(\alpha) < 0$, then $\psi(t)$ has at least one zero t^* in $[0, \alpha)$, since $\psi(0) = \frac{\eta}{\beta} \geq 0$ and $\psi(t)$ is continuous. In addition, as $\psi(t)$ is nonincreasing in $[0, \alpha)$, we have that t^* is the unique zero of $\psi(t)$ in $[0, \alpha)$.

On the other hand, if $\psi(\alpha) = 0$, then α is a double root of $\psi(t) = 0$ and $t^* = \alpha$. ■

After knowing what the conditions are to be satisfied so that the function $\psi(t)$ has a zero, we guarantee the convergence of the Newton sequence $\{t_n = N_\psi(t_{n-1})\}_{n\in\mathbb{N}}$ in the following result.

Theorem 4. *Suppose that there exists a solution $\alpha \in (0, +\infty)$ of $\psi'(t) = 0$ such that $\psi(\alpha) \leq 0$. Then, the Newton sequence $\{t_n = N_\psi(t_{n-1})\}_{n\in\mathbb{N}}$ is nondecreasing and converges to the smallest positive solution t^* of the equation $\psi(t) = 0$.*

Proof. If $\psi(0) = \frac{\eta}{\beta} = 0$, then $t_n = 0$, for all $n \geq 0$, and $\{t_n\}$ converges to $t^* = 0$. If $\psi(0) = \frac{\eta}{\beta} > 0$, then $t^* \geq 0$. By the Mean-Value Theorem, we obtain

$$t_1 - t^* = N_\psi(0) - N_\psi(t^*) = N'_\psi(\theta_0)(0 - t^*) \quad \text{with} \quad \theta_0 \in (0, t^*),$$

so that $t_1 < t^*$, since $N'_\psi(t) = \frac{\psi(t)\psi''(t)}{\psi'(t)^2} > 0$ in $[0, t^*)$. Moreover,

$$t_1 - t_0 = -\frac{\psi(0)}{\psi'(0)} \geq 0.$$

Then, by mathematical induction on n, it follows easily $t_n < t^*$ and $t_n - t_{n-1} \geq 0$. In addition, we infer that sequence $\{t_n = N_\psi(t_{n-1})\}_{n\in\mathbb{N}}$ converges to $v \in [0, t^*]$. Moreover, since t^* is the unique root of $\psi(t) = 0$ in $[0, t^*]$, it follows that $v = t^*$. ■

Next, in the following result, we prove that $\{t_n = N_\psi(t_{n-1})\}_{n\in\mathbb{N}}$ is a majorizing sequence of the sequence $\{x_n\}$, given by (1), in the Banach space X, so that the semilocal convergence of Newton's sequence is then guaranteed in X.

Theorem 5. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \longrightarrow Y$ a nonlinear q $(q \geq 2)$ times continuously differentiable operator on a nonempty open convex domain Ω and $\psi(t)$ be function defined in (2). Suppose that conditions (P1)–(P3) are satisfied, there exists a root $\alpha > 0$ of $\psi'(t) = 0$ such that $\psi(\alpha) \leq 0$, and $B(x_0, t^*) \subset \Omega$, where t^* is the smallest positive root of $\psi(t) = 0$. Then, the sequence $\{x_n\}$, given by Newton's method (1), converges to a solution x^* of $F(x) = 0$ starting at x_0. Moreover, $x_n, x^* \in \overline{B(x_0, t^*)}$ and*

$$\|x^* - x_n\| \leq t^* - t_n, \quad n \geq 0,$$

where $t_n = N_\psi(t_{n-1})$, with $n \in \mathbb{N}$ and $t_0 = 0$.

Proof. We begin by proving that Newton's sequence (1) is well-defined and $x_n \in B(x_0, t^*)$, for all $n \geq 0$. From (P1), it follows

$$\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta = -\frac{\psi(0)}{\psi'(0)} = t_1 < t^*$$

and $x_1 \in B(x_0, t^*)$.

Next, we prove the following five recurrence relations for all $n \in \mathbb{N}$:

(I_n) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leq -\frac{1}{\psi'(t_n)}$,
(II_n) $\|F^{(i)}(x_n)\| \leq \psi^{(i)}(t_n)$, $i = 2, 3, \ldots, k-1$,
(III_n) $\|F(x_n)\| \leq \psi(t_n)$,
(IV_n) $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$,
(V_n) $x_{n+1} \in B(x_0, t^*)$.

For the last, we use mathematical induction on n. The case (I_1)–(V_1) is analogous to which we do for the inductive step, no more to consider conditions (P1)–(P3). For the inductive step, we suppose that (I_n)–(V_n) are true for $n = 1, 2, \dots, d-1$ and see that (I_d)–(V_d) are true.

To prove (I_d), we consider Taylor's series of F and write

$$\begin{aligned}
I - \Gamma_{d-1}F'(x_d) &= \Gamma_{d-1}\left(F'(x_{d-1}) - F'(x_d)\right)\\
&= \Gamma_{d-1}\left(F'(x_{d-1}) - \sum_{i=1}^{k-1}\frac{1}{(i-1)!}F^{(i)}(x_{d-1})(x_d - x_{d-1})^{i-1} - \frac{1}{(k-2)!}\int_{x_{d-1}}^{x_d}F^{(k)}(z)(x_d - z)^{k-2}\,dz\right)\\
&= -\Gamma_{d-1}\left(\sum_{i=2}^{k-1}\frac{1}{(i-1)!}F^{(i)}(x_{d-1})(x_d - x_{d-1})^{i-1} + \frac{1}{(k-2)!}\int_{x_{d-1}}^{x_d}(F^{(k)}(z) - F^{(k)}(\widetilde{x}))(x_d - z)^{k-2}\,dz\right.\\
&\qquad\left. + \frac{1}{(k-2)!}\int_{x_{d-1}}^{x_d}F^{(k)}(\widetilde{x})(x_d - z)^{k-2}\,dz\right).
\end{aligned}$$

Besides, if $z \in [x_{d-1}, x_d]$ and $s \in [t_{d-1}, t_d]$, then $z = x_{d-1} + \tau(x_d - x_{d-1})$ and $s = t_{d-1} + \tau(t_d - t_{d-1})$ with $\tau \in [0, 1]$, so that

$$\begin{aligned}
\|z - \widetilde{x}\| &\le \|z - x_{d-1}\| + \|x_{d-1} - \widetilde{x}\|\\
&\le \tau\|x_d - x_{d-1}\| + \|x_{d-1} - x_{d-2}\| + \cdots + \|x_1 - x_0\| + \|x_0 - \widetilde{x}\|\\
&\le \tau(t_d - t_{d-1}) + t_{d-1} - t_{d-2} + \cdots + t_1 - t_0 + \gamma\\
&= s - \widetilde{t}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\|I - \Gamma_{d-1}F'(x_d)\| &\le \|\Gamma_{d-1}\|\left(\sum_{i=2}^{k-1}\frac{1}{(i-1)!}\|F^{(i)}(x_{d-1})\|\|x_d - x_{d-1}\|^{i-1}\right.\\
&\quad + \frac{1}{(k-2)!}\int_0^1\|F^{(k)}(x_{d-1} + \tau(x_d - x_{d-1})) - F^{(k)}(\widetilde{x})\| \times \|x_d - x_{d-1}\|^{k-1}(1-\tau)^{k-2}\,d\tau\\
&\quad\left. + \frac{1}{(k-1)!}\|F^{(k)}(\widetilde{x})\|\|x_d - x_{d-1}\|^{k-1}\right)\\
&\le -\frac{1}{\psi'(t_{d-1})}\left(\sum_{i=2}^{k-1}\frac{\psi^{(i)}(t_{d-1})}{(i-1)!}(t_d - t_{d-1})^{i-1}\right.\\
&\quad + \frac{1}{(k-2)!}\int_0^1\left(\psi^{(k)}(t_{d-1} + \tau(t_d - t_{d-1})) - \psi^{(k)}(\widetilde{t})\right)(t_d - t_{d-1})^{k-1}(1-\tau)^{k-2}\,d\tau\\
&\quad\left. + \frac{\psi^{(k)}(\widetilde{t})}{(k-1)!}(t_d - t_{d-1})^{k-1}\right)\\
&= -\frac{1}{\psi'(t_{d-1})}\left(\sum_{i=2}^{k-1}\frac{\psi^{(i)}(t_{d-1})}{(i-1)!}(t_d - t_{d-1})^{i-1}\right.\\
&\quad\left. + \frac{1}{(k-2)!}\int_0^1\psi^{(k)}(t_{d-1} + \tau(t_d - t_{d-1}))(t_d - t_{d-1})^{k-1}(1-\tau)^{k-2}\,d\tau\right)\\
&= 1 - \frac{1}{\psi'(t_{d-1})}\left(\sum_{i=1}^{k-1}\frac{\psi^{(i)}(t_{d-1})}{(i-1)!}(t_d - t_{d-1})^{i-1} + \frac{1}{(k-2)!}\int_{t_{d-1}}^{t_d}\psi^{(k)}(u)(t_d - u)^{k-2}\,du\right)\\
&= 1 - \frac{\psi'(t_d)}{\psi'(t_{d-1})}\\
&< 1,
\end{aligned}$$

since $t_{d-1} < t_d \le t^*$, $\psi''(t) \ge 0$ in $[0, +\infty)$ and $\frac{\psi'(t_d)}{\psi'(t_{d-1})} \in (0, 1)$. Therefore, by the Banach lemma on invertible operators, it follows that there exists the operator Γ_d and $\|\Gamma_d\| \le -\frac{1}{\psi'(t_d)}$.

To prove (II_d), we distinguish two cases: $i \in \{2, 3, \dots, k-1\}$ and $i = k$. First, we choose $i \in \{2, 3, \dots, k-1\}$. Then, from Taylor's series, we have

$$\begin{aligned}F^{(i)}(x_d) &= \sum_{\ell=1}^{k-i} \frac{1}{(\ell-1)!} F^{(\ell+i-1)}(x_{d-1})(x_d - x_{d-1})^{\ell-1} \\ &\quad + \frac{1}{(k-i-1)!}\left(\int_0^1 (F^{(k)}(x_{d-1} + \tau(x_d - x_{d-1})) - F^{(k)}(\widetilde{x}))(x_d - x_{d-1})^{k-i}(1-\tau)^{k-i-1}\, dz\right. \\ &\quad \left. + \int_0^1 F^{(k)}(\widetilde{x})(x_d - x_{d-1})^{k-i}(1-\tau)^{k-i-1}\, d\tau\right).\end{aligned}$$

Taking now into account that $\|z - \widetilde{x}\| \le s - \widetilde{t}$, where $z \in [x_{d-1}, x_d]$, $s \in [t_{d-1}, t_d]$ and $\|x_d - x_{d-1}\| \le t_d - t_{d-1}$, it follows

$$\begin{aligned}\|F^{(i)}(x_d)\| &\le \sum_{\ell=1}^{k-i} \frac{1}{(\ell-1)!} \|F^{(\ell+i-1)}(x_{d-1})\| \|x_d - x_{d-1}\|^{\ell-1} \\ &\quad + \frac{1}{(k-i-1)!}\left(\int_0^1 \|F^{(k)}(x_{d-1} + \tau(x_d - x_{d-1})) - F^{(k)}(\widetilde{x})\| \|x_d - x_{d-1}\|^{k-i}(1-\tau)^{k-i-1}\, d\tau\right. \\ &\quad \left. + \int_0^1 \|F^{(k)}(\widetilde{x})\| \|x_d - x_{d-1}\|^{k-i}(1-\tau)^{k-i-1}\, d\tau\right) \\ &\le \sum_{\ell=1}^{k-i} \frac{\psi^{(\ell+i-1)}(t_{d-1})}{(\ell-1)!}(t_d - t_{d-1})^{\ell-1} \\ &\quad + \frac{1}{(k-i-1)!}\left(\int_0^1 (\psi^{(k)}(t_{d-1} + \tau(t_d - t_{d-1})) - \psi(\widetilde{t}))(t_d - t_{d-1})^{k-i}(1-\tau)^{k-i-1}\, d\tau\right. \\ &\quad \left. + \int_0^1 \psi^{(k)}(\widetilde{t})(t_d - t_{d-1})^{k-i}(1-\tau)^{k-i-1}\, d\tau\right) \\ &\le \sum_{\ell=1}^{k-i} \frac{\psi^{(\ell+i-1)}(t_{d-1})}{(\ell-1)!}(t_d - t_{d-1})^{\ell-1} + \frac{1}{(k-i-1)!}\int_{t_{d-1}}^{t_d} \psi^{(k)}(s)(t_d - s)^{k-i-1}\, ds \\ &= \psi^{(i)}(t_d).\end{aligned}$$

Second, if $i = k$, then

$$\|F^{(k)}(x_d)\| \le \|F^{(k)}(x_d) - F^{(k)}(\widetilde{x})\| + \|F^{(k)}(\widetilde{x})\| \le \psi^{(k)}(t_d),$$

since $\|x_d - \widetilde{x}\| \le t_d - \widetilde{t}$.

Item (III_d) follows in a way completely analogous to the first case of item (II_d).

Items (IV_d) and (V_d) are immediate, since

$$\|x_{d+1} - x_d\| \le \|\Gamma_d\| \|F(x_d)\| \le -\frac{\psi(t_d)}{\psi'(t_d)} = t_{d+1} - t_d,$$

$$\|x_{d+1} - x_0\| \le \sum_{\ell=1}^{d-1} \|x_\ell - x_{\ell-1}\| \le \sum_{\ell=1}^{d-1}(t_\ell - t_{\ell-1}) = t_{d+1} < t^*,$$

so that $x_{d+1} \in B(x_0, t^*)$. Thus, by mathematical induction, items (I_n)–(V_n) are true for all positive integers n. As a consequence of the above, Newton's sequence $\{x_n\}$ is well-defined and $x_n \in B(x_0, t^*)$ for all $n \ge 0$.

From item (IV_n), for all $n \ge 0$, it follows that the sequence $\{t_n\}$ majorizes the sequence $\{x_n\}$ and, as a result, $\{x_n\}$ is convergent. Then, if $x^* = \lim_n x_n$, we obtain $\|x^* - x_n\| \le t^* - t_n$, for all $n \ge 0$, since $\lim_n t_n = t^*$. Besides, from (III_n), for all $n \ge 0$, and the continuity of F, it follows $F(x^*) = 0$ by letting $n \to +\infty$. ■

4. Uniqueness of solution and order of convergence

After proving the semilocal convergence of Newton's method and locating the solution x^*, we prove the uniqueness of x^*. Before, we need the following lemma.

Lemma 6. *If the conditions of the last theorem are satisfied and $\psi(t)$ has two real zeros t^* and t^{**} such that $0 < t^* \le t^{**}$ and $x \in \overline{B(x_0, t^{**})} \cap \Omega$, then*

$$\|F''(x)\| \le \psi''(t), \quad \text{for} \quad \|x - x_0\| \le t.$$

Proof. From Taylor's series, it follows

$$\begin{aligned}
F''(x) &= \sum_{i=2}^{k-1} \frac{1}{(i-2)!} F^{(i)}(x_0)(x-x_0)^{i-2} + \frac{1}{(k-3)!} \int_{x_0}^{x} F^{(k)}(z)(x-z)^{k-3}\, dz \\
&= \sum_{i=2}^{k-1} \frac{1}{(i-2)!} F^{(i)}(x_0)(x-x_0)^{i-2} + \frac{1}{(k-3)!} \int_{x_0}^{x} \left(F^{(k)}(z) \pm F^{(k)}(\widetilde{x})\right)(x-z)^{k-3}\, dz \\
&= \sum_{i=2}^{k-1} \frac{1}{(i-2)!} F^{(i)}(x_0)(x-x_0)^{i-2} \\
&\quad + \frac{1}{(k-3)!} \int_0^1 \left(F^{(k)}\left(x_0 + \tau(x-x_0)\right) - F^{(k)}(\widetilde{x})\right)(x-x_0)^{k-2}(1-\tau)^{k-3}\, d\tau \\
&\quad + \frac{1}{(k-3)!} \int_0^1 F^{(k)}(\widetilde{x})(x-x_0)^{k-2}(1-\tau)^{k-3}\, d\tau.
\end{aligned}$$

Taking norms, we obtain, from (P1)–(P3) and for $\|x - x_0\| \le t$,

$$\begin{aligned}
\|F''(x)\| &\le \sum_{i=2}^{k-1} \frac{1}{(i-2)!} \|F^{(i)}(x_0)\| \|x-x_0\|^{i-2} \\
&\quad + \frac{1}{(k-3)!} \int_0^1 \left\|F^{(k)}\left(x_0 + \tau(x-x_0)\right) - F^{(k)}(\widetilde{x})\right\| \|x-x_0\|^{k-2}(1-\tau)^{k-3}\, d\tau \\
&\quad + \frac{1}{(k-3)!} \int_0^1 \|F^{(k)}(\widetilde{x})\| \|x-x_0\|^{k-2}(1-\tau)^{k-3}\, d\tau \\
&\le \sum_{i=2}^{k-1} \frac{\psi^{(i)}(0)}{(i-2)!} t^{i-2} + \frac{1}{(k-3)!} \int_0^1 \left(\psi^{(k)}(s) - \psi^{(k)}(\widetilde{t})\right) t^{k-2}(1-\tau)^{k-3}\, d\tau \\
&\quad + \frac{1}{(k-3)!} \int_0^1 \psi^{(k)}(\widetilde{t})\, t^{k-2}(1-\tau)^{k-3}\, d\tau \\
&= \psi''(t),
\end{aligned}$$

if $z = x_0 + \tau(x - x_0)$ and $s = \tau t$ with $\tau \in [0, 1]$ and $\|z - \widetilde{x}\| \le \|x_0 - \widetilde{x}\| + \tau\|x - x_0\| \le s - \widetilde{t}$. ■

Next, provided that $\psi(t)$ has two real zeros t^* and t^{**} such that $0 < t^* \le t^{**}$, the uniqueness of solution is established in the following theorem, which is a generalization of the result on uniqueness of solution obtained under the classical Kantorovich conditions. The proof follows exactly as in Theorem 12 of [4].

Theorem 7. *If the conditions of the last theorem are satisfied and $\psi(t)$ has two real zeros t^* and t^{**}, such that $0 < t^* \le t^{**}$, then the solution x^* is unique in $B(x_0, t^{**}) \cap \Omega$ if $t^* < t^{**}$ or in $\overline{B(x_0, t^*)}$ if $t^{**} = t^*$.*

We finish this section by seeing the quadratic convergence of Newton's method under conditions (P1)–(P3). We obtain the following theorem from Ostrowski's technique [6]. For this, we need that the scalar function $\psi(t)$ has two real zeros t^* and t^{**} such that $0 < t^* \le t^{**}$. In this case,

$$\psi(t) = (t^* - t)(t^{**} - t)g(t)$$

with $g(t^*) \ne 0$ and $g(t^{**}) \ne 0$. Following then Ostrowski's technique given in [6], we obtain next result, where a priori error estimates are provided for Newton's method and whose proof is similar to that given in [7]. Notice that the quadratic convergence of Newton's method is deduced from the result if $t^* < t^{**}$ and linear if $t^* = t^{**}$.

Theorem 8. *Suppose that (P1)–(P3) are satisfied, and $\psi(t)$ has two real zeros t^* and t^{**} such that $0 < t^* \leq t^{**}$.*

(a) *If $t^* < t^{**}$ and there exist $m_1 > 0$ and $M_1 > 0$ such that $m_1 \leq \min\{D_1(t);\, t \in [t_0, t^*]\}$ and $M_1 \geq \max\{D_1(t);\, t \in [t_0, t^*]\}$, where $D_1(t) = \frac{(t^{**}-t)g'(t)-g(t)}{(t^*-t)g'(t)-g(t)}$, then*

$$\frac{(t^{**}-t^*)\theta^{2^n}}{m_1-\theta^{2^n}} \leq t^* - t_n \leq \frac{(t^{**}-t^*)\Delta^{2^n}}{M_1-\Delta^{2^n}}, \quad n \geq 0,$$

where $\theta = \frac{t^*}{t^{**}}m_1$, $\Delta = \frac{t^*}{t^{**}}M_1$ and provided that $\theta < 1$ and $\Delta < 1$.

(b) *If $t^* = t^{**}$ and there exist $m_2 > 0$ and $M_2 > 0$ such that $m_2 \leq \min\{D_2(t);\, t \in [t_0, t^*]\}$ and $M_2 \geq \max\{D_2(t);\, t \in [t_0, t^*]\}$, where $D_2(t) = \frac{(t^*-t)g'(t)-g(t)}{(t^*-t)g'(t)-2g(t)}$, then*

$$m_2^n t^* \leq t^* - t_n \leq M_2^n t^*, \quad n \geq 0,$$

provided that $m_2 < 1$ and $M_2 < 1$.

5. Example

We illustrate the above-mentioned with the following nonlinear Fredholm integral equation

$$x(s) = s^2 + \frac{1}{2}\int_0^1 s\,t^{13}\left(x(t)^{11/3} + x(t)^5\right)dt, \tag{3}$$

where $s \in [0, 1]$ and $x(s)$ is a solution to be determined.

Observe that solving Eq. (3) is equivalent to solving $\mathcal{F}(x) = 0$, where $\mathcal{F} : \Omega \subseteq \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$ is such that

$$[\mathcal{F}(x)](s) = x(s) - s^2 - \frac{1}{2}\int_0^1 s\,t^{13}\left(x(t)^{11/3} + x(t)^5\right)dt. \tag{4}$$

Notice that a solution $x^*(s)$ of Eq. (3) always satisfies

$$\|x^*(s)\| - \|s^2\| - \frac{1}{28}\left(\|x^*(s)\|^{11/3} + \|x^*(s)\|^5\right) \leq 0,$$

which is true provided that $\|x^*(s)\| \leq \rho_1 = 1.1146\ldots$ or $\|x^*(s)\| \geq \rho_2 = 1.6364\ldots$, where ρ_1 and ρ_2 are the two real positive roots of the scalar equation deduced from the last expression and given by $\frac{1}{28}\left(t^5 + t^{11/3}\right) - t + 1 = 0$. Thus, we can consider the domain

$$\Omega = \{x(s) \in \mathcal{C}([0,1]) :\ \|x(s)\| < \rho,\ s \in [0,1]\},$$

with $\rho \in (\rho_1, \rho_2)$, as domain for the operator $\mathcal{F}$.

Besides, as

$$[\mathcal{F}'(x)y](s) = y(s) - \frac{1}{2}\int_0^1 s\,t^{13}\left(\frac{11}{3}x(t)^{8/3} + 5x(t)^4\right)y(t)\,dt,$$

$$[\mathcal{F}''(x)(yz)](s) = -\frac{1}{2}\int_0^1 s\,t^{13}\left(\frac{88}{9}x(t)^{5/3} + 20x(t)^3\right)z(t)y(t)\,dt,$$

$$[\mathcal{F}'''(x)(yzw)](s) = -\frac{1}{2}\int_0^1 s\,t^{13}\left(\frac{440}{27}x(t)^{2/3} + 60x(t)^2\right)w(t)z(t)y(t)\,dt,$$

we have

$$\|\mathcal{F}'''(x) - \mathcal{F}'''(\widetilde{x})\| \leq \frac{1}{28}\left(\frac{240}{70}\left(\rho^{1/3} + \|\widetilde{x}\|^{1/3}\right)\|x-\widetilde{x}\|^{1/3} + 60\left(\rho + \|\widetilde{x}\|\right)\|x-\widetilde{x}\|\right),$$

so that $\mathcal{F}'''(x)$ is center ω-Lipschitz continuous at $\widetilde{x}$ with

$$\omega(t) = \frac{1}{28}\left(\frac{240}{70}\left(\rho^{1/3} + \|\widetilde{x}\|^{1/3}\right)t^{1/3} + 60\left(\rho + \|\widetilde{x}\|\right)t\right)$$

and we can apply Theorem 5 for guaranteeing the convergence of the method.

As the kernel of (3), $s\,t^{13}$, is separable, we can then determine the corresponding operator $[\mathcal{F}'(x)]^{-1}$. For this, we write $[\mathcal{F}'(x)y](s) = \ell(s)$, so that, if there exists $[\mathcal{F}'(x)]^{-1}$, we have

$$[\mathcal{F}'(x)]^{-1}\ell(s) = y(s) = \ell(s) + \frac{s}{2}\int_0^1 t^{13}\left(\frac{11}{3}x(t)^{8/3} + 5x(t)^4\right)y(t)\,dt.$$

If we now denote $\int_0^1 t^{13}\left(\frac{11}{3}x(t)^{8/3}+5x(t)^4\right)y(t)\,dt = \mathcal{I}$, multiply next-to-last equality by $s^{13}\left(\frac{11}{3}x(s)^{8/3}+5x(s)^4\right)$ and integrate it between 0 and 1, we obtain

$$\mathcal{I}=\frac{\int_0^1 s^{13}\left(\frac{11}{3}x(s)^{8/3}+5x(s)^4\right)\ell(s)\,ds}{1-\int_0^1 s^{14}\left(\frac{11}{3}x(s)^{8/3}+5x(s)^4\right)ds}$$

provided that

$$\int_0^1 s^{14}\left(\frac{11}{3}x(s)^{8/3}+5x(s)^4\right)ds \neq 1. \tag{5}$$

Therefore,

$$y(s)=[\mathcal{F}'(x)]^{-1}\ell(s)=\ell(s)+\frac{s}{2}\frac{\int_0^1 t^{13}\left(\frac{11}{3}x(t)^{8/3}+5x(t)^4\right)\ell(t)\,dt}{1-\int_0^1 t^{14}\left(\frac{11}{3}x(t)^{8/3}+5x(t)^4\right)dt}.$$

After that, if we consider, as it is usually done, the starting point $x_0(s)=s^2$ for Newton's method, it follows that $[\mathcal{F}'(x_0)]^{-1}$ exists, since condition (5) is satisfied at $x_0(s)=s^2$, $\left\|[\mathcal{F}'(x_0)]^{-1}\right\|\le 1.2602\ldots=\beta$, $\left\|[\mathcal{F}'(x_0)]^{-1}\mathcal{F}(x_0)\right\|\le 0.0557\ldots=\eta$ and $b_2=0.7820$.

Next, if we choose $\rho=3/2$ and $\widetilde{x}(s)=x_0(s)$, then $\gamma=0$, $\delta=2.1980\ldots$,

$$\psi(t)=(0.0442\ldots)-(0.7935\ldots)t+(0.3910\ldots)t^2+(0.3663\ldots)t^3+(0.1203\ldots)t^{10/3}+(0.2232\ldots)t^4,$$

$\alpha=0.4777\ldots$, $\psi(\alpha)=-0.1837\ldots<0$ and $B(x_0,s^*)\subset\Omega$, where $s^*=0.0575\ldots$ is the smallest positive zero of $\psi(t)$. Therefore, the convergence of Newton's method is guaranteed by Theorem 5 and taking into account the point $\widetilde{x}(s)$, where $\mathcal{F}'''(x)$ is center ω-Lipschitz continuous, as starting point for the method.

In addition, we can also guarantee the convergence of Newton's method starting at other points different from the point $\widetilde{x}(s)$ where $\mathcal{F}'''(x)$ is center ω-Lipschitz continuous, so that the domain of starting points is then increased when center conditions are required. For example, if we choose the starting point $x_0(s)=\frac{7}{10}s^2$, that also satisfies condition (5), then $\gamma=\|\widetilde{x}(s)-x_0(s)\|=3/10$, $\beta=1.0680\ldots$, $\eta=0.3309\ldots$, $b_2=0.3271\ldots$,

$$\begin{aligned}\psi(t)=&(0.3076\ldots)-(0.9604\ldots)t+(0.0695\ldots)t^2+(0.6342\ldots)t^3+(0.2232\ldots)t^4\\&+(0.0015\ldots)\sqrt[3]{10t+3}+(0.0150\ldots)\,t\,\sqrt[3]{10t+3}+(0.0502\ldots)\,t^2\,\sqrt[3]{10t+3}\\&+(0.0260\ldots)\,t^3\,\sqrt[3]{10t+3}+(0.0164\ldots)\,t^3\,\sqrt[3]{60t+18}\,t^3,\end{aligned}$$

$\alpha=0.5187\ldots$, $\psi(\alpha)=-0.0053\ldots<0$ and $B(x_0,t^*)\subset\Omega$, where $t^*=0.4627\ldots$ is the smallest positive zero of the last $\psi(t)$. Therefore, the convergence of Newton's method can be also guaranteed when the method starts at $x_0(s)\neq\widetilde{x}(s)$. Moreover, from Theorems 5 and 7, the domains of existence and uniqueness of solution are respectively

$$\{\nu\in\Omega:\|\nu(s)-x_0(s)\|\le 0.4627\ldots\}\quad\text{and}\quad\{\nu\in\Omega:\|\nu(s)-x_0(s)\|<0.5725\ldots\}.$$

On the other hand, we notice that condition (5) is satisfied in $B(x_0,t^*)$ for all $x(s)$, so that the sequence $\{x_n(s)\}$, given by Newton's method, is well-defined, so that we can apply the method for approximating a solution of Eq. (3). So, the direct application of Newton's method is

$$x_{n+1}(s)=x_n(s)-[\mathcal{F}'(x_n)]^{-1}[\mathcal{F}(x_n)](s)=s^2+\frac{s}{2}\frac{\mathcal{A}_n-\mathcal{B}_n+\mathcal{E}_n}{1-\mathcal{D}_n},$$

where

$$\mathcal{A}_n=\int_0^1 t^{13}\left(x_n(t)^{11/3}+x_n(t)^5\right)dt,\qquad \mathcal{B}_n=\int_0^1 t^{13}\left(\frac{11}{3}x_n(t)^{8/3}+5x_n(t)^4\right)x_n(t)\,dt,$$

$$\mathcal{D}_n=\int_0^1 t^{14}\left(\frac{11}{3}x_n(t)^{8/3}+5x_n(t)^4\right)dt,\qquad \mathcal{E}_n=\int_0^1 t^{15}\left(\frac{11}{3}x_n(t)^{8/3}+5x_n(t)^4\right)dt,$$

and $x^*(s)=(0.0567\ldots)s+s^2$ is the approximated solution obtained after five iterations and shown in Table 1 with stopping criterion $\|x_n(s)-x_{n-1}(s)\|<10^{-16}$. In Table 1, we can also see errors $\|x^*(s)-x_n(s)\|$ and sequence $\{\|[\mathcal{F}(x_n)](s)\|\}$. From the last, observe that $x^*(s)$ is a good approximation of a solution of (3). Moreover, see Fig. 1, the approximated solution $x^*(s)$ lies within the existence domain of solution obtained above.

6. Particular cases

Next, we see two particular cases of the last analysis: $\omega(t)=Qt$ and $\omega(t)=Qt^p$ with $Q>0$ and $p\in[0,1]$. In the first case, we say that the operator $F^{(k)}(x)$ is center Lipschitz continuous at $\widetilde{x}$ and, in the second, $F^{(k)}(x)$ is center Hölder continuous at $\widetilde{x}$.

Table 1
Approximated solution $x^*(s)$ of (3), absolute errors and $\{\|[\mathcal{F}(x_n)](s)\|\}$.

n	$x_n(s)$	$\|x^*(s) - x_n(s)\|$	$\|[\mathcal{F}(x_n)](s)\|$
1	$(0.02907312167342\ldots)s + s^2$	$2.7681\ldots \times 10^{-2}$	$2.1285\ldots \times 10^{-2}$
2	$(0.05637087524162\ldots)s + s^2$	$3.8408\ldots \times 10^{-4}$	$2.9114\ldots \times 10^{-4}$
3	$(0.05675488103969\ldots)s + s^2$	$7.9640\ldots \times 10^{-8}$	$6.0357\ldots \times 10^{-8}$
4	$(0.05675496068006\ldots)s + s^2$	$3.4277\ldots \times 10^{-15}$	$2.7825\ldots \times 10^{-15}$
5	$(0.05675496068006\ldots)s + s^2$		

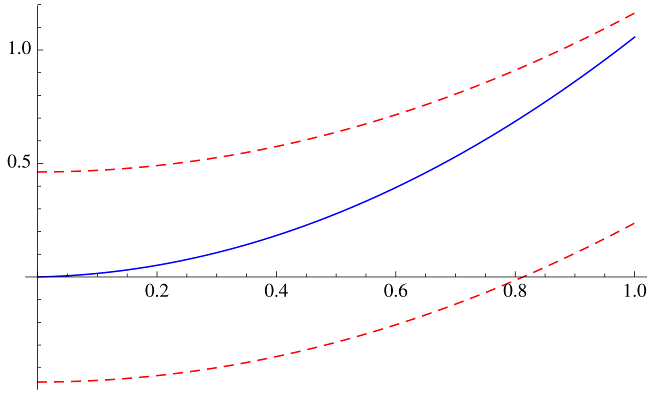

Fig. 1. Graph (the solid line) of the approximated solution $x^*(s)$ of Eq. (3).

6.1. $F^{(k)}$ is center Lipschitz continuous at an auxiliary point

We consider that conditions (P1)–(P3) are relaxed, respectively, to the following conditions:

(R1) There exists the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ with $\|\Gamma_0\| \le \beta$ and $\|\Gamma_0 F(x_0)\| \le \eta$; moreover, $\|F^{(i)}(x_0)\| \le b_i$ with $i = 2, 3, \ldots, k-1$ and $k \ge 3$.
(R2) There exists $\widetilde{x} \in \Omega$ such that $\|x_0 - \widetilde{x}\| = \gamma$, where $x_0 \in \Omega$, and $\|F^{(k)}(\widetilde{x})\| \le \delta$.
(R3) There exists $Q > 0$ such that $\|F^{(k)}(x) - F^{(k)}(\widetilde{x})\| \le Q\|x - \widetilde{x}\|$ for $x \in \Omega$.

Observe that $\omega(t) = Qt$, with $Q > 0$, in this case, so that we can find a scalar function $\widehat{\psi}(t)$ satisfying the hypotheses of Theorem 5 by solving the following initial value problem

$$\begin{cases} y^{(k)}(t) = \delta + Q(t - t_0 + \gamma), \\ y(t_0) = \dfrac{\eta}{\beta}, \quad y'(t_0) = -\dfrac{1}{\beta}, \\ y''(t_0) = b_2, \quad y'''(t_0) = b_3, \quad \ldots, \quad y^{(k-1)}(t_0) = b_{k-1}, \end{cases}$$

since we can choose $-\frac{1}{\widehat{\psi}'(t_0)} = \beta$, $-\frac{\widehat{\psi}(t_0)}{\widehat{\psi}'(t_0)} = \eta$ and $\widehat{\psi}^{(i)}(t_0) = b_i$, for $i = 2, 3, \ldots, k-1$, from (P1) and (R1). In addition, the next result is established.

Theorem 9. *The last initial value problem has only one solution $\widehat{\phi}(t) \in \mathcal{C}^j([t_0 - \gamma, +\infty))$, with $j \ge k \ge 3$, which is:*

$$\widehat{\phi}(t) = \frac{Q}{(k+1)!}(t - t_0)^{k+1} + \frac{1}{k!}(\delta + Q\gamma)(t - t_0)^k + \sum_{i=2}^{k-1} \frac{b_i}{i!}(t - t_0)^i - \frac{t - t_0}{\beta} + \frac{\eta}{\beta},$$

where Q, γ, δ, $\beta \ne 0$, η and $b_2, b_3, \ldots, b_{k-1}$ are nonnegative real numbers. In addition, $\widehat{\phi}(t)$ satisfies conditions (P1)–(P3).

Notice that function $\widehat{\phi}(t)$ satisfies $\phi(t + t_0) = \phi(t)$ with

$$\phi(t) = \frac{Q}{(k+1)!}t^{k+1} + \frac{1}{k!}(\delta + Q\gamma)t^k + \sum_{i=2}^{k-1} \frac{b_i}{i!}t^i - \frac{t}{\beta} + \frac{\eta}{\beta}, \tag{6}$$

so that we can choose, in practice, $t_0 = 0$, as we have done before for $\widehat{\psi}(t)$.

Observe also that function $\psi(t)$ is reduced to function $\phi(t)$ if $\omega(t) = Qt$.

As a consequence of the above, the semilocal convergence of Newton's method is guaranteed in the Banach space X, since function $\phi(t)$ satisfies the conditions of Theorem 5, as we see in next result.

Theorem 10. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \longrightarrow Y$ a nonlinear q ($q \geq 2$) times continuously differentiable operator on a nonempty open convex domain Ω and $\phi(t)$ be polynomial defined in (6). Suppose that conditions (R1)–(R3) are satisfied, there exists a root $\alpha > 0$ of $\phi'(t) = 0$ such that $\phi(\alpha) \leq 0$, and $B(x_0, t^*) \subset \Omega$, where t^* is the smallest positive root of $\phi(t) = 0$. Then, the sequence $\{x_n\}$, given by Newton's method (1), converges to a solution x^* of $F(x) = 0$ starting at x_0. Moreover, $x_n, x^* \in \overline{B(x_0, t^*)}$ and*

$$\|x^* - x_n\| \leq t^* - t_n, \quad n \geq 0,$$

where $t_n = N_\phi(t_{n-1})$, with $n \in \mathbb{N}$ and $t_0 = 0$.

6.2. Example

We illustrate Section 6.1 with the following nonlinear Fredholm integral equation

$$x(s) = \frac{1}{2}\sin(\pi s) + \int_0^1 \cos(\pi s)\sin(\pi t)x(t)^6\,dt, \tag{7}$$

where $s \in [0, 1]$ and $x(s)$ is a solution to be determined.

Observe that solving Eq. (7) is equivalent to solving $\mathcal{F}(x) = 0$, where $\mathcal{F} : \Omega \subseteq \mathcal{C}([0, 1]) \longrightarrow \mathcal{C}([0, 1])$ is such that

$$[\mathcal{F}(x)](s) = x(s) - \frac{1}{2}\sin(\pi s) - \int_0^1 \cos(\pi s)\sin(\pi t)x(t)^6\,dt.$$

In addition, a solution $x^*(s)$ of Eq. (7) always satisfies

$$\|x^*(s)\| - \frac{1}{2}\|\sin(\pi s)\| - \frac{2}{\pi}\|x^*(s)\|^6 \leq 0,$$

which is true provided that $\|x^*(s)\| \leq \rho_1 = 0.5113\ldots$ or $\|x^*(s)\| \geq \rho_2 = 0.9404\ldots$, where ρ_1 and ρ_2 are the two real positive roots of the scalar equation deduced from the last expression and given by $\frac{2}{\pi}t^6 - t + \frac{1}{2} = 0$. Thus, we can consider the domain

$$\Omega = \{x(s) \in \mathcal{C}([0, 1]) : \|x(s)\| < \rho,\ s \in [0, 1]\},$$

with $\rho \in (\rho_1, \rho_2)$, as domain for the operator $\mathcal{F}$.

Besides, as

$$[\mathcal{F}'(x)y](s) = y(s) - 6\int_0^1 \cos(\pi s)\sin(\pi t)x(t)^5 y(t)\,dt,$$

$$[\mathcal{F}''(x)(yz)](s) = -30\int_0^1 \cos(\pi s)\sin(\pi t)x(t)^4 z(t)y(t)\,dt,$$

$$[\mathcal{F}'''(x)(yzw)](s) = -120\int_0^1 \cos(\pi s)\sin(\pi t)x(t)^3 w(t)z(t)y(t)\,dt,$$

$$[\mathcal{F}^{(iv)}(x)(yzwd)](s) = -360\int_0^1 \cos(\pi s)\sin(\pi t)x(t)^2 d(t)w(t)z(t)y(t)\,dt,$$

we have

$$\|\mathcal{F}^{(iv)}(x) - \mathcal{F}^{(iv)}(\widetilde{x})\| \leq \frac{720}{\pi}(\rho + \|\widetilde{x}\|)\,\|x - \widetilde{x}\|,$$

so that $\mathcal{F}^{(iv)}(x)$ is center Lipschitz continuous at $\widetilde{x}$ with $Q = \frac{720}{\pi}(\rho + \|\widetilde{x}\|)$ and we can then apply Theorem 10 for guaranteeing the convergence of the method.

Hence, if we consider, as it is usually done, the starting point $x_0(s) = \frac{1}{2}\sin(\pi s)$ for Newton's method, we have $\beta = 1.0585\ldots$, $\eta = 0.0048\ldots$, $b_2 = 2/\pi$ and $b_3 = 45/8$. If we now choose $\rho = 3/4$ and $\widetilde{x}(s) = x_0(s)$, then $\gamma = 0$, $\delta = \frac{120}{\pi}$,

$$\phi(t) = (0.0045\ldots) - (0.9446\ldots)t + (0.3183\ldots)t^2 + (0.9375\ldots)t^3 + (1.5915\ldots)t^4 + (2.3873\ldots)t^5,$$

$\alpha = 0.3389\ldots$, $\phi(\alpha) = -0.2108\ldots < 0$ and $B(x_0, s^*) \subset \Omega$, where $s^* = 0.0048\ldots$ is the smallest positive zero of $\phi(t)$. Therefore, the convergence of Newton's method is guaranteed by Theorem 10 and taking into account the point $\widetilde{x}(s)$, where $\mathcal{F}^{(iv)}(x)$ is center Lipschitz continuous, as starting point for the method.

In addition, we can also guarantee the convergence of Newton's method starting at other points different from the point $\widetilde{x}(s)$ where $\mathcal{F}^{(iv)}(x)$ is center Lipschitz continuous, so that the domain of starting points is then increased when center conditions are required. For example, if we choose the starting point $x_0(s) = \frac{9}{20}\sin(\pi s)$, then $\gamma = \|\widetilde{x}(s) - x_0(s)\| = \frac{1}{20}$, $\beta = 1.0346\ldots$, $\eta = 0.0025\ldots$, $b_2 = 0.4176\ldots$, $b_3 = 4.1006\ldots$,

$$\phi(t) = (0.0024\ldots) - (0.9665\ldots)t + (0.2088\ldots)t^2 + (0.6834\ldots)t^3 + (2.1883\ldots)t^4 + (2.3873\ldots)t^5,$$

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Table 2
Absolute errors and $\{\|[\mathcal{F}(x_n)](s)\|\}$ for (7).

n	$\|x^*(s)-x_n(s)\|$	$\|[\mathcal{F}(x_n)](s)\|$
1	$5.2053\ldots \times 10^{-4}$	$5.2033\ldots \times 10^{-4}$
2	$1.2329\ldots \times 10^{-8}$	$1.2324\ldots \times 10^{-8}$

$\alpha = 0.3520\ldots$, $\phi(\alpha) = -0.2356\ldots < 0$ and $B(x_0, t^*) \subset \Omega$, where $t^* = 0.0025\ldots$ is the smallest positive zero of the last $\phi(t)$. Thus, the convergence of Newton's method can be also guaranteed when the method starts at $x_0(s) \neq \widetilde{x}(s)$. Moreover, the domains of existence and uniqueness of solution are respectively

$$\{\nu \in \Omega : \|\nu(s) - x_0(s)\| \leq 0.0025\ldots\} \quad \text{and} \quad \{\nu \in \Omega : \|\nu(s) - x_0(s)\| < 0.5625\ldots\}.$$

After that, we apply Newton's method from $x_0(s) = \frac{9}{20}\sin(\pi s)$ to approximate a solution $x^*(s)$ of integral equation (7) and obtain the approximation

$$x^*(s) = (0.0045\ldots)\cos \pi s + \frac{1}{2}\sin \pi s$$

after three iterations with stopping criterion $\|x_n(s) - x_{n-1}(s)\| < 10^{-16}$. In Table 2, we show errors $\|x^*(s) - x_n(s)\|$ and sequence $\{\|[\mathcal{F}(x_n)](s)\|\}$. From the last, observe that $x^*(s)$ is a good approximation of a solution of Eq. (7).

6.3. $F^{(k)}$ is center Hölder continuous at an auxiliary point

We consider that conditions (P1)–(P3) are now relaxed, respectively, to next conditions:

(S1) There exists the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$; moreover, $\|F^{(i)}(x_0)\| \leq b_i$ with $i = 2, 3, \ldots, k-1$ and $k \geq 3$.
(S2) There exists $\widetilde{x} \in \Omega$ such that $\|x_0 - \widetilde{x}\| = \gamma$, where $x_0 \in \Omega$, and $\|F^{(k)}(\widetilde{x})\| \leq \delta$.
(S3) There exist $Q > 0$ and $p \in [0, 1]$ such that $\|F^{(k)}(x) - F^{(k)}(\widetilde{x})\| \leq Q\|x - \widetilde{x}\|^p$ for $x \in \Omega$.

As in the previous case and taking now into account $\omega(t) = Qt^p$ with $Q > 0$ and $p \in [0, 1]$, we can find, from conditions (S1)–(S3) and Theorem 5, a real function $\widehat{\psi}(t)$ by solving next initial value problem:

$$\begin{cases} y^{(k)}(t) = \delta + Q(t - t_0 + \gamma)^p, \\ y(t_0) = \dfrac{\eta}{\beta}, \quad y'(t_0) = -\dfrac{1}{\beta}, \\ y''(t_0) = b_2, \quad y'''(t_0) = b_3, \quad \ldots, \quad y^{(k-1)}(t_0) = b_{k-1}. \end{cases}$$

So, the following result is then established.

Theorem 11. *The last initial value problem has only one solution $\widehat{\varphi}(t) \in \mathcal{C}^j([t_0 - \gamma, +\infty))$, with $j \geq k \geq 3$, which is:*

$$\begin{aligned} \widehat{\varphi}(t) = {} & \frac{Q}{(1+p)(2+p)\cdots(k+p)}(t - t_0 + \gamma)^{k+p} \\ & + \frac{\delta}{k!}(t - t_0)^k \\ & + \sum_{i=2}^{k-1} \frac{1}{i!}\left(b_i - \frac{Q\gamma^{k-i+p}}{(1+p)(2+p)\cdots(k-i+p)}\right)(t - t_0)^i \\ & - \left(\frac{1}{\beta} + \frac{Q\gamma^{k-1+p}}{(1+p)(2+p)\cdots(k-1+p)}\right)(t - t_0) \\ & + \frac{\eta}{\beta} - \frac{Q\gamma^{k+p}}{(1+p)(2+p)\cdots(k+p)}, \end{aligned} \tag{8}$$

where Q, $p \in [0, 1]$, γ, δ, $\beta \neq 0$, η and $b_2, b_3, \ldots, b_{k-1}$ are nonnegative real numbers. In addition, $\widehat{\varphi}(t)$ satisfies conditions (P1)–(P3).

Taking into account that $\widehat{\varphi}(t)$ satisfies $\widehat{\varphi}(t + t_0) = \varphi(t)$ with

$$\begin{aligned} \varphi(t) = {} & \frac{Q}{(1+p)(2+p)\cdots(k+p)}(t + \gamma)^{k+p} \\ & + \frac{\delta}{k!}t^k \end{aligned}$$

$$+\sum_{i=2}^{k-1}\frac{1}{i!}\left(b_i-\frac{Q\gamma^{k-i+p}}{(1+p)(2+p)\cdots(k-i+p)}\right)t^i$$
$$-\left(\frac{1}{\beta}+\frac{Q\gamma^{k-1+p}}{(1+p)(2+p)\cdots(k-1+p)}\right)t$$
$$+\frac{\eta}{\beta}-\frac{Q\gamma^{k+p}}{(1+p)(2+p)\cdots(k+p)},$$

we can then choose, in practice, $t_0=0$, as we have done before for $\widehat{\psi}(t)$ and $\widehat{\phi}(t)$.

Observe that, if $\omega(t)=Qt^p$, then function $\psi(t)$ is reduced to function $\varphi(t)$ and it is reduced to $\phi(t)$ if $p=1$.

Finally, the semilocal convergence of Newton's method is then guaranteed in the Banach space X, since function $\varphi(t)$ satisfies the conditions of Theorem 5, as we see in the following.

Theorem 12. *Let X and Y be two Banach spaces and $F:\Omega\subseteq X\longrightarrow Y$ a nonlinear q ($q\geq 2$) times continuously differentiable operator on a nonempty open convex domain Ω and $\varphi(t)$ be polynomial defined in (8). Suppose that conditions (S1)–(S3) are satisfied, there exists a root $\alpha>0$ of $\varphi'(t)=0$ such that $\varphi(\alpha)\leq 0$, and $B(x_0,t^*)\subset\Omega$, where t^* is the smallest positive root of $\varphi(t)=0$. Then, the sequence $\{x_n\}$, given by Newton's method (1), converges to a solution x^* of $F(x)=0$ starting at x_0. Moreover, $x_n, x^*\in\overline{B(x_0,t^*)}$ and*

$$\|x^*-x_n\|\leq t^*-t_n,\quad n\geq 0,$$

where $t_n=N_\varphi(t_{n-1})$, with $n\in\mathbb{N}$ and $t_0=0$.

6.4. Example

In our last example, we illustrate the analysis given in this section with the following nonlinear Fredholm integral equation

$$x(s)=\frac{s}{2}+\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{e}^{st}x(t)^{19/5}dt, \tag{9}$$

where $s\in\left[-\frac{1}{2},\frac{1}{2}\right]$ and $x(s)$ is a solution to be determined.

Observe that, in this case, kernel $\mathcal{K}(s,t)=\mathrm{e}^{st}$ is nonseparable, so that the application of Newton's method to solve Eq. (9) is not easy. To solve this difficulty, we first approximate $\mathcal{K}(s,t)=\mathrm{e}^{st}$ by Taylor's series. So,

$$\mathcal{K}(s,t)=\mathrm{e}^{st}=\widetilde{\mathcal{K}}(s,t)+\mathcal{R}(\epsilon,s,t);\quad \widetilde{\mathcal{K}}(s,t)=\sum_{i=0}^{j-1}\frac{s^i t^i}{i!},\quad \mathcal{R}(\epsilon,s,t)=\frac{\mathrm{e}^{s\epsilon}}{j!}s^j t^j,$$

where $\epsilon\in(\min\{0,t\},\max\{0,t\})$, consider the integral equation

$$x(s)=\frac{s}{2}+\int_{-\frac{1}{2}}^{\frac{1}{2}}\widetilde{\mathcal{K}}(s,t)x(t)^{\frac{19}{5}}\,dt,\quad s\in\left[-\frac{1}{2},\frac{1}{2}\right] \tag{10}$$

and solve it by Newton's method.

For this, we take into account that solving Eq. (10) is equivalent to solving $\mathcal{F}(x)=0$, where $\mathcal{F}:\Omega\subseteq\mathcal{C}\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)\longrightarrow\mathcal{C}\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)$ is such that

$$[\mathcal{F}(x)](s)=x(s)-\frac{s}{2}-\int_{-\frac{1}{2}}^{\frac{1}{2}}\widetilde{\mathcal{K}}(s,t)x(t)^{\frac{19}{5}}\,dt.$$

Besides, a solution $x^*(s)$ of Eq. (10) always satisfies

$$\|x^*(s)\|-\left\|\frac{s}{2}\right\|-(1.0104\ldots)\|x^*(s)\|^{19/5}\leq 0,$$

which is true provided that $\|x^*(s)\|\leq\rho_1=0.2556\ldots$ or $\|x^*(s)\|\geq\rho_2=0.8848\ldots$, where ρ_1 and ρ_2 are the two real positive roots of the scalar equation deduced from the last expression and given by $(1.0104\ldots)t^{19/5}-t+\frac{1}{4}=0$. Thus, we can consider the domain

$$\Omega=\left\{x(s)\in\mathcal{C}\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right):\ \|x(s)\|<\rho,\ s\in\left[-\frac{1}{2},\frac{1}{2}\right]\right\},$$

with $\rho\in(\rho_1,\rho_2)$, as domain for the operator $\mathcal{F}$.

Table 3
Absolute errors for (10) and $\{\|[\mathcal{F}_*(x_n)](s)\|\}$.

n	$\|x^*(s)-x_n(s)\|$	$\|[\mathcal{F}_*(x_n)](s)\|$
1	$1.0748\ldots \times 10^{-5}$	$1.0711\ldots \times 10^{-5}$
2	$2.2038\ldots \times 10^{-12}$	$7.5081\ldots \times 10^{-12}$

In addition,

$$[\mathcal{F}'(x)y](s) = y(s) - \frac{19}{5}\int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{\mathcal{K}}(s,t)x(t)^{14/5}y(t)\,dt,$$

$$[\mathcal{F}''(x)(yz)](s) = -\frac{266}{25}\int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{\mathcal{K}}(s,t)x(t)^{9/5}z(t)y(t)\,dt,$$

$$[\mathcal{F}'''(x)(yzw)](s) = -\frac{2394}{125}\int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{\mathcal{K}}(s,t)x(t)^{4/5}w(t)z(t)y(t)\,dt$$

and

$$\|\mathcal{F}'''(x) - \mathcal{F}'''(\widetilde{x})\| \le (19.3512\ldots)\left(\rho^{3/5} + \rho^{2/5}\|\widetilde{x}\|^{1/5} + \rho^{1/5}\|\widetilde{x}\|^{2/5} + \|\widetilde{x}\|^{3/5}\right)\|x-\widetilde{x}\|^{1/5}.$$

Thus, $\mathcal{F}'''(x)$ is center Hölder continuous at $\widetilde{x}$ with

$$Q = (19.3512\ldots)\left(\rho^{3/5} + \rho^{2/5}\|\widetilde{x}\|^{1/5} + \rho^{1/5}\|\widetilde{x}\|^{2/5} + \|\widetilde{x}\|^{3/5}\right) \quad \text{and} \quad p = \frac{1}{5},$$

so that we can apply Theorem 12 for guaranteeing the convergence of the method.

If we consider, as it is usually done, the starting point $x_0(s) = \frac{s}{2}$ for Newton's method, we have $\beta = 1.0859\ldots$, $\eta = 0.0056\ldots$ $b_2 = 0.8866\ldots$. Moreover, if we choose $\rho = 3/4$ and $\widetilde{x}(s) = x_0(s)$, then $\gamma = 0$, $\delta = 6.3838\ldots$,

$$\varphi(t) = (0.0052\ldots) - (0.9208\ldots)t + (0.4433\ldots)t^2 + (1.0639\ldots)t^3 + (5.7141\ldots)t^{16/5},$$

$\alpha = 0.2104\ldots$, $\varphi(\alpha) = -0.1200\ldots < 0$ and $B(x_0, s^*) \subset \Omega$, where $s^* = 0.0056\ldots$ is the smallest positive zero of $\varphi(t)$. Therefore, the convergence of Newton's method is guaranteed by Theorem 12 and taking into account the point $\widetilde{x}(s)$, where $\mathcal{F}'''(x)$ is center Hölder continuous, as starting point for the method.

Furthermore, we can also guarantee the convergence of Newton's method starting at other points different from the point $\widetilde{x}(s)$ where $\mathcal{F}'''(x)$ is center Hölder continuous, so that the domain of starting points is then increased when center conditions are required. For example, if we choose the starting point $x_0(s) = \frac{9}{20}s$, then $\gamma = \|\widetilde{x}(s) - x_0(s)\| = 1/40$, $\beta = 1.0626\ldots$, $\eta = 0.0302\ldots$, $b_2 = 0.7334\ldots$,

$$\begin{aligned}\varphi(t) &= (0.0284\ldots) - (0.9465\ldots)t + (0.1262\ldots)t^2 + (1.0639\ldots)t^3 \\ &\quad + ((0.00004\ldots) + (0.0051\ldots)t + (0.2049\ldots)t^2 + (2.7323\ldots))\sqrt[5]{1+40t}\end{aligned}$$

$\alpha = 0.2096\ldots$, $\varphi(\alpha) = -0.0993\ldots < 0$ and $B(x_0, t^*) \subset \Omega$, where $t^* = 0.0308\ldots$ is the smallest positive zero of the last $\varphi(t)$. Therefore, the convergence of Newton's method can be also guaranteed when the method starts at $x_0(s) \neq \widetilde{x}(s)$. Besides, the domains of existence and uniqueness of solution are respectively

$$\{\nu \in \Omega : \|\nu(s) - x_0(s)\| \le 0.0308\ldots\} \quad \text{and} \quad \{\nu \in \Omega : \|\nu(s) - x_0(s)\| < 0.3463\ldots\}.$$

Next, we apply Newton's method from $x_0(s) = \frac{9}{20}s$ to approximate a solution $x^*(s)$ of integral equation (10) and obtain the approximation

$$x^*(s) = (5.0044\ldots \times 10^{-1})s + (1.3813\ldots \times 10^{-5})s^3 + (1.3743\ldots \times 10^{-7})s^5$$

after three iterations with stopping criterion $\|x_n(s) - x_{n-1}(s)\| < 10^{-16}$. In Table 3, we show errors $\|x^*(s) - x_n(s)\|$ and sequence $\{\|[\mathcal{F}_*(x_n)](s)\|\}$, where $\mathcal{F}_*$ is the operator $\mathcal{F}_* : \mathcal{C}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \longrightarrow \mathcal{C}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ associated with (9),

$$[\mathcal{F}_*(x)](s) = x(s) - \frac{s}{2} - \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{st}x(t)^{\frac{19}{5}}\,dt.$$

From the last, observe that $x^*(s)$ is a good approximation of a solution of (9).

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